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THE GLOBAL ATTRACTIVITY OF THE RATIONAL DIFFERENCE EQUATION $y_n = A + \left(\frac{y_{n-k}}{y_{n-m}}\right)^p$

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ABSTRACT. This paper studies the behavior of positive solutions of the recursive equation

$$y_n = A + \left(\frac{y_{n-k}}{y_{n-m}}\right)^p, \quad n = 0, 1, 2, \dots,$$

with $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \dots\}$, where $s = \max\{k, m\}$. We prove that if $\gcd(k, m) = 1$, and $p \le \min\{1, (A+1)/2\}$, then y_n tends to $A + 1$. This complements several results in the recent literature, including the main result in K. S. Berenhaut, J. D. Foley and S. Stević, The global attractivity of the rational difference equation $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$, *Proc. Amer. Math. Soc.*, 135 (2007) 1133–1140.

1. Introduction

This paper studies the behavior of positive solutions of the recursive equation

$$y_n = A + \left(\frac{y_{n-k}}{y_{n-m}}\right)^p, \ n = 0, 1, \dots, \tag{1}$$

with $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \dots\}$, where $s = \max\{k, m\}$.

The case $k = m$ is trivial, so from now on we will assume that $k \neq m$.

Note that if $g = \gcd(k, m) > 1$, then $\{y_i\}$ can be separated into g different equations of the form

$$y_n^{(j)} = A + \left(\frac{y^{(j)}_{n-\frac{k}{g}}}{y^{(j)}_{n-\frac{m}{g}}}\right)^p, \tag{2}$$

where $j \in \{1, 2, \dots, g\}$. Hence, we may assume that $\gcd(k, m) = 1$.

The study of properties of rational and nonlinear difference equations has been an area of intense interest in recent years; cf. [1]–[25] and the references therein.

There is a relatively long history in studying equation (1). For example, for $p = 1$, the case $k = 2$, $m = 1$ was studied in [2] by Amleh *et al.*, the case $k \in \mathbb{N}$, $m = 1$ was studied by DeVault *et al.* in [11], and the case $A > 1$, $k = 1$, $m \in \mathbb{N}$ was studied by Stević in [20]. The investigation of global stability and periodicity of

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positive solutions of equation (1), for the case $p = A = 1$, $k, m \in \mathbb{N}$ was completed by results in [3] and [15]; see also [17] and [21].

The study of the case $p > 1$ was suggested in [14], where the authors noted that some results from [2] for the case $p = 1$, $k = 2$, $m = 1$, can be translated to the case $p > 1$, $k = 2$, $m = 1$. The first results for the case $p < 1$ were given in [23]. The existence of monotone solutions, for the case $p > 0$ and $A > -1$ was shown in [5] by developing the technique from [6, 7, 8, 9, 10, 24] and [25]. Equations in papers [4] and [12] were investigated by transforming them into some special cases of equation (1).

The linearized equation associated with equation (1) for the case $k = 2$ and $m = 1$ is

$$(A+1)z_n + pz_{n-1} - pz_{n-2} = 0,$$

and its characteristic roots are

$$\lambda_1 = \frac{-p + \sqrt{p^2 + 4p(A+1)}}{2(A+1)} \quad \text{and} \quad \lambda_2 = \frac{-p - \sqrt{p^2 + 4p(A+1)}}{2(A+1)}.$$

By some simple calculation we obtain

$$|\lambda_1| = \frac{2p}{p + \sqrt{p^2 + 4p(A+1)}} < 1,$$

for every $p, A > 0$.

On the other hand, we have that

$$|\lambda_2| < 1 \iff 2p < A + 1.$$

Hence, when $2p < A + 1$ equation (1) for the case $k = 2$ and $m = 1$ is locally asymptotically stable by the Linearized Stability Theorem.

Motivated by this local stability result, in [22] Stević has posed the following conjecture.

Conjecture 1. *If $k = 2$, $m = 1$ and $p, A \in (0, 1)$ are such that $p < (A+1)/2$, then every positive solution of equation (1) converges to the unique equilibrium $A+1$.*

Among other results, here we confirm the conjecture by proving that the following holds true for every $k, m \in \mathbb{N}$, $0 < A < 1$ and $0 < p \leq (A+1)/2$.

