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THE GEOMETRY OF THREE-FORMS IN SIX DIMENSIONS

NIGEL HITCHIN

Abstract

We study the special algebraic properties of alternating 3-forms in 6 dimensions and introduce a diffeomorphism-invariant functional on the space of differential 3-forms on a closed 6-manifold M. Restricting the functional to a de Rham cohomology class in $H^3(M, \mathbf{R})$, we find that a critical point which is generic in a suitable sense defines a complex threefold with trivial canonical bundle. This approach gives a direct method of showing that an open set in $H^3(M, \mathbf{R})$ is a local moduli space for this structure and introduces in a natural way the special pseudo-Kähler structure on it.

1. Introduction

This paper arose from the author's interest in the geometry of 3-forms on a manifold. Exterior differential forms of degree 2 are much studied, in particular a symplectic manifold is defined by a closed non-degenerate 2-form. Our starting point here is the fact that in dimension 6 the notion of non-degeneracy for *three-forms* makes sense too. By this we mean that if W is a real vector space of dimension 6 then the group $GL(W)$ has an open orbit, a fact that has been known for a long time [10]. The orbits we are particularly interested in have stabilizer conjugate to $SL(3, \mathbf{C})$. In this way a single 3-form Ω on a manifold M defines a reduction of the structure group of M to $SL(3, \mathbf{C})$. When we look closely at the mechanism for this reduction, there appears naturally a volume form algebraically defined by any 3-form. Integrating this form gives a diffeomorphism-invariant functional Φ on the space of 3-forms, and this is our main object of study.

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Following the analogy with Hodge theory, we restrict this functional to *closed* forms on M in a given de Rham cohomology class and look for critical points. What we find is that if the critical point Ω is a 3-form which lies in the open orbit everywhere, then the reduction of structure group is integrable in the sense that we obtain a complex threefold with trivial canonical bundle. This geometrical structure is therefore simply a critical point of the functional.

Because the functional is diffeomorphism-invariant, every critical point lies on an orbit of critical points and so Φ is never a Morse function. However, we show that formally it is a Morse-Bott function – its Hessian is non-degenerate transverse to the orbits of $\mathrm{Diff}(M)$. This requires the assumption that the complex threefold satisfies the $\partial\bar{\partial}$-lemma. This nondegeneracy can be used, together with a standard use of the Banach space implicit function theorem, to give a direct proof that the moduli space of complex structures together with non-vanishing holomorphic 3-forms on a 6-manifold is locally an open set in $H^3(M, \mathbf{R})$. In particular we see that the moduli space of complex structures is unobstructed. The novelty of our approach is that the flat structure on this moduli space is apparent from the very beginning, and the complex structure is defined in a secondary manner. This is the opposite point of view from the conventional use of Kodaira-Spencer theory as in the work of Tian and Todorov [11],[12]. For us the complex structure on the moduli space carries with it the natural special pseudo-Kähler structure whose existence is an important ingredient in mirror symmetry [1].

There is an analogous story for 3-forms on 7-manifolds, which leads to G_2 structures and their moduli, but we leave that for another paper, where the duality between 3-forms and 4-forms plays a role which is not present in 6 dimensions.

The structure of the paper is as follows. In Section 2 we consider the linear algebra of the vector space $\Lambda^3 W^*$ where W is 6-dimensional. The essential point is that over the complex numbers, a generic 3-form is the sum of two decomposable ones. In Section 3, we see this algebra from the point of view of symplectic geometry, regarding $\Lambda^3 W^*$ as a symplectic vector space under the action of $SL(W)$. This viewpoint is extremely useful for studying the variational problem. Section 4 is a detour into the realm of self-duality. With an inner product of signature $(5, 1)$ on W, we can define self-dual and anti-self-dual 3-forms, and some of the linear algebra assumes a much more concrete form. Moreover, we see a setting here for the equation of motion for a self-interacting self-dual tensor, a nonlinear equation of some current interest to physicists.

In Section 5 we introduce the invariant functional on 3-forms on a 6-manifold, and relate the critical points to integrable complex structures. In Section 6 we prove Morse-Bott nondegeneracy formally, and then use a Sobolev space model to prove rigorously that an open set in $H^3(M,\mathbf{R})$ is a local moduli space.

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2. Linear algebra

2.1 The complex case

Let V be a 6-dimensional complex vector space and Λ^3V^* the 20-dimensional vector space of alternating multilinear 3-forms on V. Take $\Omega\in\Lambda^3V^*$ and $v\in V$ and the interior product $\iota(v)\Omega\in\Lambda^2V^*$. Then $\iota(v)\Omega\wedge\Omega\in\Lambda^5V^*$. On the other hand, the natural exterior product pairing $V^*\otimes\Lambda^5V^*\to\Lambda^6V^*$ provides an isomorphism

$$A:\Lambda^5V^*\cong V\otimes\Lambda^6V^*$$

and using this we define a linear transformation $K_\Omega : V\to V\otimes\Lambda^6V^*$ by

$$K_\Omega(v)=A(\iota(v)\Omega\wedge\Omega). \tag{1}$$

Definition 1. Define $\lambda(\Omega)\in(\Lambda^6V^*)^2$ by

$$\lambda(\Omega)=\frac{1}{6}\operatorname{tr}K_\Omega^2.$$

Using $\lambda(\Omega)$ we have the following characterization of "non-degenerate" elements in Λ^3V^*:

Proposition 1. *For $\Omega\in\Lambda^3V^*$, $\lambda(\Omega)\neq 0$ if and only if $\Omega=\alpha+\beta$ where α,β are decomposable and $\alpha\wedge\beta\neq 0$. The 3-forms α,β are unique up to ordering.*

Proof. Let $v_1,\dots,v_6$ be a basis of V and $\theta_1,\dots,\theta_6\in V^*$ the dual basis. Take

$$\varphi=\ \theta_1\wedge\theta_2\wedge\theta_3\ +\ \theta_4\wedge\theta_5\wedge\theta_6.$$

Let $\epsilon = \theta_1 \wedge \cdots \wedge \theta_6$ be the associated basis vector for $\Lambda^6 V^*$. We find easily that

$$K_\varphi v_i = v_i \epsilon \quad (i = 1,2,3), \qquad K_\varphi v_i = -v_i \epsilon \quad (i = 4,5,6), \tag{2}$$

and

$$\lambda(\varphi) = \epsilon^2. \tag{3}$$

Now if $\Omega = \alpha + \beta$ where α, β are decomposable then $\alpha = \xi_1 \wedge \xi_2 \wedge \xi_3$ and $\beta = \eta_1 \wedge \eta_2 \wedge \eta_3$ and the condition $\alpha \wedge \beta \neq 0$ implies that $\{\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3\}$ forms a basis for V^*. Thus by an element of $GL(V)$, Ω can be transformed to φ. Since $\lambda(\varphi)$ is non-zero from (3), so is $\lambda(\Omega)$.

If we transform φ by K_φ we find

$$K_\varphi^* \varphi = (\theta_1 \wedge \theta_2 \wedge \theta_3 - \theta_4 \wedge \theta_5 \wedge \theta_6)\epsilon^3$$

and so

$$K_\varphi^* \varphi + \epsilon^3 \varphi = 2(\theta_1 \wedge \theta_2 \wedge \theta_3)\epsilon^3.$$

Thus we see that $K_\varphi^* \varphi + \lambda(\varphi)^{3/2}\varphi \in \Lambda^3 V^* \otimes (\Lambda^6 V^*)^3$ is decomposable, for each choice of square root of $\lambda(\varphi)$. Moreover, we have

$$K_\varphi^* \varphi \wedge \varphi = 2\epsilon^4 = 2\lambda(\varphi)^2. \tag{4}$$

Let $G \subset GL(V)$ be the stabilizer of φ. Each element of G commutes with K_φ and so preserves or interchanges the two subspaces spanned by v_1, v_2, v_3 and v_4, v_5, v_6 respectively. The identity component G_0 preserves them, and the nonvanishing 3-forms $\theta_1 \wedge \theta_2 \wedge \theta_3$ and $\theta_4 \wedge \theta_5 \wedge \theta_6$ defined on them, and so is isomorphic to a subgroup of $SL(3, \mathbf{C}) \times SL(3, \mathbf{C})$. Thus

$$\dim G_0 \leq \dim(SL(3, \mathbf{C}) \times SL(3, \mathbf{C})) = 16.$$

But $\dim GL(V) = 36$ so the dimension of the orbit is at least $36 - 16 = 20$. Since $\dim \Lambda^3 V^* = 20$, the orbit is open and the stabilizer must actually be equal to $SL(3, \mathbf{C}) \times SL(3, \mathbf{C})$.

