The Gibbs–Appell equations of motion

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A particularly simple and direct derivation of the Gibbs–Appell equations of motion is given. In addition to the conventional results, a relatively unknown but elegant and useful form of the equations of motion is also obtained. The role of virtual displacements in generating generalized equations of motion is discussed. The relationship between the Gibbs–Appell equations of motion and Lagrange's equations of motion is discussed. Auxiliary results that facilitate the application of the Gibbs–Appell equations of motion to rigid bodies are presented. The theory is demonstrated by generating equations of motion for a disk rolling on a horizontal plane.

I. INTRODUCTION

One of the most neglected, misunderstood, and mistreated methods for determining the motion of mechanical systems is the Gibbs–Appell method, which was first discovered by Gibbs[1] in 1879 and independently discovered and developed by Appell[2] in 1899.

A limited number of textbooks contain discussions of the Gibbs–Appell equations of motion.[3] Most of these books present the Gibbs–Appell method as a secondary method that is useful but not necessary for the solution of certain problems.

Recently, we have become convinced that the Gibbs–Appell equations of motion deserve a much more central place in the theoretical hierarchy of generalized equations of motion than they have hitherto received.

At the same time, we have also become aware of conflicting positions on the basic principles and techniques involved in the derivation of generalized equations of motion. In particular, we are bothered by the disparate views concerning the use of the concept of a virtual displacement, and the associated concept of virtual work, which range from the position held by some that the concept of a virtual displacement is "ill-defined, nebulous, and hence objectionable,"[4] to the position held by others that this concept is "an absolutely essential requirement for the entire structure of analytical dynamics."[4] Our own position on this subject is that the concept of a virtual displacement is a very simple and very useful concept, which has played a significant role in the historical development of generalized equations of motion, but which, although extremely helpful, is not strictly necessary.

In defense of these views, our objective in this article is (1) to show that it is possible to derive the Gibbs–Appell equations of motion in a very simple, straightforward manner without making use of virtual quantities or variational principles; (2) to show that the Gibbs–Appell equations of motion are more versatile, more general, and easier to derive than Lagrange's equations of motion; (3) to present a little known but useful and elegant form of the Gibbs–Appell equations of motion; (4) to show the role of virtual displacements as a convenient but not necessary adjunct in the derivation and application of the Gibbs–Appell equations of motion; (5) to show that Lagrange's equations of motion are a special case of the Gibbs–Appell equations of motion; (6) to present some auxiliary results that facilitate the application of the Gibbs–Appell equations of motion; (7) to illustrate the power of the Gibbs–Appell method by deriving equations of motion for a disk rolling on a horizontal plane; and (8) to argue on the basis of the above results that the Gibbs–Appell equations of motion constitute a marvelous theoretical and pedagogical starting point for the unification, presentation, and generation of generalized equations of motion.

II. DERIVATION OF THE GIBBS–APPELL EQUATIONS OF MOTION

Consider a system consisting of N particles, which is subject to a set of known forces, M holonomic constraints, and L anholonomic constraints.

If we represent the configuration of the system by a single point $\mathbf{x} \equiv x_1, x_2, \ldots, x_{3N}$ in a $3N$-dimensional Cartesian configuration space,[5] and if, in this space, we let $\mathbf{f}$ be the $3N$-dimensional resultant of the given forces acting on the system and $\mathbf{F}$ be the $3N$-dimensional resultant of all the constraint forces, then Newton's equations of motion for the system are

$$m_i \ddot{x}_i = f_i + F_i, \quad i = 1,2,\ldots,3N. \tag{1}$$

If we let $\mathbf{q} = q_1, q_2, \ldots, q_{3N-M}$ be any $3N - M$ coordinates which together with the holonomic constraint conditions uniquely determine the configuration of the system, then the effect of the holonomic constraints on the possible configurations of the system can be described by the following relations:

$$x_i = x_i(\mathbf{q}, t), \quad i = 1,2,\ldots,3N. \tag{2}$$

If we now let $\dot{\mathbf{r}} \equiv \dot{r}_1, \dot{r}_2, \ldots, \dot{r}_{3N-M-L}$ be any $3N - M - L$ quantities linear in $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{3N-M}$ which together with the anholonomic constraint conditions uniquely determine the values of $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{3N-M}$, then the effect of the anholonomic constraints on the possible motions of the system can be described by the following relations:

$$\dot{q}_i = \sum_j a_{ij}(\mathbf{q}, t)\dot{r}_j + b_i(\mathbf{q}, t), \quad i = 1,2,\ldots,3N - M. \tag{3}$$

