Progress of Theoretical Physics, Vol. 48, No. 3, September 1972

The Ginzburg-Landau Equation for $^3P_2$ Pairing

——*Superfluidity in Neutron Stars*——

Toshimitsu FUJITA*) and Toshihiko TSUNETO

Department of Physics, Kyoto University, Kyoto

(Received February 23, 1972)

The purpose of this work is to study superfluid properties of a neutral fermion system with pairing in a non-zero angular momentum state, in particular, with $^3P_2$ pairing expected in neutron stars. The gap equation and the equation of the Ginzburg-Landau (G-L) type are derived for general pairing. We then specialize to the case of $^3P_2$ pairing and present the G-L equation for the five component order parameter in the form of a non-linear field equation for a spin 2 field. The expressions for the free energy, the current density and the angular momentum density are found in terms of the order parameter. With the help of the G-L equation some features of vortex states are discussed.

§ 1. Introduction

Recently superconducting or superfluid states of a fermion system with non-zero angular momentum pairing have received renewed attention in connection with the possible occurrence of superfluidity in neutron stars. It is believed that the interior of a typical neutron star beneath its solid crust is composed of three degenerate quantum liquids of neutrons, protons and electrons, the number density of protons being roughly several per cent of that of neutrons.[1] The neutron liquid, and probably the proton liquid as well, are expected to be in a superfluid state, the character of which depends on the density of nucleons. According to the detailed analysis using nuclear interactions obtained from nucleon-nucleon scattering data,[2] the superfluidity of neutrons most favourable at densities above 1.6×10^{14}g/cm^3 is due to $^3P_2$ pairing rather than to $^1S_0$ pairing familiar in the BCS theory of superconductivity.

In the past superfluid states with non-zero angular momentum pairing have been studied with regard to the superconductivity of transition metals and especially to the possibility of superfluid He^3 with $^1D_2$ pairing.[3] Little is known, however, about the hydrodynamical properties of such a superfluid. Would it behave in much the same way as superconductors with $^1S_0$ pairing or as superfluid He^4 inspite of the fact that it has anisotropic energy gap and its pairs possess internal angular momentum? This is an important question since the superfluidity may

*) Present address: Department of Physics, Faculty of Education, Wakayama University, Wakayama.

explain some interesting behaviour of pulsars, which most likely are rotating neutron stars.[4] For example, Baym, Pethick, Pines and Ruderman invoked the superfluidity of neutrons to explain the slowing-down rate after spin-up of the Vela pulsar. They supposed the neutron superfluid to be of the $^1S_0$ type and hence to behave like the ordinary superfluid. The main purpose of the present work is to study this question from a microscopic point of view.

In the next section we first set up the gap equation for a general pairing state using the generalized Hartree-Fock approximation and then derive the corresponding Ginzburg-Landau equation. Although this equation is valid only in the limited region of temperature close to the transition temperature T_c, we expect that it would describe at least qualitatively important characteristics of the superfluid, as is the case for superconductors or for superfluid He^4. In § 3, specializing to the case of $^3P_2$ pairing, we rewrite the G-L equation in a more convenient form of a set of nonlinear equations for a spin 2 field. In the succeeding section we obtain the expressions for the free energy, the current density and the angular momentum density in terms of the field quantities. As an application of the theory we discuss in § 5 the solutions corresponding to states with a single vortex line.

§ 2. Equations for the order parameter

In this section we shall derive in the BCS-Gor'kov approximation the equation for the order parameter or the so-called gap equation for a system of neutral fermions interacting through an attractive two-body potential V. Let us describe our system by the following hamiltonian:

$$\mathcal{H}=\sum_{\boldsymbol{k}\alpha}\xi(k)\,a^\dagger_{\boldsymbol{k}\alpha}a_{\boldsymbol{k}\alpha}+\frac{1}{2}\sum_{\boldsymbol{k}_1\alpha,\,\boldsymbol{k}_2\beta,\,\boldsymbol{k}_3\gamma,\,\boldsymbol{k}_4\delta}\langle\boldsymbol{k}_1\alpha,\,\boldsymbol{k}_2\beta|V|\boldsymbol{k}_4\delta,\,\boldsymbol{k}_3\gamma\rangle a^\dagger_{\boldsymbol{k}_1\alpha}a^\dagger_{\boldsymbol{k}_2\beta}a_{\boldsymbol{k}_3\gamma}a_{\boldsymbol{k}_4\delta}\,,\tag{1}$$

where $\xi(k)=\boldsymbol{k}^2/2m-\mu$ and μ is the fermi energy of the system. Since the interaction conserves the total momentum, its matrix element can be written in the form

$$\langle\boldsymbol{k}_1\alpha,\,\boldsymbol{k}_2\beta|V|\boldsymbol{k}_4\delta,\,\boldsymbol{k}_3\gamma\rangle=\Omega^{-1}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\,\boldsymbol{k}_3+\boldsymbol{k}_4}V_{\alpha\beta\delta\gamma}\left(\frac{\boldsymbol{k}_1-\boldsymbol{k}_2}{2},\,\frac{\boldsymbol{k}_4-\boldsymbol{k}_3}{2}\right),\tag{2}$$

where Ω is the volume of the system. In what follows we shall use the technique of thermal Green's function. In the presence of the pair condensation it is convenient to introduce the anomalous Green's functions

$$\mathcal{F}^\dagger_{\alpha\beta}(\boldsymbol{k}_1,\,\boldsymbol{k}_2;\,\tau_1-\tau_2)=\langle T_\tau[a^\dagger_{\boldsymbol{k}_1\alpha}(\tau_1)\,a^\dagger_{\boldsymbol{k}_2\beta}(\tau_2)]\rangle,\tag{3}$$

as well as the ordinary Green's function $\mathcal{G}^\omega_{\alpha\beta}(\boldsymbol{k}_1,\,\boldsymbol{k}_2)$ defined as usual. The equations of motion for these functions in the generalized Hartree-Fock approximation are the well-known Gor'kov equations. When the pairing is in an arbitrary state, we have

$$\{i\omega-\xi(k_1)\}\mathcal{G}^{\omega}_{\alpha\beta}(\boldsymbol{k}_1,\boldsymbol{k}_2)+\sum_{q\gamma}\Delta_{\alpha\gamma}(\boldsymbol{k}_1,\boldsymbol{q})\mathcal{F}^{\dagger\omega}_{\gamma\beta}(\boldsymbol{q},\boldsymbol{k}_2)=\delta_{\alpha,\beta}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_2},$$
$$\{i\omega+\xi(k_1)\}\mathcal{F}^{\dagger\omega}_{\alpha\beta}(\boldsymbol{k}_1,\boldsymbol{k}_2)+\sum_{q\gamma}\Delta^{\dagger}_{\alpha\gamma}(\boldsymbol{q},\boldsymbol{k}_1)\mathcal{G}^{\omega}_{\gamma\beta}(\boldsymbol{q},\boldsymbol{k}_2)=0, \tag{4}$$

where the order parameters are defined by

$$\Delta_{\alpha\beta}(\boldsymbol{k}_1,\boldsymbol{k}_2)=-T\sum_{\omega q K\gamma\delta}\Omega^{-1}V_{\alpha\beta,\gamma\delta}\left(\frac{\boldsymbol{k}_1-\boldsymbol{k}_2}{2},\boldsymbol{q}\right)\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{K}}\mathcal{F}^{\omega}_{\gamma\delta}\left(\boldsymbol{q}+\frac{\boldsymbol{K}}{2},-\boldsymbol{q}+\frac{\boldsymbol{K}}{2}\right),$$
$$\Delta^{\dagger}_{\alpha\beta}(\boldsymbol{k}_1,\boldsymbol{k}_2)=\Delta^{*}_{\beta\alpha}(\boldsymbol{k}_1,\boldsymbol{k}_2). \tag{5}$$

