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The Global Structure of Traveling Waves in Spatially Discrete Dynamical Systems

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We obtain existence of traveling wave solutions for a class of spatially discrete systems, namely, lattice differential equations. Uniqueness of the wave speed c, and uniqueness of the solution with $c \neq 0$, are also shown. More generally, the global structure of the set of all traveling wave solutions is shown to be a smooth manifold where $c \neq 0$. Convergence results for solutions are obtained at the singular perturbation limit $c \to 0$.

KEY WORDS: Traveling waves; spatially discrete systems; lattice differential equations; continuation methods; heteroclinic orbits; Lin's method; Mel'nikov method.

1. INTRODUCTION

We are interested in lattice differential equations, namely, infinite systems of ordinary differential equations indexed by points on a spatial lattice, such as the D-dimensional integer lattice $\mathbf{Z}^D$. Our focus in this paper is the global structure of the set of traveling wave solutions for such systems. This entails results on existence and uniqueness, and on continuous (or smooth) dependence of traveling waves and their speeds on parameters, as well as some delicate convergence results in the singular perturbation case $c \to 0$ of the wave speed tending to zero. We believe such results generally to be crucial in the analysis and understanding of many problems, such as those involving non-planar waves (with wave speed and direction varying throughout the lattice), and also problems in which the characteristics of the system, such as the coupling parameters, vary from point to point in the lattice.

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Lattice differential equations arise in a wide variety of models of systems in which the spatial structure has a discrete character. Models occur, for example, in chemical reaction theory [39, 55], in image processing and pattern recognition [27–29, 42, 77, 86], in material science [13, 30, 48], and in biology [10, 11, 36, 37, 49, 50, 53, 87–89]. Much is already known about chains of coupled oscillators, such as can arise in biology or electronics (Josephson junctions); see, for example, Refs. 5, 6, 36, 37, 39, 52–54, 68, 70–72, 90–92 and additional references in these papers. See also the papers [61, 65, 66], and [67]. For papers on the closely related subject of coupled-map lattices, see, for example, Refs. 1, 2, 26, and 83.

One source of inspiration for us is the above-referenced works of L. Chua and M. Hasler and their co-workers on image processing. They consider the equations of a so-called **Cellular Neural Network** (CNN), which are related in spirit to systems such as (1.9) below. Their work combines both numerical simulations and experimental results. We are also strongly motivated by the numerical simulations of lattice differential equations by Cahn *et al.* [14], in which moving interfaces between spatial patterns are clearly visible. The present work can be thought of, in part, as an attempt to provide a theoretical framework for these experimental and numerical results. We mention several general surveys [21, 25, 62, 64], in which some of the results here are outlined. Other recent developments in lattice differential equations are found, for example, in Refs. 15 and 22–24.

Traveling waves have been extensively studied for partial differential equations. The reaction diffusion equation

$$u_t = \Delta u - f(u), \qquad u = u(t, \zeta), \qquad \zeta \in \mathbf{R}^D \tag{1.1}$$

was proposed in 1937 by Sir Ronald Fisher [43] to describe the spread of an advantageous gene population in an infinite region. Rigorous results were obtained for this equation by Kolmogoroff *et al.* [51] in the same year. The scalar variable u is constrained to a bounded interval, which we normalize to $[-1, 1]$. The conditions $f(-1) = f(1) = 0$ which are imposed ensure that this constraint is maintained for solutions. Traveling waves are solutions of the form $u(t, \zeta) = x(v \cdot \zeta - ct)$ with $x: \mathbf{R} \to \mathbf{R}$, and where the given vector $v \in \mathbf{R}^D$ with $|v| = 1$ denotes the direction of motion of the wave. Here $c \in \mathbf{R}$ is the unknown wave speed to be determined as part of the solution. Setting $\xi = v \cdot \zeta - ct$ leads to the second-order ordinary differential equation

$$-cx'(\xi) = x''(\xi) - f(x(\xi)), \qquad \xi \in \mathbf{R} \tag{1.2}$$

for which one typically imposes boundary conditions at $\pm\infty$, for example,

$$\lim_{\xi \to -\infty} x(\xi) = -1, \qquad \lim_{\xi \to \infty} x(\xi) = 1 \tag{1.3}$$

if one seeks a wave which joins the two equilibrium states $u = \pm 1$. Note that Eq. (1.2) is independent of v.