For the converse, note that the algebraic condition (4) and the decomposability of $K_\Omega^* \Omega + \lambda(\Omega)^{3/2}\Omega$ hold on the open orbit and therefore hold everywhere. Thus in general we have decomposable (possibly zero) forms $\alpha, \beta \in \Lambda^3 V^*$ with

$$\begin{aligned} \lambda(\Omega)^{3/2}\Omega + K_\Omega^* \Omega &= 2\lambda(\Omega)^{3/2}\alpha \\ \lambda(\Omega)^{3/2}\Omega - K_\Omega^* \Omega &= 2\lambda(\Omega)^{3/2}\beta \end{aligned}$$

and so if $\lambda(\Omega) \neq 0$

$$\Omega = \alpha + \beta, \qquad K_\Omega^* \Omega = \lambda(\Omega)^{3/2}(\alpha - \beta). \tag{5}$$

From (4), $K_\Omega^* \Omega \wedge \Omega = 2\lambda(\Omega)^2 \neq 0$ but also from (5)

$$K_\Omega^* \Omega \wedge \Omega = 2\lambda(\Omega)^{3/2} \alpha \wedge \beta$$

so $\alpha \wedge \beta \neq 0$ as required.

By construction, α and β are unique given the choice of square root of $\lambda(\Omega)$. q.e.d.

Remark. The proposition tells us that the open set $\lambda(\Omega) \neq 0$ is the orbit of the 3-form $\varphi = \theta_1 \wedge \theta_2 \wedge \theta_3 + \theta_4 \wedge \theta_5 \wedge \theta_6$ under the action of $GL(V)$. We can therefore deduce properties of Ω from those of φ. For example, from (2) and (3) it follows that

$$\operatorname{tr} K_\Omega = 0 \tag{6}$$

$$K_\Omega^2 = \lambda(\Omega) 1. \tag{7}$$

2.2 The real case

Now suppose that W is a *real* 6-dimensional vector space and $\Omega \in \Lambda^3 W^*$. In this case $\lambda(\Omega) \in (\Lambda^6 W^*)^2$ is real. If L is a real one-dimensional vector space, we say that a vector $u \in L \otimes L = L^2$, is positive ($u > 0$) if $u = s \otimes s$ for some $s \in L$ and negative if $-u > 0$.

Proposition 2. *Suppose that $\lambda(\Omega) \neq 0$ for $\Omega \in \Lambda^3 W^*$. Then*

- $\lambda(\Omega) > 0$ *if and only if* $\Omega = \alpha + \beta$ *where* α, β *are real decomposable* 3*-forms and* $\alpha \wedge \beta \neq 0$

- $\lambda(\Omega) < 0$ *if and only if* $\Omega = \alpha + \overline{\alpha}$ *where* $\alpha \in \Lambda^3(W^* \otimes \mathbf{C})$ *is a complex decomposable* 3*-form and* $\alpha \wedge \overline{\alpha} \neq 0$

Proof. Let $V = W \otimes \mathbf{C}$ be the complexification of W. From Proposition 1, $\Omega = \alpha + \beta$ for decomposable complex 3-forms. Since Ω is real, and α, β are unique up to ordering, complex conjugation must preserve the pair and there are only two possibilities: either α and β are both real or $\beta = \overline{\alpha}$.

To decide which holds, recall that the definition of α and β in the proof of the proposition was

$$\begin{aligned}\Omega + \lambda(\Omega)^{-3/2} K_\Omega^* \Omega &= 2\alpha \\ \Omega - \lambda(\Omega)^{-3/2} K_\Omega^* \Omega &= 2\beta.\end{aligned}$$

Thus if $\lambda(\Omega) > 0$, $\lambda(\Omega)^{1/2}$ is real and α and β are real, and if $\lambda(\Omega) < 0$, the square root is imaginary and α and β are complex conjugate.

q.e.d.

We deduce from Proposition 2 that if Ω is real and $\lambda(\Omega) > 0$, then it lies in the $GL(W)$ orbit of

$$\varphi = \theta_1 \wedge \theta_2 \wedge \theta_3 + \theta_4 \wedge \theta_5 \wedge \theta_6$$

for a basis $\theta_1, \ldots, \theta_6$ of W^*, and if $\lambda(\Omega) < 0$ in the orbit of

$$\varphi = \alpha + \overline{\alpha}, \quad \alpha = (\theta_1 + i\theta_2) \wedge (\theta_3 + i\theta_4) \wedge (\theta_5 + i\theta_6). \tag{8}$$

Thus the 20-dimensional real vector space $\Lambda^3 W^*$ contains an invariant quartic hypersurface $\lambda(\Omega) = 0$ which divides $\Lambda^3 W^*$ into two open sets: $\lambda(\Omega) > 0$ and $\lambda(\Omega) < 0$. The identity component of the stabilizer of a 3-form lying in the former is conjugate to $SL(3, \mathbf{R}) \times SL(3, \mathbf{R})$ and in the latter to $SL(3, \mathbf{C})$.

Choose now an orientation on W: this is a class of bases for the one-dimensional vector space $\Lambda^6 W^*$. We then have a distinguished ordering of α and β or α and $\overline{\alpha}$, by the condition that $\alpha \wedge \beta$ or $i\alpha \wedge \overline{\alpha}$ lies in the orientation class.

Definition 2. Let W be oriented and $\Omega \in \Lambda^3 W^*$ be such that $\lambda(\Omega) \neq 0$. Then, writing Ω in terms of decomposables ordered by the orientation, we define $\hat{\Omega} \in \Lambda^3 W^*$ by:

- If $\lambda(\Omega) > 0$, and $\Omega = \alpha + \beta$ then $\hat{\Omega} = \alpha - \beta$.
- If $\lambda(\Omega) < 0$, and $\Omega = \alpha + \overline{\alpha}$ then $\hat{\Omega} = i(\overline{\alpha} - \alpha)$.

Note that $\hat{\hat{\Omega}} = -\Omega$ in both cases. The complementary 3-form $\hat{\Omega}$ has the defining property that if $\lambda(\Omega) > 0$ then $\Omega + \hat{\Omega}$ is decomposable and if $\lambda(\Omega) < 0$, the complex form $\Omega + i\hat{\Omega}$ is decomposable.

In this paper we shall be mainly concerned with the open set

$$U = \{\Omega \in \Lambda^6 W^* : \lambda(\Omega) < 0\}.$$

As we have seen, the stabilizer of $\Omega \in U$ is conjugate to $SL(3, \mathbf{C})$ so U is just the homogeneous space $GL^{+}(6, \mathbf{R})/SL(3, \mathbf{C})$. This is the space of complex structures on $\mathbf{R}^6$ together with a complex-linear 3-form. Here it appears, exceptionally, not as a homogeneous space but as an open set in a vector space. In concrete terms, the real 3-form Ω determines the structure of a complex vector space with a complex 3-form on the real vector space W as follows.

From (7) we have $K_\Omega^2 = \lambda(\Omega)1$, so if $\lambda(\Omega) < 0$, we define the complex structure I_Ω on W by

$$I_\Omega = \frac{1}{\sqrt{-\lambda(\Omega)}} K_\Omega. \tag{9}$$

The real 3-form Ω is then the real part of the complex form of type $(3, 0)$

$$\Omega^c = \Omega + i\hat{\Omega}. \tag{10}$$

We also have

$$\Omega \wedge \hat{\Omega} = 2\sqrt{-\lambda(\Omega)}. \tag{11}$$

These statements can be proved simply by checking for φ in (8) since U is an orbit.

We have described here the linear algebra of 3-forms in 6 dimensions. It will be useful also to describe the same objects in *symplectic* terms, which we shall do next.

3. Symplectic geometry

3.1 The moment map

Choose a non-zero vector $\epsilon \in \Lambda^6 W^*$, and consider the group of linear transformations $SL(W) \subset GL(W)$ preserving ϵ.