As can easily be seen,

$$\Delta_{\alpha\beta}(\boldsymbol{k}_1,\boldsymbol{k}_2)=-\Delta_{\beta\alpha}(\boldsymbol{k}_2,\boldsymbol{k}_1). \tag{6}$$

Let us now consider the case where the center of mass of all the pairs are moving with momentum $\boldsymbol{K}$, that is, when there is a uniform flow of the superfluid. The Green's functions in this case take the following form:[5]

$$\mathcal{G}^{\omega}_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2)=e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)/2}\mathcal{G}^{\omega}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{r}_1-\boldsymbol{r}_2),$$
$$\mathcal{F}^{\dagger\omega}_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2)=e^{-i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_2)/2}\mathcal{F}^{\dagger\omega}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{r}_1-\boldsymbol{r}_2) \tag{7}$$

and in terms of their fourier components, e.g.,

$$\mathcal{F}^{\dagger\omega}_{\alpha\beta}(\boldsymbol{k}_1,\boldsymbol{k}_2)=\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{K}}\mathcal{F}^{\dagger\omega}_{\alpha\beta,\boldsymbol{K}}\left(-\frac{\boldsymbol{k}_1-\boldsymbol{k}_2}{2}\right),$$

we can write the Gor'kov equations (4) as

$$\{i\omega-\xi(p^+)\}\mathcal{G}^{\omega}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p})+\sum_{\gamma}\Delta_{\alpha\gamma,\boldsymbol{K}}(\boldsymbol{p})\mathcal{F}^{\dagger\omega}_{\gamma\beta,\boldsymbol{K}}(\boldsymbol{p})=\delta_{\alpha,\beta},$$
$$\{i\omega+\xi(-p^-)\}\mathcal{F}^{\dagger\omega}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p})+\sum_{\gamma}\Delta^{\dagger}_{\alpha\gamma,\boldsymbol{K}}(\boldsymbol{p})\mathcal{G}^{\omega}_{\gamma\beta,\boldsymbol{K}}(\boldsymbol{p})=0, \tag{8}$$

where $\boldsymbol{p}^{\pm}=\boldsymbol{p}\pm\boldsymbol{K}/2$ and

$$\Delta^{\dagger}_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k})=-T\sum_{p\alpha\beta}\Omega^{-1}V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k})\mathcal{F}^{\dagger\omega}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p}). \tag{9}$$

Eliminating the anomalous functions, we obtain

$$\{i\omega-\xi(p^+)\}\{i\omega+\xi(p^-)\}\mathcal{G}^{\omega}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p})-\sum_{\gamma\delta}\Delta_{\alpha\gamma,\boldsymbol{K}}(\boldsymbol{p})$$
$$\times\Delta^{\dagger}_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{p})\mathcal{G}^{\omega}_{\delta\beta,\boldsymbol{K}}(\boldsymbol{p})=\{i\omega+\xi(p^-)\}\delta_{\alpha,\beta}. \tag{10}$$

Unlike the case of ${}^1S_0$ pairing, the second term in the above equation related to the energy gap in the excitation spectrum is in general not diagonal with respect to the spin indices. For simplicity we write down the gap equation only for the case where the order parameters satisfy the condition

$$\sum_{\gamma}\Delta_{\alpha\gamma,\boldsymbol{K}}(\boldsymbol{p})\Delta^{\dagger}_{\gamma\beta,\boldsymbol{K}}(\boldsymbol{p})\equiv\delta_{\alpha,\beta}\Delta_{\boldsymbol{K}}^2(\boldsymbol{p}) \tag{11}$$

and $\mathcal{G}_{\alpha\beta}$ is diagonal. This implies that the superfluid state under consideration retains time reversal symmetry and does not have, for example, spin polarization.

In this case we can easily obtain from (9) the equations for the order parameters as follows:

$$\Delta^\dagger_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k}) = T\sum_{\omega \boldsymbol{p}\alpha\beta} \Omega^{-1} V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k}) \Delta^\dagger_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p}) [\{i\omega+\xi(p^-)\}\{i\omega-\xi(p^+)\} - \Delta_{\boldsymbol{K}}{}^2(\boldsymbol{p})]^{-1}$$

$$= -\sum_{\boldsymbol{p}\alpha\beta} \Omega^{-1} V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k}) \Delta^\dagger_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p}) \frac{1}{4\varepsilon_{\boldsymbol{K}}(\boldsymbol{p})}$$

$$\times \left\{\tanh\frac{\varepsilon_{\boldsymbol{K}}{}^+(\boldsymbol{p})}{2T} + \tanh\frac{\varepsilon_{\boldsymbol{K}}{}^-(\boldsymbol{p})}{2T}\right\}, \qquad (12)$$

where

$$\varepsilon_{\boldsymbol{K}}(\boldsymbol{p}) = [\{\xi(p) + K^2/8m\}^2 + \Delta_{\boldsymbol{K}}{}^2(\boldsymbol{p})]^{1/2},$$

$$\varepsilon_{\boldsymbol{K}}{}^{\pm}(\boldsymbol{p}) = \varepsilon_{\boldsymbol{K}}(\boldsymbol{p}) \pm \boldsymbol{K}\cdot\boldsymbol{p}/2m\,.$$

Since we are primarily interested in the study of superfluid properties of our system and since Eqs. (4) are too complicated when there is a spatial variation of the order parameters, we next try to derive the approximate equations for the order parameters which in the case of $^1S_0$ pairing is the well-known Ginzburg-Landau equation. When temperature is close to T_c the order parameters are small and the expansion in powers of them becomes valid. From Eqs. (4) and (5) it is easy to obtain by iteration the following equation valid to third order in Δ's:

$$\Delta^\dagger_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k}) = A_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k}) + B_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k}), \qquad (13)$$

where

$$A_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k}) = -T\sum_{\boldsymbol{p}\alpha\beta} \Omega^{-1} V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k}) \bar{\mathcal{G}}^{-\omega}\left(-\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\right) \bar{\mathcal{G}}^{\omega}\left(\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\right) \Delta^\dagger_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p}), \qquad (14)$$

and

$$B_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k}) = T\sum_{\boldsymbol{p}\boldsymbol{Q}_1\boldsymbol{Q}_2\boldsymbol{Q}_3\alpha\beta\mu_1\mu_2} \Omega^{-1} V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k}) \delta_{\boldsymbol{Q}_1+\boldsymbol{Q}_2,\boldsymbol{Q}_3+\boldsymbol{K}}$$

$$\times \bar{\mathcal{G}}^{-\omega}\left(-\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\right) \bar{\mathcal{G}}^{\omega}\left(\boldsymbol{p}+\boldsymbol{Q}_1-\frac{\boldsymbol{K}}{2}\right) \bar{\mathcal{G}}^{-\omega}\left(-\boldsymbol{p}+\boldsymbol{Q}_2-\frac{\boldsymbol{K}}{2}\right) \bar{\mathcal{G}}^{\omega}\left(\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\right)$$

$$\times \Delta^\dagger_{\alpha\mu_1,\boldsymbol{Q}_1}\left(\boldsymbol{p}-\frac{\boldsymbol{K}-\boldsymbol{Q}_1}{2}\right) \Delta_{\mu_1\mu_2,\boldsymbol{Q}_3}\left(\boldsymbol{p}+\frac{\boldsymbol{Q}_1-\boldsymbol{Q}_2}{2}\right) \Delta^\dagger_{\mu_2\beta,\boldsymbol{Q}_2}\left(\boldsymbol{p}+\frac{\boldsymbol{K}-\boldsymbol{Q}_2}{2}\right). \qquad (15)$$