Aronson and Weinberger [7] distinguished three cases of interest based on the form of the nonlinearity $f: \mathbf{R} \to \mathbf{R}$, namely,

Heterozygote intermediate: $\quad f(u) > 0, \quad -1 < u < 1$

Heterozygote superior: $\quad \begin{cases} f(u) < 0, & -1 < u < q \\ f(u) > 0, & q < u < 1 \end{cases}$

for some $\quad q \in (-1, 1)$

Heterozygote inferior: $\quad \begin{cases} f(u) > 0, & -1 < u < q \\ f(u) < 0, & q < u < 1 \end{cases}$

for some $\quad q \in (-1, 1)$

[A nondegeneracy condition, that $f'(u) \neq 0$ at certain points where $f(u) = 0$, is also assumed. We note here that our normalizations above differ slightly from those of Ref. 7, but this is mostly a matter of notation.] In the heterozygote intermediate case they established the existence of $c^* > 0$ such that there exists for every $c \geqslant c^*$, but not for any $c < c^*$, a monotone solution to Eq. (1.2) with (1.3). For the heterozygote superior case they established a similar result, except that the first condition in (1.3) is replaced by $\lim_{\xi \to -\infty} x(\xi) = q$ (in fact this case easily reduces to the heterozygote intermediate case by a simple change of variables taking the interval $q \leqslant u \leqslant 1$ to the interval $(-1 \leqslant u \leqslant 1)$. Finally, for the heterozygote inferior case they proved under the additional assumption

$$\int_{-1}^{1} f(u)\, du > 0$$

that there exists a canonically defined $c^* > 0$ such that (1.2), (1.3), has a monotone solution for $c = c^*$. They showed, further, in all these cases, that c^* was the asymptotic speed of propagation of a broad class of disturbances from $u = 1$.

From a dynamical systems point of view, the boundary conditions (1.3) in the heterozygote intermediate case correspond, when $c > 0$, to finding a heteroclinic trajectory joining a saddle (at $u = -1$) to a stable node or stable spiral (at $u=1$). In this case it is not unexpected that there exists an interval of values $c \geqslant c^*$ for which there is a monotone connection. The same is true for the connection from $u=q$ to $u=1$ in the heterozygote superior case. On the other hand, the conditions (1.3) in the heterozygote inferior case correspond to joining two saddles, at $u=\pm 1$, and here one expects this to occur only for isolated values $c=c^*$. An added difficulty in the heterozygote inferior case is the presence of the third equilibrium $u=q$ between the two $u=\pm 1$ which are being joined. In fact, in the present paper, which deals with an infinite-dimensional analog of the heterozygote inferior case, overcoming the difficulties presented by the third equilibrium is a significant task.

Weinberger [87, 88], developed an abstract and very general framework for studying a large class of problems primarily of heterozygote intermediate and superior type. His setting was rich enough to include both partial differential equations as well as higher-dimensional lattice differential equations such as (1.5) below, and both continuous- and discrete-time problems, and his results included precise statements on the existence of direction-dependent traveling waves and the asymptotic shape of propagation of initial disturbances. See also the papers [58, 59], in this spirit. Other results on the propagation of initial disturbances for various equations of heterozygote intermediate type are given in Refs. 4, 33, 76, 82, 84, and 85. One finds further results on traveling waves for problems of heterozygote intermediate type in Refs. 8, 32, 44, 74, 75, and 95.

The classic work of Fife and McLeod [40], on the other hand, is concerned with the heterozygote inferior case of Eq. (1.1) in dimension $D=1$. It too contains precise results on the propagation of initial disturbances.