Equations (2) and (3) define the restrictions imposed by the constraints on the configuration and the motion of the system. In addition to these restrictions, we shall assume as an added requirement that the components F_i of the constraint force satisfy the following relation:

$$\sum_i F_i s_{ij} = 0, \quad j = 1,2,\ldots,3N - M - L, \tag{4}$$

where

$$s_{ij} \equiv \frac{\partial \dot{x}_i(\mathbf{q}, \dot{\mathbf{r}}, t)}{\partial \dot{r}_j} = \frac{\partial \ddot{x}_i(\mathbf{q}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t)}{\partial \ddot{r}_j}. \tag{5}$$

For convenience and for the purposes of this article only we will refer to Eq. (4) as Lagrange's principle. Its justification will be considered in Sec. III.

Equations (1)–(4) provide us with $12N - 2M - L$ equations in the $12N - 2M - L$ unknowns $x_1, x_2, \ldots, x_{3N}$, $q_1, q_2, \ldots, q_{3N-M}, \dot{r}_1, \dot{r}_2, \ldots, \dot{r}_{3N-M-L}, F_1, F_2, \ldots$, and F_{3N}. By judicious combination of the above equations, the number of equations and unknowns we have to worry about can be dramatically decreased.

If we multiply Eq. (1) by s_{ij}, sum over i, and make use of Eq. (4), we obtain

$$\sum_i m_i \ddot{x}_i s_{ij} = \sum_i f_i s_{ij}. \tag{6}$$

If we now define

$$S \equiv \frac{1}{2} \sum_i m_i \ddot{x}_i^2 \tag{7}$$

and

$$Q_j \equiv \sum_i f_i s_{ij}, \tag{8}$$

then Eq. (6) can be written in the following form:

$$\frac{\partial S(\mathbf{q}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t)}{\partial \ddot{r}_j} = Q_j. \tag{9}$$

The $3N - M - L$ equations (9) are the Gibbs–Appell equations of motion. These equations together with the $3N - M$ equations (3) provide us with $6N - 2M - L$ equations in the $6N - 2M - L$ unknowns $q_1, q_2, \ldots, q_{3N-M}, \dot{r}_1, \dot{r}_2, \ldots, \dot{r}_{3N-M-L}$.

An even simpler and more elegant form of the Gibbs–Appell equations can be obtained if we define

$$U \equiv \sum_i f_i \ddot{x}_i, \tag{10}$$

$$R \equiv S - U. \tag{11}$$

It then follows that

$$Q_j = \frac{\partial U(\mathbf{q}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t)}{\partial \ddot{r}_j} \tag{12}$$

and the Gibbs–Appell equations assume the form

$$\frac{\partial R(\mathbf{q}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t)}{\partial \ddot{r}_j} = 0. \tag{13}$$

This form of the equations of motion was introduced by Appell[6] but is generally ignored in most treatments of the Gibbs–Appell method. We have found it quite useful. When we wish to make a distinction, we shall refer to Eq. (9) as the first form of the Gibbs–Appell equations of motion and to Eq. (13) as the second form.

The Gibbs–Appell equations of motion are not affected by additive terms of the form $\phi(\mathbf{q}, \dot{\mathbf{r}}, t)$, which occur in R, S, or U; that is, terms that when expressed as a function of $\mathbf{q}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}$, and t do not contain at least one of the accelerations $\ddot{r}_i$. Hence, in what follows, we will assume that the equality of two different values of either R, S, or U means that they differ at most by additive terms of the above form, and we will freely drop from expressions for R, S, or U terms of this form. It is also interesting to note that we could multiply $R(\mathbf{q}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, t)$ by any function of the above form without altering the equations of motion.