In the above expressions we have used the same notation as (9) and $\bar{\mathcal{G}}^{\omega}(\boldsymbol{p})$ is the Green's function in the normal state, $\bar{\mathcal{G}}^{\omega}(\boldsymbol{p}) = \{i\omega - \xi(p)\}^{-1}$. We note that in contrast to A the second term B cannot be obtained by simply expanding the right-hand side of (12) since it involves coupling between Δ's with different values of the total momentum. Now we further assume that the order parameters vary slowly in space so that $\boldsymbol{K}$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$ and $\boldsymbol{Q}_3$ in the above expression are negligibly small compared to $|\boldsymbol{p}|$ which is of the order of the fermi momentum p_f. Expanding the normal Green's functions in (14) and (15) with respect to the center-of-mass momenta and keeping only the second order terms for A and the first order for B, we obtain

$$A=-T\sum_{\omega p\alpha\beta}\Omega^{-1}V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k})\Big[\overline{\mathcal{G}}^{-\omega}(p)\overline{\mathcal{G}}^{\omega}(p)+\frac{(\boldsymbol{V}_p\cdot\boldsymbol{K})^2}{4}$$
$$\times\{2(\overline{\mathcal{G}}^{-\omega}(p))^3\overline{\mathcal{G}}^{\omega}(p)-(\overline{\mathcal{G}}^{-\omega}(p)\overline{\mathcal{G}}^{\omega}(p))^2\}\Big]\Delta^{\dagger}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p}), \tag{16}$$

$$B=T\sum_{\omega pQ_1Q_2Q_3\alpha\beta\mu_1\mu_2}\Omega^{-1}V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k})\,(\overline{\mathcal{G}}^{-\omega}(p)\overline{\mathcal{G}}^{\omega}(p))^2$$
$$\times\Delta^{\dagger}_{\alpha\mu_1,Q_1}(\boldsymbol{p})\Delta_{\mu_1\mu_2,Q_3}(\boldsymbol{p})\Delta^{\dagger}_{\mu_2\beta,Q_2}(\boldsymbol{p})\delta_{Q_1+Q_2,Q_3+\boldsymbol{K}}, \tag{17}$$

where $\boldsymbol{v}_p=\hat{\boldsymbol{p}}p_f/m$ ($\hat{\boldsymbol{a}}$ is a unit vector along $\boldsymbol{a}$; $\hat{\boldsymbol{a}}=\boldsymbol{a}/|\boldsymbol{a}|$). When summed over ω, they reduce to

$$A=-(2\pi)^{-3}\sum_{\alpha\beta}\int d\boldsymbol{p}V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k})\Delta^{\dagger}_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{p})$$
$$\times\left\{\frac{1}{2\xi(p)}\tanh\frac{\xi(p)}{2T}-\frac{(\boldsymbol{V}_p\cdot\boldsymbol{K})^2}{8\xi^3(p)}\tanh\frac{\xi(p)}{2T}\right\}, \tag{18}$$

$$B=(2\pi)^{-3}\sum_{\alpha\beta\mu_1\mu_2Q_1Q_2Q_3}\int d\boldsymbol{p}V_{\beta\alpha,\delta\gamma}(\boldsymbol{p},\boldsymbol{k})\Delta^{\dagger}_{\alpha\mu_1,Q_1}(\boldsymbol{p})$$
$$\times\Delta_{\mu_1\mu_2,Q_3}(\boldsymbol{p})\Delta^{\dagger}_{\mu_2\beta,Q_2}(\boldsymbol{p})\frac{1}{4\xi^3(p)}\tanh\frac{\xi(p)}{2T}\delta_{Q_1+Q_2,Q_3+\boldsymbol{K}}. \tag{19}$$

When we make fourier transformation of these expressions from $\boldsymbol{K}$ to the center-of-mass coordinate $\boldsymbol{R}$, we shall finally arrive at the generalized G-L equation.

§ 3. $^3P_2$ pairing

So far we have used the representation of the states of the condensed pair in terms of its total momentum $\boldsymbol{K}$, its relative momentum $\boldsymbol{k}$ and its spin quantum numbers α, β. Thus the order parameters $\Delta_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{k})$ can be regarded as the projection of the pair state on the ket $|\alpha\rangle_1|\beta\rangle_2|\boldsymbol{K},\boldsymbol{k}\rangle$ where $|\alpha\rangle_1$ and $|\beta\rangle_2$ are the basis of the spin space of the particle 1 and 2, respectively. Instead of this representation one can use the projection onto $|\alpha\rangle_1|\beta\rangle_2|\boldsymbol{K},klm\rangle$ where l and m are the angular momentum quantum numbers of the relative motion of the pair. The relation between the two representations is simply given by

$$\Delta_{\gamma\delta,\boldsymbol{K}}(\boldsymbol{k})=\sum_{lm}\Delta^{(lm)}_{\gamma\delta,\boldsymbol{K}}(k)\,Y_l^m(\theta_k,\varphi_k). \tag{20}$$

Since the magnitude J of the internal angular momentum of the pair has in general a definite value, it is more convenient to adopt yet another representation obtained by the projection of the angular momentum state onto $|(\frac{1}{2}\frac{1}{2})S,l;JM\rangle$, where S is the total spin quantum number and M the z component of the total angular momentum. In other words we write the pair state as

$$\sum_{SlJM} \Psi_{SlJM,\boldsymbol{K}}(k) \,|\, (\tfrac{1}{2}\tfrac{1}{2})S, l; JM\rangle\,. \tag{21}$$

Then the coefficients Ψ's are related to Δ's by

$$\Delta^{(lm)}_{\gamma\delta,\boldsymbol{K}}(k) = \sum_{S\sigma JM} \Psi_{SlJM,\boldsymbol{K}}(k) \langle \tfrac{1}{2}\tfrac{1}{2}\gamma\delta | S\sigma\rangle \langle Sl\sigma m | JM\rangle. \tag{22}$$