First note that $\Lambda^3 W^*$ is a symplectic vector space with the symplectic form ω defined by

$$\omega(\Omega_1, \Omega_2)\epsilon = \Omega_1 \wedge \Omega_2 \in \Lambda^6 W^*. \tag{12}$$

This is invariant under the action of $SL(W)$, which is simple, and so we have a well-defined moment map $\mu : \Lambda^3 W^* \to \mathfrak{sl}(W)^*$ where $\mathfrak{sl}(W)$, the

Lie algebra of $SL(W)$, is the space of trace zero endomorphisms of W. If we identify the Lie algebra $\mathfrak{sl}(W)$ with its dual using the bi-invariant form $\mathrm{tr}(XY)$, then we can characterize the linear transformation K_Ω defined in (1) as follows:

Proposition 3. *The moment map for $SL(W)$ acting on $\Lambda^3 W^*$ is given by*

$$\mu(\Omega) = K_\Omega.$$

Proof. If V is a symplectic vector space with symplectic form ω, the Lie algebra $\mathfrak{sp}(V)$ of $Sp(V)$ is $Sym^2(V^*)$ the space of homogeneous quadratic polynomials on V. This is the space of Hamiltonian functions for $\mathfrak{sp}(V)$. Concretely, given $a \in \mathfrak{sp}(V)$, we have the corresponding function

$$\mu_a(v) = \omega(a(v), v) \tag{13}$$

and the moment map μ is defined by

$$\mathrm{tr}(\mu(v)a) = \mu_a(v).$$

In our case $a \in \mathfrak{sl}(W)$ defines $\rho(a) \in \mathfrak{sp}(\Lambda^3 W^*)$ via the exterior power representation and from (12) and (13)

$$\mu_a(\Omega)\epsilon = \rho(a)\Omega \wedge \Omega.$$

Using a basis $w_1, \ldots, w_6$ of W and its dual basis $\theta_1, \ldots, \theta_6$ we write

$$a = \sum_{i,j} a^i_j w_i \otimes \theta_j$$

and then the Lie algebra action is

$$\rho(a)\Omega = \sum_{i,j} a^i_j \theta_j \wedge \iota(w_i)\Omega \tag{14}$$

so that

$$\rho(a)\Omega \wedge \Omega = \sum_{i,j} a^i_j \theta_j \wedge \iota(w_i)\Omega \wedge \Omega. \tag{15}$$

But from the definition of K_Ω in (1),

$$\theta \wedge \iota(w)\Omega \wedge \Omega = \theta(K_\Omega(w))\epsilon$$

for any $\theta \in W^*$ so from (15),

$$\rho(a)\Omega \wedge \Omega = \sum_{i,j} a^i_j \theta_j(K_\Omega(w_i)) = \mathrm{tr}(aK_\Omega)$$

which proves the proposition. q.e.d.

Remark. Note that $\lambda(\Omega)$, which (by trivializing $\Lambda^6 W^*$ with ϵ) is now an $SL(W)$-invariant function, is just given by

$$\lambda(\Omega) = \frac{1}{6}\,\mathrm{tr}(\mu(\Omega)^2)$$

and thus $\lambda(\Omega)$ has a natural symplectic interpretation. Its exceptional property is that $SL(W)$ acts transitively on the generic hypersurface $\mathrm{tr}(\mu(\Omega)^2) = const$. If we ask this of a general symplectic representation we obtain a finite list of possibilities which appears in the classification of symplectic holonomy groups in [8]. This is the symplectic analogue of the fact that irreducible Riemannian holonomy groups are the compact groups which act transitively on spheres. We hope to study the analogous special geometry associated to these other cases in subsequent papers.

3.2 The Hamiltonian function ϕ

The symplectic form restricts to the open set U to define the structure of a symplectic manifold. On U, $\lambda(\Omega) < 0$, and we define

$$\phi(\Omega) = \sqrt{-\lambda(\Omega)}$$

so that ϕ is a smooth function here, homogeneous of degree 2. We can then rewrite (9) as

$$K_\Omega = \phi I_\Omega$$

and (11) as

$$\Omega \wedge \hat{\Omega} = 2\phi\epsilon. \tag{16}$$

Note that from (16) we have

$$2\phi(\hat{\Omega})\epsilon = \hat{\Omega} \wedge \hat{\hat{\Omega}} = -\hat{\Omega} \wedge \Omega = \Omega \wedge \hat{\Omega}$$

and so

$$\phi(\hat{\Omega}) = \phi(\Omega).$$

The function ϕ defines a Hamiltonian vector field X_ϕ on U. Since U is an open set in the vector space $\Lambda^3 W^*$, we may consider the vector field as a function

$$X_\phi : U \to \Lambda^3 W^*$$

and then we have:

Proposition 4. *Let* $\Omega \mapsto \hat{\Omega}$ *be the transformation of Definition* 2. *Then*

$$X_\phi(\Omega) = -\hat{\Omega}.$$

Proof. Write $\Omega = \alpha + \overline{\alpha}$ in terms of complex decomposables. Then from (16)

$$2\phi\epsilon = \Omega \wedge \hat{\Omega} = 2i\alpha \wedge \overline{\alpha}$$

and so

$$\dot{\phi}\epsilon = i\dot{\alpha} \wedge \overline{\alpha} + i\alpha \wedge \dot{\overline{\alpha}}. \tag{17}$$

Now if $\alpha = \theta_1 \wedge \theta_2 \wedge \theta_3$ is a smooth curve of decomposables, on differentiation we see that

$$\dot{\alpha} = \dot{\theta}_1 \wedge \theta_2 \wedge \theta_3 + \theta_1 \wedge \dot{\theta}_2 \wedge \theta_3 + \theta_1 \wedge \theta_2 \wedge \dot{\theta}_3$$

and so $\dot{\alpha} \wedge \alpha = 0$. Hence

$$\omega(i(\alpha - \overline{\alpha}), \dot{\alpha} + \dot{\overline{\alpha}})\epsilon = i(\alpha - \overline{\alpha}) \wedge (\dot{\alpha} + \dot{\overline{\alpha}}) = i\dot{\alpha} \wedge \overline{\alpha} + i\alpha \wedge \dot{\overline{\alpha}}$$

and so from (17) we can write

$$\dot{\phi}\epsilon = \omega(i(\alpha - \overline{\alpha}), \dot{\alpha} + \dot{\overline{\alpha}})\epsilon = -\omega(\hat{\Omega}, \dot{\Omega})\epsilon.$$

Since the Hamiltonian vector field is defined by the property $\iota(X_\phi)\omega = d\phi$ we have our result. q.e.d.

Remark. If we regard U as parametrizing complex structures on W together with $(3,0)$ forms then we have an obvious circle action $\Omega + i\hat{\Omega} \mapsto e^{i\theta}(\Omega + i\hat{\Omega})$. The real point of view we are following here describes this as an action on the real parts which is

$$\Omega \mapsto \cos\theta\,\Omega - \sin\theta\,\hat{\Omega}$$

and differentiating at $\theta = 0$ we see that this action is generated by the vector field $X_\phi = -\hat{\Omega}$. The position vector (or Euler vector field) Ω integrates to give the $\mathbf{R}^*$-action $\Omega \mapsto \lambda\Omega$ and the two define the action of multiplying the complex form $\Omega + i\hat{\Omega}$ by $\mathbf{C}^*$.

3.3 The complex structure on U

Proposition 5. *The derivative* $J = DX_\phi \in \mathrm{End}(\Lambda^3 W^*)$ *of* $X_\phi : U \to \Lambda^3 W^*$ *at* Ω *satisfies* $J^2 = -1$ *and defines an integrable complex structure on* U*.*

Proof. We have seen that $\hat{\hat{\Omega}} = -\Omega$ and $X_\phi = -\hat{\Omega}$, so it follows that $(DX_\phi)^2 = -1$. This is an almost complex structure.

To prove integrability, it is easiest to use flat coordinates $x_1, \ldots, x_{20}$ on U. The symplectic form can be written $\omega = \sum_{\alpha,\beta} \omega_{\alpha\beta} dx_\alpha \wedge dx_\beta$ with $\omega_{\alpha\beta}$ constant. Let $\omega^{\alpha\beta}$ be the inverse matrix. In these coordinates, since X_ϕ is the Hamiltonian vector field of ϕ, the matrix of $J = DX_\phi$ is

$$J^\alpha_\beta = \sum_\gamma \omega^{\alpha\gamma} \frac{\partial^2 \phi}{\partial x_\gamma \partial x_\beta}. \tag{18}$$

Following [4], we define the complex functions $z_1, \ldots, z_{20}$ by

$$z_\alpha = x_\alpha - i \sum_\beta \omega^{\alpha\beta} \frac{\partial \phi}{\partial x_\beta}.$$

Then

$$dz_\alpha = dx_\alpha - i \sum_{\beta,\gamma} \omega^{\alpha\beta} \frac{\partial^2 \phi}{\partial x_\beta \partial x_\gamma} dx_\gamma = dx_\alpha - i \sum_l J^\alpha_\gamma dx_\gamma.$$

Now

$$J(dz_\alpha) = \sum_\beta J^\alpha_\beta (dx_\beta - i \sum_l J^\beta_\gamma dx_\gamma) = \sum_\beta J^\alpha_\beta dx_\beta + i dx_\alpha = i dz_\alpha$$

so dz_α is of type $(1,0)$. Since $dz_\alpha + d\bar{z}_\alpha = 2dx_\alpha$ spans the cotangent space at each point, we can find 10 independent functions amongst these which are local holomorphic coordinates for U. q.e.d.