Our interest in this paper is with the lattice differential equation analog of the heterozygote inferior case. This is also called the **bistable** case in view of the fact that for the associated ordinary differential equation $\dot{u}=-f(u)$ both the equilibria $u=\pm 1$ are stable, the equilibrium $u=q$ being unstable. Perhaps the simplest such nonlinearity is the cubic polynomial

$$f(u)=(u-q)(u^2-1), \qquad -1<q<1 \tag{1.4}$$

To illustrate the problems of interest here, consider the infinite system of ordinary differential equations

$$\dot{u}_{i,j}=\alpha(u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j})-f(u_{i,j}), \qquad (i,j)\in \mathbf{Z}^2 \tag{1.5}$$

on the lattice $\mathbf{Z}^2$. Here α is a real parameter (the coupling parameter) multiplying a discrete Laplacian, with typically a bistable nonlinearity such as (1.4). Let $\theta \in \mathbf{R}$ be given, and consider solutions of the system (1.5) of the form

$$u_{i,j}(t) = x(i \cos \theta + j \sin \theta - ct) \tag{1.6}$$

for some unknown function $x\colon \mathbf{R} \to \mathbf{R}$, and some unknown real number c. A solution of the form (1.6) of the system (1.5) is called a **traveling wave solution** of (1.5). The wave speed c, as before to be determined as part of the solution, can be either zero or nonzero. The parameter θ, which is given, represents the direction of motion of the wave, and so $v = (\cos \theta, \sin \theta) \in \mathbf{R}^2$ in the above notation. The so-called detuning parameter q, in the nonlinearity (1.4), is also given. We again seek a wave satisfying (1.3), that is, joining the equilibria $u = \pm 1$.

For the one-dimensional case [equivalently, with $\theta = 0$ in (1.6)], the existence of a traveling wave solution (1.6) satisfying (1.3), to a class of equations of heterozygote inferior type including (1.5), was shown by Zinner [94] and by Hankerson and Zinner [47] (see also Ref. 93). Their methods, however, rely on nonconstructive fixed-point theorems. By contrast, we employ a continuation (homotopy) method and obtain precise information on uniqueness and dependence of solutions on parameters. Specifically, a general Fredholm alternative [63] for linearized traveling wave equations places a large class of such problems within the framework of classical bifurcation theory. We then construct a homotopy between the given system, such as (1.5), and a model system for which the result is known. This approach was taken by the author with Chow and Diekmann [18] and with Chow and Lin [20] for a classes of integral and delay differential equations, and more recently by others including Ermentrout and McLeod [38] and Bated *et al.* [9]. The very recent paper by Carpio *et al.* [16] also uses a homotopy approach for a special case (linear symmetric couplings) of the problem we consider here. While some of their proofs are simpler than ours, their method does not extend to the general class of nonlinearities considered here.

Traveling waves have been studied for other nonlocal evolution equations of bistable type [9, 17, 31, 41], for spatially varying systems [3], and in the context of numerical discretizations [12], although generally the nonlocal character of the systems is spatially distributed (modeled by a smooth convolution) rather than by point masses located on a lattice. The lattice structure in our treatment leads to additional difficulties and subtleties. These manifest themselves as real parts of eigenvalues clustering at zero, $\operatorname{Re} \lambda_n \to 0$, for the linearized traveling wave equation around an equilibrium for the wave speed $c = 0$. The resulting lack of uniform hyperbolicity necessitates great care in obtaining the required asymptotic estimates on the solutions, and also in proving convergence of the solutions in the singular perturbation limit $c \to 0$.

Issues connected with the commensurability of the wave motion with the lattice (that is, whether the direction of wave motion has rational or irrational slope with respect to the lattice) can also arise. Some of the more subtle issues in this connection are more fully explored in Ref. 15.