Combining (20) and (22) we can express $\Delta_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{k})$ in terms of Ψ's. For example, in the case of the $^3P_0$ pairing we have

$$\tilde{\Delta}_{\boldsymbol{K}}(\boldsymbol{k}) = \frac{\Psi_0}{\sqrt{3}} \begin{pmatrix} Y_1^{-1}(\hat{\boldsymbol{k}}) & -Y_1^0(\hat{\boldsymbol{k}})/\sqrt{2} \\ -Y_1^0(\hat{\boldsymbol{k}})/\sqrt{2} & -Y_1^1(\hat{\boldsymbol{k}}) \end{pmatrix},$$

where $\Psi_0 \equiv \Psi_{1100,\boldsymbol{K}}(k)$ and $\tilde{\Delta}_{\boldsymbol{K}}(\boldsymbol{k})$ denotes $\Delta_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{k})$ in a matrix form. Similarly in the case of the $^3P_2$ pairing we have

$$\tilde{\Delta}_{\boldsymbol{K}}(\boldsymbol{k}) = \begin{pmatrix} \Psi_2 Y_1^1 + \dfrac{\Psi_1}{\sqrt{2}} Y_1^0 + \dfrac{\Psi_0}{\sqrt{6}} Y_1^{-1} & \dfrac{\Psi_1}{2} Y_1^1 + \dfrac{\Psi_0}{\sqrt{3}} Y_1^0 + \dfrac{\Psi_{-1}}{2} Y_1^{-1} \\ \dfrac{\Psi_1}{2} Y_1^1 + \dfrac{\Psi_0}{\sqrt{3}} Y_1^0 + \dfrac{\Psi_{-1}}{2} Y_1^{-1} & \dfrac{\Psi_0}{\sqrt{6}} Y_1^1 + \dfrac{\Psi_{-1}}{\sqrt{2}} Y_1^0 + \Psi_{-2} Y_1^{-1} \end{pmatrix}, \tag{23}$$

where $\Psi_M \equiv \Psi_{112M,\boldsymbol{K}}(k)$. Correspondingly we can express the matrix element of the interaction operator in the two representations as follows:

$$\begin{aligned} V_{\alpha\beta,\gamma\delta}(\boldsymbol{p}, \boldsymbol{k}) &= \sum_{lml'm'} \langle \alpha\beta lm | V_{p,k} | \gamma\delta l'm'\rangle Y_l^m(\hat{\boldsymbol{p}}) Y_{l'}^{m'*}(\hat{\boldsymbol{k}}) \\ &= \sum_{\substack{lml'm' \\ S\sigma JMS'\sigma'J'M'}} \langle \tfrac{1}{2}\tfrac{1}{2}\alpha\beta | S\sigma\rangle \langle Sl\sigma m | JM\rangle Y_l^m(\hat{\boldsymbol{p}}) \\ &\quad \times \langle \tfrac{1}{2}\tfrac{1}{2}\gamma\delta | S'\sigma'\rangle \langle S'l'\sigma'm' | J'M'\rangle Y_{l'}^{m'*}(\hat{\boldsymbol{k}}) \\ &\quad \times \langle SlJM | V_{p,k} | S'l'J'M'\rangle\,. \end{aligned} \tag{24}$$

We note that, because of the rotation invariance of the interaction, $\langle SlJM | V_{p,k} | S'l'J'M'\rangle$ is diagonal in J and M and does not depend on M. We also remark that in the absence of a magnetic field or rotation of the system the state of the pair itself may be invariant against time reversal, in which case the Ψ's satisfy

$$\Psi_{SlJM} = (-)^{J+M} \Psi^*_{SlJM}\,. \tag{25}$$

Let us now specialize to the case of the $^3P_2$ pairing. In other words we assume that the effective interaction is most attractive for $S=1$, $l=1$ and $J=2$, and that there is no tensor coupling or other interactions which give rise to off-diagonal elements connecting the states with different S and l. When the condition (25) of the time reversal invariance is satisfied, we can readily write down the gap equation (12), retaining only the terms with $J=2$ in (24):

$$\Psi^*_{M,\boldsymbol{K}}(k) = -\sum_{\alpha\beta\sigma mm'M} \int (2\pi)^{-3} d\boldsymbol{p}\, V(p, k) \frac{\Psi^*_{M',\boldsymbol{K}}(p)}{4\varepsilon_{\boldsymbol{K}}(\boldsymbol{p})}$$

$$\times\left\{\tanh\frac{\varepsilon_{\boldsymbol{K}}{}^{+}(\boldsymbol{p})}{2T}+\tanh\frac{\varepsilon_{\boldsymbol{K}}{}^{-}(\boldsymbol{p})}{2T}\right\}\langle\tfrac{1}{2}\,\tfrac{1}{2}\alpha\beta|1\sigma\rangle$$

$$\times\langle 11\sigma m|2M\rangle\langle\tfrac{1}{2}\,\tfrac{1}{2}\alpha\beta|1\sigma\rangle\langle 11\sigma m'|2M'\rangle Y_1{}^{m}(\hat{\boldsymbol{p}})\,Y_1{}^{m'*}(\hat{\boldsymbol{k}}),\qquad(26)$$

where $V(p,k)\equiv\langle 112M|\,V_{p,k}|112M\rangle$. In the limit of $\boldsymbol{K}\to 0$ and $T\to 0$, this equation agrees with the one derived by Tamagaki.[2] Since it is not our main concern here to study possible solutions of this equation, we proceed to the discussion of the G-L equation. Since it is obviously more convenient to use the representation in terms of Ψ's, we try to write Eqs. (13) in the form

$$\Psi^{*}_{M,\boldsymbol{K}}(k)=\mathcal{A}^{(1)}_{M,\boldsymbol{K}}(k)+\mathcal{A}^{(2)}_{M,\boldsymbol{K}}(k)+\mathcal{B}_{M,\boldsymbol{K}}(k),\qquad(27)$$

where $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ come from A given in (18), $\mathcal{A}^{(1)}$ corresponding to the zeroth order term in $\boldsymbol{K}$ and $\mathcal{A}^{(2)}$ to the second order, respectively, and $\mathcal{B}$ from B of (14). The fourier transform of this equation will give the G-L equation in real space. Let us briefly explain how to obtain these three terms of (27).

a) $\mathcal{A}^{(1)}_{MK}$. From the first term of (18) together with (24) it is easy to find

$$\mathcal{A}^{(1)}_{M,\boldsymbol{K}}(k)=-\int\frac{p^2dp}{(2\pi)^3}V(p,k)\frac{\tanh(\xi(p)/2T)}{2\xi(p)}\Psi^{*}_{M,\boldsymbol{K}}(p)\qquad(28)$$

and its fourier transform

$$\mathcal{A}_{M}{}^{(1)}(\boldsymbol{R};k)=-\int\frac{p^2dp}{(2\pi)^3}V(p,k)\frac{\tanh(\xi(p)/2T)}{2\xi(p)}\Psi_{M}{}^{*}(\boldsymbol{R};p).\qquad(29)$$

b) $\mathcal{A}^{(2)}_{MK}$. Substituting (22), (24) and the expression

$$(\hat{\boldsymbol{p}}\cdot\boldsymbol{K})^2=\frac{K^2}{3}\left\{1+\sum_{m}\frac{8\pi}{5}Y_2{}^{m}(\hat{\boldsymbol{K}})\,Y_2{}^{m*}(\hat{\boldsymbol{p}})\right\}$$

into the second term of (18) and integrating with respect to the solid angle $\hat{\boldsymbol{p}}$, we get, with the help of the table of Clebsch-Gordan coefficient,

$$\mathcal{A}^{(2)}_{M,\boldsymbol{K}}(k)=\frac{p_f{}^2}{24m^2}\int\frac{p^2dp}{(2\pi)^3}V(p,k)\frac{\tanh(\xi(p)/2T)}{\xi^3(p)}\sum_{M'}\mathcal{D}_{MM'}(\boldsymbol{K})\,\Psi^{*}_{M',\boldsymbol{K}}(p),\qquad(30)$$