Theorem 1. *Suppose that $m, k \geq 1$, and that p, A are positive numbers satisfying $0 < A < 1$ and $0 < p \leq (A+1)/2$. If the sequence $\{y_i\}$ satisfies (1) with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ where $s = \max\{m, k\}$, then, $\{y_i\}$ converges to the unique equilibrium $A + 1$.*

Remark 1 (The case k even and m odd). Note that the general characteristic equation associated with the linearized equation for equation (1) is

$$(A+1)\lambda^s + p\lambda^{s-m} - p\lambda^{s-k} = 0, \tag{3}$$

and for k even and m odd, equation (3) has a real root $\lambda_0 < -1$, when $p > (A+1)/2$. To see this, suppose that $p > (A+1)/2$, and set

$$f(\lambda) = A + 1 + \frac{p}{\lambda^m} - \frac{p}{\lambda^k}. \tag{4}$$

Now, note that $f(-1) = A + 1 - 2p < 0$, and for $\lambda < -1$,

$$f(\lambda) = A + 1 - \frac{p}{|\lambda|^m} - \frac{p}{|\lambda|^k} > A + 1 - \frac{2p}{|\lambda|^{\min\{m,k\}}} > 0, \tag{5}$$

for sufficiently large $|\lambda|$.

Hence, by the continuity of the function f on the interval $(-\infty,-1)$ it follows that $f(\lambda)=0$ for some $\lambda\in(-\infty,-1)$, as required.

Thus, by the Linearized Stability Theorem, the positive equilibrium $\bar{y}=A+1$ of equation (1) is not stable, in this case. This fact in conjunction with Theorem 1 gives a full characterization of stability for the case k even and m odd, for $A,p\in(0,1)$. □

The paper proceeds as follows. In Section 2, we introduce some preliminary lemmas and notation. Section 3 is devoted to global stability, where among other results we give a proof of Theorem 1.

2. Preliminaries and notation

In this section, we introduce some preliminary lemmas and notation.

First, consider the simple transformed sequence $\{z_i\}$ defined by $z_n=y_n-A$, for $n\geq -s$. Then, equation (1) becomes

$$z_n=\left(\frac{A+z_{n-k}}{A+z_{n-m}}\right)^p, \tag{6}$$

for $n\geq 0$.

Now, define $\{z_i^*\}$ by

$$z_i^*=\begin{cases} z_i, & \text{if } z_i\geq 1,\\ \frac{1}{z_i}, & \text{otherwise.}\end{cases} \tag{7}$$

The following elementary lemma will be useful.

Lemma 1. *If $x>1$ and $0<A<1$, then*

$$\left(\frac{A+x}{A+1/x}\right)^{\frac{A+1}{2}}\leq x, \tag{8}$$

with the inequality if and only if $x=1$, and if $x\geq 1$ and $A>1$, then the reverse inequality to inequality (8) *holds.*

Proof. Assume first that $A\in(0,1)$. Then the inequality in (8) is equivalent to

$$g_A(x)\overset{def}{=}(A+1)\ln\left(\frac{A+x}{Ax+1}\right)-(1-A)\ln x\leq 0. \tag{9}$$

Note that

$$\lim_{x\to+\infty} g_A(x)=-\infty \text{ and } g_A(1)=0. \tag{10}$$

By some simple calculations we obtain that

$$g_A'(x) \quad = \quad -\frac{A\,(x-1)^2\,(1-A)}{(A+x)\,(Ax+1)\,x}<0, \tag{11}$$

when $x\neq 1$, since $A\in(0,1)$.

Hence, $g_A(x)$ is decreasing, and thus by (10) is negative on the interval $(1,\infty)$.

Now, assume that $A>1$. Then $\lim_{x\to+\infty} g_A(x)=+\infty$ and $g_A(1)=0$. On the other hand, by (11) it follows that $g_A'(x)>0$, from which the desired inequality follows. □

Remark 2. Note that if $A=1$ the inequality in (8) becomes equality.

Next we prove a contraction lemma which will be helpful in showing convergence of solutions in the transformed space obtained through (7).

Lemma 2. *Suppose* $\{z_i\}$ *satisfies* (6) *with* $p \leq (A+1)/2$ *and* $A \in (0,1]$. *Then,*

$$1 \leq z_n^* \leq \max\{z_{n-k}^*, z_{n-m}^*\}, \tag{12}$$

for all $n \geq s$.

