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THE GEOMETRY OF TOTALLY GEODESIC FOLIATIONS ADMITTING KILLING FIELD

CARLOS CURRÀS-BOSCH

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Abstract. We study the differential geometry of codimension-one totally geodesic foliations admitting Killing field and as applications we prove, among others, that any Killing field preserves a codimension-one totally geodesic foliation of a manifold of dimension 2 or 3, under certain topological conditions on the leaves.

Introduction. In 1980 Johnson-Witt (see [6]) proved that any Killing field preserves a codimension-one totally geodesic foliation by compact leaves. In 1983 Oshikiri [7] proved a similar result when the manifold is compact and recently Oshikiri [9] has generalized this result to Killing fields with bounded length.

In this paper we shall prove, in § 3, that any Killing field preserves a geodesic flow on a complete Riemannian 2-manifold, if there is at least one compact leaf or there is some non-closed leaf (non-closed as subspace). These results are based on a study of the relative topology of the leaves of such a foliation, developped in § 2 and also in § 3.

§ 4 is devoted to the study of codimension-one totally geodesic foliations; we give another proof of Oshikiri's result and establish some results which are used in § 5, where is proved that any Killing field preserves a codimension-one totally geodesic foliation on a 3-manifold if there is at least one compact leaf.

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1. Preliminaries. This section is concerned with the study of some properties of complete Riemannian manifolds (M, g) with a codimension-one totally geodesic foliation $\mathscr{L}$.

We recall (see [1] or [7]), that if $(\tilde{M}, \pi)$ denotes the universal covering of M and $\tilde{\mathscr{L}}$ the canonical lifting of $\mathscr{L}$, then $\tilde{M}$ is isometric to a trivially

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foliated Riemannian manifold $\tilde{L}\times \boldsymbol{R}$, where $\tilde{L}$ is the universal covering of any leaf of $\mathscr{L}$ and the metric is

$$g = ds^2_{\tilde{L}} + f^2 dt^2 .$$

Here f is a smooth positive function on $\tilde{M}$ and $ds^2_{\tilde{L}}$ is the metric on $\tilde{L}$, induced by any inclusion ν_{t_0}

$$\nu_{t_0}\colon \tilde{L} \to \tilde{M} = \tilde{L}\times \boldsymbol{R} , \quad z \to (z, t_0)$$

and dt^2 is the canonical metric on $\boldsymbol{R}$.

We remark that any deck-transformation preserves $\tilde{\mathscr{L}}$ and our goal in this paper will be to prove that, under suitable hypothesis, any Killing field X preserves $\mathscr{L}$ and we shall do so by proving that the Killing field $\tilde{X}$ on $\tilde{M}$, π-related to X, preserves $\tilde{\mathscr{L}}$. Thus we begin studying some properties of the Killing fields on $\tilde{M}$.

All the manifolds considered in this paper will be connected and complete. We work in the C^∞-category.

PROPOSITION 1.1. *Any Killing field $\tilde{X}$ on $\tilde{M}$, is of the form $Y + \phi\partial_t$, where*

(i) *Y is a Killing field on $\tilde{L}\times t$, with respect to $ds^2_{\tilde{L}}$;*

(ii) *$Y(f) = -(\phi f)'$ ($'$ denotes the derivation with respect to t);*

(iii) *For any vector field T, tangent to $\tilde{\mathscr{L}}$, with $[T, \partial_t] = 0$, we have $T(\phi)f^2 = g(T, [Y, \partial_t])$.*

PROOF. Let T_i, T_j be two orthonormal vector fields tangent to $\tilde{\mathscr{L}}$, with $[T_i, \partial_t] = [T_j, \partial_t] = 0$. Then from $(L_{Y+\phi\partial_t}(g))(T_i, T_j) = 0$ we obtain (i). The same argument applied to $g((1/f)\partial_t, (1/f)\partial_t)$ and to $g((1/f)\partial_t, T_i)$ proves (ii) and (iii). Note that $g([Y, \partial_t], \partial_t) = 0$.

PROPOSITION 1.2. *On $\tilde{M}$ any Killing field preserving $\tilde{\mathscr{L}}$ can be expressed as $Y + \phi\partial_t$, verifying* (i), (ii) *and* (iii) *of Proposition* 1.1 *and either*

(iv) *$T(\phi) = 0$, for all T tangent to $\tilde{\mathscr{L}}$, or*

(iv′) $[Y, \partial_t] = 0$.

PROOF. If $Y + \phi\partial_t$ preserves $\tilde{\mathscr{L}}$, taking any T tangent to $\tilde{\mathscr{L}}$, with $[T, \partial_t] = 0$, we know that $[Y + \phi\partial_t, T]$ must be tangent to $\tilde{\mathscr{L}}$, so $T(\phi) = 0$ and from (iii) this is equivalent to $[Y, \partial_t] = 0$.

From (iv′) of Proposition 1.2 we see that any Killing field orthogonal to $\tilde{\mathscr{L}}$ preserves $\tilde{\mathscr{L}}$ and the same is true on $(M, g, \mathscr{L})$. Furthermore it is well-known that a codimension-one foliation which admits an orthogonal Killing field must be totally geodesic (see for instance [2]).

Now we study briefly such foliations.

PROPOSITION 1.3. *If a Killing field preserving a codimension-one totally geodesic foliation $\mathscr{L}$ vanishes at some point, then it is tangent to $\mathscr{L}$ everywhere.*

PROOF. We consider the Killing field $\tilde{X} = Y + \phi\partial_t$, induced on the universal covering $\tilde{M}$. Let $(l_o, 0)$ be a point on $\tilde{M}$, where $\tilde{X}$ vanishes. Thus $Y_{(l_o,0)} = 0$. Since $[Y, \partial_t] = 0$, Y vanishes along $\{(l_o, t)\}_{t\in \boldsymbol{R}}$, hence from (ii) of Proposition 1.1 we have $0 = -(\phi f)'_{(l_o,t)}$, i.e., (ϕf) remains constant along (l_o, t). Since $\phi_{(l_o,0)} = 0$, we have $\phi_{(l_o,t)} = 0$, for all $t \in \boldsymbol{R}$.