III. LAGRANGE'S PRINCIPLE

In this section we consider the validity of Eq. (4), which as indicated earlier we refer to as Lagrange's principle.

If at time t the given system is in a configuration $\mathbf{q}$, then from Eqs. (2) and (3) it can be shown that the *possible* subsequent displacements $d\mathbf{x}$ of the system are those that satisfy the equations

$$dx_i = \sum_j s_{ij}\, dr_j + \alpha_i\, dt \tag{14}$$

for some set of values of $d\mathbf{r}$ and dt, where s_{ij} is defined by Eq. (5), and α_i is a known function of $\mathbf{q}$ and t, the exact form of which we will not need. The *actual* displacement will of course depend on the applied forces and the initial conditions.

If we examine individually the various constraint forces that are responsible for limiting the possible displacements to the above set of displacements, we find in general that they satisfy Lagrange's principle. For example, if our system consists of a single particle that is confined to a smooth surface, which may or may not be moving, then from Eq. (14) it follows that the vector $\mathbf{s}(j)$, whose components are s_{1j}, s_{2j}, and s_{3j}, will be tangent to the surface. Since the surface is smooth, the constraint force $\mathbf{F}$ must be perpendicular to the surface, hence $\mathbf{F}\cdot\mathbf{s}(j) = 0$ and Lagrange's principle is satisfied. As a second example, if our system consists of a disk or sphere rolling without slipping on either a stationary or moving surface, then the constraint force responsible for keeping the system from slipping is applied to the particle in the system that is in contact with the surface; hence the only nonvanishing components of the $3N$-dimensional constraint force $\mathbf{F}$ are those associated with this particle. But for those components of $\mathbf{x}$ associated with this particle $s_{ij} = 0$, it follows that Lagrange's principle is valid for this constraint force. In a similar fashion, it is possible to show that an action–reaction pair of forces that maintain a fixed distance between two particles, or the forces that smoothly hinge two bodies together, will satisfy Lagrange's principle. We can proceed in this fashion and confirm for each of the constraint forces acting on a given system the validity of Lagrange's principle.

Though we can thus verify Lagrange's principle by considering the constraint forces one by one, there is no general proof of the principle.

IV. VIRTUAL DISPLACEMENTS

The process of verifying Lagrange's principle can frequently be simplified if we introduce the concept of a virtual displacement. We define a virtual displacement to be one of the possible displacements, as given by Eq. (14), for which $dt = 0$; or equivalently a displacement $\delta\mathbf{x}$ that satisfies the equations

$$\delta x_i = \sum_j s_{ij}\, \delta r_j \tag{15}$$

for some value of $\delta\mathbf{r}$. Thus, given a system in a configuration $\mathbf{q}$ at time t, the set of all virtual displacements is a particular subset of the set of all possible displacements. Physically such displacements are the displacements that would occur if the system was frozen in its motion at time t, and the system was then moved without violating any of the constraints operating on the system.

A definition of a virtual displacement is sometimes made which allows the displacement to violate one of the constraints. Our definition excludes such displacements. It is also possible to consider finite as well as infinitesimal virtual displacements. We restrict our consideration to infinitesimal virtual displacements. Hence, whenever the term "a virtual displacement" is used in this article, it is to be read by those who employ the term in the broader sense as "an infinitesimal virtual displacement that does not violate any of the constraints."

From the above definition, Eq. (15), it follows that the work done by the constraint force $\mathbf{F}$ in a virtual displacement is given by

$$\delta W = \sum_i F_i\, \delta x_i = \sum_i F_i\Big(\sum_j s_{ij}\, \delta r_j\Big) = \sum_j \Big(\sum_i F_i s_{ij}\Big)\delta r_j. \tag{16}$$

A necessary and sufficient condition that the above quantity vanishes for arbitrary values of $\delta\mathbf{r}$ is that Lagrange's principle is satisfied. Hence assuming Lagrange's principle to be true is equivalent to assuming that the work done by the constraint forces in an arbitrary virtual displacement is zero. If we are interested in determining whether or not Lagrange's principle is satisfied for a particular constraint, it is generally easier to simply confirm that the work done by the force in an arbitrary virtual displacement is zero rather than to try to prove Lagrange's principle directly.