At each Ω we have the natural complex structure J on the vector space $\Lambda^3 W^*$ but W itself is also complex with respect to I_Ω. Take the type decomposition of $\Lambda^3 W^* \otimes \mathbf{C}$ with respect to I_Ω:

$$\Lambda^3 W^* \otimes \mathbf{C} = \Lambda^{3,0} \oplus \Lambda^{2,1} \oplus \Lambda^{1,2} \oplus \Lambda^{0,3}.$$

Proposition 6. *The complex structure* $J = i$ *on* $\Lambda^{3,0} \oplus \Lambda^{2,1}$ *and* $-i$ *on* $\Lambda^{1,2} \oplus \Lambda^{0,3}$*.*

Proof. Given $a \in \mathfrak{sl}(W)$ let X_a be the vector field on $\Lambda^3 W^*$ induced by the action. This is a linear action. This means that $DX_a = \rho(a)$ where ρ is the representation at the Lie algebra level, and $X_a = \rho(a)\Omega$ where Ω is the position vector.

Now ϕ is invariant under $SL(W)$ so $[X_a, X_\phi] = 0$, thus

$$J(X_a) = DX_\phi(X_a) = DX_a(X_\phi) = \rho(a)X_\phi. \tag{19}$$

But $X_\phi = -\hat{\Omega}$ from Proposition 4, and $X_a = \rho(a)\Omega$, thus (19) can be written

$$J(\rho(a)\Omega) = -\rho(a)\hat{\Omega}$$

and since $J^2 = -1$,

$$J(\rho(a)(\Omega + i\hat{\Omega})) = -\rho(a)(\hat{\Omega} - i\Omega) = i\rho(a)(\Omega + i\hat{\Omega}). \tag{20}$$

From (14)

$$\rho(a)(\Omega + i\hat{\Omega}) = \sum_{i,j} a^i_j \theta_j \wedge \iota(w_i)(\Omega + i\hat{\Omega})$$

and since $\Omega + i\hat{\Omega}$ is of type $(3,0)$, this is of type $(3,0) + (2,1)$. The complex structure J acts on it as i from (20).

Consider J acting on the vector field Ω. We use (18) to calculate

$$J(\Omega)_\alpha = \sum_{\beta,\gamma} \omega^{\alpha\gamma} x_\beta \frac{\partial^2 \phi}{\partial x_\gamma \partial x_\beta} = \sum_\gamma \omega^{\alpha\gamma} \frac{\partial \phi}{\partial x_\gamma} = -\hat{\Omega}_\alpha$$

since $\phi(\Omega)$ is homogeneous of degree 2 and so $\partial\phi/\partial x_\alpha$ has degree 1. It follows that $J(\Omega + i\hat{\Omega}) = i(\Omega + i\hat{\Omega})$ so J acts as i on forms of type $(3,0)$.

Now $SL(W)$ has orbits of real codimension one, and the position vector field Ω is transverse to the orbit so any vector of type $(3,0)+(2,1)$ is a complex linear combination of a vector $\rho(a)(\Omega + i\hat{\Omega})$ and $\Omega + i\hat{\Omega}$ and hence is acted on by J as i. q.e.d.

3.4 Special geometry of U

We have actually derived here a certain geometric structure on the open set U — a *special pseudo-Kähler metric*. Recall from [2] the definition:

Definition 3. A special Kähler manifold is a complex manifold M with complex structure $J \in \Omega^1(T)$ such that:

1. There is a Kähler metric g with Kähler form ω.

2. There is a flat torsion-free connection D such that:
   (a) $D\omega = 0$.
   (b) $d_D J = 0 \in \Omega^2(M, T)$.

Proposition 7. *The open set $U \subset \Lambda^3 W^*$ has an $SL(W)$-invariant special pseudo-Kähler structure of Hermitian signature $(1,9)$ and X_ϕ is an infinitesimal isometry of the metric.*

Proof. Firstly, U is given as an open set in a symplectic vector space, so we take the flat torsion-free connection as the ordinary derivative D. The complex structure J is given by $J = DX_\phi = d_D X_\phi$, so

$$d_D J = d_D^2 X_\phi = 0$$

since D is flat. For a pseudo-Kähler metric, the metric and symplectic form are related by

$$g(X, Y) = \omega(JX, Y).$$

But $\omega(J\Omega_1, \Omega_2)$ is given in flat coordinates by

$$\sum_\alpha J^\alpha_\beta \omega_{\alpha\gamma} = \frac{\partial^2 \phi}{\partial x_\gamma \partial x_\beta}$$

from (18). This is symmetric and hence does indeed define a pseudo-Kähler metric

$$g = \sum_{\beta,\gamma} \frac{\partial^2 \phi}{\partial x_\beta \partial x_\gamma} dx_\beta dx_\gamma.$$

We need to determine the signature of the corresponding Hermitian form.

From Proposition 6, the $(1,0)$ vectors for J are $\Lambda^{3,0} \oplus \Lambda^{2,1}$ and the symplectic form ω is

$$\omega(\Omega_1, \Omega_2)\epsilon = \Omega_1 \wedge \Omega_2.$$

Now if $\theta_1, \theta_2, \theta_3$ form a basis for $\Lambda^{1,0}$,

$$\theta_1 \wedge \theta_2 \wedge \overline{\theta}_3 \wedge \overline{\theta}_1 \wedge \overline{\theta}_2 \wedge \theta_3 = -\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \overline{\theta}_1 \wedge \overline{\theta}_2 \wedge \overline{\theta}_3$$

so the Hermitian form on $\Lambda^{2,1}$ has the opposite sign to that on the 1-dimensional space $\Lambda^{3,0}$. But the metric applied to the position vector field Ω (which is the real part of a vector in $\Lambda^{3,0}$) is

$$g(\Omega, \Omega) = \sum \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} x_\alpha x_\beta = 2\phi$$

since ϕ is homogeneous of degree 2. Since ϕ is positive, the metric is positive on $\Lambda^{3,0}$ and hence has signature $(1,9)$ on the whole tangent space.

The vector field X_ϕ integrates, as we have seen, to the circle action $\Omega + i\hat{\Omega} \mapsto e^{i\theta}(\Omega + i\hat{\Omega})$. This changes the complex 3-form, but not the underlying complex structure I_Ω. From Proposition 6, the complex structure J on U only depends on I_Ω, so that the circle action preserves J. By definition X_ϕ is symplectic, so it preserves the metric too. q.e.d.

Remark. The $\mathbf{C}^*$ action generated by the vector fields $X_\phi = \hat{\Omega}$ and the position vector field Ω is holomorphic and the quotient has (see [2]) a *projective special Kähler structure.* This space in our case is the space of complex structures on $\mathbf{R}^6$ compatible with a given orientation, the homogeneous space $GL^+(6,\mathbf{R})/GL(3,\mathbf{C})$.

The symplectic approach of this section will be especially useful in analyzing the variational problem, but we take a short detour first to investigate a little more the linear algebra of 3-forms in 6 dimensions.

4. Self-duality of 3-forms

4.1 Lorentzian self-duality

We point out here how the geometry of 3-forms in six dimensions can be interpreted in the presence of a Lorentzian metric. This will shed some light on the nonlinear map $\Omega \mapsto \hat{\Omega}$ and also explain the setting for a nonlinear equation of some current interest in theoretical physics.

Suppose then that the 6-dimensional real vector space W has a metric of signature $(5,1)$. The Hodge star operator

$$* : \Lambda^3 W^* \to \Lambda^3 W^*$$

is defined by the basic property

$$\alpha \wedge *\beta = (\alpha, \beta)\epsilon \tag{21}$$

where $\epsilon \in \Lambda^6 W^*$ is the volume form.

Example. If $e_0, e_1, \ldots, e_5$ is an orthogonal basis of W^* with $(e_0, e_0) = -1$ and $(e_i, e_i) = 1$ for $1 \leq i \leq 5$, then with $\epsilon = e_0 \wedge e_1 \wedge \cdots \wedge e_5$, we have

$$\begin{aligned} *e_0 \wedge e_1 \wedge e_2 &= -e_3 \wedge e_4 \wedge e_5 \\ *e_3 \wedge e_4 \wedge e_5 &= -e_0 \wedge e_1 \wedge e_2. \end{aligned}$$

The star operator just defined has the property $*^2 = 1$ and we can decompose $\Lambda^3 W^*$ into the ± 1 eigenspaces of $*$ — the self-dual and anti-self-dual 3-forms:

$$\Lambda^3 W^* = \Lambda_+ \oplus \Lambda_-.$$

If $\beta = \theta_1 \wedge \theta_2 \wedge \theta_3$ is a non-zero decomposable 3-form, let $\varphi_1, \varphi_2, \varphi_3$ be a basis for the 3-dimensional space orthogonal to that spanned by $\theta_1, \theta_2, \theta_3$. Then from the definition of the Hodge star (21), we see that $\alpha \wedge *\beta = 0$ for any 3-form α of the form $\varphi_i \wedge \rho$. This means that $*\beta$ is a multiple of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$. Thus, as in the example above, the Hodge star takes *decomposable* 3*-forms to decomposable* 3*-forms.* Geometrically it transforms a volume form on a 3-dimensional subspace to a volume form on the orthogonal space.