We remark that any Killing field orthogonal to $\tilde{\mathscr{L}}$ everywhere never vanishes, because otherwise this vector field should be tangent to $\tilde{\mathscr{L}}$ everywhere, a contradiction. So $\phi \not\equiv 0$ for such a vector field.

THEOREM 1.1. *Let $(M, g, \mathscr{L})$ be a simply connected Riemannian manifold with a codimension-one foliation such that there exists a Killing field X, orthogonal to $\mathscr{L}$ everywhere. Then M is a warped product $L \times_{\psi} I$, for a function ψ defined on L, where I is either an open interval or a half-line or $\boldsymbol{R}$.*

PROOF. We know that $\mathscr{L}$ is totally geodesic and $M = L \times \boldsymbol{R}$, $g = g_L + f^2dt^2$. From (ii) of Proposition 1.1, we have $0 = -(\phi f)'$ so (ϕf) is a function on L. Set $\psi = \phi f$. We get $f = \psi/\phi$ (we recall that $\phi \not\equiv 0$). Because of (iii) of Proposition 1.1, we have $T(\phi)f^2 = 0$ for any T tangent to $\mathscr{L}$, so $\phi = \phi(t)$. In a new parametrization s of $\boldsymbol{R}$ with $ds = (1/\phi)dt$, the Riemannian metric reads as $g_L + \psi^2ds^2$.

In general for a Riemannian manifold $(M, g, \mathscr{L})$ with a codimension-one foliation one can obtain by well-known arguments, a k-fold covering $(\hat{M}, g, \hat{\mathscr{L}})$ of M, where $\hat{\mathscr{L}}$ is the induced foliation, such that $\hat{M}$ can be oriented and $\hat{\mathscr{L}}$ transversally oriented. k can be 1 (trivial case), 2 or 4.

In § 2 we shall be interested in proving, under appropriate hypothesis, that the closure of any leaf contains some compact leaf. Since $k = 1$, 2 or 4, as above, it suffices to do so on $\hat{M}$.

2. Closure of a leaf. In this section we state some results concerning the closure of any leaf of a codimension-one totally geodesic foliation. These results will be used in § 3 and § 4.

Due to the well-known facts pointed out at the end of § 1, we may assume that the manifold is orientable and the foliation transversally orientable.

PROPOSITION 2.1 (see [5, p. 18], for a more general result). *The* *closure of any leaf L of $\mathscr{L}$ is saturated, i.e., is the union of a certain number of leaves.*

PROOF. Let $q \in \bar{L} \setminus L$. Then q lies in another leaf L_s. We know that there is an open neighborhood U_q of q, which is isometric to a trivially foliated Riemannian manifold $V_q \times (-\varepsilon, \varepsilon)$, where V_q is an open neighborhood of q in L_s and the metric is of the from $ds^2_{V_q} + f^2 dt^2$ for a smooth positive function f on $V_q \times (-\varepsilon, \varepsilon)$.

Since $q \in \bar{L}$, the connected component of $L_s \cap U_q$ containing q also lies in $\bar{L}$, so $\bar{L} \cap L_s$ is an open set in L_s. Since it is obviously closed, we have $\bar{L} \supset L_s$, hence

$$\bar{L} = \bigcup_{s \in T} L_s .$$

In the next section we shall apply this result in the following form: If a point $q \in \bar{L} \setminus L$, then the leaf along q also lies in $\bar{L} \setminus L$.

Let p_1 and p_2 be the natural projections from $\tilde{M} = \tilde{L} \times \boldsymbol{R}$ onto $\tilde{L}$ and $\boldsymbol{R}$ respectively.

LEMMA 2.1. *For any leaf L_r of $\mathscr{L}$, $p_2(\pi^{-1}(\bar{L}_r))$ is a closed subset of $\boldsymbol{R}$.*

PROOF. $p_2^{-1}(p_2(\pi^{-1}(\bar{L}_r))) = \pi^{-1}(\bar{L}_r)$ is closed and p_2 is a quotient map.

PROPOSITION 2.2. *If $\mathscr{L}$ has at least one compact leaf, the closure of any leaf contains some compact leaf.*

PROOF. Suppose that L_r is a non-compact leaf such that L_r does not contain any compact leaf. Let $\tilde{L} \times r \in \pi^{-1}(L_r)$ and suppose that $\pi(\tilde{L} \times 0) = L_0$ is compact.

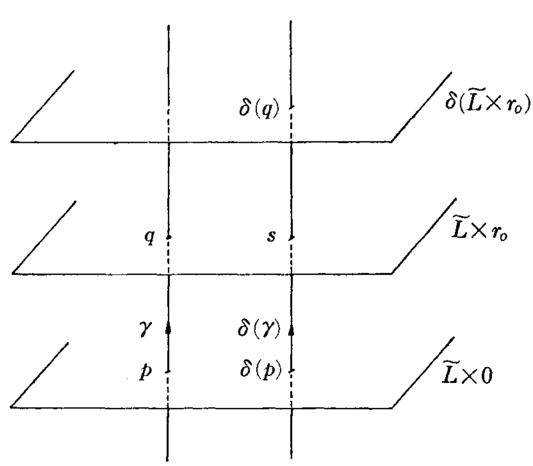

FIGURE 1

Without loss of generality we can assume $r > 0$. Recall that we are assuming M oriented and $\mathscr{L}$ transversally oriented.

