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The Geometry of the Super KP Flows

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Abstract. A supersymmetric generalization of the Krichever map is used to construct algebro-geometric solutions to the various super Kadomtsev-Petviashvili (SKP) hierarchies. The geometric data required consist of a suitable algebraic supercurve of genus g (generally *not* a super Riemann surface) with a distinguished point and local coordinates (z, θ) there, and a generic line bundle of degree $g-1$ with a local trivialization near the point. The resulting solutions to the Manin-Radul SKP system describe coupled deformations of the line bundle and the supercurve itself, in contrast to the ordinary KP system which deforms line bundles but not curves. Two new SKP systems are introduced: an integrable "Jacobian" system whose solutions describe genuine Jacobian flows, deforming the bundle but not the curve; and a nonintegrable "maximal" system describing independent deformations of bundle and curve. The Kac-van de Leur SKP system describes the same deformations as the maximal system, but in a different parametrization.

1. Introduction

The theory of the generalized KdV equations, or the KP hierarchy, stands at the crossroads of several flourishing branches of modern mathematics and physics: Riemann surfaces, algebraic geometry, integrable systems, loop groups, conformal field theory, string theory, and quantum gravity. The centerpiece of the theory is the construction of algebro-geometric solutions to this infinite system of nonlinear differential equations from geometric data, and the dual interpretation of the solutions as flows in the moduli space of geometric data or in an infinite-dimensional Grassmannian [1–6]. The geometric "Krichever" data consist of a Riemann surface with a choice of local coordinate near a distinguished point, and a generic line bundle of degree $g-1$ with a choice of local trivialization, and the flows deform the line bundle. The Krichever construction which produces the solutions has become a basic tool in the operator formalism of conformal field theory because of its utility for describing deformations of the geometric data. In particular it is closely connected to the action of the Virasoro algebra.

Various supersymmetric generalizations of the KP hierarchy have been proposed. Of these, the SKP hierarchy of Manin and Radul (MRSKP hierarchy) has attracted the most attention [7]. The algebraic theory of this hierarchy is now well understood, particularly its integrability and the conditions for the unique solvability of its initial value problem [8]. The interpretation of the hierarchy in terms of flows on a super Grassmannian has also been discussed [9]. Although it has been generally assumed that algebro-geometric solutions must arise from super Riemann surfaces in some way (because of the appearance of the supersymmetric derivative operator D throughout the theory), this has never been demonstrated even though the elements of the super Krichever construction have been developed [10–13]. Some solutions have been obtained by Mulase [8] and by Pakuliak [14] in terms of the super elliptic functions [15] on supertori, indicating that such an algebro-geometric construction should exist in genus 1. In contrast, Radul [16] showed that solutions of the MRSKP hierarchy can be obtained from pairs of solutions to ordinary KP, raising the question of how two sets of ordinary Krichever data are to be related to the presumed super Krichever data.

A supersymmetric generalization of the approach to the KP hierarchy via loop groups, affine Lie algebras, and bosonization has been given by Kac and van de Leur, who arrived at a quite different formulation of a SKP hierarchy (KVSKP hierarchy) as Hirota bilinear equations [17, 18]. This KVSKP hierarchy has been studied by Bergvelt [19], who explained its geometric interpretation in terms of orbits of the general linear supergroup action on a super Grassmannian. This had been a confusing issue because the ordinary KP theory makes use of the projective embedding of the Grassmannian given by the Plücker coordinates, which does not generalize to the super Grassmannian. However, the relation of this KVSKP hierarchy to that of Manin and Radul, and the possibility of constructing algebro-geometric solutions, remained unclear. A similar formulation of a SKP hierarchy using the language of superconformal field theory and a different, supersymmetric bosonization was given by LeClair [20].

In this paper we will explain the relation between these different SKP hierarchies, and use the super Krichever construction to obtain the algebro-geometric solutions. The geometric Krichever data which generate a solution consist of a $(1|1)$-dimensional supermanifold (supercurve), which is *not* a super Riemann surface except in the case of genus 1, with given local coordinates near a distinguished point, and a line bundle satisfying certain cohomology conditions (these restrict its degree to be $g-1$ but also give a constraint on the supercurve, unlike the KP case) and with a given local trivialization near the point. In contrast to the ordinary case, the SKP hierarchies describe deformations of the supermanifold as well as the line bundle. Thus, these flows take place not in the Picard variety of a fixed supermanifold, but in the universal Picard bundle over the moduli space of supercurves, whose fiber at any curve is its Picard variety. Arbitrary deformations of the bundle are possible, but only those deformations of the supercurve which preserve projectedness: changes in the patching of the odd coordinate but not the even one. The KVSKP hierarchy includes all deformations of these types, which are not integrable since deformations of the supermanifold and bundle do not generally commute. The MRSKP hierarchy describes a special subset of the deformations in which changes in the supermanifold are coupled to changes in the bundle in such a way that the resulting flows do commute. Neither hierarchy is a precise geometric analogue of the ordinary KP hierarchy which, by deforming the line bundle only, can be used to solve the Schottky problem of characterizing the Jacobian varieties (here viewed as the Picard varieties of line bundles) of Riemann surfaces. We will introduce a new, integrable SKP system which provides this missing analogue. We also generalize the theory to families of Krichever data, which is to say nonsplit supermanifolds, and observe that in this case the geometric construction of the solutions requires not just a local trivialization, but actually a choice of a particular transition function, for the line bundle. Equivalently, this amounts to a choice of connection in the universal Picard bundle.

The paper is organized as follows. Section 2 is a review of the Krichever theory of algebro-geometric solutions to the ordinary KP hierarchy, with emphasis on the formulation in terms of pseudodifferential operators, commutative rings of ordinary differential operators, and their deformations rather than the alternative treatment in terms of tau functions. Section 3 contains the generalization to the super case, with the focus on the MRSKP hierarchy. We discuss the cohomology conditions to be imposed on the geometric data and their consequences, the definition of the super Grassmannian, the way in which the supersymmetry relation $D^2=\partial_x$ leads to deformations of the supermanifold as well as the line bundle on it, and the extension to (nonsplit) families of geometric data. Section 4 modifies the construction so that only the line bundles are deformed by the flows, leading to the new "Jacobian" SKP hierarchy introduced here. Section 5 further generalizes the construction to include all deformations of the type described above. The resulting "maximal" SKP hierarchy is shown to be equivalent to the KVSKP hierarchy. Section 6 contains the conclusions and directions for further research.

2. The KP Hierarchy and the Krichever Construction

We begin with a review of the KP hierarchy and the construction of algebro-geometric solutions by means of the Krichever map. The KP hierarchy is a set of equations for the deformation of a pseudodifferential operator

$$L=\partial+u_2\partial^{-1}+u_3\partial^{-2}+\dots. \tag{1}$$

Here $\partial=d/dx$, and the coefficients $u_i(x,t)$ should be regarded as formal power series in x and in the infinitely many deformation parameters $t_1,t_2,\dots$ (the question of convergence of these series is not of central importance in the geometric theory). Such pseudodifferential operators are multiplied by using the generalized Leibniz rule

$$\partial^n\cdot f(x)=\sum_{i=0}^{\infty}\binom{n}{i}(\partial^i f)\partial^{n-i}, \tag{2}$$

$$\binom{n}{i}=\frac{\Gamma(n+1)}{\Gamma(i+1)\Gamma(n-i+1)}. \tag{3}$$

The absence of a term $u_1\partial^0$ in L is necessary and sufficient for the existence of a pseudodifferential operator

$$S=1+s_1\partial^{-1}+s_2\partial^{-2}+\dots \tag{4}$$

with

$$S^{-1}LS=\partial. \tag{5}$$

The KP hierarchy is the infinite set of commuting flows on the space of operators L given by

$$\frac{\partial L}{\partial t_n}=[L^n_+,L]=-[L^n_-,L],\tag{6}$$

where L^n_+ denotes the differential operator part of L^n, obtained by dropping all terms containing negative powers of ∂, and $L^n_-=L^n-L^n_+$ consists of the terms dropped. These equations are equivalent to