It also follows from Eqs. (5), (8), and (15) and the definition of work that the work done by the given force $\mathbf{f}$ in a virtual displacement is given by

$$\delta W = \sum_i Q_i\, \delta r_i. \tag{17}$$

It is often much easier to determine the values of the Q_i by exploiting the above equation than by directly using the definition given by Eq. (8).

It should be noted that the introduction of the concept of a virtual displacement is a convenience and not a necessity. We were very careful in our derivation of the Gibbs–Appell equations of motion not to use this concept nor to employ any variational principles. This has been done to counteract the impression conveyed by many mechanics texts that any advanced technique used in mechanics requires the introduction of virtual quantities and the employment of variational principles.

V. LAGRANGE'S EQUATIONS OF MOTION

If there are no anholonomic constraints, and we let $\dot{r}_i = \dot{q}_i$, then it is relatively easy to show that

$$\frac{\partial S(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},t)}{\partial \ddot{q}_i} = \frac{d}{dt}\left(\frac{\partial T(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_i}\right) - \frac{\partial T(\mathbf{q},\dot{\mathbf{q}},t)}{\partial q_i} \tag{18}$$

and

$$Q_j \equiv \sum_i f_i\, \frac{\partial \dot{x}_i(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_j} = \sum_i f_i\, \frac{\partial x_i(\mathbf{q},t)}{\partial q_j}, \tag{19}$$

where T is the kinetic energy of the system and the definition of Q_j goes over to the usual definition of the generalized component of the force associated with the generalized coordinate q_j. If we use Eqs. (18) and (19) in the Gibbs–Appell equations, Eq. (9), we obtain Lagrange's equations of motion. It follows that Lagrange's equations of motion can be considered as a special case of the Gibbs–Appell equations of motion.

It is possible starting from the Gibbs–Appell equations to obtain Lagrange's equations for quasicoordinates but these equations are usually harder to apply than the Gibbs–Appell equations and will not be considered here.

VI. GIBBS–APPELL VERSUS LAGRANGE

There are two features of the Gibbs–Appell method that in principle if not in practice make it superior to Lagrange's method: (1) The Gibbs–Appell method handles anholonomic constraints in a simple, direct fashion whereas Lagrange's method requires additional machinery to handle such constraints; (2) the Gibbs–Appell method allows the introduction of motional coordinates that are not simply the time derivatives of the configurational coordinates, thus giving it a versatility not shared by Lagrange's method without substantial modification. These features allow certain systems to be handled far more economically using the Gibbs–Appell method than using Lagrange's method. The problem of finding equations of motion for a disk rolling on a flat surface is a good illustrative example and will be considered later.

If we are dealing with holonomic systems and there is no immediate advantage to introducing motional coordinates that are other than the time derivatives of the configurational coordinates, then Lagrange's equations are usually simpler to use than the Gibbs–Appell equations. Frequently, however, we are unnecessarily deterred from using the Gibbs–Appell method for such problems by the fact that thoughtless evaluation of the function $S(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},t)$ frequently leads to expressions containing a discouraging number of terms. Suppose, for instance, we wish to obtain the equations of motion of an unconstrained particle in terms of the spherical coordinates r, θ, and ϕ. A straightforward substitution of $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$ into the expression $S = \frac{1}{2}m(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)$ will result in 36 distinct terms in $\ddot{x}^2$, 36 distinct terms in $\ddot{y}^2$, and 6 distinct terms in $\ddot{z}^2$. By contrast, Lagrange's method requires us to obtain the kinetic energy function $T(\mathbf{q},\dot{\mathbf{q}},t)$ and, in the case considered above, a straightforward substitution of $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$ into the expression $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ will result in 6 distinct terms in $\dot{x}^2$, 6 distinct terms in $\dot{y}^2$, and 3 distinct terms in $\dot{z}^2$. Hence it would appear that for this problem the Lagrange method is considerably easier than the Gibbs–Appell method. The operational disparity between the two methods in problems such as this is not however as great as the above comments imply, for the following reasons: (1) Terms in S that do not contain accelerations can be dropped. (2) There are a number of analytical relations, which are available or which frequent use of the Gibbs–Appell method would generate, that can be used to simplify the evaluation of S; for example, the following relation