If β (real or complex) is decomposable and $*\beta = \beta$ then the space spanned by $\theta_1, \theta_2, \theta_3$ is orthogonal to itself and hence isotropic.

We see next that real self-dual 3-forms lie on one side of the hypersurface $\lambda(\Omega) = 0$:

Proposition 8. *Let $\Omega \in \Lambda^3 W^*$ be self-dual. Then $\lambda(\Omega) \geq 0$, and if $\lambda(\Omega) > 0$, then $\Omega = \alpha + *\alpha$ where $\alpha \in \Lambda^3 W^*$ is decomposable.*

Proof. Suppose $\lambda(\Omega) < 0$, then from Proposition 1, $\Omega = \alpha + \overline{\alpha}$ with $\alpha = \theta_1 \wedge \theta_2 \wedge \theta_3$ and θ_i complex. These span the $(1, 0)$ forms for the complex structure I_Ω. Since $*$ is linear and $\alpha, \overline{\alpha}$ are unique up to ordering, the self-duality condition implies that either

$$*\alpha = \alpha \quad \text{or} \quad *\alpha = \overline{\alpha}.$$

In the first case, the space of $(1, 0)$ forms is isotropic, so we have a pseudo-hermitian metric. This must have real signature of the form $(2k, 6 - 2k)$ and not $(5, 1)$. In the second case the θ_i are orthogonal to the $\overline{\theta}_j$ so the metric is of type $(2, 0) + (0, 2)$ and so the signature must be $(3, 3)$ and not $(5, 1)$.

We conclude that $\lambda(\Omega) > 0$ and from Proposition 1, $\Omega = \alpha + \beta$ for real decomposable forms α, β. Since $*\Omega = \Omega$ we must have either $*\alpha = \alpha$ and $*\beta = \beta$ or $*\alpha = \beta$. The first implies that there is a real isotropic 3-dimensional subspace which is impossible in signature $(5,1)$ so the second condition holds, as required.

Note that since $\alpha \wedge \beta \neq 0$, the two 3-dimensional subspaces of W determined by α and β span W. Since they are orthogonal, one must have signature $(2,1)$ and the other $(3,0)$. q.e.d.

Remark. We see from the proposition that $\Omega = \alpha + *\alpha$ where α and $*\alpha$ are decomposable. The signature of the two 3-dimensional subspaces gives us a means of distinguishing α and $*\alpha$ and hence an ordering. The formula (21) tells us that

$$\alpha \wedge *\alpha = (\alpha, \alpha)\epsilon$$

so if the metric on the 3-dimensional subspace of W determined by α is of signature $(2,1)$ (or equivalently $(\alpha,\alpha) > 0$) then this is the ordering determined as in Section 2 by the orientation.

Since for a self-dual form $\lambda(\Omega) > 0$, and $\Omega = \alpha + *\alpha$, by the definition of the map $\Omega \mapsto \hat{\Omega}$ we have

$$\hat{\Omega} = \alpha - *\alpha,$$

hence:

Proposition 9. *If Ω is self-dual, then $\hat{\Omega}$ is anti-self-dual.*

Thus $\Omega \mapsto \hat{\Omega}$ is a nonlinear map from an open set of the space Λ_+ to Λ_-. We define in this case

$$\phi(\Omega) = \sqrt{\lambda(\Omega)}$$

to be the positive square root.

4.2 Lagrangian aspects

There is a symplectic interpretation also to self-duality. First recall that the symplectic form ω on $\Lambda^3 W^*$ is defined by $\omega(\Omega_1, \Omega_2)\epsilon = \Omega_1 \wedge \Omega_2$. We have:

Proposition 10. *The subspaces $\Lambda_+, \Lambda_- \subset \Lambda^3 W^*$ of self-dual and anti-self-dual 3-forms are Lagrangian.*

Proof. Suppose Ω_1, Ω_2 are self-dual then

$$\Omega_1 \wedge \Omega_2 = \Omega_1 \wedge *\Omega_2 = (\Omega_1, \Omega_2)\epsilon$$

and similarly $\Omega_2 \wedge \Omega_1 = (\Omega_2, \Omega_1)\epsilon$. But $(\Omega_1, \Omega_2) = (\Omega_2, \Omega_1)$ and $\Omega_1 \wedge \Omega_2 = -\Omega_1 \wedge \Omega_2$ so $\Omega_1 \wedge \Omega_2 = 0$. The same argument gives the anti-self-dual version. q.e.d.

From this result, the symplectic pairing identifies Λ_- with the dual space Λ_+^*. We then obtain:

Proposition 11. *The map $\Omega \mapsto \hat{\Omega}$ from Λ_+ to $\Lambda_- \cong \Lambda_+^*$ is the derivative of $-\phi$ restricted to Λ_+.*

Proof. We saw in Proposition 4 that $-\hat{\Omega} = X_\phi$ the Hamiltonian vector field generated by the function ϕ. Let $\dot{\Omega}$ be tangent to Λ_+. Then

$$D\phi(\dot{\Omega}) = \omega(X_\phi, \dot{\Omega}).$$

But this pairing identifies Λ_-, in which $X_\phi = -\hat{\Omega}$ lies, with Λ_+^*, hence the result. q.e.d.

From (10) and (11) we see that the subset

$$L = \{\Omega + \hat{\Omega} \in \Lambda_+ \oplus \Lambda_- : \lambda(\Omega) > 0\}$$

is the Lagrangian submanifold generated by the invariant function $-\phi$. If $f(x)$ is an *arbitrary* smooth function, then the form

$$\Omega + f(\phi)\hat{\Omega}$$

where Ω is self-dual, lies on the Lagrangian submanifold generated by $g(\phi)$ where $g'(x) = f(x)$. This is the setting for a nonlinear equation — the equation of motion for a self-interacting self-dual tensor in six dimensions — which appears in the physics literature [5] in the context of M-theory five-branes. Explicitly these equations are:

$$dH = d(H_+ + H_-) = 0$$

where H_+, H_- are the self-dual and anti-self-dual components of the 3-form H, and

$$\begin{aligned} (H_+)_{abc} &= Q^{-1} h_{abc} \\ (H_-)_{abc} &= Q^{-1} k_a^d h_{dbc}, \end{aligned}$$

where

$$k_a^b = h_{acd} h^{bcd}$$

and

$$Q = 1 - \frac{2}{3}\operatorname{tr} k^2.$$

The equations say in our language that H_- is proportional to $\hat{H}_+$. They can be put in the form

$$d(\Omega + f(\phi)\hat{\Omega}) = 0$$

for a particular function $f(\phi)$.

If we replace the nonlinear Lagrangian submanifold which defines this equation by the linear one Λ_+ then we just obtain the self-dual "Maxwell equations" in six dimensions.

4.3 Spinor formulation

For self-dual forms, the function ϕ and the nonlinear map $\Omega \mapsto \hat{\Omega}$ also have a concrete representation using 2-component spinors. We recall the special isomorphism

$$Spin(5,1) \cong SL(2,\mathbf{H}).$$

The two spin representations in this signature are dual 4-dimensional complex vector spaces S and S^* with a quaternionic structure, and then

$$\Lambda_+ \cong Sym^2 S^*, \quad \Lambda_- \cong Sym^2 S$$

are the real 10-dimensional spaces of self-dual and anti-self-dual 3-forms. Thus a self-dual 3-form can be interpreted as a symmetric linear map

$$A : S \to S^*.$$

The quartic invariant function λ must be a multiple of the only $SL(4,\mathbf{C})$ quartic invariant of A, namely the determinant:

$$\lambda(A) = 4 \det A.$$

We find now the following simple expression for $\hat{\Omega}$:

Proposition 12.