$$\frac{\partial S}{\partial t_n}=-L^n_-S=-(S\partial^n S^{-1})_-S.\tag{7}$$

Algebro-geometric solutions to the KP hierarchy are constructed from sets of Krichever data $(M,p,\mathscr{L},z,\phi)$, where M is a compact connected Riemann surface (more generally, an irreducible algebraic curve) of some genus g, p is a (smooth) point of M, z is a local coordinate vanishing at p, $\mathscr{L}$ is a line bundle (more generally, a torsion free rank 1 sheaf) on M satisfying the cohomology conditions $H^0(M,\mathscr{L})=H^1(M,\mathscr{L})=0$, and ϕ is a local trivialization of $\mathscr{L}$ over a neighborhood, say $U=\{|z|<1\}$, of p. The cohomology conditions on $\mathscr{L}$, together with the Riemann-Roch theorem

$$\dim H^0(M,\mathscr{L})-\dim H^1(M,\mathscr{L})=\deg\mathscr{L}+1-g,\tag{8}$$

imply that $\deg\mathscr{L}=g-1$, so that $\mathscr{L}$ belongs to the same connected component $\mathrm{Pic}^{g-1}M$ of $\mathrm{Pic}M$ as the spin bundles $K^{1/2}$, K being the canonical bundle of M. More importantly, the Riemann-Roch theorem for $\mathscr{L}\otimes\mathscr{O}(p)$ gives

$$\dim H^0(M,\mathscr{L}\otimes\mathscr{O}(p))-\dim H^1(M,\mathscr{L}\otimes\mathscr{O}(p))=1,\tag{9}$$

so that although $\mathscr{L}$ has no holomorphic sections, it does have a unique (up to normalization) section holomorphic except for a simple pole at p. (Here we use the fact that tensoring with $\mathscr{O}(p)$ increases $\dim H^0$ by at most unity.) Using the trivialization ϕ we can represent this section as a function $s(z)$ in the chart U: $s(z)=z^{-1}+\text{holomorphic}$. Similarly, $\dim H^0(M,\mathscr{L}\otimes\mathscr{O}(np))=n$ and $H^1(M,\mathscr{L}\otimes\mathscr{O}(np))=0$ for all $n>0$, so that there are unique (up to normalization and linear combination with sections having lower-order poles) sections holomorphic except for a pole at p of any positive order. Via the trivialization ϕ, the space $H^0(M\backslash p,\mathscr{L})$ spanned by such sections becomes a space of functions on the circle $|z|=1$ bounding U, in fact a point W of the Grassmannian Gr consisting of all closed subspaces of $L^2(S^1)$ for which the projection onto $H_+=\mathrm{span}\{z^{-1},z^{-2},\dots\}$ is Fredholm of index zero. W actually belongs to the big-cell of Gr, consisting of subspaces for which the projection is an isomorphism. (In keeping with our general philosophy we will usually view the elements of W as formal Laurent series rather than functions.)

If M is covered by the Stein patches U and $M\backslash p$, then $\mathscr{L}$ is trivial on each patch and is completely described by its transition function $h(z)$ on the intersection $U\backslash p$. We now deform $\mathscr{L}$ to a (formal) family of bundles $\mathscr{L}(x,t)$ having the transition function

$$\exp\left(xz^{-1}+\sum_{n=1}^{\infty}t_nz^{-n}\right)h(z)\equiv G(z,x,t)h(z).\tag{10}$$

At least formally, this family actually parametrizes the entire connected component $\mathrm{Pic}^{g-1}M$, with considerable redundancy: the deformation is trivial whenever

$G(z, x, t)$ extends to a holomorphic function on $M \backslash p$. It will be crucial that $\exp xz^{-1}$ is an eigenfunction of $\partial = d/dx$ with eigenvalue z^{-1}, hence also an eigenfunction of $\exp \sum_{n=1}^{\infty} t_n \partial^n$ with eigenvalue $\exp \sum_{n=1}^{\infty} t_n z^{-n}$. In the language of quantum field theory, x will play the role of a source for generating "insertions" of z^{-1}. By the semicontinuity theorem [21] this family of bundles generically continues to enjoy the cohomology properties $H^0 = H^1 = 0$, so that there is a function $s(z, x, t) = z^{-1}$ + holomorphic in z, representing via ϕ in U the unique section with a simple pole at p. In the other chart of the covering, $M \backslash p$, this section is represented by the function restricting to $G(z, x, t)h(z)s(z, x, t)$ on $U \backslash p$. However, since the "unperturbed" transition function $h(z)$ plays no role in the analysis, it is customary to drop it and represent the section by the "wave function" or "Baker-Akhiezer function" $w(z, x, t) = G(z, x, t)s(z, x, t)$. The x-dependence of the Baker-Akhiezer function provides a convenient basis for the space of sections W, since $\partial^k w(z, x, t) = G(z, x, t)(z^{-k-1} + \text{higher powers of } z)$. Thus $w(z, x, t)$ and its derivatives (more precisely, $G(z, x, t)^{-1}$ times these) form a basis for the entire space of sections W.

Now let $H^0(M \backslash p, \mathcal{O})$ be the ring of functions on M holomorphic except for a pole of any finite order at p. For any such function $f(z)$, $f(z)w(z, x, t)$ represents a section of $\mathcal{L}$ with a pole only at p, so using the basis just described it can be written as $P(f)w(z, x, t)$ with $P(f)$ a differential operator in x with coefficients depending on x and t. For each t, the association of $P(f)$ to f gives a commutative ring $R(t)$ of differential operators isomorphic to $H^0(M \backslash p, \mathcal{O})$; more precisely, we obtain an isospectral family of commutative rings of differential operators.

Now let $S = 1 + s_1(x, t)\partial^{-1} + s_2(x, t)\partial^{-2} + \dots$ be the unique pseudodifferential operator such that

$$s(z, x, t) \exp xz^{-1} = z^{-1} S \exp xz^{-1}, \tag{11}$$

or, equivalently,

$$w(z, x, t) = z^{-1} S G(z, x, t). \tag{12}$$

(The successive derivatives in S produce the successive terms in w viewed as a Laurent series in z.) A short calculation will show that S, or equivalently $L \equiv S\partial S^{-1}$, is a solution to the KP hierarchy. Each derivative $\partial w/\partial t_n$ represents a section with a pole of finite order at p, and expressing it in terms of the basis of x-derivatives of w shows that

$$\frac{\partial w}{\partial t_n} = B_n(t) w \tag{13}$$

for some differential operator $B_n(t)$. Combining this with the derivative of (12),

$$\begin{aligned} \frac{\partial w}{\partial t_n} &= z^{-1} \frac{\partial S}{\partial t_n} G + z^{-1} S z^{-n} G \\ &= z^{-1} \frac{\partial S}{\partial t_n} S^{-1} S G + z^{-1} S \partial^n S^{-1} S G \\ &= \frac{\partial S}{\partial t_n} S^{-1} w + S \partial^n S^{-1} w, \end{aligned} \tag{14}$$

yields

$$B_n = \frac{\partial S}{\partial t_n} S^{-1} + S \partial^n S^{-1}. \tag{15}$$

Because the first term on the right contains only negative powers of ∂, the differential operator B_n must be the differential operator part of the second term, that is, L^n_+. Therefore, rearranging (15),

$$\frac{\partial S}{\partial t_n}=(B_n-L^n)S=-L^n_-S\,, \tag{16}$$

which is the KP hierarchy. It should be stressed that the geometric deformations of the Krichever data make sense for arbitrary line bundles on Riemann surfaces; the cohomology conditions on the bundle are required only in order to describe these flows by their effect on the operator S and thereby produce the KP hierarchy.