Let $C = p_2(\pi^{-1}(\bar{L}_r)) \cap [0, r]$, which is a compact subset of $[0, r]$. Since $0 \notin C$, set $r_o = \inf C$, which obviously belongs to C, and $\pi(\tilde{L} \times r_o)$ is a non-compact leaf. Let δ be any deck-transformation such that $\delta(\tilde{L} \times 0) = \tilde{L} \times 0$ (recall that L_0 is compact) and $\delta(\tilde{L} \times r_o) \neq \tilde{L} \times r_o$. Such a deck-transformation exists because if not $\pi(\tilde{L} \times r_o)$ should be compact (see Figure 1). Let γ be the orthogonal segment to $\tilde{\mathscr{L}}$ from p to $\tilde{L} \times r_o$. Since $\delta(\tilde{L} \times r_o) \neq \tilde{L} \times r_o$ and $r_o = \inf C$, we get $p_2(\delta(q)) > r_o$. Now applying δ^{-1} to $\delta(\gamma)$ from $\delta(p)$ to s, we see by the same argument that $p_2(\delta^{-1}(s)) < r_o$, which is absurd.

3. Preservation of geodesic flows. In this section (M, g) denotes a two-dimensional complete Riemannian manifold and $\mathscr{L}$ a geodesic flow. We know that the universal covering $(\tilde{M}, \pi)$ of M, is trivially foliated as $\tilde{M} = \boldsymbol{R} \times \boldsymbol{R}$ and the Riemannian metric can be written as

$$g = dx^2 + f(x, t)^2 dt^2 ,$$

where f is a smooth positive function on $\boldsymbol{R} \times \boldsymbol{R}$.

Any Killing field on $\tilde{M}$ is of the form $Y + \phi\partial_t$, where $Y = \alpha(t)\partial_x$ is a Killing field on $(\boldsymbol{R}, dx^2)$.

From (iii) of Proposition 1.1, we know that

$$\text{(A)} \qquad \partial_x(\phi) f^2 = -\alpha' \quad (' \text{ denotes the derivation with respect to } t) .$$

From now on we suppose that X is a Killing field on M and $\tilde{X} = \alpha(t)\partial_x + \phi\partial_t$ is the Killing field on $\tilde{M}$ π-related to X.

PROPOSITION 3.1. *If $\pi(\boldsymbol{R} \times r)$ is a compact leaf, then $\alpha'(r) = 0$ and ϕ remains constant on $\boldsymbol{R} \times r$.*

PROOF. We can suppose $r = 0$. Since $\pi(\boldsymbol{R} \times 0)$ is compact, there exists a deck-transformation δ such that $\delta(\boldsymbol{R} \times 0) = \boldsymbol{R} \times 0$ and the subgroup of deck-transformations preserving $\boldsymbol{R} \times 0$ admits δ as a generator.

Let $(x_o, 0) = \delta(0, 0)$. Since δ is an isometry we have:

$$\delta(\partial_x)_{(x_o,0)} = (\partial_x)_{(x_o,0)}$$

$$\delta(\partial_t)_{(x_o,0)} = (f(0, 0)/f(x_o, 0))(\partial_t)_{(x_o,0)} .$$

Since $\tilde{X}$ is δ-invariant, at $(x_o, 0)$ we have:

$$\delta(\alpha\partial_x + \phi\partial_t) = \alpha\partial_x + \phi(f(0, 0)/f(x_o, 0))\partial_t .$$

Thus

$$\text{(B)} \qquad \phi(x_o, 0) f(x_o, 0) = \phi(0, 0) f(0, 0) .$$

Since $Y = \alpha\partial_x$ and $(1/f)\partial_t$ are δ-invariant and since $[Y, (1/f)\partial_t] = Y(1/f)\partial_t - (1/f)\alpha'\partial_x$, we may apply δ to $[Y, (1/f)\partial_t]$ and we see that if $\alpha'(0) \neq 0$, then $f(0, 0) = f(x_o, 0)$, so $\phi(x_o, 0) = \phi(0, 0)$. But from (A) we get $(\partial_x\phi)f^2 = -\alpha'$ so $\partial_x\phi$ has constant sign along $\boldsymbol{R}\times 0$ and $\phi(x_o, 0) \neq \phi(0, 0)$.

Hence $\alpha'(0) = 0$ and $\partial_x(\phi) = 0$.

Now suppose that $L_o = \pi(\boldsymbol{R}\times 0)$ is non-compact. We know that $\bar{L}_o$ is the union of some leaves. Let $L_r \subset \bar{L}_o$ and $\pi(\boldsymbol{R}\times r) = L_r$, $\pi(\boldsymbol{R}\times 0) = L_o$, $r > 0$.

LEMMA 3.1. *If $L_r \subset \bar{L}_o$, then $\alpha'(r) = 0$ and further if $L_o \not\subset \bar{L}_r$, then L_r is compact.*

PROOF. Each point of $\boldsymbol{R}\times r$, for instance $(0, r)$, is the limit of a sequence $(0, r_n)$, with $r_n \uparrow r$ and $\boldsymbol{R}\times r_n = \delta_n(\boldsymbol{R}\times 0)$, where each δ_n is a deck-transformation.

$\delta_n(0, 0) = (x_n, r_n)$ and as in Proposition 3.1 we have

$$\delta_n(\partial_x)_{(x_n,r_n)} = (\partial_x)_{(x_n,r_n)} \quad \text{and}$$

$$\delta_n(\partial_t)_{(x_n,r_n)} = (f(0, 0)/f(x_n, r_n))(\partial_t)_{(x_n,r_n)} .$$

Since Y is δ_n-preserved we see that

$$\alpha(0) = \alpha(r_n)$$

$$\phi(x_n, r_n)f(x_n, r_n) = \phi(0, 0)f(0, 0) .$$

Since α is C^∞ with respect to t, we have $\alpha'(r) = \cdots = \alpha^m(r) = \cdots = 0$.

If $L_o \not\subset \bar{L}_r$, we continue considering $\boldsymbol{R}\times 0$ and $\boldsymbol{R}\times r$ on $\tilde{M}$, π-related to L_o and L_r respectively $(r > 0)$ such that there exists a sequence of deck-transformations δ_n, verifying $p_2(\delta_n(\boldsymbol{R}\times 0)) \uparrow r$.