Proof. Suppose that $z_{n-k} > z_{n-m}$ and set $x = \max\{z_{n-k}^*, z_{n-m}^*\}$. Note that if $z_{n-k} \geq 1$, then $1 \leq z_{n-k} \leq x$ and consequently

$$1/x \leq z_{n-k} \leq x, \tag{13}$$

and if $z_{n-k} < 1$, then $1/z_{n-k} = z_{n-k}^* \leq x$ from which (13) also holds. Similarly, we have that

$$1/x \leq z_{n-m} \leq x. \tag{14}$$

Then, from (6), (13) and (14), for $n \geq s$, we have that

$$z_n^* = z_n = \left(\frac{A+z_{n-k}}{A+z_{n-m}}\right)^p \leq \left(\frac{A+z_{n-k}}{A+z_{n-m}}\right)^{\frac{A+1}{2}} \leq \left(\frac{A+x}{A+\frac{1}{x}}\right)^{\frac{A+1}{2}} \leq x, \tag{15}$$

where the final inequality in (15) follows from Lemma 1. Similarly, if $z_{n-k} \leq z_{n-m}$, then

$$z_n^* = \frac{1}{z_n} = \left(\frac{A+z_{n-m}}{A+z_{n-k}}\right)^p \leq \left(\frac{A+z_{n-m}}{A+z_{n-k}}\right)^{\frac{A+1}{2}} \leq \left(\frac{A+x}{A+\frac{1}{x}}\right)^{\frac{A+1}{2}} \leq x, \tag{16}$$

and the lemma is proved. □

Now, set

$$D_n = \max_{n-s \leq i \leq n-1}\{z_i^*\}, \tag{17}$$

for $n \geq s$.

The following result is a simple consequence of Lemma 2 and (17).

Lemma 3. *The sequence* $\{D_i\}$ *is monotonically nonincreasing in* i, *for* $i \geq s$.

Since $D_i \geq 1$ for $i \geq s$, Lemma 3 implies that, as i tends to infinity, the sequence $\{D_i\}$ converges to some limit, say D, where $D \geq 1$.

Next, we have the following lemma concerning boundedness of solutions to equation (1).

Lemma 4. *If* $p \in (0,1)$, *then every positive solution of equation* (1) *is bounded.*

Proof. First, note that each $n \in \mathbb{N}_0$ can be written in the form $lk+i$, for some $l \in \mathbb{N}_0$ and $i \in \{0,1,\dots,k-1\}$. Let $l_0 = l_0(i)$ be the smallest element of $\mathbb{N}_0$ such that $l_0k+i \geq m$. From (1) and since $y_n > A$ for every $n \geq 0$, we have that

$$y_{lk+i} = A + \frac{y_{(l-1)k+i}^p}{y_{lk+i-m}^p} < A + \frac{y_{(l-1)k+i}^p}{A^p}, \tag{18}$$

for every $l \in \mathbb{N}_0$ and $i \in \{0,1,\dots,k-1\}$ such that $lk+i \geq m$. Let $(u_l^{(i)})_{l\in\mathbb{N}}$ be the solution of the difference equation

$$u_l^{(i)} = A + \frac{u_{l-1}^{(i)}}{A^p}, \qquad u_{l_0}^{(i)} = y_{i(l_0-1)+i}.$$

By (18) and induction we see that $y_{(l-1)k+i} \le u_l^{(i)}$, $l \ge l_0$. Hence it is enough to prove that the sequences $(u_l^{(i)})_{l \ge l_0}$, $i \in \{0, 1, \dots, k-1\}$, are bounded.

Since the function

$$f(x) = A + \frac{x^p}{A^p}, \quad x \in (0, \infty),$$

is increasing and concave for $p \in (0,1)$ it follows that there is a unique fixed point $\bar{x}$ of the equation $f(x) = x$ and that the function f satisfies the condition

$$(f(x) - x)(x - \bar{x}) < 0, \quad x \in (0, \infty).$$

Using this fact it is easy to see that if $u_l^{(i)} \in (0, \bar{x}]$ the sequence is nondecreasing and bounded above by $\bar{x}$ and if $u_l^{(i)} \ge \bar{x}$, it is nonincreasing and bounded below by $\bar{x}$. Hence for every $u_{l_0}^{(i)} \in (0, \infty)$, each of the sequences $u_l^{(i)}$, $i \in \{0, 1, \dots, k-1\}$, is bounded, from which the result follows. □

3. Convergence of solutions to equation (1)

In this section, we study the global attractivity of the positive solutions of equation (1). First, we give a proof of Theorem 1.

Proof of Theorem 1. Note that it suffices to show that the transformed sequence $\{z_i^*\}$ converges to 1.