Not all solutions to the KP hierarchy are obtained by this construction. If the local coordinate z is chosen so that z^{-k} extends to a holomorphic function on $M\backslash p$, then the ring $R(t)$ must contain a differential operator $P(z^{-k})$, which must in fact be L^k in view of $L^kw=S\partial^kS^{-1}z^{-1}SG=z^{-k}w$. Therefore, up to an equivalence relation reflecting changes of the local coordinate, the solutions obtained all have the property that some power L^k is a pure differential operator. Furthermore, the ring $R(t)$ contains an operator of each sufficiently high order. It is possible to reconstruct all the geometric data $(M,p,z,\mathscr{L},\phi)$ from such a solution S. S determines w, which, together with its derivatives, gives a basis for the space of sections W. The ring of functions $H^0(M\backslash p,\mathcal{O})$ (restricted to U and expressed in terms of the local coordinate z) is the maximal stabilizer of W, the maximal set A_W of formal Laurent series in z with $A_WW\subset W$. The affine part $M\backslash p$ of the algebraic curve M is then Spec A_W, and from the module W over A_W one constructs the sheaf $\tilde{W}$ on Spec A_W which is just the family of bundles $\mathscr{L}$ [21]. Because both the functions in A_W and the sections in W come with information about their pole orders at p, we know how to form local functions or sections holomorphic at p by taking quotients, in particular a function having a simple zero at p which can serve as a local uniformizing parameter. This enables us to glue onto $M\backslash p$ a standard disk U with this uniformizing parameter, whose relation to z is known, and this gives the extensions of the curve and the sheaf to the point at infinity p. Because we have a realization of the ring of functions A_W as a ring of commuting differential operators $R(0)$, we can give a more elementary description of these algebraic constructions. Choose a pair of operators P and Q of relatively prime order from $R(0)$ and consider their simultaneous eigenspaces, $P\psi=\lambda\psi$ and $Q\psi=\mu\psi$. One shows that the eigenspaces are one-dimensional (so they are the simultaneous eigenspaces of all the operators in the ring) and that the operators P and Q, or their eigenvalues, satisfy a polynomial relation $F(P,Q)=0$ or $F(\lambda,\mu)=0$ which is just the equation of the affine curve $M\backslash p$ in $\mathbf{C}^2$ [22]. $\mathscr{L}$ is the bundle whose fiber at any point of this curve is the eigenspace for the given eigenvalues. The maximal ideal in Spec$R(0)$ associated to such a point consists of all operators in the ring having eigenvalue zero on that eigenspace.

3. The Manin-Radul SKP Hierarchy

The MRSKP hierarchy is a set of flow equations deforming a pseudosuperdifferential operator of the form

$$L=D+u_1+u_2D^{-1}+u_3D^{-2}+\dots, \tag{17}$$

where $D=\partial_\xi+\xi\partial_x$, and the coefficients $u_i(x,\xi,t)$ are formal power series in the even variables x, t_{2n} and the odd variables ξ, t_{2n-1}, $n=1,2,\dots$. Here we assume that

$Du_1 + 2u_2 = 0$, which is necessary and sufficient for the existence of a pseudosuper-differential operator

$$S = 1 + s_1 D^{-1} + s_2 D^{-2} + \dots \tag{18}$$

with

$$S^{-1} L S = D. \tag{19}$$

The generalized Leibniz rule here can be deduced from $D^{-1} = D\partial_x^{-1}$, which follows from $D^2 = \partial_x$. The MRSKP hierarchy reads

$$\frac{\partial L}{\partial t_{2n}} = [L_+^{2n}, L] = -[L_-^{2n}, L], \tag{20}$$

$$\begin{aligned} \frac{\partial L}{\partial t_{2n-1}} &= [L_+^{2n-1}, L] - 2L^{2n} + \sum_{k=1}^{\infty} t_{2k-1} [L_+^{2n+2k-2}, L] \\ &= -[L_-^{2n-1}, L] + \sum_{k=1}^{\infty} t_{2k-1} \frac{\partial L}{\partial t_{2n+2k-2}}, \end{aligned} \tag{21}$$

where the brackets are supercommutators and $L^n = L_+^n + L_-^n$ is the decomposition into nonnegative and negative powers of D. The sign conventions adopted here for the flow parameters t_n are those of Mulase [8] and differ slightly from those of Manin and Radul and of Ueno et al. [7, 9]. The equivalent system for the operator S is

$$\frac{\partial S}{\partial t_{2n}} = -L_-^{2n} S, \tag{22}$$

$$\frac{\partial S}{\partial t_{2n-1}} = -\left(L_-^{2n-1} + \sum_{k=1}^{\infty} t_{2k-1} L_-^{2n+2k-2} \right) S, \tag{23}$$

where $L^n = SD^n S^{-1}$.

We will now use a supersymmetric generalization of the Krichever map [10–13] to construct solutions to the MRSKP hierarchy from a set of geometric data $(M, p, \mathscr{L}, z, \theta, \phi)$. Here M is a compact connected complex supermanifold [23] of dimension $(1|1)$, *not* necessarily a super Riemann surface, p is an irreducible divisor on M (its body p_{red} is a single point of M_{red}), $\mathscr{L}$ is a line bundle on M satisfying $H^0(M, \mathscr{L}) = H^1(M, \mathscr{L}) = 0$, (z, θ) are local coordinates on M near p such that p is defined by the equation $z = 0$, and ϕ is a trivialization of $\mathscr{L}$ in the neighborhood $U = \{|z| < 1\}$ of p.

The significance of the cohomology conditions on $\mathscr{L}$ again follows from the super Riemann-Roch theorem [24]. This creates a potential problem because the super Riemann-Roch theorem is not universally valid. It certainly holds when M is split, with no nilpotent parameters in its structure sheaf besides θ, so we consider this case first. More precisely, M is assumed to be a single supermanifold of dimension $(1|1)$ for now; shortly we will consider families of such supermanifolds over parameter spaces of the form $\mathrm{Spec} \wedge (\beta_1, \beta_2, \dots, \beta_N)$, so that functions on M may depend on N globally defined odd parameters β_i as well as on θ. The super Riemann-Roch theorem reads

$$\dim H^0(M, \mathscr{L}) - \dim H^1(M, \mathscr{L}) = (\deg \mathscr{L} + 1 - g | \deg \mathscr{L} + \deg \mathscr{E} + 1 - g). \tag{24}$$

On a split supermanifold such as M, an expression $f(z)+\theta g(z)$ extends to a global function exactly when $f(z)$ extends to a global function on M_{red} and $g(z)$ extends to a global section of a certain line bundle on M_{red}; this is the bundle denoted $\mathscr{E}$ in (24). For a super Riemann surface, $\mathscr{E}$ is a spin bundle, of degree $g-1$, but here the cohomology conditions clearly imply $\deg\mathscr{L}=g-1$ and $\deg\mathscr{E}=0$. Therefore supermanifolds M satisfying these conditions cannot be super Riemann surfaces except in the case $g=1$, which neatly allows the solutions to MRSKP in terms of super elliptic functions obtained by Pakuliak [14] and by Mulase [8].

The super Riemann-Roch theorem for $\mathscr{L}\otimes\mathscr{O}(p)$ gives

$$\dim H^0(M,\mathscr{L}\otimes\mathscr{O}(p))-\dim H^1(M,\mathscr{L}\otimes\mathscr{O}(p))=(1|1), \tag{25}$$

so that there are unique (up to normalization) even and odd sections of $\mathscr{L}$ holomorphic except for simple poles at p. Similarly, $H^1(M,\mathscr{L}\otimes\mathscr{O}(np))=0$ and $\dim H^0(M,\mathscr{L}\otimes\mathscr{O}(np))=(n|n)$ for all $n>0$, and there are unique (up to normalization and linear combination with sections having lower-order poles) even and odd sections with behavior z^{-n} and θz^{-n} near p, for any $n>0$.