- *If $A \in Sym^2 S^*$ represents a self-dual 3-form Ω then*

$$\hat{\Omega} = \sqrt{\det A}\, A^{-1} \in Sym^2 S.$$

- *If $B \in Sym^2 S$ represents an anti-self-dual* 3*-form* Ω *then*

$$\hat{\Omega} = -\sqrt{\det A} A^{-1} \in Sym^2 S.$$

Proof. We use Proposition 11. We have the standard identity for differentiating the determinant:

$$\frac{1}{\det A} d(\det A)(\dot{A}) = d(\log \det A)(\dot{A}) = \operatorname{tr}(A^{-1}\dot{A})$$

and so since $\phi = \sqrt{\lambda} = 2\sqrt{\det A}$,

$$d\phi(\dot{A}) = d(2\sqrt{\det A})(\dot{A}) = \operatorname{tr}(\sqrt{\det A} A^{-1}\dot{A}).$$

Taking $\operatorname{tr}(AB)$ to be the natural pairing between $Sym^2 S$ and $Sym^2 S^*$ we have the result for self-dual forms. The second result follows from $\hat{\hat{\Omega}} = -\Omega$. q.e.d.

The isomorphism $Spin(5,1) \cong SL(2, \mathbf{H})$ is fundamental in the twistor theory of the 4-sphere. The group $SO(5,1)$ acts as the conformal transformations of S^4 defined by the quadric $x_0^2 = x_1^2 + \cdots + x_5^2$ in $\mathbf{R}\mathrm{P}^5 = P(W)$ and $SL(2, \mathbf{H})$ as the projective transformations of the twistor space $\mathbf{C}\mathrm{P}^3 = P(S)$ which commute with the real structure. The 4-sphere parametrizes this way the real lines in $\mathbf{C}\mathrm{P}^3$. Now a self-dual 3-form with $\lambda(\Omega) > 0$ can be written $\Omega = \alpha + *\alpha$ where α defines a 3-dimensional subspace of W with signature $(2,1)$. The corresponding plane in $\mathbf{R}\mathrm{P}^5$ intersects S^4 in a circle. On the other hand the quadratic form A representing Ω defines a quadric in $\mathbf{C}\mathrm{P}^3$. This geometry comes to the fore in particular in the study of the moduli space of charge 2 instantons [3].

Note that in the twistor interpretation, a quadratic form A with $\det(A) \neq 0$ defines a nonsingular quadric in $\mathbf{C}\mathrm{P}^3$. From (12) the map $\Omega \mapsto \hat{\Omega}$ is then nothing more than replacing a quadric by the dual quadric.

5. An invariant functional

5.1 Critical points

We return now to the mainstream of our development. Let M be a closed, oriented 6-manifold. A global 3-form Ω on M defines at each point a vector in $\Lambda^3 T^*$ and so a global section $\lambda(\Omega)$ of $(\Lambda^6 T^*)^2$. We take $|\lambda(\Omega)|$, which is a non-negative continuous section of $(\Lambda^6 T^*)^2$, and the square root (a section of $\Lambda^6 T^*$) which lies in the given orientation class. Thus $\sqrt{|\lambda(\Omega)|}$ is a continuous 6-form on M, which is smooth wherever $\lambda(\Omega)$ is non-zero. We define a functional Φ on $C^\infty(\Lambda^3 T^*)$ by

$$\Phi(\Omega) = \int_M \sqrt{|\lambda(\Omega)|}. \tag{22}$$

Since $\lambda : \Lambda^3 T^* \to (\Lambda^6 T^*)^2$ is $GL(6, \mathbf{R})$-invariant, Φ is clearly invariant under the action of orientation-preserving diffeomorphisms.

The functional Φ is homogeneous of degree 2 in Ω, just like the norm square of a form using a Riemannian metric and it is natural to try and apply "non-linear Hodge theory" — to look for critical points of Φ on a cohomology class of closed 3-forms on M. We find the following:

Theorem 13. *Let M be a compact complex 3-manifold with trivial canonical bundle and Ω the real part of a non-vanishing holomorphic 3-form. Then Ω is a critical point of the functional Φ restricted to the cohomology class $[\Omega] \in H^3(M, \mathbf{R})$.*

Conversely, if Ω is a critical point on a cohomology class of an oriented closed 6-manifold M and $\lambda(\Omega) < 0$ everywhere, then Ω defines on M the structure of a complex manifold such that Ω is the real part of a non-vanishing holomorphic 3-form.

Proof. Choose a non-vanishing 6-form ϵ on M and assume that Ω is a closed 3-form with $\lambda(\Omega) < 0$ (as is the case for the real part of a holomorphic 3-form). Writing

$$\lambda(\Omega) = -\phi(\Omega)^2 \epsilon^2$$

we have

$$\Phi(\Omega) = \int_M \phi(\Omega)\epsilon.$$

Since $\lambda(\Omega) \neq 0$, ϕ is smooth so take the first variation of this functional:

$$\delta\Phi(\dot{\Omega}) = \int_M D\phi(\dot{\Omega})\epsilon. \tag{23}$$

From the symplectic interpretation, $D\phi(\dot{\Omega}) = \omega(X_\phi, \dot{\Omega})$ and $X_\phi = -\hat{\Omega}$, so that

$$D\phi(\dot{\Omega})\epsilon = -\hat{\Omega} \wedge \dot{\Omega}$$

and therefore (23) gives

$$\delta\Phi(\dot{\Omega}) = -\int_M \hat{\Omega} \wedge \dot{\Omega}. \tag{24}$$

We are varying in a fixed cohomology class, so $\dot{\Omega} = d\varphi$ for some 2-form φ. Putting this in the integral, we have

$$\delta\Phi(\dot{\Omega}) = -\int_M \hat{\Omega} \wedge d\varphi = -\int_M d\hat{\Omega} \wedge \varphi$$

by Stokes' theorem. Thus $\delta\Phi = 0$ at Ω for all φ if and only if

$$d\hat{\Omega} = 0.$$

If M is a complex manifold with a non-vanishing holomorphic 3-form $\Omega + i\hat{\Omega}$ then since a holomorphic 3-form is closed, $d\Omega = d\hat{\Omega} = 0$ and so Ω is a critical point for Φ.

Conversely, assume that $d(\Omega + i\hat{\Omega}) = 0$ and $\lambda(\Omega) < 0$. Then a complex 1-form θ is of type $(1,0)$ with respect to the almost complex structure I_Ω if and only if

$$\theta \wedge (\Omega + i\hat{\Omega}) = 0.$$

Taking the exterior derivative

$$d\theta \wedge (\Omega + i\hat{\Omega}) = 0$$

which means that $d\theta$ has no $(0,2)$ component. But from the Newlander-Nirenberg theorem this means that I_Ω is integrable. Since $(\Omega + i\hat{\Omega})$ is of type $(3,0)$,

$$0 = d(\Omega + i\hat{\Omega}) = \overline{\partial}(\Omega + i\hat{\Omega})$$

and so is holomorphic. q.e.d.

Example. A Calabi-Yau threefold is by definition a Kähler manifold with a covariant constant (and hence nonvanishing) holomorphic 3-form, and so its complex structure appears as a critical point of the functional. These are perhaps the most interesting and plentiful examples. On the other hand, at the opposite extreme from the Kähler case are the non-Kähler complex threefolds with trivial canonical bundle which are diffeomorphic to connected sums of $k \geq 2$ copies of $S^3 \times S^3$ (see [7]).

5.2 Nondegeneracy

Since the functional Φ is diffeomorphism-invariant, any critical point lies on a Diff(M)-orbit of critical points and so cannot be nondegenerate. We can ask however if it is formally a Morse-Bott critical point i.e., if its Hessian is nondegenerate transverse to the action of Diff(M). The next proposition gives conditions under which this is true:

Proposition 14. *Let M be a compact complex 3-manifold with non-vanishing holomorphic 3-form $\Omega + i\hat{\Omega}$. Suppose M satisfies the $\partial\overline{\partial}$-lemma, then the Hessian of Φ is nondegenerate transverse to the action of* Diff(M) *at Ω.*

(Recall that the $\partial\overline{\partial}$-lemma holds if each exact p-form α satisfying $\partial\alpha = 0$ and $\overline{\partial}\alpha = 0$ can be written as $\alpha = \partial\overline{\partial}\beta$ for some $(p-2)$-form β. This is true for any Kähler manifold, but also for the non-Kähler examples above.)