where

$$\mathcal{D}(\boldsymbol{K})=K^2I+\frac{\sqrt{16\pi}}{5\sqrt{5}}K^2\left(\begin{array}{ccccc}-Y_2{}^{0} & \sqrt{\frac{3}{2}}Y_2{}^{1} & -Y_2{}^{2} & 0 & 0\\ -\sqrt{\frac{3}{2}}Y_2{}^{-1} & \frac{1}{2}Y_2{}^{0} & \frac{1}{2}Y_2{}^{1} & -\sqrt{\frac{3}{2}}Y_2{}^{2} & 0\\ -Y_2{}^{-2} & -\frac{1}{2}Y_2{}^{-1} & Y_2{}^{0} & -\frac{1}{2}Y_2{}^{1} & -Y_2{}^{2}\end{array}\right.$$

$$\begin{pmatrix} 0 & -\sqrt{\frac{3}{2}}Y_2^{-2} & \frac{1}{2}Y_2^{-1} & \frac{1}{2}Y_2^{0} & -\sqrt{\frac{3}{2}}Y_2^{1} \\ 0 & 0 & -Y_2^{-2} & \sqrt{\frac{3}{2}}Y_2^{-1} & -Y_2^{0} \end{pmatrix}$$

with $Y_2^M = Y_2^M(\hat{\boldsymbol{K}})$ and I being the unit matrix. The fourier transform gives the following result:

$$\mathcal{A}_M^{(2)}(\boldsymbol{R};p) = \frac{p_f^2}{24m^2}\int\frac{p^2dp}{(2\pi)^3}V(p,k)\frac{\tanh(\xi(p)/2T)}{\xi^3(p)}\sum_{M'}\mathcal{D}_{MM'}(\boldsymbol{R})\Psi^*_{M'}(\boldsymbol{R};p), \tag{31}$$

where $\mathcal{D}_{MM'}(\boldsymbol{R})$ is the differential operator

$$\mathcal{D}(\boldsymbol{R}) = -\frac{1}{5}$$

$$\times\begin{pmatrix} 3(\partial_z^2+2\partial_+\partial_-) & -3\partial_z\partial_+ & -\sqrt{\frac{3}{2}}\partial_+^2 & 0 & 0 \\ -3\partial_z\partial_- & \frac{3}{2}(4\partial_z^2+3\partial_+\partial_-) & -\sqrt{\frac{3}{2}}\partial_z\partial_+ & -\frac{3}{2}\partial_+^2 & 0 \\ -\sqrt{\frac{3}{2}}\partial_-^2 & -\sqrt{\frac{3}{2}}\partial_z\partial_- & 7(\partial_z^2+4\partial_+\partial_-) & \sqrt{\frac{3}{2}}\partial_z\partial_+ & -\sqrt{\frac{3}{2}}\partial_+^2 \\ 0 & -\frac{3}{2}\partial_-^2 & \sqrt{\frac{3}{2}}\partial_z\partial_- & \frac{3}{2}(4\partial_z^2+3\partial_+\partial_-) & 3\partial_z\partial_+ \\ 0 & 0 & -\sqrt{\frac{3}{2}}\partial_-^2 & 3\partial_z\partial_- & 3(\partial_z^2+2\partial_+\partial_-) \end{pmatrix} \tag{32}$$

with

$$\partial_\pm = \partial_x \pm i\partial_y\,.$$

c) $\mathcal{B}_{M,\boldsymbol{K}}$. This requires a straight-forward but tedious calculation involving Clebsch-Gordan coefficients. Performing the integration with respect to $\hat{\boldsymbol{p}}$ and fourier transforming to the $\boldsymbol{R}$ space, we get

$$\begin{aligned}\mathcal{B}_M(\boldsymbol{R};p) = \frac{1}{4}\sum_{\substack{\alpha\beta\mu_1\mu_2mm'm''m'''\\ \sigma\sigma'\sigma''\sigma'''M'M''M'''l}}&\int\frac{p^2dp}{(2\pi)^3}V(p,k)\frac{\tanh(\xi(p)/2T)}{\xi^3(p)}\\ &\times\{\langle\tfrac{1}{2}\tfrac{1}{2}\beta\alpha|1\sigma\rangle\langle 11\sigma m|2M\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\mu_1\alpha|1\sigma'\rangle\langle 11\sigma'm'|2M'\rangle\\ &\times\langle\tfrac{1}{2}\tfrac{1}{2}\beta\mu_2|1\sigma''\rangle\langle 11\sigma''m''|2M''\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\beta\mu_2|1\sigma'''\rangle\langle 11\sigma'''m'''|2M'''\rangle\}\\ &\times\frac{9\delta_{m'+m''',m+m''}}{4\pi(2l+1)}\{\langle 11m'm'''|lm'+m'''\rangle\langle 11mm''|lm+m''\rangle\langle 1100|l0\rangle^2\}\\ &\times\Psi^*_{M'}(\boldsymbol{R};p)\Psi_{M''}(\boldsymbol{R};p)\Psi^*_{M'''}(\boldsymbol{R};p).\end{aligned} \tag{33}$$

Carrying out this complicated summation, we obtain as a result

$$\mathcal{B}_M(\boldsymbol{R};p)=\frac{1}{4}\int\frac{p^2dp}{(2\pi)^3}V(p,k)\frac{\tanh(\xi(p)/2T)}{\xi^3(p)}\sum_{M'}\mathcal{N}_{MM'}(\boldsymbol{R};p)\Psi^*_{M'}(\boldsymbol{R};p) \tag{34}$$

with the matrix

$$\mathcal{N}(\boldsymbol{R};p)=\frac{3}{20\pi}$$

$$\times\begin{bmatrix} \lambda+\xi_2 & \frac{1}{\sqrt{6}}\eta_1+\eta_2 & \frac{1}{3}\zeta & 0 & 0 \\ \frac{1}{\sqrt{6}}\eta_1^*+\eta_2^* & \lambda+\frac{1}{2}\xi_1 & \frac{5}{6}\eta_1+\frac{1}{\sqrt{6}}\eta_2 & 0 & 0 \\ \frac{1}{3}\zeta^* & \frac{5}{6}\eta_1^*+\frac{1}{\sqrt{6}}\eta_2^* & \lambda & \frac{5}{6}\eta_1+\frac{1}{\sqrt{6}}\eta_2 & -\frac{1}{3}\zeta \\ 0 & 0 & \frac{5}{6}\eta_1^*+\frac{1}{\sqrt{6}}\eta_2^* & \lambda-\frac{1}{2}\xi_1 & \frac{1}{\sqrt{6}}\eta_1+\eta_2 \\ 0 & 0 & -\frac{1}{3}\zeta^* & \frac{1}{\sqrt{6}}\eta_1^*+\eta_2^* & \lambda-\xi_2 \end{bmatrix}, \tag{35}$$

where

$$\begin{aligned} \lambda&=\sum_M\Psi_M{}^*\Psi_M\,,\\ \xi_1&=\Psi_1{}^*\Psi_1-\Psi^*_{-1}\Psi_{-1}\,,\\ \xi_2&=\Psi_2{}^*\Psi_2-\Psi^*_{-2}\Psi_{-2}\,,\\ \eta_1&=\Psi_1{}^*\Psi_0+\Psi_0{}^*\Psi_{-1}\,,\\ \eta_2&=\Psi_2{}^*\Psi_1+\Psi^*_{-1}\Psi_{-2}\,,\\ \zeta&=\Psi_2{}^*\Psi_0-\Psi_0{}^*\Psi_{-2}\,. \end{aligned} \tag{36}$$

This form of the nonlinear matrix is not unique since it also contains Ψ^*'s.