Since the MRSKP flow deforms the bundle $\mathscr{L}$ and, as we will see, the supermanifold M, producing a (nonsplit) family depending on the odd parameters t_{2n-1}, it is necessary to reexamine the super Riemann-Roch theorem in the nonsplit case. We consider families over Spec $\wedge(\beta_1,\beta_2,\ldots,\beta_N)$; the case of infinitely many odd parameters can be treated as a direct limit. As above, the consequence of the super Riemann-Roch theorem which we need is the fact that $\dim H^0(M,\mathscr{L}\otimes\mathscr{O}(np))=(n|n)$. In the nonsplit case the dimension here should be the dimension over the parameter space $\wedge(\beta_1,\beta_2,\ldots,\beta_N)$, and the result can fail if $H^0(M,\mathscr{L}\otimes\mathscr{O}(np))$ is not a freely generated module over this ring [24, 25]. Associated to a family of supermanifolds $(M,\mathscr{O})$ and bundles $\mathscr{L}$ there is a split supermanifold $(M,\mathscr{O}_s)$ and bundle $\mathscr{L}_s$ obtained by quotienting out the ideal $\mathscr{I}$ generated by the β_i in all sheaves. The conclusion we need will follow if we can show that the quotient map $H^0(M\backslash p,\mathscr{L})\to H^0(M\backslash p,\mathscr{L}_s)$ is surjective, so that each section in the split case extends to a section over the family. We will show that this follows from the conditions $H^i(M,\mathscr{L})=0$ which are satisfied by our Krichever data. First, these conditions imply that $H^i(M,\mathscr{L}_s)=0$ as well, since if c were any nontrivial cocycle here then $\beta_1\beta_2\ldots\beta_N c$ would be a nontrivial cocycle in $H^i(M,\mathscr{L})$. From our analysis in the split case, this implies that $H^1(M,(\mathscr{L}\otimes\mathscr{O}(np))_s)=0$ for all $n>0$. We now apply the exact sequence

$$0\to\wedge^j(\beta_1,\beta_2,\ldots,\beta_N)\mathscr{F}_s\to\mathscr{F}/\mathscr{I}^{j+1}\to\mathscr{F}/\mathscr{I}^j\to 0 \tag{26}$$

and its consequence

$$H^0(M,\mathscr{F}/\mathscr{I}^{j+1})\to H^0(M,\mathscr{F}/\mathscr{I}^j)\to\wedge^j(\beta_1,\beta_2,\ldots,\beta_N)H^1(M,\mathscr{F}_s) \tag{27}$$

to $\mathscr{F}=\mathscr{L}\otimes\mathscr{O}(np)$ to conclude inductively in j that sections in the split case do extend. Therefore, the cohomology conditions on the Krichever data are sufficient to guarantee that the space of sections W is freely generated over any purely odd parameter space.

In contrast, the space of functions $H^0(M\backslash p,\mathscr{O})$ need not be freely generated. However, because $H^1(M,\mathscr{O}(np))=0$ for n sufficiently large, the spaces $H^0(M\backslash p,\mathscr{O})/H^0(M,\mathscr{O}(np))$ of functions having sufficiently high pole order $n+1$ must be freely generated by the same argument as above.

The space $H^0(M\backslash p,\mathscr{L})$ of sections of $\mathscr{L}$ holomorphic except for finite-order poles at p can be viewed as a point W of the big-cell of a super Grassmannian Gr.

Points of Gr are subspaces of the space of formal Laurent series $\sum_{n \geqslant -\infty}^{\infty} (a_n + \theta b_n) z^n$ for which the projection onto $H_+ = \text{span}\{z^{-n}, \theta z^{-n}, n=1,2,\ldots\}$ is Fredholm of index zero [11–13]. In the split case these are vector subspaces over $\mathbf{C}$, while in the nonsplit case "subspaces" means freely generated modules over the finite- or infinite-dimensional odd parameter space. The coefficients a_n, b_n are valued in the parameter space. The big-cell consists of subspaces for which the projection is an isomorphism. The results above show that the space W of sections belongs to the big-cell even in the nonsplit case.

We can obtain a convenient basis for W by deforming the transition function $h(z)$[1] of $\mathscr{L}$ by $\exp(xz^{-1} + \xi\theta)$ and acting with differential operators (x and ξ acting as sources for z^{-1} and θ), but now this expression is not an eigenfunction of D:

$$D \exp(xz^{-1} + \xi\theta) = (\theta + \xi z^{-1}) \exp(xz^{-1} + \xi\theta), \tag{28}$$

where $\theta + \xi z^{-1}$ is not an eigenvalue because it contains the variable ξ on which D acts. In fact, D is a nonintegrable vector field in view of $[D, D] = 2D^2 = 2\partial_x \neq 0$, so it has no nontrivial eigenfunctions at all. The equation $D\psi(x,\xi) = \lambda\psi(x,\xi)$, with λ an odd parameter, implies $D^2\psi = -\lambda^2\psi = 0$, which is readily seen to imply that ψ is constant. Nevertheless, D can be considered to have "operator-valued eigenvalues":

$$D \exp(xz^{-1} + \xi\theta) = (\theta - z^{-1}\partial_\theta) \exp(xz^{-1} + \xi\theta), \tag{29}$$

where the "eigenvalue" is indeed independent of the variables on which D acts. This motivates the further deformation of the bundle $\mathscr{L}$ by multiplying its transition function with the "eigenvalue" of the operator $\exp \sum_{n=1}^{\infty} t_n D^n$ which plays a central role in the algebraic theory of the MRSKP hierarchy [8]. The "transition function" of the resulting family of bundles $\mathscr{L}(x, \xi, t)$ will be

$$\exp\left[\sum_{n=1}^{\infty} t_{2n} z^{-n} + \sum_{n=1}^{\infty} t_{2n-1}(\theta z^{-n+1} - z^{-n}\partial_\theta)\right] \exp(xz^{-1} + \xi\theta) h(z). \tag{30}$$

This can be simplified by using $e^{A+B} = e^A e^B e^{-[A,B]/2}$, which holds when A and B commute with $[A, B]$. Here

$$A = \sum_{m=1}^{\infty} t_{2m-1}\theta z^{-m+1}, \qquad B = -\sum_{n=1}^{\infty} t_{2n-1} z^{-n}\partial_\theta, \tag{31}$$

and the commutator

$$[A, B] = \sum_{m,n=1}^{\infty} t_{2m-1} t_{2n-1} z^{-m-n+1} = 0 \tag{32}$$

vanishes due to its symmetry in the odd parameters t_{2n-1}. The "transition function" is then

$$\exp\left[\sum_{n=1}^{\infty} (t_{2n} z^{-n} + t_{2n-1}\theta z^{-n+1})\right] \exp\left(-\sum_{n=1}^{\infty} t_{2n-1} z^{-n}\partial_\theta\right) \exp(xz^{-1} + \xi\theta) h(z). \tag{33}$$

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[1] Note that in the split case this transition function cannot depend on θ, since it must be even and no other odd parameters are available to form even products

If we let $s(z, \theta, x, \xi, t)$ be the expression in terms of the trivialization ϕ in U of the unique section of this family of bundles with the behavior z^{-1} + holomorphic in z, and let the Baker-Akhiezer function $w(z, \theta, x, \xi, t)$ be the restriction to $U\backslash p$ of this section in the other chart $M\backslash p$ of the covering (times $h(z)^{-1}$), then the relation between these will be

$$w(z,\theta,x,\xi,t)=\exp\left[\sum_{n=1}^{\infty}(t_{2n}z^{-n}+t_{2n-1}\theta z^{-n+1})\right]$$
$$\times\exp\left[xz^{-1}+\xi\left(\theta-\sum_{k=1}^{\infty}t_{2k-1}z^{-k}\right)\right]s\left(z,\theta-\sum_{k=1}^{\infty}t_{2k-1}z^{-k},x,\xi,t\right). \quad (34)$$