$$(\overset{\cdot\cdot}{\overline{\alpha\sin\beta}})^2 + (\overset{\cdot\cdot}{\overline{\alpha\cos\beta}})^2 = \ddot{\alpha}^2 + \alpha^2\ddot{\beta}^2 - 2\alpha\ddot{\alpha}\dot{\beta}^2 + 4\alpha\dot{\alpha}\dot{\beta}\ddot{\beta} + \mathrm{fcn}(\alpha,\beta,\dot{\alpha},\dot{\beta}) \tag{20}$$

greatly simplifies the evaluation of S in the above example. (3) Once we have obtained S and T, it is easier to obtain the Gibbs–Appell equations from S than to obtain Lagrange's equations from T. (4) It is not always necessary or even advisable to complete the squares in S in order to use it in the Gibbs–Appell equations of motion. In general, we will find that if terms of the form $(\phi_1 + \phi_2 + \dots)^2$ occur in S, then, depending on the problem, determination of $\partial S/\partial\ddot{r}_i$ may in one case be best achieved by completing the squares, combining terms, and then taking the derivative; while in another case it may best be achieved by not completing the square but rather working with the derivative in the form $2(\phi_1 + \phi_2 + \dots)[\partial(\phi_1 + \phi_2 + \dots)/\partial\ddot{r}_i]$.

Although certain classes of problems are generally better handled with one method rather than another, it is difficult to definitively determine that one method is clearly superior to another for all problems of a certain class. For example, using the Gibbs–Appell method, it is possible with limited analytical effort to obtain equations of motion for a multiple planar pendulum containing an arbitrary number of components,[7] apparently more readily than with Lagrange's method, despite the fact that the system is a holonomic system and the motional coordinates chosen are the time derivatives of the configurational coordinates.

VII. SYSTEMS OF PARTICLES

For a system of particles, it can be shown that the value of S with respect to an arbitrary point a is given by

$$S(a) = \tfrac{1}{2}M\,\mathbf{A}\cdot\mathbf{A} + S(c), \tag{21}$$

where M is the total mass of the system, $\mathbf{A}$ is the acceleration of the center of mass with respect to the point a, and $S(c)$ is the value of S with respect to the center of mass.

It can also be shown that for a rigid body the value of S with respect to a point fixed in the rigid body is given by

$$S = \frac{1}{2}\sum_i\sum_j I_{ij}\dot{\omega}_i\dot{\omega}_j + \sum_i\sum_j\sum_k\sum_l \epsilon_{ijk}I_{kl}\dot{\omega}_i\omega_j\omega_l - \sum_i\sum_j\sum_k\sum_l \epsilon_{ijk}I_{kl}\omega_i\Omega_j\dot{\omega}_l, \tag{22}$$

where the ω_i and I_{ij} are the components with respect to an arbitrary frame 0123 of the angular velocity and inertia tensor of the body, respectively; the Ω_j are components with respect to the frame 0123 of the angular velocity of the frame 0123 with respect to either an inertial frame or a frame fixed in the body; and $\epsilon_{ijk} = +1, -1$, or 0 depending on whether ijk is an even permutation of 123, an odd permutation of 123, or neither.

The last term on the right-hand side of Eq. (22) is erroneously missing from the expression for S, which appears in the textbook by Desloge.[8] The error in Desloge's result occurs because he mistakenly assumes in his derivation of the theorem that the components of $\dot{\boldsymbol{\omega}}$ with respect to the arbitrary frame are given by $\dot{\omega}_i$ rather than by the correct values, which are $\dot{\omega}_i + \Sigma_j\,\Sigma_k\,\epsilon_{ijk}\Omega_j\omega_k$. If the $\dot{\omega}_i$ in his final result are replaced by $\dot{\omega}_i + \Sigma_j\,\Sigma_k\,\epsilon_{ijk}\Omega_j\omega_k$ and terms not containing accelerations are dropped, where it is assumed that the Ω_i are not functions of the acceleration, then the correct expression above is obtained. The theorem that is stated in the text by Desloge would only be true if the arbitrary frame were the inertial frame or the body frame. This is the case in the examples he gives and in his corollary.