Since $L_o \notin \bar{L}_r$, let $r_o = \inf\{p_2(\pi^{-1}(\bar{L}_r)) \cap [0, r]\}$, $r_o \in p_2(\pi^{-1}(\bar{L}_r))$ and $L_{r_o} \subset \bar{L}_r \subset \bar{L}_o$, see Figure 2.

By considering the above sequence δ_n, one sees immediately that L_r lies in the closure of any leaf $\pi(\boldsymbol{R}\times s)$, $0 \leqq s \leqq r_o$, so $\pi(\boldsymbol{R}\times r_o)$ can not be compact, unless $r_o = r$. No one leaf $\boldsymbol{R}\times s$, $0 < s \leqq r_o$ can be of the form $\delta(R\times 0)$, for some deck-transformation δ, because otherwise $p_2(\delta^{-1}(\boldsymbol{R}\times r_o)) \in (0, r_o)$, which is absurd. Hence any leaf $\delta(\boldsymbol{R}\times 0)$ with $p_2(\delta(\boldsymbol{R}\times 0)) \in (0, r)$, verifies $p_2(\delta(\boldsymbol{R}\times 0)) \in (r_o, r)$.

Since any transversal through any point of $\boldsymbol{R}\times r$ intersects leaves of the form $\rho(\boldsymbol{R}\times r)$, with ρ belonging to the group of deck-transformations, one sees that L_r is not a proper leaf (see [4, p. 45]). Since we are in the C^∞-category we have that L_r is locally dense, i.e., there is a transversal c_o through any point of $\boldsymbol{R}\times r$ such that $c_o \cap \bar{L}_r$ is a closed

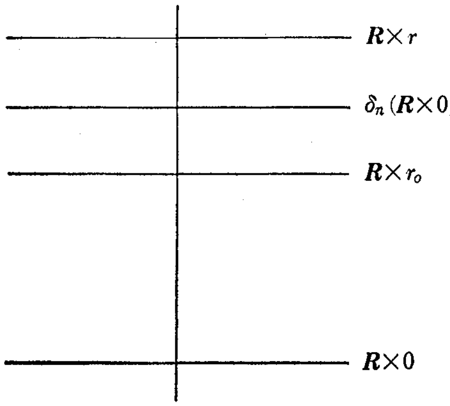

FIGURE 2

interval. But as $p_2(\delta_n(\boldsymbol{R}\times 0))\uparrow r$ we have $L_0\in\bar{L}_r$, which is absurd, so necessarily $r_0 = r$ and L_r is compact.

THEOREM 3.1. *Let (M, g) be a compact Riemannian two-manifold with a geodesic flow $\mathscr{L}$. Then any Killing field preserves $\mathscr{L}$.*

THEOREM 3.2. *Let (M, g) be a complete Riemannian two-manifold with a geodesic flow $\mathscr{L}$ such that there exists at least one compact leaf. Then any Killing field preserves $\mathscr{L}$.*

PROOF OF THEOREM 3.1. As above, if X is a Killing field on M, $\tilde{X} = Y + \phi\partial_t$ denotes the Killing field on $\tilde{M}$, π-related to X, where $Y = \alpha(t)\partial_x$.

We show $\alpha' = 0$ everywhere so $[Y, \partial_t] = 0$. Since $\tilde{X}$ preserves $\tilde{\mathscr{L}}$, X preserves $\mathscr{L}$.

Let L_0 be any leaf, and suppose $\pi(\boldsymbol{R}\times 0) = L_0$. If L_0 is compact we know that $\alpha'(0) = 0$, so we may assume that L_0 is not compact. Let $L_r\subset\bar{L}_0$. If $L_0\subset\bar{L}_r$, the argument of Lemma 3.1 shows that $\alpha'(0)=0$, so from now on we assume that $L_0\not\subset\bar{L}_r$, and then we know that L_r is compact. Let r be such that $\pi(\boldsymbol{R}\times r) = L_r$. There exists a sequence of deck-transformations δ_n such that $p_2(\delta_n(\boldsymbol{R}\times 0))\uparrow r$. By Lemma 3.1, we have $\delta_n(\boldsymbol{R}\times r) = \boldsymbol{R}\times r$. Since L_r is compact, we may assume that $\delta_n = \delta^n$, with $\delta = \delta_1$.

Now one sees immediately that if $\delta(\boldsymbol{R}\times 0) = \boldsymbol{R}\times r_1$, for any $t\in[0, r_1)$, then $L_r\subset\overline{\pi(\boldsymbol{R}\times t)}$. Since α is the same for any two leaves on $\tilde{\mathscr{L}}$, which are π-related, and $\alpha(r) = \lim\alpha(t)$ (as $t\to r$), we have $\alpha(r) = \alpha(t)$ for all $t\in[0, r_1)$, so α is locally constant and $\alpha'(0) = 0$.

PROOF OF THEOREM 3.2. As above we shall prove that $\alpha' = 0$. Let L_0 be any leaf and suppose that it is non-compact. Let $\pi(\boldsymbol{R}\times 0) = L_0$, we know from Proposition 2.2 that there is a compact leaf $L_r \subset \bar{L}_o$. Let $\pi(\boldsymbol{R} \times r) = L_r$. By the same argument as in Theorem 3.1 we see that $\alpha' = 0$.

Finally we give sufficient conditions not related to compactness, which ensure the preservation of geodesic flows. These conditions refer to some topological properties of the leaves and along this study we shall obtain some interesting properties of the topology of the leaves.

From now on we suppose that $(M, g, \mathscr{L})$ is a complete Riemannian two-manifold, with a geodesic flow $\mathscr{L}$ and furthermore we suppose that there is not any compact leaf.

As a corollary to Lemma 3.1, we have:

LEMMA 3.2. *If* $\bar{L}_i = \cup_{j \in \tau_i} L_j$, *then* $\bar{L}_j \supset L_i$ *for any* $j \in \tau_i$.