Upon substitution of (1.6) into (1.5), we obtain

$$\begin{aligned} -cx'(\xi) = \alpha(x(\xi+\cos\theta) + x(\xi-\cos\theta) + x(\xi+\sin\theta) \\ + x(\xi-\sin\theta) - 4x(\xi)) - f(x(\xi)) \end{aligned} \tag{1.7}$$

which if $c \neq 0$ is a functional differential equation of mixed type, where "mixed" refers to the fact that the equation involves both forward and backward translates of the argument of the solution x. In contrast to the case of the partial differential equation above, Eq. (1.7) depends on the direction parameter θ, reflecting the anisotropy of the lattice. Except for early work of Rustichini [79, 80], not much of a general nature is known about such mixed functional differential equations.

The limit $c \to 0$ in Eq. (1.7) results in a singular perturbation problem. Indeed, when $c = 0$ Eq. (1.7) is in fact a difference equation, not a differential equation. In this case, the solution need only be defined for ξ in the set $\mathscr{X} \subseteq \mathbf{R}$ given by

$$\mathscr{X} = \{ i\cos\theta + j\sin\theta \mid (i, j) \in \mathbf{Z}^2 \}$$

The set $\mathscr{X}$ is either a discrete subset, or a countable dense subset of $\mathbf{R}$, depending on whether the quantity $\tan\theta$ is rational or irrational. Moreover, if $\tan\theta$ is rational, then Eq. (1.7) with $c = 0$ is equivalent to a mapping of finite (but possibly high) dimension. For example, if $\theta = 0$, then by denoting $x_n = x(n)$, this system can be written as

$$x_{n+1} = 2x_n - x_{n-1} + \alpha^{-1} f(x_n) \tag{1.8}$$

a map of the plane. When $\tan\theta$ is irrational, however, with $c = 0$, no such reduction of (1.7) to a finite-dimensional map seems possible, and indeed, in this case Eq. (1.7) seems most challenging.

More general lattice systems can be considered, for example,

$$\dot{u}_{i,j} = \sum_{k=2}^{N} \alpha_k (u_{i+a_k, j+b_k} - u_{i,j}) - f(u_{i,j}), \qquad (i, j) \in \mathbf{Z}^2 \tag{1.9}$$

where $a_k, b_k \in \mathbf{Z}$ are given integers, and $\alpha_k \in \mathbf{R}$ are given coupling coefficients, for $2 \leqslant k \leqslant N$ (beginning the sum at $k = 2$ leads to more consistent notation below). In the system (1.9) the strengths of the couplings therefore depend on the relative placements of the two lattice points $(i, j) \in \mathbf{Z}^2$ and $(i+a_k, j+b_k) \in \mathbf{Z}^2$. With the Ansatz (1.6), Eq. (1.9) leads to the system

$$\begin{gathered} -cx'(\xi) = \sum_{k=2}^{N} \alpha_k(x(\xi + r_k) - x(\xi)) - f(x(\xi)) \\ r_k = a_k \cos\theta + b_k \sin\theta, \qquad 2 \leqslant k \leqslant N \end{gathered} \tag{1.10}$$

which we may write in the general form

$$-cx'(\xi) = F(x(\xi + r_1), x(\xi + r_2),..., (\xi + r_N)) \tag{1.11}$$

with the convention that $r_1 = 0$ and with c the wave speed as before. In fact, Eq. (1.11) arises from many systems besides (1.9), for example from those with nonlinear couplings between lattice sites. Other lattices, such as the hexagonal lattice in the plane, the crystallographic lattices lattices in $\mathbf{R}^3$, and the integer lattice $\mathbf{Z}^D \subseteq \mathbf{R}^D$ in arbitrary dimensions, can be considered, and again one arrives at an equation of the form (1.11).

In this paper we shall develop some general theory and tools to analyze a broad class of functional differential equations of mixed type, including many of the form (1.11). In our treatment, we employ a parameter ρ in the nonlinearity F, as we are concerned with how solutions vary. In addition, ρ functions as a homotopy parameter in our existence proof.