If the frame 0123 is one with respect to which the inertia tensor is diagonal, then

$$S = \frac{1}{2}\sum_i I_i\,\dot{\omega}_i^2 + \sum_i\sum_j\sum_k \epsilon_{ijk}I_k\dot{\omega}_i\omega_j\omega_k + \sum_i\sum_j\sum_k \epsilon_{ijk}I_i\dot{\omega}_i\Omega_j\omega_k. \tag{23}$$

VIII. THE ROLLING DISK

In order to illustrate the above results, we consider the problem of finding equations of motion for a homogeneous disk of mass m and radius a which is free to roll on a rough horizontal plane. This is a problem that can be more easily handled with the Gibbs–Appell equations than with Lagrange's equations.

A solution to this problem using the Gibbs–Appell method can be found in the textbook by Pars.[9] We consider the problem with a few additional streamlining techniques: (1) We use the second form of the Gibbs–Appell equation, Eq. (13), rather than the more conventional form, Eq. (9); and (2) we use the technique of suppression of constants.[10]

Let $AXYZ$ be a Cartesian frame whose origin A and XY plane lie on the fixed surface; 0123, a Cartesian frame with its origin 0 at the center of the disk, its 3 axis perpendicular to the disk, and its 1 axis in the direction $\mathbf{e}_z \times \mathbf{e}_3$ where $\mathbf{e}_z$ is a unit vector in the AZ direction, and $\mathbf{e}_3$ is a unit vector in the 03 direction; X, Y, and Z are the coordinates of the center 0 of the disk with respect to the frame $AXYZ$; θ is the angle that the 3 axis makes with the Z axis; ϕ is the angle that the 1 axis makes with the X axis; P is a point fixed on the periphery of the disk; ψ is the angle that the line OP makes with the 1 axis; ω_1, ω_2, and ω_3 are the components with respect to the 0123 frame of the angular velocity of the disk; Ω_1, Ω_2, and Ω_3 are the components with respect to the 0123 frame of the angular velocity of the 0123 frame with respect to the disk; and I_1, I_2, and I_3 are the moments of inertia of the disk with respect to the 1, 2, and 3 axes, respectively.

In terms of the quantities introduced above,

$$R = \frac{1}{2} m(\ddot{X}^2 + \ddot{Y}^2 + \ddot{Z}^2) + \frac{1}{2} \sum_i I_i \dot{\omega}_i^2 + \sum_i \sum_j \sum_k \epsilon_{ijk} I_k \dot{\omega}_i \omega_j \omega_k + \sum_i \sum_j \sum_k \epsilon_{ijk} I_i \dot{\omega}_i \Omega_j \omega_k + mg\ddot{Z}. \tag{24}$$

The moments of inertia are given by

$$I_1 = I_2 = \tfrac{1}{4} ma^2, \tag{25}$$

$$I_3 = \tfrac{1}{2} ma^2. \tag{26}$$

The components with respect to the 0123 frame of the angular velocity of the 0123 frame with respect to the disk are

$$\Omega_1 = \Omega_2 = 0, \tag{27}$$

$$\Omega_3 = -\dot{\psi} = \omega_2 \cot\theta - \omega_3. \tag{28}$$

We will suppress the constants m, g, and a by employing a system of units in which

$$m = g = a = 1. \tag{29}$$

Substituting Eqs. (25)–(29) into Eq. (24) we obtain

$$R = \tfrac{1}{2}(\ddot{X}^2 + \ddot{Y}^2 + \ddot{Z}^2) + \tfrac{1}{8}\dot{\omega}_1^2 + \tfrac{1}{8}\dot{\omega}_2^2 + \tfrac{1}{4}\dot{\omega}_3^2 + \tfrac{1}{2}\dot{\omega}_1\omega_2\omega_3 - \tfrac{1}{2}\omega_1\dot{\omega}_2\omega_3 - \tfrac{1}{4}\dot{\omega}_1\omega_2^2\cot\theta + \tfrac{1}{4}\omega_1\dot{\omega}_2\omega_2\cot\theta + \ddot{Z}. \tag{30}$$