From this lemma we see that for any two leaves L_i, L_j, their closures $\bar{L}_i$, $\bar{L}_j$ coincide or are disjoint, because if $L_k \subset \bar{L}_i \cap \bar{L}_j$, then $\bar{L}_k = \bar{L}_i = \bar{L}_j$. According to [4], the closure of any leaf is a minimal set.

From now on whenever we say that a leaf is closed, it means that it is closed as a subspace ($\bar{L} = L$), but non-compact, because from now on we are supposing that there is not any compact leaf.

LEMMA 3.3. *If there is some non-closed leaf, then any leaf is non-closed.*

PROOF. Let L_o be non-closed. Then $\bar{L}_o = \cup_{i \in \tau_o} L_i$, with $\# \tau_o > 1$. Suppose that L_{i_0} is a closed leaf. Note that $L_{i_0} \not\subset \bar{L}_o$ by Lemma 3.2.

On $\tilde{M}$ one can find leaves, $\boldsymbol{R} \times i_o$, $\boldsymbol{R} \times 0$, $\boldsymbol{R} \times k_o$, π-related to L_{i_0}, L_o, L_{k_0}, respectively, with $L_{k_0} \subset \bar{L}_o$ such that there is not any leaf π-related to L_{i_0} between $\boldsymbol{R} \times i_o$ and $\boldsymbol{R} \times k_o$. Furthermore we can find a sequence of deck-transformations ε_n with $p_2(\varepsilon_n(\boldsymbol{R} \times 0)) \downarrow k_o$. See Figure 3.

Since L_{i_0} is non-compact, $\varepsilon_n(\gamma)$ must be as described in Figure 3 and

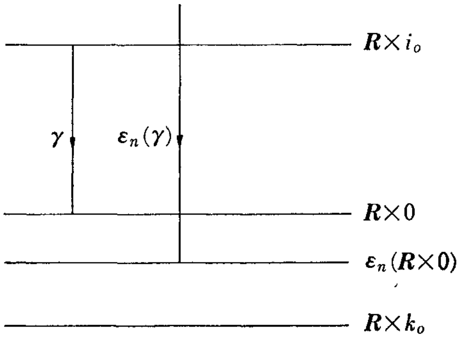

FIGURE 3

applying ε_n^{-1}, we see that $\varepsilon_n^{-1}(\boldsymbol{R}\times i_o)$ lies between $\boldsymbol{R}\times i_o$ and $\boldsymbol{R}\times k_o$, which is absurd.

THEOREM 3.3. *Let $(M, g, \mathscr{L})$ be a complete non-compact Riemannian two-manifold with a geodesic flow $\mathscr{L}$ but without compact leaves. If there is some non-closed leaf, then any leaf is dense on M, i.e., $\bar{L} = M$.*

PROOF. If $\bar{L}_i \subsetneqq M$, we can find L_j with $\bar{L}_i \cap \bar{L}_j = \varnothing$. On $\tilde{M}$ one can find $\boldsymbol{R}\times i$, $\boldsymbol{R}\times j$, $\boldsymbol{R}\times j'$, π-related to L_i, L_j, $L_{j'}$, respectively, such that $\bar{L}_j = \bar{L}_{j'}$ with $i > j > j'$ and without any leaf π-related neither to any leaf of $\bar{L}_i$ nor to any leaf of $\bar{L}_j$ between $\boldsymbol{R}\times i$ and $\boldsymbol{R}\times j$. Since there is a sequence of deck-transformations ε_n with $p_2(\varepsilon_n(\boldsymbol{R}\times j')) \uparrow j$, we obtain a contradiction as in Lemma 3.3.

Finally we observe that in the conditions of Theorem 3.3, any Killing field X on M, induces, as is well known, $\tilde{X}$ on $\tilde{M}$, with $\tilde{X} = \alpha(t)\partial_x + \phi\partial_t$ and as no one leaf is closed we deduce from the proofs of Theorem 3.1 and Theorem 3.2, that $\alpha' = 0$ everywhere, so we have:

THEOREM 3.4. *Let $(M, g, \mathscr{L})$ be a complete non-compact Riemannian two-manifold with a geodesic flow $\mathscr{L}$ but without compact leaves. If there is some non-closed leaf, then any Killing field preserves $\mathscr{L}$.*

For complete non-compact Riemannian two-manifolds, it remains only to consider the case whenever all the leaves are closed, but in this case it is possible to have Killing fields not preserving the foliation. For instance on $\boldsymbol{R}^2 = \boldsymbol{R}\times\boldsymbol{R}$ consider the standard flat metric and the foliation $\{\boldsymbol{R}\times t\}_{t\in\boldsymbol{R}}$, any non-parallel Killing field does not preserve this foliation.

4. Codimension-one foliations. In this section $(M, g, \mathscr{L})$ denotes a complete Riemannian n-manifold, with a codimension-one totally geodesic foliation $\mathscr{L}$. We know that the universal covering $(\tilde{M}, \pi)$ of M, is trivially foliated as $\tilde{L}\times\boldsymbol{R}$, where $\tilde{L}$ is the universal covering of any leaf of $\mathscr{L}$ and the metric is of the form $ds_{\tilde{L}}^2 + f^2dt^2$.

Any Killing field $\tilde{X}$ on $\tilde{M}$ can be written as $Y + \phi\partial_t$, where Y and ϕ satisfy the conditions of Proposition 1.1. In particular from (iii) of Proposition 1.1, for any T tangent to $\tilde{\mathscr{L}}$ with $[T, \partial_t] = 0$, we get

$$T(\phi)f^2 = g(T, [Y, \partial_t]) . \tag{C}$$

We now prove that any Killing field preserves $\mathscr{L}$ if M is compact. At the same time we obtain some technical results which will be used in §5.

LEMMA 4.1. *For any Killing field $\tilde{X}$ on $\tilde{M}$, $[Y, \partial_t]$ is tangent to $\tilde{\mathscr{L}}$*

everywhere and is a Killing field on each leaf.