It is now clear that the family of "transition functions" under discussion involves deformations of the supermanifold M as well as the bundle $\mathscr{L}$. The transition function $h(z)$ of $\mathscr{L}$ has indeed been multiplied by the factor

$$\exp\left[\sum_{n=1}^{\infty}(t_{2n}z^{-n}+t_{2n-1}\theta z^{-n+1})\right]\exp(xz^{-1}+\xi\theta). \quad (35)$$

However, a "Schiffer deformation" [26] has simultaneously been performed on the supermanifold M itself. That is, the disk U has been cut out of M, and then reattached with the identification of θ on its boundary and $\theta-\sum_{k=1}^{\infty}t_{2k-1}z^{-k}$ on the boundary circle of $M\backslash U$. Because this shift of the θ coordinate does not extend holomorphically throughout the interior of U, it cannot be removed by a redefinition of this coordinate but rather induces a nontrivial change in the complex supermanifold structure of M. Such variations of moduli are familiar in the operator formalism in (super) conformal field theory [10, 27], where they are generated by the stress tensor of the theory. Here the variation in moduli is coupled to the deformation of the bundle by the use of the t_{2n-1} to parametrize both deformations. An important consequence of this coupling was the vanishing of the commutator (32). These deformations of supercurve and bundle, but not more general ones with independent parameters, actually commute[2]. The MRSKP system thus describes a flow of the Krichever data, not in the Picard variety of M only, but in the universal Picard bundle whose fiber over any point M in the moduli space of supermanifolds having $\deg\mathscr{E}=0$ is $\mathrm{Pic}^{g-1}M$. This is the true significance of the relation $D^2=\partial_x$, which had led most investigators to expect a relation between SKP and super Riemann surfaces. Instead, the nonintegrability of D requires the presence of both θ and ∂_θ in its "eigenvalues," which lead respectively to deformations of bundle and curve, with identical parameters.

We can now complete the verification that these geometric flows produce solutions to the MRSKP hierarchy. As before, w and its derivatives $D^n w$ provide a basis of sections of $\mathscr{L}(x, \xi, t)$. The even order derivatives $D^{2n}w$ give even sections having leading poles z^{-n-1}, while odd order derivatives $D^{2n-1}w$ give odd sections with leading poles θz^{-n}. We introduce the wave operator S by

$$s(z,\theta,x,\xi,t)\exp(xz^{-1}+\xi\theta)=z^{-1}S\exp(xz^{-1}+\xi\theta), \quad (36)$$

[2] One could object here that the θ variable in the term $\exp(xz^{-1}+\xi\theta)$ was certainly shifted as a result of the Schiffer deformation, showing a failure of commutativity. This term should be viewed as merely an auxiliary deformation introduced to provide a convenient basis for the space of sections via its derivatives with respect to x and ξ. The significant deformations are the t-dependent ones at $x=\xi=0$

which, using (34), translates into

$$w(z,\theta,x,\xi,t)=z^{-1}S\exp\left[\sum_{n=1}^{\infty}(t_{2n}z^{-n}+t_{2n-1}\theta z^{-n+1})\right.$$
$$\left.+xz^{-1}+\xi\left(\theta-\sum_{k=1}^{\infty}t_{2k-1}z^{-k}\right)\right]$$
$$\equiv z^{-1}SG(z,\theta,x,\xi,t)\,. \tag{37}$$

Modulo the difference in sign conventions this is the same relation between wave function and wave operator found in the purely algebraic study of the MRSKP hierarchy [9], and it leads directly to the MRSKP equations (22, 23). Expressing the derivatives of w with respect to the t_n in terms of the basis of sections $D^n w$ gives

$$\frac{\partial w}{\partial t_n}=B_n(t)w\,, \tag{38}$$

with the B_n superdifferential operators. $\Big($The B_n with odd subscripts are actually superdifferential operators of infinite order, as is clear from the explicit formulas below. Formally this causes no trouble, but it could be avoided by redefining B_{2n-1} in terms of $\frac{\partial w}{\partial t_{2n-1}}-\sum_{k=1}^{\infty}t_{2k-1}\frac{\partial w}{\partial t_{2n+2k-2}}\Big)$. Comparing this with the derivatives of (37) yields

$$B_{2n}=\frac{\partial S}{\partial t_{2n}}S^{-1}+SD^{2n}S^{-1}\,, \tag{39}$$

and

$$B_{2n-1}=\frac{\partial S}{\partial t_{2n-1}}S^{-1}+SD^{2n-1}S^{-1}+\sum_{k=1}^{\infty}t_{2k-1}SD^{2n+2k-2}S^{-1}\,. \tag{40}$$

The infinite sum in this last equation corrects for the fact that $\partial/\partial t_{2n-1}$ brings down an unshifted θ from $G(z,\theta,x,\xi,t)$ while D^{2n-1} involves $\partial/\partial\xi$, which brings down the shifted θ. These equations imply that

$$B_{2n}=L_+^{2n}\,, \tag{41}$$

$$B_{2n-1}=L_+^{2n-1}+\sum_{k=1}^{\infty}t_{2k-1}L_+^{2n+2k-2}\,. \tag{42}$$

Inserting these expressions in (39, 40) and rearranging immediately produces the MRSKP equations in the form (22, 23).

The connection between Krichever data and commutative rings of differential operators also generalizes to the super case. As before, consider the supercommutative ring $H^0(M\backslash p,\mathcal{O})$ of functions on M holomorphic except for poles at p. Assign a superdifferential operator $P(f)$ to any such function f by $f(z,\theta)w(z,\theta,x,\xi,t)|_{t=0}=P(f)w(z,\theta,x,\xi,t)|_{t=0}$, where the right side is the expression of the section fw in the basis of derivatives of w. The result is a supercommutative ring $R(0)$ of superdifferential operators isomorphic to $H^0(M\backslash p,\mathcal{O})$. To show the supercommutativity, let f and g be two functions of definite $\mathbf{Z}_2$ parities $\tilde{f},\tilde{g}$. Then $fgw=fP(g)w=(-1)^{\tilde{f}\tilde{g}}P(g)fw=(-1)^{\tilde{f}\tilde{g}}P(g)P(f)w$.

But also, $fgw=(-1)^{\tilde{f}\tilde{g}}gfw=P(f)P(g)w$. Since the operators are uniquely determined by their action on w, it follows that $P(f)P(g)=(-1)^{\tilde{f}\tilde{g}}P(g)P(f)$. Repeating the construction for nonzero t will realize the MRSK flow as a (nonisospectral) family of supercommutative rings $R(t)$. Because the function $f(z,\theta)$ is defined inside U, fs is the restriction of a section of $\mathscr{L}$ to U. Transforming to the representative fw of this section in the other chart $M\backslash p$ introduces the shift of θ, so the correct correspondence between functions f and superdifferential operators $P(f)$ is

$$f\left(z,\theta-\sum_{k=1}^{\infty}t_{2k-1}z^{-k}\right)w(z,\theta,x,\xi,t)=P(f)w(z,\theta,x,\xi,t)\,. \tag{43}$$

If the local coordinate z was chosen so that z^{-k} extends to a global function on $M\backslash p$, then the corresponding operator is $P(z^{-k})=L^{2k}$. Therefore the solutions we obtain are all such that some even power of L is a pure differential operator. Furthermore, since the MRSKP flow does not alter the patching of the z coordinate, this property will be preserved under the flow. However, no odd power L^{2n-1} belongs to the ring $R(0)$, since in a supercommutative ring any odd operator must have square zero, whereas L is conjugate to the nonintegrable vector field D. If the local coordinate θ is chosen so that θz^{-l} extends to a global function, then the corresponding operator is $P(\theta z^{-l})=S\partial_\xi\partial_x^l S^{-1}$. Since the flow does change the patching of θ, this property is not preserved under the flow; θz^{-l} will not extend to a global function on the deformed supercurve, and $S\partial_\xi\partial_x^l S^{-1}$ will not be a pure superdifferential operator for nonzero t.