The disk has five configurational degrees of freedom but only three motional degrees of freedom. Hence, to apply the Gibbs–Appell method, we need to express R as a function of five configuration coordinates q_i and three motion coordinates $\dot{r}_i$. We shall choose $q_1 = X$, $q_2 = Y$, $q_3 = \theta$, $q_4 = \phi$, $q_5 = \psi$, $\dot{r}_1 = \omega_1$, $\dot{r}_2 = \omega_2$, and $\dot{r}_3 = \omega_3$.

Using the holonomic constraint condition,

$$Z = \sin\theta, \tag{31}$$

the two anholonomic constraint conditions

$$\dot{X} = -\omega_3\cos\phi + \omega_1\sin\phi\sin\theta, \tag{32}$$

$$\dot{Y} = -\omega_3\sin\phi - \omega_1\cos\phi\sin\theta, \tag{33}$$

and the relations

$$\omega_1 = \dot{\theta}, \tag{34}$$

$$\omega_2 = \dot{\phi}\sin\theta, \tag{35}$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}, \tag{36}$$

we can determine $\ddot{X}$, $\ddot{Y}$, and $\ddot{Z}$ as functions of X, Y, θ, ϕ, ψ, ω_1, ω_2, ω_3, and t. Thus

$$\ddot{X} = \dot{\omega}_1\sin\theta\sin\phi - \dot{\omega}_3\cos\phi + \omega_1^2\cos\theta\sin\phi + \omega_1\omega_2\cos\phi + \omega_2\omega_3\csc\theta\sin\phi, \tag{37}$$

$$\ddot{Y} = -\dot{\omega}_1\sin\theta\cos\phi - \dot{\omega}_3\sin\phi + \omega_1^2\cos\theta\cos\phi + \omega_1\omega_2\sin\phi + \omega_2\omega_3\csc\theta\cos\phi, \tag{38}$$

$$\ddot{Z} = \dot{\omega}_1\cos\theta - \omega_1^2\sin\theta. \tag{39}$$

Using the above results in Eq. (30) and noting that

$$\ddot{X}^2 + \ddot{Y}^2 + \ddot{Z}^2 = \dot{\omega}_1^2 + \dot{\omega}_3^2 + 2\dot{\omega}_1\omega_2\omega_3 - 2\omega_1\omega_2\dot{\omega}_3 + \mathrm{fcn}(\theta,\phi,\omega_1,\omega_2,\omega_3), \tag{40}$$

we obtain

$$R = \tfrac{5}{8}\dot{\omega}_1^2 + \tfrac{1}{8}\dot{\omega}_2^2 + \tfrac{3}{4}\dot{\omega}_3^2 + \tfrac{3}{2}\dot{\omega}_1\omega_2\omega_3 - \tfrac{1}{2}\omega_1\dot{\omega}_2\omega_3 - \omega_1\omega_2\dot{\omega}_3 - \tfrac{1}{4}\dot{\omega}_1\omega_2^2\cot\theta + \tfrac{1}{4}\omega_1\dot{\omega}_2\omega_2\cot\theta + \dot{\omega}_1\cos\theta. \tag{41}$$

The Gibbs–Appell equations of motion for the system are thus

$$\frac{\partial R}{\partial\dot{\omega}_1} = \frac{5}{4}\dot{\omega}_1 + \frac{3}{2}\omega_2\omega_3 - \frac{1}{4}\omega_2^2\cot\theta + \cos\theta = 0, \tag{42}$$

$$\frac{\partial R}{\partial\dot{\omega}_2} = \frac{1}{4}\dot{\omega}_2 - \frac{1}{2}\omega_1\omega_3 + \frac{1}{4}\omega_1\omega_2\cot\theta = 0, \tag{43}$$

$$\frac{\partial R}{\partial\dot{\omega}_3} = \frac{3}{2}\dot{\omega}_3 - \omega_1\omega_2 = 0. \tag{44}$$