Let us outline several principal results we prove for (1.11). We shall always take $r_1 = 0$, while $r_j \neq 0$ for $j \neq 1$. Under the basic assumption that the nonlinearity $F(u_1, u_2,..., u_N)$ is increasing in the "shifted" variables,

$$\frac{\partial F}{\partial u_j} > 0, \qquad 2 \leqslant j \leqslant N$$

[which for (1.9) means that each α_k is positive], and that $\Phi(x) = F(x, x,..., x)$ is such that $-\Phi$ is an N-shaped function, such as the cubic f in (1.4), or generally of heterozygote inferior type, we have the following for Eq. (1.11) with the boundary conditions (1.3).

- Existence of $c \in \mathbf{R}$ and of a monotone solution $x = P(\xi)$ (Theorem 2.1).
- Uniqueness of c among the class of monotone solutions (Theorem 2.1 and Proposition 6.5).
- Uniqueness of the solution $P(\xi)$ when $c \neq 0$ (Theorem 2.1).

- Continuous dependence of c on system parameters (such as q in (1.4)), and smooth dependence of c and $P(\xi)$ on parameters when $c \neq 0$ (Theorem 2.1 and Proposition 6.4).
- A convergence result for $P(\xi)$ for the case of the singular perturbation limit $c \to 0$ (Theorem 2.3).
- Monotone increasing dependence of c on the detuning parameter q for a class of systems including (1.10), with (1.4), with strict monotonicity when $c \neq 0$ (Proposition 2.4).
- Sufficient conditions (so-called coercivity at ± 1) for $c > 0$ and for $c < 0$. In particular, for (1.10), with (1.4), we have $c > 0$ for q sufficiently near $+1$ if

$$\sum_{k=2}^{N} r_k \alpha_k \leqslant 0 \tag{1.12}$$

while $c < 0$ for q near -1 if the opposite inequality in (1.12) holds (Theorem 2.6).
- An asymptotic description of $P(\xi)$ as $\xi \to \pm\infty$, both for $c \neq 0$ and $c = 0$, including the difficult case of shifts r_j which are not rationally related (Theorem 2.2).

One also expects that generically the following will hold.

- Nonuniqueness of the solution $P(\xi)$ when $c = 0$.
- The existence of a nontrivial interval

$$q_- \leqslant q \leqslant q_+ \tag{1.13}$$

of the detuning parameter q in which $c = 0$ identically (so-called **propagation failure**).

In fact, the last two properties can easily be established in great generality. For $\theta = 0$ (or more generally for rational values of $\tan \theta$), Eq. (1.11) with $c = 0$ reduces to a finite-dimensional map such as (1.8). As with (1.8) and the nonlinearity (1.4), the unstable manifolds of the two equilibria ± 1 have the same dimension, and generically the stable and unstable manifolds $W^s(1)$ and $W^u(-1)$ intersect transversely. Such an intersection is stable under perturbations of the nonlinearity, resulting thereby in propagation failure, and quite generally there are an even number of distinct orbits passing monotonically from -1 to $+1$, yielding nonuniqueness of the solution. In fact, the range (1.13) of propagation failure in Eq. (1.7) with (1.4) satisfies $q_\pm \to \pm 1$ as $\alpha \to 0$, as was shown by MacKay and Sepulchre [60] using an implicit function technique.

For early work on propagation failure see Keener [49, 50], and for its appearance in both theory and applications, see for example Refs. 9, 55, 60, and 72. Recently, with Cahn and Van Vleck [15], we have completed a study both of the existence and uniqueness of traveling waves, and as well propagation failure, for the system (1.5) with the idealized nonlinearity

$$f(u) = \begin{cases} u+1, & u < q \\ u-1, & u > q \end{cases} \tag{1.14}$$

The function f in (1.14) is a cartoon of the cubic polynomial (1.4), and the piecewise linear nature of (1.14) allows for very explicit calculations to be made. In particular, a detailed study of the relation between the direction parameter θ and the onset of propagation failure was made in Ref. 15. Let us also mention the extensive numerical calculations of traveling wave solutions of Elmer and Van Vleck [34], [35], in this direction.