We now make the simplifying assumption, as often done in the study of superconductivity, that the matrix element of the interaction, which is to be understood as the effective interaction or the relevant T-matrix, is constant in the neighbourhood of the fermi surface, that is, $V(p,k)=-v\phi(p)\phi(k)$, with the cut-off function $\phi(p)$ defined by $\phi(p)=1$ for $|\xi(p)|<\tilde{\omega}$ and $=0$ otherwise. Clearly our order parameters have the form $\Psi^*_{M,K}(k)=\Psi^*_{M,K}\phi(k)$. Then the integration with respect to p can be carried out as usual and we arrive at the final result for the G-L equation:

$$\left(\ln\frac{T_c}{T}\right)\Psi_M{}^*(\boldsymbol{R})-\frac{7\zeta(3)p_f{}^2}{48(\pi Tm)^2}\sum_{M'}\mathcal{D}_{MM'}(\boldsymbol{R})\Psi_{M'}^*(\boldsymbol{R})$$
$$-\frac{7\zeta(3)}{8(\pi T)^2}\sum_{M'}\mathcal{N}_{MM'}(\boldsymbol{R})\Psi_{M'}^*(\boldsymbol{R})=0\,, \tag{37}$$

where

$$T_c=\frac{2\tilde{\omega}\gamma}{\pi}\exp\left[-\frac{8\pi^3}{mp_fv}\right]. \tag{38}$$

(We note that a factor 4π is absorbed in the constant v as compared to the coupling constant in the BCS theory.)

§ 4. Free energy, current density and angular momentum density

Since there exist many solutions, even without spatial variations, of the G-L equation in the present case, it is necessary to select the most stable solution as one that has the lowest value of the free energy. For this reason and also for the purpose of deriving the expression for the current density we first find the free energy of our system. The excess free energy, that is, the difference between the free energy of the superfluid state F_s and that of the normal state F_n, $F=F_s-F_n$, can be obtained, within the same approximation as we have used to derive Eq. (3), from the well-known formula[6]

$$\frac{\partial F}{\partial g}=-\frac{T}{2g}\sum_{\omega Kk\alpha\beta}\Delta_{\alpha\beta,\boldsymbol{K}}(\boldsymbol{k})\mathcal{F}^{\dagger\omega}_{\beta\alpha,\boldsymbol{K}}(\boldsymbol{k}), \tag{39}$$

where g is a parameter multiplying the interaction energy. Since, however, we are interested in the region of temperature where the G-L equation is valid, we can determine F in the following manner. Noting that the G-L equation is nothing but the minimization condition of F with respect to the field quantities Ψ's, we can readily write down the corresponding form for F in terms of Ψ's as follows:

$$F=C\int d\boldsymbol{R}\Big[-\left(\ln\frac{T_c}{T}\right)\lambda+\frac{21\zeta(3)}{160\pi(\pi T)^2}\Big(\frac{1}{2}\lambda^2+\frac{1}{4}\xi_1{}^2+\frac{1}{2}\xi_2{}^2$$
$$+\left|\eta_2+\frac{1}{\sqrt{6}}\eta_1\right|^2+\frac{2}{3}\left|\eta_1\right|^2+\frac{1}{3}\left|\zeta\right|^2\Big)+\frac{7\zeta(3)p_f{}^2}{48(\pi Tm)^2}\sum_{MM'}\Psi_M\mathcal{D}_{MM'}\Psi_{M'}^*\Big]. \tag{40}$$

We have only to determine the constant. Let us now consider a simple case that there is no spatial variation and that Ψ's satisfy the time reversal symmetry. Then it is easy to show that Eq. (39) reduces to

$$\frac{\partial F}{\partial g}=-\frac{\Omega}{2g^2v}\sum_M\Psi_M{}^*\Psi_M=-\frac{\Omega}{2g^2v}\lambda\,. \tag{41}$$

The G-L equation in this special case is simply a single equation

$$\ln(T_c/T) = \{21\zeta(3)/160\pi(\pi T)^2\}\lambda$$

with

$$T_c = (2\tilde{\omega}\gamma/\pi)\exp[-(2\pi)^3/mp_f gv].$$

Using this equation one can integrate (41) as

$$\frac{F}{\Omega} = \int \frac{d(1/g)}{d\lambda}\frac{\lambda}{2v}d\lambda = -\frac{mp_f}{16\pi^3}\cdot\frac{21\zeta(3)}{32\pi(\pi T)^2}\lambda^2.$$

Comparing this result with the expression (40) for this case, we obtain $C = mp_f/16\pi^3$.

The minimum value of the free energy is thus given by

$$F_{\min} = \frac{mp_f}{32\pi^3}\int d\boldsymbol{R}\left[-\left(\ln\frac{T_c}{T}\right)\lambda + \frac{7\zeta(3)p_f^2}{48(\pi Tm)^2}\sum_{MM'}\Psi_M \mathcal{D}_{MM'}\Psi_{M'}^*\right] \tag{42}$$

because of the G-L equation. When there is no spatial variation, the excess free energy is proportional to λ, and since λ takes the maximum value when $\xi_1 = \xi_2 = \eta_1 = \eta_2 = \zeta = 0$, the states satisfying this condition are most favourable with the free energy

$$\frac{F}{\Omega} = -\frac{5(\pi T)^2 mp_f}{21\pi^2\zeta(3)}\left(\ln\frac{T_c}{T}\right)^2. \tag{43}$$

The solutions satisfying the condition of the time reversal invariance belong to this class. It must be remarked here that there are many uniform solutions satisfying this condition and that they are degenerate in the present approximation; to decide which one among them is most favourable, one must keep higher order terms in Ψ's in the free energy.

At this point we would like to make an important remark that our theory is invariant against spatial rotations. Needless to say, while a particular state of the superfluid may not be invariant, the theory itself must be invariant against coordinate transformations. The invariance, for example, of the free energy as given in the form (40) is not evident at all. Note that our order parameters Ψ_M form a spinor of rank 5 and may be regarded as a spin 2 field. Associated with this field there are 25 independent tensor operators. It is shown in the Appendix that we can indeed rewrite the free energy in a manifestly invariant form in terms of the tensor operators. Here we only mention that the term in the free energy involving spatial derivatives can be written in a compact form

$$\frac{7p_f^3\zeta(3)}{3840\pi^3 m(\pi T)^2}\sum_{MM'ij}(-)\partial_i\Psi_M[S_iS_j - 7\delta_{i,j}I]_{MM'}\partial_j\Psi_{M'}^*,$$

where S_i is the angular momentum operator.

From the above expression we can readily find the density of mass current:

$$J_i(\boldsymbol{R}) = \mathcal{R}e(-i)\frac{7p_f^3\zeta(3)}{960\pi^3(\pi T)^2}\sum_{MM'j}\Psi_M\left[\frac{S_iS_j+S_jS_i}{2}-7\delta_{i,j}I\right]_{MM'}\partial_j\Psi^*_{M'}. \tag{44}$$

It would seem rather strange that the pairs moving in one direction can produce a current in other directions. We conjecture that the solutions of the G-L equations corresponding to the pairs moving with a constant momentum $\boldsymbol{K}$ gives rise only to current in the direction of $\boldsymbol{K}$. In general, however, it is impossible to define the superfluid density as a scalar quantity because of the tensor appearing in the expression (44).