PROOF. Locally $Y = \alpha^i \partial_{x_i}$, for a local frame $\{x_1, \cdots, x_n\}$ in $\tilde{L}$. $[Y, \partial_t] = -\alpha'^i \partial_{x_i}$, which is tangent to $\tilde{\mathscr{L}}$. $L_{[Y,\partial_t]} ds_{\tilde{L}}^2 = L_Y(L_{\partial_t}(ds_{\tilde{L}}^2)) - L_{\partial_t}(L_Y(ds_{\tilde{L}}^2))$, which is zero because $ds_{\tilde{L}}^2$ does not depend on t and Y is a Killing field on $(\tilde{L}, ds_{\tilde{L}}^2)$.

From now on X denotes a Killing field on M and $\tilde{X}$ the Killing field on $\tilde{M}$, π-related to X. Under certain hypothesis we shall prove that X preserves $\mathscr{L}$ by proving that $[Y, \partial_t] = 0$, which amounts (see Proposition 1.2) to proving that $\tilde{X}$ preserves $\tilde{\mathscr{L}}$.

Recall that we may assume $\mathscr{L}$ to be transversally oriented, so the field $[Y, (1/f)\partial_t]$ can be projected to M. Since

$$[Y, (1/f)\partial_t] = Y(1/f)\partial_t + (1/f)[Y, \partial_t]$$

and $(1/f)[Y, \partial_t]$ is the $\tilde{\mathscr{L}}$-tangent component of $[Y, (1/f)\partial_t]$, we see that $(1/f)[Y, \partial_t]$ can be projected to M.

PROPOSITION 4.1. *If $[Y, \partial_t]$ vanishes at some point $p \in \tilde{M}$, then it vanishes along the leaf through p.*

PROOF. Let T_p be any vector in p tangent to $\tilde{\mathscr{L}}$ and let $\gamma(t)$ be the geodesic verifying $\gamma(0) = p$, $\dot{\gamma}(0) = T_p$. Let $T = \dot{\gamma}$.

One can extend T to a suitable neighborhood in such a way that $[T, \partial_t] = 0$. Since $[Y, \partial_t]$ is a Killing field on each leaf of $\tilde{\mathscr{L}}$, it is a Jacobi field along γ. It is known that $\dot{\gamma}(g(\dot{\gamma}, [Y, \partial_t])) = 0$ and we have $g(\dot{\gamma}(0), [Y, \partial_t](p)) = 0$. Thus $g(T, [Y, \partial_t]) = 0$ along γ and hence $T(\phi) = 0$ along γ by (C).

Since T_p was arbitrary we have proved that ϕ is constant on this leaf, so at any point of this leaf we have $T(\phi)f^2 = 0 = g(T, [Y, \partial_t])$. Hence $[Y, \partial_t] = 0$ on this leaf.

Now looking at the leaves where $[Y, \partial_t] \neq 0$, we see that ϕ verifies $d\phi \not\equiv 0$. Since $[Y, \partial_t]$ is a Killing field and so a Jacobi field, the level hypersurfaces of ϕ are totally geodesic and the Killing field (on the leaf) $[Y, \partial_t]$ is orthogonal to them. Because of Theorem 1.1 any leaf, where $[Y, \partial_t] \neq 0$, is a warped product $\tilde{S} \times_\psi I$. If s is the canonical parameter of I, then $[Y, \partial_t] = k\partial_s$ with k constant along the leaf.

LEMMA 4.2. *Let δ be any deck-transformation of $(\tilde{M}, \pi)$. For any p of $\tilde{M}$, we have*

$$f(p)\phi(p) = f(\delta(p))\phi(\delta(p)) .$$

PROOF. Since $\tilde{X}$ can be projected to M, we have $\delta(\tilde{X}) = \tilde{X}$. Since

δ preserves $\tilde{\mathscr{L}}$ we see that $\delta(Y) = Y$ and $\delta(\phi\partial_t) = \phi\partial_t$. We are done, since δ is an isometry.

THEOREM 4.1. *Let $(M, g, \mathscr{L})$ be a compact Riemannian manifold with a codimension-one totally geodesic foliation $\mathscr{L}$. Any Killing field preserves $\mathscr{L}$.*

PROOF. We shall prove that $\tilde{X}$ preserves $\tilde{\mathscr{L}}$.

Let p be a point of $\tilde{L}\times t = \tilde{S}\times I$. We may assume $p = (y_0, 0)$ and $\phi(p) \geqq 0$; if $\phi < 0$ at all the points of $\tilde{L}\times t$, replace $\tilde{X}$ by $-\tilde{X}$. Let γ be the geodesic with $\gamma(0) = p$, $\dot{\gamma}(0) = [Y, \partial_t]$ and let u be the parameter of γ. From (C) we have

$$(\partial\phi/\partial u)f^2 = g(\dot{\gamma}, [Y, \partial_t]) . \tag{D}$$

Since $[Y, \partial_t]$ is a Killing field on $\tilde{L}\times t$, we know that $g(\dot{\gamma}, [Y, \partial_t])$ remains constant along γ and equal to $\|[Y, \partial_t]\|^2_{\gamma(0)} = h$. Since $(\partial\phi/\partial u) = (1/f^2)g(\dot{\gamma}, [Y, \partial_t])$ along γ, we get $(\partial\phi/\partial u) = (1/f^2)h \geqq 0$.

Then suppose there exists some sequence $u_n \to \infty$ with $(1/f(\gamma(u_n))^2) \to 0$. Since ϕ is strictly increasing, we get $f(\gamma(u_n))\phi(\gamma(u_n)) \to \infty$, which is absurd, because $f\phi$ can be projected to M and M is compact.