Once again it is possible to reconstruct all the geometric data from the corresponding wave operator $S(t=0)$. S determines w, which along with its derivatives gives a basis for the space of sections W in the super Grassmannian Gr. The rest of the argument was developed in [13] where the invertibility of the super Krichever functor was shown. Once again the ring of functions $H^0(M\backslash p,\mathcal{O})$ is obtained as the maximal set A_W of formal Laurent series in z and θ such that $A_W W \subset W$. The affine curve $M\backslash p$ should be $\operatorname{Spec} A_W$, and indeed there is a natural notion of Spec of a supercommutative ring having this property. Writing $A_W=(A_W)_0\oplus(A_W)_1$, the reduced space of $M\backslash p$ will be $\operatorname{Spec}(A_W)_0$, while the odd part of the structure sheaf of $M\backslash p$ is the sheaf $(\widetilde{A_W})_1$. The sheaf $\widetilde{W}$ constructed from the module W over A_W is then $\mathscr{L}$ restricted to the affine curve. As before, these sheaves extend over the point p because the information about the pole orders of their sections at p allows us to form quotients holomorphic at p, and local uniformizing parameters there, so that we know how to glue in U with a sheaf of germs of sections holomorphic at p. The pair of Riemann surfaces and bundles used by Radul to construct algebro-geometric solutions in the split case can be made visible in the same way. Writing $W=W_0\oplus\theta W_1$, we can view the two W_i as points of an ordinary Grassmannian. If their maximal stabilizers in the space of ordinary Laurent series are A_{W_i}, then Radul's curves are the $\operatorname{Spec} A_{W_i}$ and his bundles are the sheaves $\widetilde{W}_i$. In fact, both stabilizers coincide, and the common curve is the body (reduced space) of M. Our construction in the split case therefore gives only the subset of Radul's solutions for which the two curves are the same. If it is true that our construction gives all finite-dimensional orbits of the MRSKP flows (see Sect. 4), the implication is that Radul's solutions with distinct curves are infinite-dimensional orbits.

Although the deformation of Krichever data described by the MRSKP hierarchy produces a nonsplit family of data, the discussion above is still restricted to the case in which the initial data is split. The discussion must still be generalized to include initial data which is itself a nonsplit family over Spec $\wedge(\beta_1, \beta_2, \ldots, \beta_N)$. Since the cohomology conditions imposed on this data are sufficient to guarantee that all spaces of sections are freely generated over this parameter space, most of the discussion carries over unchanged except for the additional dependence of the wave function and wave operator on the additional parameters. However, the fact that the transition function $h(z, \theta)$ of the initial line bundle $\mathscr{L}$ may now depend on θ introduces a major conceptual difference. In writing the relation between the wave function and the wave operator, the initial transition function $h(z)$ was omitted, which was permissible because this factor is not changed by the flow. Because the flow shifts θ, however, a factor $h(z, \theta)$ will change under the flow. We can simply redefine the geometric flows to include an additional deformation of the line bundle so as to keep this factor constant, which then allows us to omit it and obtain the MRSKP system via the same computation as before. However, we then encounter the problem that the transition function for a given bundle is arbitrary up to a cocycle. If a different initial transition function $g(z, \theta)$ is chosen, so that the ratio $g(z, \theta)/h(z, \theta)$ extends to a holomorphic function in $M \backslash p$, this ratio will generally not extend holomorphically after a shift of θ. This means that the geometric flow on the Krichever data, and the resulting solution to the MRSKP hierarchy, are not uniquely determined by the initial data but depend also on the choice of a particular transition function for the initial bundle $\mathscr{L}$. Equivalently, one must choose a trivialization of $\mathscr{L}$ in $M \backslash p$ as well as in U. This is in agreement with the fact that the unique solvability of the initial value problem for the MRSKP hierarchy can be proven only in the split case [8].

The geometric picture of the flows makes it clear that this situation was to be expected. The integral curves of these flows are paths in the universal Picard bundle over the moduli space of supercurves. Although there is a well-defined notion of a flow along a fiber of such a Picard bundle, deforming $\mathscr{L}$ but not M, there is no invariant notion of a horizontal flow deforming M but not $\mathscr{L}$. Given a choice of transition function $h(z, \theta)$, we can define a horizontal flow by keeping the same transition function as M changes, but the definition obviously depends on the choice of $h(z, \theta)$. The MRSKP flow is then a specific diagonal flow in the universal Picard bundle, the diagonal direction being defined relative to the horizontal direction specified by the choice of $h(z, \theta)$. Thus the choice of initial transition function is roughly equivalent to a choice of a connection in the universal Picard bundle. It would be interesting to investigate the geometry of this situation in more detail.

The reconstruction of the geometric data from a solution in the nonsplit case is no more difficult than in the split case. The only difference is in the decomposition $A_W = (A_W)_0 \oplus (A_W)_1$, where due to the presence of the parameters β_i, $(A_W)_0$ now contains even nilpotents. Spec$(A_W)_0$ now gives the reduced space of $M \backslash p$ already equipped with the even part of the structure sheaf of $M \backslash p$ itself, and $(\widetilde{A_W})_1$ provides the odd part of the full structure sheaf. Because A_W still contains functions with behavior z^{-n} and θz^{-n} for all sufficiently large n, it is still possible to obtain local uniformizing parameters at p as quotients of its elements, and thereby to complete the affine supercurve $M \backslash p$ to M.

As in the KP case, the geometric flows on the Krichever data make sense for arbitrary supercurves M and bundles $\mathscr{L}$. The cohomology conditions on these objects serve only to constrain the structure of W so as to allow its description and that of the flows in terms of the wave operator S. Unlike the KP case, however, the cohomology conditions are actually needed to guarantee that W is freely generated and so can be viewed as a point of Gr in the nonsplit case. This means that the description of the flows on arbitrary Krichever data will actually require a significantly generalized notion of Grassmannian.

4. The Jacobian SKP Hierarchy

Given the geometric understanding of the MRSKP hierarchy, it is easy to construct new SKP hierarchies which describe alternative deformations of the geometric Krichever data. In particular, we can construct one which describes deformations of the line bundle $\mathscr{L}$ on a fixed supermanifold M by simply omitting the Schiffer deformation from the formula relating the wave function and the wave operator. Since these flows on the universal Picard bundle are purely vertical, no choice of connection will be necessary in the nonsplit case. This new SKP hierarchy will be a more natural supersymmetric generalization of ordinary KP from the geometric point of view than is the Manin-Radul hierarchy. We will refer to it as the Jacobian SKP hierarchy. The new relation replacing (37) will be

$$w(z,\theta,x,\xi,t)=z^{-1}S\exp\left[\sum_{n=1}^{\infty}(t_{2n}z^{-n}+t_{2n-1}\theta z^{-n+1})+xz^{-1}+\xi\theta\right], \tag{44}$$

which easily leads to the new SKP hierarchy

$$\frac{\partial S}{\partial t_{2n}}=-(S\partial_x^n S^{-1})_-S=-(SD^{2n}S^{-1})_-S, \tag{45}$$

$$\frac{\partial S}{\partial t_{2n-1}}=-(S\partial_\xi\partial_x^{n-1}S^{-1})_-S=-[S(D^{2n-1}-\xi D^{2n})S^{-1}]_-S. \tag{46}$$

Note that $S\xi D^{2n}S^{-1}\neq\xi SD^{2n}S^{-1}=\xi L^{2n}$, because the operator S contains ∂_ξ and so does not commute with ξ. This means that there is no simple way to rewrite the Jacobian SKP system completely in terms of L rather than S. The same will be true of the "maximal" SKP hierarchy discussed in Sect. 5. Thus, although the Manin-Radul hierarchy is not the most geometrically natural supersymmetric generalization of ordinary KP, it is distinguished as the only simple generalization which can be written in Lax form as a flow on L. Since the connection between the KP hierarchy and 2d quantum gravity is made via the Lax formalism [28–30], it is the MRSKP hierarchy which is expected to be relevant for 2d quantum supergravity. Indeed, its interpretation in this context has recently been investigated in [31]. However, since the odd flows have not been interpreted, and indeed seem incompatible with any reasonable string equation, the alternative odd flows of the Jacobian hierarchy should also be examined in this context. These Jacobian flows can of course be realized as an isospectral family of supercommutative rings $R(t)$ via the correspondence between functions and superdifferential operators discussed in Sect. 3.