Much of our analysis concerns the linear equation

$$-cx'(\xi) = \sum_{j=1}^{N} A_j(\xi)\, x(\xi + r_j) \tag{1.15}$$

for $c \neq 0$, and the related linear operator Λ given by

$$(\Lambda x)(\xi) = -cx'(\xi) - \sum_{j=1}^{N} A_j(\xi)\, x(\xi + r_j)$$

In particular, Eq. (1.15) arises as the linearization of (1.11) around specific solutions. For the system (1.15) with bounded coefficients A_j we have the following results.

- The Fredholm alternative, namely that Λ is a Fredholm operator from $W^{1,p}$ to L^p, with a characterization of the range of Λ in terms of the adjoint operator, provided that (1.15) is asymptotically hyperbolic (Theorem A of Ref. 63).
- Construction of the Green's function for Λ in the case of a constant coefficient hyperbolic system (Theorem 4.1 of Ref. 63).
- The one-dimensionality of the kernel of Λ and the cokernel of Λ (the complement of the range) in the case of a scalar Eq. (1.15), when the shifted coefficients are uniformly positive

$$\inf_{\xi \in \mathbf{R}} A_j(\xi) > 0, \qquad 2 \leqslant j \leqslant N$$

and when the kernel of Λ contains a nonnegative element, that is, when Eq. (1.15) has a nontrivial bounded nonnegative solution (Theorem 4.1 of the present paper).

The results of Theorem A of Ref. 63 are related, at least in spirit, to the theory of exponential dichotomies (see, for example, Refs. 45, 73, 81, and the references therein), and they allow one to analyze heteroclinic solutions of the nonlinear Eq. (1.11) using a Lyapunov–Schmidt approach. In fact, under monotonicity conditions on the function F as above, with Theorem 4.1, such an approach will be used to construct a global continuation which is the basis of our proof of existence of traveling waves.

While the above-mentioned Lyapunov–Schmidt approach for studying heteroclinic orbits sometimes has the name Mel'nikov associated with it, it is in fact quite distinct from, and technically simpler than, the so-called Mel'nikov method [69]. The latter seeks to detect heteroclinic orbits by directly measuring the separation between the stable and unstable manifolds of two equilibria. By contrast, our Lyapunov–Schmidt approach examines candidates $x\colon \mathbf{R} \to \mathbf{R}$ for heteroclinic solutions of (1.11), using the function

$$h(\xi) = cx'(\xi) + F(x(\xi + r_1), x(\xi + r_2),..., x(\xi + r_N))$$

as a measure of the degree to which x fails to satisfy (1.11). By means of a Lyapunov–Schmidt reduction only a finite number of functionals of h are needed to determine whether the candidate x is a solution, and so the problem of determining heteroclinic orbits reduces to the analysis of a finite-dimensional bifurcation equation.

The advantage of this Lyapunov–Schmidt approach over the Mel'nikov method is that it does not rely on the existence of stable or unstable manifolds for the differential equation. Indeed, their existence is not know for the Eq. (1.11), which does not generate a dynamical system, and one would expect here that both the stable and unstable manifolds would be infinite-dimensional if they do exist. Generally, the Lyapunov–Schmidt method applies to a broader class of equations than does the Mel'nikov method. Such a Lyapunov–Schmidt approach was first employed in Ref. 19 to study both homoclinic and subharmonic periodic solutions of Duffing's equation. Subsequently, X.-B. Lin developed and applied the method extensively (see, for example, Ref. 57), and it has sometimes been known as "Lin's method."

Before closing this section, let us construct, in an explicit form, a traveling wave solution for a particular nonlinearity. This solution will serve as the starting point for the homotopy used to prove the existence of traveling waves for general nonlinearities. Let $k>0$ be fixed and for any $q \in [-1, 1]$ let

$$f(u, q) = \frac{\gamma(u-q)(u^2-1)}{1-\gamma u}, \qquad \gamma = \tanh k \tag{1.16}$$

at least for $|u| \leqslant 1$. For u outside this range modify the function f to be C^1 and nonzero, and also so that the derivative $D_1 f(u, q)$ with respect to the first argument u is locally lipschitz in u [this technical condition will ensure that condition (i) in Section 2 is fulfilled]. Note that the nonlinearity (1.16) is qualitatively similar to the bistable nonlinearity (1.4) with the detuning parameter q. Consider now the system