If $(1/f^2) \geqq (1/k^2)$ (as $u \to \infty$) for some constant k, then $f \leqq k$. In this case from (D), ϕ goes to infinity as $u \to \infty$. Since $[Y, (1/f)\partial_t]$ can be projected to M, we have

$$(h/f) = (1/f)g([Y,\partial_t],\dot{\gamma}) = g([Y,(1/f)\partial_t],\dot{\gamma}) \leqq |\dot{\gamma}|m = \sqrt{h}\cdot m \quad (m = \text{constant})$$

Since $g([Y, \partial_t], \dot{\gamma})$ is constant along γ, we have $f \geqq \sqrt{h}/m$, along γ. Thus $f\phi$ goes to infinity as $u \to \infty$, which is also absurd.

REMARK. The arguments involved in the proof of Theorem 4.1, show that if L_0 is a compact leaf, $[Y, \partial_t]$ vanishes on each leaf of $\pi^{-1}(L_0)$.

From Lemma 4.2 we see that if $\phi \neq 0$ on some leaf of $\pi^{-1}(L_0)$ (L_0 compact), the same is true on all the leaves of $\pi^{-1}(L_0)$ and f remains bounded on any-one of these leaves.

5. 3-dimensional case.

5. 3-dimensional case. Our purpose now is to prove the validity of a result similar to Theorem 3.2 for three-manifolds.

THEOREM 5.1. *Let $(M, g, \mathscr{L})$ be a 3-dimensional complete Riemannian manifold, with a codimension-one totally geodesic foliation $\mathscr{L}$, such that there exists at least one compact leaf. Then any Killing field preserves $\mathscr{L}$.*

PROOF. We know that the universal covering $(\tilde{M}, \pi)$ is $\tilde{L}\times \boldsymbol{R}$ and $g = ds^2_{\tilde{L}} + f^2dt^2$. If $\tilde{X}$ denotes the Killing field on $\tilde{M}$, π-related to X, then $\tilde{X} = Y + \phi\partial_t$.

Recall (see (§ 4)) that if $\pi(\tilde{L}\times t)$ is a compact leaf $[Y, \partial_t]$ vanishes on $\tilde{L}\times t$; we prove that $[Y, \partial_t]$ is zero everywhere. In order to do so let $L_{t_o} = \pi(\tilde{L}\times t_o)$ be a non-compact leaf of $\mathscr{L}$ and $L_o = \pi(\tilde{L}\times 0)$ a compact leaf with $L_o \subset \bar{L}_{t_o}$; our claim now is $[Y, \partial_t]_{|\tilde{L}\times t_o} = 0$.

Indeed, suppose $[Y, \partial_t]_{|\tilde{L}\times t_o} \neq 0$. Since $[Y, \partial_t]$ does not vanish on $\tilde{L}\times t_o$, this leaf is a warped product $\tilde{S}\times_\psi I$, where $\tilde{S} = \boldsymbol{R}$ and I is an interval, a half-line or the whole $\boldsymbol{R}$. Since $q \in \bar{L}_{t_o}$ for any point $q \in L_o$, there is a sequence $p_n \in L_{t_o}$, such that $p_n \to q$. Translating it to $\tilde{M}$ we may assume $\pi(l, 0) = q$ and $\pi(z_n, t_o) = p_n$, $t_o < 0$, and there exists a sequence of deck-transformations ε_n such that $\varepsilon_n(z_n, t_o) = (l, t_n)$, with $t_n \uparrow 0$. Since L_o is compact we may also assume that $\tilde{L}\times 0$ remains fixed by ε_n and without loss of generality that $\varepsilon_n = \varepsilon^n$ for some deck-transformation ε and there is not any leaf π-related to L_o between $\tilde{L}\times t_o$ and $\tilde{L}\times 0$.

We firstly observe that $L_o \in \pi(\tilde{L}\times t)$ for any leaf $\tilde{L}\times t$ with $t \in [t_o, 0)$.

LEMMA 5.1. *$f(z_n, t_o) \to 0$, as $n \to \infty$.*

PROOF. Since $\varepsilon^n(\tilde{L}\times 0) = \tilde{L}\times 0$ and ε^n preserves the curves orthogonal to the foliation, we have $\varepsilon^n(z_n, 0) = (l, 0)$.

Recall that $(1/f)[Y, \partial_t]$ can be projected, hence $\varepsilon^n((1/f)[Y, \partial_t]) = (1/f)[Y, \partial_t]$, so that for any $(r, t_{n+1}) \in \tilde{L}\times t_{n+1} = \varepsilon^{n+1}(\tilde{L}\times t_o)$, we have

$$\varepsilon([Y, \partial_t])_{(r,t_{n+1})} = (f(\varepsilon^{-1}(r, t_{n+1}))/f(r, t_{n+1}))[Y, \partial_t]_{(r,t_{n+1})} ,$$

but as $\varepsilon([Y, \partial_t])$ is a Killing field on $\tilde{L}\times t_{n+1}$, $f(\varepsilon^{-1}(r, t_{n+1}))/f(r, t_{n+1})$ does not depend on $r \in L$, thus it is equal to $f(z_1, t_n)/f(l, t_{n+1})$. Since

$$\varepsilon^{n'}(\varepsilon^{n''}([Y, \partial_t])) = \varepsilon^{n'+n''}([Y, \partial_t]) ,$$

one sees that $f(z_1, t_n)/f(l, t_{n+1})$ does not depend on n.

Let $\rho(t_o) = \rho(t_n) = f(z_1, t_n)/f(l, t_{n+1})$. We have $f(z_n, t_o) = f(l, t_n)\cdot\rho(t_o)^n$. By a continuity argument we have $\rho(t_o) = f(z_1, 0)/f(l, 0)$, thus for any leaf $\tilde{L}\times t$, $t \in [t_o, 0)$, with $[Y, \partial_t]|_{\tilde{L}\times t} \neq 0$, we have $\rho(t) = f(z_1, 0)/f(l, 0) = \rho(t_o)$.

If $\rho(t_o) \geqq 1$, then $\int_{t_o}^{t_1} f(z_n, t)dt$ is lower bounded away from zero, which is absurd because this integral gives the length of the orthogonal curve to $\tilde{\mathscr{L}}$ from (z_n, t_o) to (z_n, t_1), but this length is preserved by ε^n so $\int_{t_o}^{t_1} f(z_n, t)dt \to 0$, as $n \to \infty$, hence $\rho(t_o) < 1$ and $f(z_n, t_o) \to 0$.