By the definition in (17), the values of D_i are taken on by entries in the sequence $\{z_j^*\}$, and as well, by Lemma 2, $z_i^* \in [1, D_i]$ for $i \ge s$. Now, for any $\epsilon > 0$, we can find an N such that

$$z_N^* \in [D, D + \epsilon], \tag{19}$$

and for $i \ge N - s$,

$$z_i^* \in [1, D + \epsilon]. \tag{20}$$

Note that, similar to (13) and (14), (20) implies that

$$\frac{1}{D + \epsilon} \le z_{N-m}, z_{N-k} \le D + \epsilon. \tag{21}$$

We will show that $D = 1$, and from this, (7), (17) and the definition of D, the result follows.

Now, suppose $D > 1$, and note that (19) implies that $z_N \ne 1$.

First, consider the case $z_N > 1$. Then, from (19), we have that

$$z_N = z_N^* \in [D, D + \epsilon]. \tag{22}$$

Solving for z_{n-k} in (6), and employing (22) and (21), gives

$$\begin{aligned} D + \epsilon \quad &\ge \quad z_{N-k} = z_N^{1/p}\left(A + z_{N-m}\right) - A \\ &\ge \quad D^{1/p}\left(A + \frac{1}{D + \epsilon}\right) - A \\ &\ge \quad D^{\frac{2}{A+1}}\left(A + \frac{1}{D + \epsilon}\right) - A. \end{aligned} \tag{23}$$

This implies that

$$\left(\frac{A + D + \epsilon}{A + \frac{1}{D+\epsilon}}\right) \ge D^{\frac{2}{A+1}}. \tag{24}$$

Assume now that $z_N < 1$. Then,

$$\frac{1}{z_N} = z_N^* \in [D, D+\epsilon]. \tag{25}$$

From (6), and employing (21) and (25), it follows that

$$\begin{aligned} D+\epsilon &\geq z_{N-m} = (z_N^*)^{1/p}\,(A + z_{N-k}) - A \\ &\geq D^{1/p}\left(A + \frac{1}{D+\epsilon}\right) - A \\ &\geq D^{\frac{2}{A+1}}\left(A + \frac{1}{D+\epsilon}\right) - A. \end{aligned} \tag{26}$$

From (26) we have that (24) holds in this case, as well. Since $\epsilon > 0$ was arbitrary and $D > 1$, by Lemma 1 we arrive at a contradiction, which implies that $D = 1$, and the theorem follows. □

Remark 3. Note that the above argument breaks down when $p > (A+1)/2$. In particular, we have that for $p > (A+1)/2$,

$$\left(\frac{A+x}{A+1/x}\right)^p > x, \tag{27}$$

for $x = 1+\epsilon$, for sufficiently small $\epsilon > 0$. To see this, similar to (9), set

$$h_{A,p}(x) = p\big(\ln(A+x) - \ln(Ax+1)\big) - (1-p)\ln x, \tag{28}$$

and note that the condition $h_{A,p}(x) > 0$ is equivalent to (27). Now, $h_{A,p}(1) = 0$ and

$$h'_{A,p}(1) = \frac{2p - (A+1)}{(A+1)} > 0. \tag{29}$$

Hence for sufficiently small $\epsilon > 0$, $h_{A,p}(1+\epsilon) > 0$.

The next theorem is devoted to the case $p \in (0,1)$ and $A \geq 1$. It is simpler than Theorem 1 and is essentially a consequence of the boundedness result in Lemma 4.

Theorem 2. *Suppose that $m, k \geq 1$, $p \in (0,1)$ and $A \geq 1$. If the sequence $\{y_i\}$ satisfies* (1) *with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0,\infty)$ where $s = \max\{m,k\}$, then, $\{y_i\}$ converges to the unique equilibrium $A+1$.*

Proof. By Lemma 4, every solution $\{y_n\}$ of equation (1) is bounded which implies that there are finite $\liminf y_n = \lambda$ and $\limsup y_n = \Lambda$. Assume to the contrary that $\lambda < \Lambda$. Taking the lim inf and lim sup in (1) it follows that

$$A + \frac{\lambda^p}{\Lambda^p} \leq \lambda < \Lambda \leq A + \frac{\Lambda^p}{\lambda^p}.$$

From this and since $p \in (0,1)$, it follows that

$$A\Lambda^p + \lambda^p \leq \Lambda^p\lambda < \Lambda\lambda^p \leq A\lambda^p + \Lambda^p;$$

i.e.,

$$(A-1)\Lambda^p < (A-1)\lambda^p.$$

Since $A \geq 1$, this is impossible. Therefore we have that $\lambda = \Lambda$, which implies the result. □

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