If we substitute Eqs. (34) and (35) in Eqs. (42)–(44) and restore the constants m, g, and a we obtain

$$5\ddot{\theta} + 6\omega_3\dot{\phi}\sin\theta - \dot{\phi}^2\sin\theta\cos\theta + (4g/a)\cos\theta = 0 \tag{45}$$

$$\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta - 2\omega_3\dot{\theta} = 0, \tag{46}$$

$$3\dot{\omega}_3 - 2\dot{\phi}\dot{\theta}\sin\theta = 0. \tag{47}$$

We thus have three equations in the three unknowns ϕ, θ, and ω_3. The results obtained by solving Eqs. (45)–(47) for $\phi(t)$, $\theta(t)$, and $\omega_3(t)$ can be used in Eqs. (32)–(36) to obtain $X(t)$, $Y(t)$, $\psi(t)$, $\omega_1(t)$, and $\omega_2(t)$.

IX. CONCLUSION

From the above it follows that not only are the Gibbs–Appell equations of motion formally more elegant and simpler to derive than Lagrange's equations of motion, but they are also more powerful and more versatile. Anholonomic constraints can be handled in a more straightforward manner with the Gibbs–Appell equations than with Lagrange's equations; and the Gibbs–Appell use of motional coordinates $\dot{r}$, which are other than simple derivatives of the configurational coordinates q opens up possibilities not available in the Lagrangian approach.

Despite the general superiority of the Gibbs–Appell equations of motion over Lagrange's equations of motion, the majority of routine problems can be handled more easily with Lagrange's equations rather than with the Gibbs–Appell equations.

However, even if one ignores those problems for which the Gibbs–Appell method is clearly superior to Lagrange's method, it is still true, from a theoretical and pedagogical point of view, that the Gibbs–Appell equations of motion constitute a marvelous starting point for the unification, presentation, and generation of generalized equations of motion. They are easy to derive and once derived lead in a very natural fashion to Lagrange's equations of motion.

[1]J. W. Gibbs, Am. J. Math. **2**, 49 (1879).

[2]P. Appell, C. R. Acad. Sci. (Paris) **129**, 317, 423, 459 (1899); J. Math. Pures Appl. **6**, 5 (1900); J. Reine Angew. Math. **121**, 310 (1900).

[3]See, for example, E. J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies* (MacMillan, London, 1905; Dover, New York, 1955), Art. 430; J. S. Ames and F. D. Murnaghan, *Theoretical Mechanics* (Ginn, Lexington, MA, 1929; Dover, New York, 1958), pp. 329–332; W. D. MacMillan, *Dynamics of Rigid Bodies* (McGraw-Hill, New York, 1936), pp. 341–346; E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge U. P., London, 1937), Art. 107; L. A. Pars, *A Treatise on Analytical Dynamics* (Wiley, New York, 1965), Chaps. 12–13; F. Gantmacher, *Lectures in Analytical Mechanics* (Mir, Moscow, 1970), pp. 57–65; Ju. I. Neimark, and N. A. Fufaev, *Dynamics of Nonholonomic Systems* (American Mathematical Society, Providence, RI, 1972), pp. 147–159; E. A. Desloge, *Classical Mechanics, Volume* 2 (Wiley, New York, 1982), Chaps. 69–70.

[4]Each of these statements was made by an outstanding and respected academic working in the area of engineering dynamics. Both names are withheld because the purpose of these quotations is simply to emphasize the controversial nature of the subject, and not to associate the two polar positions with particular individuals.

[5]See, for example, E. A. Desloge, *Classical Mechanics* (Wiley, New York, 1982), Vol. 1, Chap. 31.

[6]P. Appell, C. R. Acad. Sci. (Paris) **129**, 459 (1899).

[7]E. A. Desloge, unpublished result.

[8]E. A. Desloge, *Classical Dynamics* (Wiley, New York, 1982), Vol. 2, p. 728, Theorem 2.

[9]L. A. Pars, *A Treatise on Analytical Dynamics* (Wiley, New York, 1965), Chap. 13.

[10]See, for example, E. A. Desloge, Am. J. Phys. **52**, 312 (1984).