REMARK. We deduce from the proof of this Lemma that ϕ must vanish on $\tilde{L}\times 0$ (in general on any leaf π-related to a compact leaf), because if $\phi \neq 0$ on this leaf $(f\phi)(z_n, 0) \to 0$, but as L_o is compact and $f\phi$ can be projected. ϕ must vanish at some point, recalling that ϕ is constant on such a leaf we have $\phi = 0$.

One can obtain the same result by applying a Proposition due to Oshikiri (see [9, p. 355]), where is proved that any flow-generating Killing field maps a compact leaf in another one also compact obviously, so necessarily $\phi = 0$ on $\tilde{L}\times 0$.

LEMMA 5.2. *The foliation $\tilde{S}$ is ∂_t-invariant at all the leaves $\tilde{L}\times t$, for all $t \in [t_o, 0)$.*

PROOF. On any leaf $\tilde{L}\times t$, with $[Y, \partial_t]|_{\tilde{L}\times t} \neq 0$, we know that $\tilde{L}\times t \cong \tilde{S}_{(t)}\times_{\psi_t} I_{(t)}$. We may assume the sequence $\varepsilon^n(z_n, t) = (l, t'_n)$, with $t'_n \uparrow 0$.

Let γ_n be a normalized minimal geodesic in $\tilde{L}\times t$, from $\gamma_n(0) = (l, t)$ to (z_n, t). Recall that $(1/f(z_n, t))\|[Y, \partial_t]\| \to 0$, as $n \to \infty$, and since $f(z_n, t) \to 0$, we have $\|[Y, \partial_t]\|_{(z_n,t)} \to 0$, i.e., $\psi_t(z_n, t) \to 0$, as $n \to \infty$.

Since $[Y, \partial_t]$ is a Killing field on $\tilde{L}\times t$ and so a Jacobi field, $\dot{\gamma}_n(0)$ goes to a unit vector $\dot{\gamma}(0)$ on $T_{(l,t)}(\tilde{S}_t)$. Since $\dot{\gamma}(0)$ does not depend on t, because (z_n, t) and so γ_n can be defined for any t, $\tilde{S}_t$ is ∂_t-invariant and so is the function ψ_t.

REMARK. From the above proof we see that $\tilde{S}$ is also defined for $\tilde{L}\times 0$.

LEMMA 5.3. *Any deck-transformation preserving $\tilde{L}\times 0$ must preserve $\tilde{S}$.*

PROOF. Since $(1/f)[Y, \partial_t]$ can be projected its direction is preserved by any deck-transformation and so is its orthogonal distribution.

Since $\tilde{L}\times t = \tilde{S}\times_\psi I = \boldsymbol{R}\times_\psi I$, for all $t \in [t_o, 0)$, we can take global coordinates (x, y) and Y is of the form

$$Y = \lambda\partial_x + \mu\partial_y ,$$

with $\lambda = \lambda(x, y, t)$ and $\mu = \mu(x, y, t)$. Since $\lambda\partial_x$ is a Killing field for any leaf of $\tilde{S}$, we have $\lambda = \lambda(y, t)$. Since $[Y, \partial_t] = r\partial_y$ (r = constant, on any leaf $\tilde{L}\times t$), we have $\lambda = \lambda(y)$ and $\mu = \theta(x, y) - \rho(t)$. Since $\tilde{L}\times 0 = \tilde{S}\times I$ and L_o is compact, from Theorem 4.1 we have that Y preserves $\tilde{S}$ on $\tilde{L}\times 0$. Since $Y_{(l,t)} = Y_{(l,0)} - (\rho_{(t)} - \rho_{(0)})\partial_y$ whose last term obviously preserves $\tilde{S}$, we see that Y preserves $\tilde{S}$ on $\tilde{L}\times t$, thus

$$[Y, \partial_x] = 0 \quad \text{and} \quad [Y, (1/\psi)\partial_y] = 0 ,$$

and we obtain $\theta = \theta(y)$, $\lambda = \lambda_o = \text{const.}$ and $-\lambda_o(\psi_x/\psi^2) - (1/\psi)\theta_y = 0$.

If $\lambda_o \neq 0$, then $\psi_x/\psi = (-1/\lambda_o)\theta_y = k = \text{const.}$, because the first term depends only on x and the second one only on y, thus $\theta(y) = (-\lambda_o k)y + \mu_o$ and $\psi = \psi_o \exp(kx)$. But $k \neq 0$, because otherwise $\psi = \psi_o = \text{const.}$, in contradiction to $\psi(z_n, t_o) \to 0$.

Thus $k \neq 0$ and $Y = \lambda_o \partial_x + ((-\lambda_o k)y + \mu_o - \rho(t))\partial_y$. Since L_o is compact we have that $\|Y\|$ must be bounded on $\tilde{L} \times 0$, $\lambda_o = 0$ and we get $Y = -\rho(t)\partial_y$, with $\rho(0) = 0$. So we have $Y|_{\tilde{L}\times 0} = 0$ and $[Y, \partial_t]|_{\tilde{L}\times 0} = \rho'(0) = 0$.

As we pointed out at the remark to Lemma 5.1, $\phi = 0$ on $\tilde{L} \times 0$, so $\tilde{X}$ vanishes on $\tilde{L} \times 0$ and as $\rho'(0) = 0$, $A_{\tilde{X}} = L_{\tilde{X}} - \nabla_{\tilde{X}} = 0$, on $\tilde{L} \times 0$, so that $\tilde{X} \equiv 0$. Hence $[Y, \partial_t] \equiv 0$ everywhere.

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DEPARTMENT D'ÀLGEBRA I GEOMETRIA
FACULTAT DE MATEMÀTIQUES
UNIVERSITAT DE BARCELONA
GRAN VIA 585
08007 BARCELONA
SPAIN