To conclude this section we make a brief comment on the angular momentum density of the system, which seems to be necessary in view of the fact that the pairs have internal angular momentum J in the present case. The operator of the angular momentum density at a point $\boldsymbol{r}$ with respect to the origin is given by

$$\psi^\dagger(\boldsymbol{r})\boldsymbol{\sigma}\psi(\boldsymbol{r})+\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r}). \tag{45}$$

When we calculate the average of this operator, we simply get $\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r})$, that is, the contribution from the orbital motion. That the contribution from the spin and the relative motion of the pairs vanish can be understood if one realizes that the pairing is actually that of the particles above the fermi surface and that of the holes below. As a simple example, consider a uniform solution with only one component, say, Ψ_2 non-vanishing. From (23) the only non-vanishing element of $\tilde{\Delta}(\boldsymbol{k})$ is then $\Delta_{11}(\boldsymbol{k})$. Consequently, the spin down particles stay in the normal state and the number of the spin up particles is clearly not changed by the pairing, which one can see explicitly by solving Eq. (10) for this particular case.

§ 5. Vortex state

As an application of the G-L equations we consider a single vortex line in the $^3P_2$ superfluid and attempt to see if there exist solutions of the G-L equations representing such a state.[7] We can rewrite the G-L equations in cylindrical coordinates with the z-axis coinciding with the quantization axis of the angular momentum of the pairs simply by substituting

$$\partial_\pm = e^{\pm i\varphi}(\partial_r \pm i\partial_\varphi/r)$$

in the matrix operator $\mathcal{D}_{MM'}$. From the equations thus obtained it is apparent that any solution with axial symmetry, except spatially constant solutions, must have the form

$$\Psi_M^*(r,\varphi,z) = \exp[-i(J-M)\varphi]\Psi_M^*(r,z). \tag{46}$$

This means that the z-component of the total angular momentum, that is, the sum of the internal angular momentum and the orbital (vortex) angular momentum, is a good quantum number and is equal to J for all the pairs. Hence a vortex state is characterized by this quantum number.

When the variations of the order parameters Ψ along the z direction vanish as is the case for a rectilinear vortex, the differential operator $\mathscr{D}_{MM'}$ has non-vanishing elements only between the even or the odd components. Therefore, the states with only the even or the odd components non-vanishing are possible because they are then not coupled by the non-linear terms either ($\eta_1=\eta_2=0$). Since our main interest here is to study the existence of the vortex solutions, we restrict our discussion to the even solutions, assuming $\Psi_1=\Psi_{-1}=0$. Let us introduce

$$\begin{aligned}&\xi=\sqrt{a}\,r\,,\\&\Psi_M{}^*(r,\varphi)=\sqrt{a/b}\exp[-i(J-M)\varphi]\psi_M(\xi),\end{aligned}\tag{47}$$

where

$$\begin{aligned}&a=240(\pi Tm)^2\ln(T_c/T)/7\zeta(3)p_f{}^2,\\&b=9m^2/2\pi p_f{}^2.\end{aligned}$$

In terms of these variables the G-L equations now take the following form:

$$\begin{aligned}&\psi_2-\left\{2\psi_2{}^3+\frac{4}{3}\psi_0{}^2\psi_2-\frac{1}{3}\psi_0{}^2\psi_{-2}\right\}+6\left\{\partial_\xi{}^2+\frac{\partial_\xi}{\xi}-\frac{(J-2)^2}{\xi^2}\right\}\psi_2\\&\quad-\sqrt{\frac{3}{2}}\left\{\partial_\xi{}^2+\frac{2J-1}{\xi}\partial_\xi+\frac{J(J-2)}{\xi^2}\right\}\psi_0=0\,,\\&\psi_0-\left\{\frac{4}{3}\psi_2{}^2\psi_0+\psi_0{}^3+\frac{4}{3}\psi_{-2}^2\psi_0-\frac{2}{3}\psi_0\psi_2\psi_{-2}\right\}+4\left\{\partial_\xi{}^2+\frac{\partial_\xi}{\xi}-\frac{J^2}{\xi^2}\right\}\psi_0\\&\quad-\sqrt{\frac{3}{2}}\left\{\partial_\xi{}^2+\frac{2J+3}{\xi}\partial_\xi+\frac{J(J+2)}{\xi^2}\right\}\psi_{-2}-\sqrt{\frac{3}{2}}\left\{\partial_\xi{}^2-\frac{2J-3}{\xi}\partial_\xi+\frac{J(J-2)}{\xi^2}\right\}\psi_2=0\,,\\&\psi_{-2}-\left\{2\psi_{-2}^3+\frac{4}{3}\psi_0{}^2\psi_{-2}-\frac{1}{3}\psi_0{}^2\psi_2\right\}+6\left\{\partial_\xi{}^2+\frac{\partial_\xi}{\xi}-\frac{(J+2)^2}{\xi^2}\right\}\psi_{-2}\\&\quad-\sqrt{\frac{3}{2}}\left\{\partial_\xi{}^2-\frac{2J+1}{\xi}+\frac{J(J+2)}{\xi^2}\right\}\psi_0=0\,.\end{aligned}\tag{48}$$

Close to the center of the vortex we can expand ψ's in powers of ξ and easily find that each ψ must have the form

$$\psi_M=C^{(M)}_{|J-M|}\xi^{|J-M|}+C^{(M)}_{|J-M|+2}\xi^{|J-M|+2}+\cdots,\tag{49}$$

where the coefficients in the second and the following terms can be expressed by the first ones. Far from the axis they must approach uniform values. Unlike the case of the $^1S_0$ pairing there exist many solutions corresponding to uniform states. Consequently we expect that there are many possible solutions for a vortex state with a given quantum number J, at least in the present approximation. As an example the result of numerical calculation for a vortex state with quantum number 1 is given in Fig. 1. The Runge-Kutta-Gill method is used

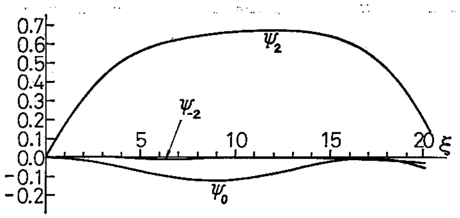

Fig. 1. One of the numerical solutions of Eq. (48) with vortex quantum number 1. The integration is started with $\xi=0.01$ with the initial condition given by (49) up to the third order in ξ.

with a starting point at a finite value of ξ and with the initial conditions given in the form (49). We vary the coefficients C's so as to make ψ's approach constant values for large ξ. The solution given in Fig. 1 still oscillates for large values of ξ, but it suggests the existence of a stable solution.

In order to be able to discuss, for example, the superfluidity in a rotating neutron star, one has to make more detailed analysis. Specifically it is necessary to see which solution minimizes the free energy in a rotating system, $F_{\rm rot}=F-\Omega\cdot L$, where Ω is the angular velocity and L is the total angular momentum of the system. From the previous discussions, however, it seems that the essential nature of the vortex states is not drastically different from that which we find in superfluid helium or in superconductors.