The Jacobian SKP hierarchy has been discovered and discussed independently by Mulase in the split case (no odd parameters besides the t_{2n-1}) [32]. He pointed out that it is integrable, and that its initial value problem is uniquely solvable using the same super Birkhoff decomposition which gives the unique solvability for the Manin-Radul hierarchy [8]. Further, he showed that every finite-dimensional orbit of these flows is isomorphic to the suitably defined Jacobian of a (1|1) supercurve M. (The argument is essentially that the reduced orbit must be the Jacobian of a Riemann surface by ordinary KP theory, while the nilpotent parts of the flows are effectively infinitesimal and so add no global structure. One need only ensure cohomologically that only finitely many of them generate the orbit.) Although Mulase's Jacobian is defined as $H^1(M, \mathcal{O})/H^1(M, Z)$, whereas the Picard group of line bundles would normally be defined with $\mathcal{O}$ replaced by the sheaf $\mathcal{O}_0$ of even functions, this does in fact mean that all finite-dimensional orbits are obtained from deformations of line bundles in the manner discussed here. The difference reflects only Mulase's precise definition of the orbits and his restriction to split M. The cohomology group $H^1(M, \mathcal{O})$ for split M contains the same information as the group $H^1(M, \mathcal{O}_0)$ for M a family over an odd parameter space, a cocycle $f + \theta g$ in the former group corresponding to a cocycle $f + \eta\theta g$ in the latter, with η an odd parameter. Similar methods should prove that all finite-dimensional orbits of the MRSKP flows are obtained by our Krichever construction. However, the precise definition of the orbits required for such a proof should wait for a more satisfactory definition of a super Grassmannian whose points are non-freely generated modules over a parameter space of variable size.

5. The Kac-van de Leur SKP Hierarchy

We have seen that the MRSKP hierarchy describes a very specific simultaneous deformation of the Krichever data M and $\mathcal{L}$. It is natural to separate the deformations of M from those of $\mathcal{L}$, and the Jacobian SKP hierarchy introduced in Sect. 4 is a step in this direction, deforming $\mathcal{L}$ only. We can also write equations for the deformation of M alone, although as discussed previously this requires some choice of a horizontal direction in the universal Picard bundle in the nonsplit case. It is most natural to choose a specific transition function for $\mathcal{L}$ and then write a relation between wave function and wave operator including the Schiffer deformation of M but no further change in the transition function. This relation will be [cf. (37)]

$$w(z, \theta, x, \xi, \hat{t}) = z^{-1} S \exp\left[xz^{-1} + \xi\left(\theta - \sum_{n=1}^{\infty} \hat{t}_{2n-1} z^{-n}\right)\right] \tag{47}$$

and the resulting SKP flow equations for S are

$$\frac{\partial S}{\partial \hat{t}_{2n-1}} = -(S\xi D^{2n} S^{-1})_- S\,. \tag{48}$$

The Schiffer deformation arising from the MRSKP hierarchy is not the most general deformation of M which changes the patching of θ while preserving that of z. To generate all such deformations one must add those which act multiplicatively on θ, in effect deforming the bundle $\mathcal{E}$ characterizing M. These lead to the relation between wave function and wave operator

$$w(z, \theta, x, \xi, \hat{t}) = z^{-1} S \exp\left[xz^{-1} + \xi\theta \exp \sum_{n=1}^{\infty} \hat{t}_{2n} z^{-n}\right], \tag{49}$$

and the flow equations

$$\frac{\partial S}{\partial \hat{t}_{2n}} = -(S\xi D^{2n+1} S^{-1})_- S\,. \tag{50}$$

If we combine the two flow equations just derived with the two comprising the Jacobian SKP hierarchy of Sect. 4, we have a set of four equations involving two infinite sets of even and odd flow parameters which describe the most general set of independent deformations of M and $\mathscr{L}$ preserving the patching of the z coordinate. We will refer to these as the maximal SKP hierarchy. Since such independent flows do not commute, the equations are certainly not integrable, but individual flows can be exponentiated to one-parameter groups. If the initial M was a projected family of supermanifolds, this is the most general set of deformations which preserve this property. The meaning of projectedness is that there exists a projection map from M to M_{red}: algebraically, functions $f(z)$ on M_{red} pull back to functions $f(z)$ on M. Since the (S)KP theory realizes the ring of functions on the affine curve as a ring of (super)differential operators, projectedness gives an inclusion of these rings of operators, and the flows now under consideration are the most general preserving this inclusion.

The KVSKP hierarchy has two infinite sets of even and odd flow parameters, and we will argue that it is equivalent to the flows just described. In principle this could be demonstrated explicitly using the formulas given by Dolgikh and Schwarz [12], which relate the wave operator S to the super tau function appearing in the KVSKP hierarchy, but it seems extremely difficult to carry this out. Instead, we will use the work of Kac and van de Leur [18] and of Bergvelt [19], which gives a very complete description of this SKP hierarchy in terms of group actions on the super Grassmannian, to relate these group actions to our flows. We will summarize this work briefly, omitting some of the more technical points.

Subspaces W in the big-cell of the super Grassmannian Gr have a free basis over $\wedge(\beta_1, \beta_2, \ldots, \beta_N)$ formed from linear combinations of the standard basis elements $v_i = z^{-i}$, $v_{i+1/2} = \theta z^{-i}$, $i \in \mathbf{Z}$. An infinite-dimensional general linear supergroup acts on such subspaces; the infinitesimal generators of its Lie superalgebra $gl_{\infty|\infty}$ are the "elementary matrices" E_{ij} which change v_j into v_i and annihilate all other basis vectors. There is a single orbit C of this group which consists of Gr minus a hypersurface, and a natural action of the group on the module $\Gamma(C, \text{Ber}^*)$ consisting of sections of the dual Berezinian bundle over C. The super tau function as defined by Schwarz [11] is a highest weight vector σ_0 of this module. There is a more abstract construction of this module as a Fock space for superfermionic operators ψ_i in which the group generators are realized as $E_{ij} = (-1)^{2j} \psi_i \psi_j^*$. The central fact is that the module remains irreducible under a smaller "super Heisenberg" algebra s_A generated by the operators

$$\lambda(n) = \sum_{k \in \mathbf{Z}} E_{k,k+n} = z^n \left(1 - \theta \frac{\partial}{\partial \theta}\right), \tag{51}$$

$$\mu(n) = \sum_{k \in \mathbf{Z}+1/2} E_{k,k+n} = z^n \theta \frac{\partial}{\partial \theta}, \tag{52}$$

$$e(n) = \sum_{k \in \mathbf{Z}} E_{k-1/2,k+n} = \theta z^{n+1}, \tag{53}$$

$$f(n) = \sum_{k \in \mathbf{Z}} E_{k,k+n-1/2} = z^{n-1} \frac{\partial}{\partial \theta}, \tag{54}$$

$n \in \mathbf{Z}$, where the last equalities give the action on the v_i by multiplication and/or differentiation. The super boson-fermion correspondence (superbosonization) provides a representation of these operators as differential operators acting on the algebra of polynomials in infinitely many even and odd variables denoted in [18] as x_n, θ_n, $n \in \mathbf{Z}\backslash\{0\}$, and a representation of the ψ_i in terms of them as vertex operators. The KVSKP hierarchy is the bosonized representation of the bilinear equation

$$\sum_{i \in \mathbb{Z}/2} (-1)^{2i} \psi_i \otimes \psi_i^* (\tau \otimes \tau) = 0 . \tag{55}$$

Bergvelt has shown that this equation characterizes the points of the orbit C in terms of their response to the infinitesimal flows in s_A.