Proof. We need the second variation of the functional

$$\Phi(\Omega) = \int_M \phi(\Omega)\epsilon.$$

Since ϕ depends only on the value of Ω at a point and not its derivatives we obtain the Hessian

$$\delta^2\Phi(\dot{\Omega}_1, \dot{\Omega}_2) = \int_M D^2\phi(\dot{\Omega}_1, \dot{\Omega}_2)\epsilon$$

but from the Hamiltonian interpretation we can write this as

$$\delta^2\Phi(\dot{\Omega}_1, \dot{\Omega}_2) = \int_M DX_\phi(\dot{\Omega}_1) \wedge \dot{\Omega}_2 = \int_M J\dot{\Omega}_1 \wedge \dot{\Omega}_2. \tag{25}$$

We want to study the degeneracy of the Hessian, so suppose that $\delta^2\Phi(\dot{\Omega}_1, \dot{\Omega}_2) = 0$ for $\dot{\Omega}_1 = d\psi$ and *all* $\dot{\Omega}_2 = d\varphi$. Then from (25)

$$0 = \int_M Jd\psi \wedge d\varphi = -\int_M dJd\psi \wedge \varphi$$

by Stokes' theorem. This holds for all 2-forms φ, which implies

$$dJd\psi = 0.$$

Thus to prove the proposition we should deduce from this that $d\psi$ is tangent to the Diff(M) orbit through Ω.

The tangent space to the $\mathrm{Diff}(M)$ orbit is the space spanned by $\mathcal{L}_X\Omega$ for a vector field X. We have

$$\mathcal{L}_X\Omega = d(\iota(X)\Omega) + \iota(X)d\Omega$$

but since Ω is closed, these are the 3-forms $d(\iota(X)\Omega)$. Since Ω is the real part of a non-vanishing $(3,0)$ form, the 2-forms $\iota(X)\Omega$ are precisely the real sections α of $\Lambda^{2,0} \oplus \Lambda^{0,2}$.

The proposition follows from the following lemma:

Lemma 15. *Let M be a compact complex 3-manifold with non-vanishing holomorphic 3-form which satisfies the $\partial\overline{\partial}$-lemma, then $d\psi$ is tangent to the $\mathrm{Diff}(M)$ orbit of Ω if and only if $dJd\psi = 0$.*

Proof. If ψ is of type $(2,0)$ then $d\psi \in C^\infty(\Lambda^{3,0} \oplus \Lambda^{2,1})$ and then from Proposition 6, $dJd\psi = d(id\psi) = id^2\psi = 0$. Thus $dJd\psi = 0$.

Conversely, suppose $dJd\psi = 0$. Since we have just seen this holds for all ψ of type $(2,0)+(0,2)$ assume ψ has type $(1,1)$. Then $d\psi = \partial\psi + \overline{\partial}\psi$ where $\partial\psi, \overline{\partial}\psi$ are of type $(2,1), (1,2)$ respectively. From Proposition 6,

$$0 = dJd\psi = id\partial\psi - id\overline{\partial}\psi = 2i\partial\overline{\partial}\psi.$$

This means that $\partial d\psi = \overline{\partial} d\psi = 0$ and so applying the $\partial\overline{\partial}$-lemma to $d\psi$ we can write

$$d\psi = i\partial\overline{\partial}\gamma$$

for a real 1-form γ. Writing $\gamma = \theta + \overline{\theta}$ where θ is of type $(1,0)$, we have

$$d\psi = i\partial\overline{\partial}\gamma = i\partial\overline{\partial}(\theta + \overline{\theta}) = d(i(\overline{\partial\theta} - \partial\theta)).$$

Since $i(\overline{\partial\theta} - \partial\theta)$ is real and of type $(2,0) + (0,2)$ this proves the lemma. q.e.d.

At a formal level what we have proved here is that if we consider the invariant functional Φ as a function on the quotient of a cohomology class by $\mathrm{Diff}(M)$, then it has a non-degenerate critical point at a 3-form Ω which defines a complex manifold with trivial canonical bundle. We then expect that nearby cohomology classes will also have non-degenerate critical points and that an open set in $H^3(M, \mathbf{R})$ will parametrize the moduli of such complex structures, (together with holomorphic 3-forms). This is true in all dimensions by the results of A. Todorov [12] and G. Tian [11], but we shall give next a direct treatment in three dimensions from our variational point of view.

6. The moduli space

6.1 Sobolev spaces

If $f(x,t)$ is a smooth family of functions $f : \mathbf{R}^m \times \mathbf{R}^n \mapsto \mathbf{R}$ such that $f(x,0)$ has a non-degenerate critical point at $x = 0$, then there is a neighbourhood $U \times V$ of $(0,0)$ such for each $t \in V$, $f(x,t)$ has a unique non-degenerate critical point in U. To prove this, we just apply the implicit function theorem to the map

$$(x,t) \mapsto D_x f.$$

We shall use this argument next in a Banach space context to translate the formal results of the last section into a concrete construction of a moduli space.

Take a 3-form Ω on M defining a complex structure satisfying the $\partial\bar{\partial}$-lemma, and choose a Hermitian metric. We take a slice for the $\mathrm{Diff}(M)$ action at Ω by looking at the space of closed forms which are orthogonal to the orbit of Ω. Recall that the tangent space of the orbit consists of forms $d\psi$ where ψ is real and of type $(2,0)+(0,2)$, so using

$$\int_M (\alpha, d\psi) = \int_M (d^*\alpha, \psi)$$

orthogonality is the condition

$$(d^*\alpha)^{2,0} = 0.$$

We shall work with Sobolev spaces of forms $L^2_k(\Lambda^p)$, choosing k appropriately when required. So let E be the Banach space

$$E = \{\alpha \in L^2_k(\Lambda^3) : d\alpha = 0 \quad \text{and} \quad (d^*\alpha)^{2,0} = 0\}.$$

First we show that L^2 orthogonal projection onto E is well-behaved in the Sobolev norm.

Let G be the Green's operator for the Laplacian Δ on forms. Then elliptic regularity says that

$$G : L^2_k(\Lambda^p) \to L^2_{k+2}(\Lambda^p)$$

and for any form α we have a Hodge decomposition

$$\alpha = H(\alpha) + d(d^*G\alpha) + d^*(Gd\alpha) \tag{26}$$

where $H(\alpha)$ is harmonic. So given $\alpha \in L^2_k(\Lambda^3)$ define first the form $\alpha_1 \in L^2_k(\Lambda^3)$ by

$$\alpha_1 = H(\alpha) + d(d^*G\alpha).$$

This is an L^2 orthogonal projection to closed forms in the same Sobolev space. To further project onto E we want to find a form θ of type $(2,0)$ such that

$$\alpha_2 = \alpha_1 + d(\theta + \overline{\theta})$$

satisfies $(d^*\alpha_2)^{2,0} = 0$. Write

$$d^*G\alpha = \rho + \nu + \overline{\nu} \in L^2_{k+1}(\Lambda^2)$$

where ρ is of type $(1,1)$ and ν of type $(2,0)$. Then with $\psi = \theta + \nu$ we want

$$(d^*d(\rho + \psi + \overline{\psi}))^{2,0} = 0$$

or, decomposing into types,

$$\overline{\partial}^*\partial\rho + \partial^*\partial\psi + \overline{\partial}^*\overline{\partial}\psi = 0$$

or equivalently

$$(\partial^*\partial + \overline{\partial}^*\overline{\partial})\psi = -\overline{\partial}^*\partial\rho. \tag{27}$$

Now $\partial^*\partial + \overline{\partial}^*\overline{\partial}$ is elliptic. Indeed it is the sum of two non-negative second order operators $\partial^*\partial$ and $\overline{\partial}^*\overline{\partial}$ and the latter is itself elliptic on $(2,0)$ forms. The null space consists of $(2,0)$ forms ψ satisfying $\overline{\partial}\psi = \partial\psi = 0$ (the holomorphic 2-forms). It follows that for each ψ in this null-space

$$\int_M (\overline{\partial}^*\partial\rho, \psi) = \int_M (\partial\rho, \overline{\partial}\psi) = 0$$

and so given ρ, we can solve the equation (27) for ψ. Using the Green's operator for $\partial^*\partial + \overline{\partial}^*\overline{\partial}$, we find $\theta \in L^2_{k+1}(\Lambda^2)$ as required.

Thus $\alpha_2 = \alpha_1 + d(\theta + \overline{\theta}) \in L^2_k(\Lambda^3)$ lies in the Banach space E, and the map $\alpha \mapsto \alpha_2$ is a continuous projection, orthogonal in L^2.

If α is harmonic, then $d\alpha = 0$ and $d^*\alpha = 0$ and so in particular $(d^*\alpha)^{2,0} = 0$ and $\alpha \in E$. We then split $E = E_1 \oplus E_2$ where E_1 is the finite-dimensional space of harmonic forms and E_2 the exact ones in E.