§ 6. Concluding remarks

We have studied within the generalized Hartree-Fock approximation the basic properties of the $^3P_2$ superfluid using the simple model of a neutral fermion system interacting only with the attractive $^3P_2$ interaction. Even in this framework the theory is rather complicated, and its consequences must be studied in more detail before we can get a clear picture of such a superfluid. Concerning the superfluidity in a neutron star there are many interesting problems we have not discussed in this work. For example, it would be interesting to study the transition between the $^1S_0$ and the $^3P_2$ superfluid associated with change in nucleon density. Also the existence of protons would present interesting problems such as their effect on the $^3P_2$ neutron superfluidity as strongly interacting impurities or the co-existence of the two superfluids of different characters.

Appendix

Invariance under Spatial Rotation

Let the space spanned by S_x, S_y and S_z be a representation space $D^{(1)}$ of the three-dimensional rotation group, and as the basis vectors we adopt

$$\begin{aligned} S_1 &= S_x+iS_y\,, \\ S_0 &= -\sqrt{2}S_z\,, \\ S_{-1} &= -(S_x-iS_y)\,. \end{aligned} \tag{A·1}$$

These belong to the tensor operator of the first order $T_m^{(1)}$ $(m=1, 0, -1)$. Because we consider the case of $J=2$ only, the coefficients $\langle j_1 \| T^{(k)} \| j_2 \rangle$ in the usual expression may be omitted, and our irreducible tensor operator of the 1st order is given by $t_m^{(1)}=S_m(m=1, 0, -1)$, where S_m here and after are matrices of $J=2$:

$$S_1=\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_0=\begin{pmatrix} -2\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2} \end{pmatrix}, \tag{A·2}$$

$$S_{-1}=-S_1^t. \qquad \text{(suffix } t\text{: transposed matrix)}$$

The irreducible matrix of higher order $t_m^{(k)}$ can be constructed as the basis of $D^{(k)}$, which is the irreducible subspace of

$$\overbrace{D^{(1)}\times\cdots\times D^{(1)}}^{k}=D^{(k)}+\cdots+D^{(1)}+D^{(0)}.$$

a) *2nd order*

$$t_2^{(2)}=S_1S_1,$$

$$t_1^{(2)}=\frac{1}{2}(I_-t_2^{(2)})=\frac{1}{\sqrt{2}}(S_0S_1+S_1S_0),$$

$$t_0^{(2)}=\frac{1}{\sqrt{6}}(I_-t_1^{(2)})=\frac{1}{\sqrt{6}}(S_{-1}S_1+2S_0S_0+S_1S_{-1}), \tag{A·3}$$

$$t_{-1}^{(2)}=\frac{1}{\sqrt{2}}(S_{-1}S_0+S_0S_{-1}),$$

$$t_{-2}^{(2)}=S_{-1}S_{-1},$$

where I_- is the spin down operator.

b) *3rd order*

$$t_3^{(3)}=S_1S_1S_1. \tag{A·4}$$

Successive operaions of I_- produce the remainders; for example,

$$t_2^{(3)}=\frac{1}{\sqrt{6}}(I_-t_3^{(3)})=\frac{1}{\sqrt{3}}(S_0S_1S_1+S_1S_0S_1+S_1S_1S_0).$$

With the help of these operators we now show that the expression (40) for the free energy is invariant under rotations. Let $|\Psi)$ be the spinor of rank 5 which represents the state of pairs. The first term of the integrand in Eq. (40) is proportional to $(\Psi|I|\Psi)=\sum\Psi_M{}^*\Psi_M=\lambda$ which is obviously invariant. The value

$$C^{(k)}=\sum_m(-)^m(\Psi|t^{(k)}_{-m}|\Psi)(\Psi|t_m{}^{(k)}|\Psi) \tag{A·5}$$

is a scalar which is invariant under any rotation of the coordinate axis. After a simple calculation, we get the following:

$$\begin{aligned}C^{(0)}&=\lambda^2,\\ C^{(1)}&=8\{|\eta_2+\sqrt{3/2}\eta_1|^2+(\xi_2+\xi_1/2)^2\},\\ C^{(3)}&=\frac{288}{5}\{5|\eta_2|^2-5\xi_1\xi_2+5|\zeta|^2+2|\sqrt{3/2}\eta_2-\eta_1|^2\}.\end{aligned} \tag{A·6}$$

The third term of the integrand is proportional to

$$\frac{1}{2}C^{(0)}+\frac{7}{60}C^{(1)}+\frac{1}{864}C^{(3)}$$

and is also rotationally invariant.

The differential operators

$$\begin{aligned}d_1&=\partial_x+i\partial_y\,,\\ d_0&=-\sqrt{2}\partial_z\,,\\ d_{-1}&=-(\partial_x-i\partial_y)\end{aligned} \tag{A·7}$$

belong to the tensor operator of the lst order and, using the same procedure as before, we can construct the tensor operator of the 2nd order $d^{(2)}$. Then the operator

$$\tilde{D}=\frac{1}{20}\sum_m(-)^m t^{(2)}_{-m}d_m{}^{(2)} \tag{A·8}$$

is invariant under rotations and its explicit form is the following:

$$\frac{1}{5}\begin{pmatrix}(2\partial_z{}^2-\partial_+\partial_-) & 3\partial_z\partial_- & \sqrt{\frac{3}{2}}\partial_-{}^2 & 0 & 0\\ 3\partial_z\partial_+ & -\frac{1}{2}(2\partial_z{}^2-\partial_+\partial_-) & \sqrt{\frac{3}{2}}\partial_z\partial_- & \frac{3}{2}\partial_-{}^2 & 0\\ \sqrt{\frac{3}{2}}\partial_+{}^2 & \sqrt{\frac{3}{2}}\partial_z\partial_+ & -(2\partial_z{}^2-\partial_+\partial_-) & -\sqrt{\frac{3}{2}}\partial_z\partial_- & \sqrt{\frac{3}{2}}\partial_-{}^2\\ 0 & \frac{3}{2}\partial_+{}^2 & -\sqrt{\frac{3}{2}}\partial_z\partial_+ & -\frac{1}{2}(2\partial_z{}^2-\partial_+\partial_-) & -3\partial_z\partial_-\\ 0 & 0 & \sqrt{\frac{3}{2}}\partial_+{}^2 & -3\partial_z\partial_+ & (2\partial_z{}^2-\partial_+\partial_-)\end{pmatrix}.$$

Equation (40) can be rewritten in a manifestly invariant form as

$$F=\frac{mp_f}{16\pi^3}\int dR\Big[-\ln\frac{T_c}{T}(\Psi|I|\Psi)$$
$$+\frac{7\zeta(3)p_f^2}{48(\pi T)^2m^2}(\Psi|\left\{\frac{1}{20}\sum_m(-)^m t_{-m}^{(2)}d_m^{(2)}-\nabla^2 I\right\}|\Psi)$$
$$+\frac{21\zeta(3)}{160\pi(\pi T)^2}\left\{\frac{1}{2}(\Psi|I|\Psi)^2+\frac{7}{60}\sum_m(-)^m(\Psi|t_{-m}^{(1)}|\Psi)(\Psi|t_m^{(1)}|\Psi)\right.$$
$$\left.+\frac{1}{864}\sum_m(-)^m(\Psi|t_{-m}^{(3)}|\Psi)(\Psi|t_m^{(3)}|\Psi)\right\}\Big]. \qquad (A\cdot9)$$

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