Compare the infinitesimal action of our flows $\partial/\partial t_n$, $\partial/\partial \hat{t}_n$ with that of the operators in s_A. For example, the deformation of $\mathscr{L}$ by multiplying its transition function with $\exp t_{2n} z^{-n}$ acts on the subspace W by $W \to (\exp - t_{2n} z^{-n}) W$. That is, multiplying the restriction to U of a section of $\mathscr{L}$ by $\exp - t_{2n} z^{-n}$ yields a section of the deformed bundle which is unchanged in $M \backslash p$. More precisely, since such multiplication does not make sense in a space of formal Laurent series, one should say that the deformation parametrized by t_{2n} acts infinitesimally on W by multiplication with $-z^{-n}$. This infinitesimal action is the same (up to sign) as that of $\lambda(-n) + \mu(-n)$. Note that the deformation does *not* act multiplicatively on the special section s, which is defined as the one with leading pole z^{-1}: the multiplicative action on s produces a section, but not the one with this leading pole. The correct action on s is given by the SKP equations. Similarly, the deformation of $\mathscr{L}$ parametrized by $\hat{t}_{2n-1}$ acts on W by $W \to (\exp \hat{t}_{2n-1} z^{-n} \partial_\theta) W$. Infinitesimally this coincides with the action of $f(-n+1)$. The deformation parametrized by t_{2n-1} acts infinitesimally as $e(-n)$, and that parametrized by $\hat{t}_{2n}$ acts as $\mu(-n)$. Thus the flows of the maximal SKP hierarchy generate half of the super Heisenberg algebra s_A. The infinitesimal action of the remaining generators of s_A involves positive powers of z. Such flows act in a simple way not only on the subspace W, but on the specific section s as well, so that no differential equations are needed to describe the action. In terms of the Krichever data, these flows change the choices of local trivialization ϕ and local coordinate θ. For example, $\mu(n)$ for $n \geqq 0$ acts by $s \to (1 + \varepsilon z^n \theta \partial_\theta) s$, a change of the θ coordinate. The conclusion is that the KVSKP hierarchy characterizes the points of the orbit C by their response to the same deformations described by the maximal SKP hierarchy, plus others whose action can be described without the need for differential equations. Under bosonization, τ becomes a polynomial in the variables x_n, θ_n, which are therefore the flow parameters corresponding to our t_n, $\hat{t}_n$, although each KVSKP flow can be a linear combination of the flows of the maximal hierarchy. The precise relation between the two sets of flow parameters depends on the specifics of the bosonization and detailed properties of the section σ_0 (tau function) and will not be determined here.

6. Conclusions and Open Problems

We have seen that algebro-geometric solutions to all versions of the SKP hierarchy can be obtained from suitable geometric data by means of a Krichever construction. The data consist of a (1|1)-dimensional supermanifold, generally *not* a super Riemann surface, a line bundle satisfying $H^0 = H^1 = 0$, and local coordinates and trivializations. The SKP equations describe deformations of the bundle as well as deformations of the supermanifold of the type which preserve projectedness. The Manin-Radul hierarchy describes a specific combination of these deformations which is integrable and can be translated into a Lax formalism for a pseudosuperdifferential operator L. The Jacobian SKP hierarchy introduced here describes deformations of the bundle only; it is integrable but cannot be put in Lax form. The maximal SKP hierarchy describes all possible deformations of the stated types, which are also the nontrivial deformations appearing in the hierarchy of Kac and van de Leur. The geometric setting for these flows is the universal Picard bundle over the moduli space of supercurves. When the initial data is nonsplit, the unique definition of the nonvertical flows requires a choice of connection in this universal bundle.

There are several further directions to pursue. The treatment of general (nonsplit) families in this paper has not been as rigorous or as elegant as possible. We have shown that the cohomology conditions on the Krichever data guarantee that the spaces of sections we need are freely generated over finite-dimensional odd parameter spaces, and treated the infinite-dimensional case as a direct limit. The SKP flows, however, make sense for geometric data not satisfying the cohomology conditions, and infinite-dimensional parameter spaces are fundamental to the SKP theory. A more general notion of super Grassmannian should be developed which makes sense for arbitrary parameter spaces and without restriction to freely generated subspaces of the space of formal Laurent series. This more general notion will be the appropriate setting for a proof that the Krichever construction produces all finite-dimensional orbits of the MRSKP hierarchy, and for further study of the geometry of the universal Picard bundle relevant to the nonsplit case. Because the space of sections $H^0(M\backslash p, \mathscr{L})/H^0(M, \mathscr{L}\otimes\mathcal{O}(np))$ is freely generated for sufficiently large n, one might guess that Gr should be defined as the set of all submodules, of the space of formal Laurent series with coefficients from a parameter space, for which the projection onto span$\{z^{-n}, \theta z^{-n}, n>N\}$ for some N gives a free module. This definition must be supplemented by an analogue of the Fredholm condition which makes sense over an infinite-dimensional parameter space.

Given a solution S of a SKP hierarchy obtained by the Krichever construction, we have seen how to reconstruct the geometric data. We have not addressed the question of how to recognize such solutions given only S. This question is closely connected with the problem of classifying the supercommutative rings $R(0)$ of differential operators which can arise from the construction. In the ordinary KP theory it is shown that a solution comes from geometric data whenever A_W is a rank 1 algebra, and that essentially all rank 1 algebras of differential operators are obtained (the rank being the G.C.D. of the pole orders of the elements of A_W, or of the orders of the differential operators). In the split case, we showed in [13] that a rank 1 stabilizer A_W does arise from geometric data, but we did not consider the realization of A_W by differential operators or the nonsplit case. The development of a Burchnall-Chaundy theory of supercommutative rings of superdifferential operators, particularly the reconstruction of geometric data from polynomial relations among the operators in such a ring, is an important goal.

We have also not considered in detail the Krichever construction starting from a singular supercurve rather than a smooth supermanifold. Although it is clear in general that additional solutions can be obtained in this way, we have not tried to describe precisely what kinds of singularities should be allowed, or the types of nonmaximal stabilizers A_W which arise from the rings $H^0(M\backslash p, \mathcal{O})$ on such curves. This type of generalization of the Krichever map was considered in [13] in the split case, but the significant generalization will be to the case of families and the resulting solutions.

Another important question, which also arises in the ordinary KP theory when line bundles are replaced by vector bundles [33], has to do with the geometric meaning of noncommutative stabilizers of subspaces W. The discussion of the maximal SKP hierarchy shows that it is natural to consider Laurent series in z with coefficients that may be differential operators in θ. Equivalently, introducing a vector representation $[f(z), g(z)]$ for $f(z)+\theta g(z)$ [11], these become Laurent series in z with 2×2 matrix coefficients. The maximal stabilizer, in the space of such series, of a subspace W is much larger than our A_W and generally nonsupercommutative. Our A_W is a supercommutative subring of this maximal stabilizer consisting of series involving θ but not ∂_θ. Another supercommutative subring $\bar{A}_W$ consists of the series involving ∂_θ but not θ. These supercommutative stabilizers are important because the supermanifold M can be recovered via the Spec construction only from a supercommutative ring. The question arises of the relation between the various supercommutative subrings of the maximal stabilizer (what is Spec$\bar{A}_W$?), and the possible geometric interpretation of the maximal stabilizer itself. It is intriguing to note in this connection that in Theorem 1.3 of [13] the embedding of A_W in a larger nonsupercommutative stabilizer was used in an essential way to characterize those rings A_W which arise from geometric data.

Finally, the equivalence of the KVSKP hierarchy with our maximal hierarchy has only been established in general terms. It would be desirable to explicitly relate the flow parameters appearing in the two hierarchies. This should shed light on the relation between the super tau function and some putative super theta function. In principle, the formulas in [12] should provide an answer since they relate the super tau function to the wave operator S and give a bilinear identity for the wave function which should be equivalent to the KVSKP equation (55). It is not clear how explicit these formulas can be made, however.

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