The Banach space E is L^2-orthogonal to the orbit of $\mathrm{Diff}(M)$ through Ω and hence is transversal to the orbits through a neighbourhood of Ω. The functional Φ is smooth for 3-forms α for which $\lambda(\alpha) < 0$ at all points, so in order to define Φ on E we need uniform estimates on α. The Sobolev embedding theorem tells us that in 6 dimensions we can achieve this with $L^2_k(\Lambda^3)$ for $k > 3$. Moreover in this range L^2_k is a Banach algebra and so multiplication is smooth. Thus in a Sobolev neighbourhood of Ω, Φ is a smooth function on E. Its derivative at α is given (from (24)) by

$$\delta\Phi(\dot{\alpha}) = -\int_M \hat{\alpha}\wedge\dot{\alpha} = \int_M (*\hat{\alpha},\dot{\alpha})\epsilon = \int_M (P(*\hat{\alpha}),\dot{\alpha})\epsilon$$

where P is orthogonal projection onto E.

The space E is transverse to the $\mathrm{Diff}(M)$ orbits, and Φ is constant on these, so its derivative is determined by its derivative as a function on E. We want the critical points of Φ restricted to a cohomology class so if P_2 denotes orthogonal projection onto the exact forms E_2, our critical points are the zeros of the function $F : E \to E_2$ defined by

$$F(\alpha) = P_2(*\hat{\alpha}).$$

6.2 Invertibility of the derivative

To apply the implicit function theorem, we want the derivative $D_2F : E_2 \to E_2$ to be invertible, where, as we calculated in the previous section,

$$D_2F(\dot{\alpha}) = P_2(*J\dot{\alpha}).$$

The second variation calculation (25) shows that this is an injection for exact $\dot{\alpha}$.

We now prove surjectivity. Note that if β is of type $(2,0)+(0,2)$, $P_2(*Jd\beta) = 0$, so surjectivity for $d\beta \in L^2_k(\Lambda^3)$ implies surjectivity on the transversal E_2. We use the following lemma (the proof is somewhat shorter in the Kähler case):

Lemma 16. *If $\gamma \in L^2_{k+1}(\Lambda^{1,1})$ is a real form which satisfies $(d^*d\gamma)^{2,0} = 0$, then there exist a real form $\rho \in L^2_k(\Lambda^3)$ with $d^*\rho = 0$ and a complex form $\sigma \in L^2_{k+1}(\Lambda^{2,2})$ such that*

$$d\gamma = \rho + \partial^*\sigma + \overline{\partial}^*\overline{\sigma}.$$

Proof. The condition $(d^*d\gamma)^{2,0} = 0$ is equivalent to

$$\overline{\partial}^*\partial\gamma = 0. \tag{28}$$

Using the Green's operator for the $\overline{\partial}$-Laplacian, we write

$$\partial\gamma = H + \overline{\partial}G\overline{\partial}^*\partial\gamma + \overline{\partial}^*G\overline{\partial}\partial\gamma = H + \overline{\partial}^*G\overline{\partial}\partial\gamma \tag{29}$$

from (28). Here H is the $\overline{\partial}$-harmonic component.

Now if $\overline{\partial}\theta = 0$, applying the $\partial\overline{\partial}$-lemma to $d\theta$ we have $d\theta = \partial\overline{\partial}\nu$ and so $\theta - \overline{\partial}\nu$ is closed. Moreover, by imposing the condition $\overline{\partial}^*\nu = 0$, if $\theta \in L^2_k$, $\nu \in L^2_{k+1}$. Similarly if $\overline{\partial}^*\theta = 0$, there is a form ν such that $\psi = \theta - \overline{\partial}^*\nu$ is coclosed: $d^*\psi = 0$. Applying this to the harmonic part H, which satisfies $\overline{\partial}^*H = 0$, we have a coclosed form $\psi = H - \overline{\partial}^*\nu$ and so from (29) a $(2,2)$ form $\sigma = G\overline{\partial}\partial\gamma - \nu$ such that

$$\partial\gamma = \psi + \overline{\partial}^*\sigma.$$

Adding on the complex conjugate we obtain

$$d\gamma = \rho + \overline{\partial}^*\sigma + \partial^*\overline{\sigma}$$

as required. q.e.d.

To continue with surjectivity, take $d\gamma \in E_2$. Since $P_2(d\gamma) = 0$ if γ is of type $(2,0) + (0,2)$, we may assume that γ is of type $(1,1)$. From the lemma, we can write

$$\begin{aligned} d\gamma &= \rho + \overline{\partial}^*\sigma + \partial^*\overline{\sigma} \\ &= \rho + (\overline{\partial}^* + \partial^*)u + i(\overline{\partial}^* - \partial^*)v \end{aligned}$$

where $\sigma = u + iv$. Now v is of type $(2,2)$ so $i(\overline{\partial}^* - \partial^*)v = J(\overline{\partial}^* + \partial^*)v = Jd^*v$ which gives

$$Jd^*v = d\gamma - \rho - d^*u.$$

Since ρ is coclosed, $d^*(\rho + d^*u) = 0$ and thus the form $\rho + d^*u$ is orthogonal to all exact forms. Hence $P_2(Jd^*v) = P_2(d\gamma)$ or

$$P_2(*Jd*v) = P_2(d\gamma)$$

and we have surjectivity. From the open mapping theorem the derivative is now invertible.

6.3 The geometry of the moduli space

We can now apply the implicit function theorem for Banach spaces to deduce that in a sufficiently small neighbourhood U' of Ω, the subspace $f^{-1}(0) \cap U'$ is diffeomorphic to a neighbourhood of $H(\Omega) \in E_1$, the space of harmonic 3-forms. Equivalently, the natural projection $p : E \to H^3(M, \mathbf{R})$ identifies $f^{-1}(0) \cap U'$ with a neighbourhood U of the cohomology class $[\Omega]$.

Taking $k > 4$ for the Sobolev space $L^2_k(\Lambda^3)$, we have enough regularity to use the proof of the Newlander-Nirenberg theorem in [9] to deduce that the critical points of Φ on $U' \subset E$ define a family of complex structures with trivial canonical bundle on M. We now have what we need: a family of critical points parametrized by $U \subset H^3(M, \mathbf{R})$.

In this real approach to Calabi-Yau manifolds, the first thing we reach is a moduli space which is an open set in the real vector space $H^3(M, \mathbf{R})$. We expect to see a complex structure on this (though the reader might also appreciate Kodaira's recollection in [6] "At first we did not take notice of the fact that the parameter appearing in the definition of a complex manifold is in general a complex one..."). We shall see next how a complex structure arises in our formalism.

Proposition 17. *The open set $U \subset H^3(M, \mathbf{R})$ has the structure of a special pseudo-Kähler manifold of Hermitian signature $(1, h^{2,1} - 1)$.*

Proof. The construction is essentially induced from that on $\Lambda^3 W^*$ of Section 3. In fact if M is a 6-torus the two are exactly the same.

We made use of the symplectic form

$$\omega(\Omega_1, \Omega_2)\epsilon = \Omega_1 \wedge \Omega_2$$

on $\Lambda^3 W^*$ and here we use its integrated form to define a flat symplectic structure on $H^3(M, \mathbf{R})$: take closed forms Ω_1, Ω_2 with cohomology classes $[\Omega_1], [\Omega_2]$ and define

$$\omega([\Omega_1], [\Omega_2]) = \int_M \Omega_1 \wedge \Omega_2 = ([\Omega_1] \cup [\Omega_2])[M].$$

By Poincaré duality this is non-degenerate.

If Ω defines a complex structure then so does $\cos\theta\,\Omega + \sin\theta\,\hat{\Omega}$ so we may as well assume that the neighbourhood U is invariant by the circle action

$$[\Omega] \mapsto \cos\theta\,[\Omega] + \sin\theta\,[\hat{\Omega}].$$

This vector field preserves the symplectic form and has Hamiltonian function

$$\Psi = ([\Omega] \cup [\hat{\Omega}])[M].$$

From (16) this is essentially the critical value of the functional Φ at the critical point Ω.

The derivative of the map $[\Omega] \mapsto [\hat{\Omega}]$ on U defines an almost complex structure since $\hat{\hat{\Omega}} = -\Omega$, and it is integrable just as in Proposition 5.

All that remains is to determine the signature. Here we use the fact that the $\partial\bar{\partial}$-lemma implies that the cohomology has a (p, q) decomposition. This is the argument we used in proving surjectivity: each $\bar{\partial}$-closed form can be made closed by adding on a $\bar{\partial}$-exact form. The determination of the sign is then just as in Proposition 7. q.e.d.

This special pseudo-Kähler structure (see [1] for its origins) again has the property that the circle action is an isometry giving the quotient by the $\mathbf{C}^*$ action the structure of a projective special Kähler manifold. This quotient forgets the choice of holomorphic 3-form and is a complex manifold of dimension $h^{2,1}$ which by our construction parametrizes a family of complex structures on M. Since from Proposition 6, J acts as i on $H^{2,1}$, this is, as a complex manifold, the usual Kuranishi moduli space.

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Mathematical Institute, Oxford, UK