$$-cx'(\xi) = \gamma^{-1}(x(\xi-k) - x(\xi)) - f(x(\xi), q) \tag{1.17}$$

which is of the form (1.11) with $r_1 = 0$ and $r_2 = -k$. We claim that for $c = 1$ and $q = 0$ the function

$$P(\xi) = \tanh \xi, \qquad \xi \in \mathbf{R} \tag{1.18}$$

is a solution to (1.17). To prove this claim we first note that

$$P'(\xi) = \operatorname{sech}^2 \xi = 1 - \tanh^2 \xi = 1 - P(\xi)^2$$

$$P(\xi - k) = \tanh(\xi - k) = \frac{\tanh \xi - \tanh k}{1 - \tanh \xi \tanh k} = \frac{P(xi) - \gamma}{1 - \gamma P(\xi)}$$

We then have that

$$\begin{aligned}
&P'(\xi) + \gamma^{-1}(P(\xi-k) - P(\xi)) \\
&\quad = 1 - P(\xi)^2 + \gamma^{-1}\left(\frac{P(\xi) - \gamma - P(\xi) + \gamma P(\xi)^2}{1 - \gamma P(\xi)}\right) \\
&\quad = 1 - P(\xi)^2 + \frac{P(\xi)^2 - 1}{1 - \gamma P(\xi)} = \frac{(P(\xi)^2 - 1)(-1 + \gamma P(\xi) + 1)}{1 - \gamma P(\xi)} \\
&\quad = \frac{\gamma P(\xi)(P(\xi)^2 - 1)}{1 - \gamma P(\xi)} = f(P(\xi), 0)
\end{aligned}$$

which proves the claim.

The system (1.17) with (1.16), and the solution (1.18), will play a key role in the proof of Theorem 2.6 in Section 8. Specifically, we shall construct a homotopy between the parameterized family (1.17) and a general family (2.1) as given in the statement of that theorem. In so doing, we will be able to homotopically continue solutions from the system (1.17) to the general system (2.1). Let us remark here that the system (1.17) with the nonlinearity (1.16) parameterized by q is a so-called normal family, as defined in the following section. Very roughly, this means that $f(u, q) = 0$ at $u = \pm 1$ and $u = q$, and that f varies monotonically with q, just as for the cubic nonlinearity (1.4).

The present paper is organized as follows. Section 2 is devoted to statements of the main results, along with the basic assumptions on the nonlinear Eq. (1.11). Section 3 presents several technical results, including comparison principles which will be used repeatedly in proving later results. Sections 4 and 5 deal with linear Eqs. (1.15), such as would arise as the linearization about a traveling wave solution. The differential equation case $c \neq 0$ is handled in Section 4, where in Theorem 4.1 the basic Fredholm alternative result needed for the local (Lyapunov–Schmidt/Lin's method) continuation of a solution is proved. The case $c = 0$ of a difference equation is handled in Section 5. There we obtain preliminary results toward an asymptotic description of the solution. The arguments here are particularly delicate in the case of irrationally related shifts r_j. In Section 6 we return to the nonlinear Eq. (1.11), and prove uniqueness of the wave speed c, and also of the solution when $c \neq 0$. This requires the detailed information about asymptotics of solutions given in Theorem 2.2, which is also proved there. Smooth dependence of the wave on parameters, plus some monotonicity properties (including Theorem 2.4) are also proved. Section 7 deals with the singular perturbation limit $c \to 0$. Theorem 2.3, which describes the convergence of solutions under this limit, is proved there. Finally, in Section 8, the remainder of the proof of the main existence result, Theorem 2.1, is given. As well, Theorem 2.6, which gives sufficient conditions for $c \neq 0$, is proved.

2. MAIN RESULTS

Our principal interest in this paper is the nonlinear autonomous equation

$$-cx'(\xi) = F(x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N), \rho) \tag{2.1}$$

Here $\rho \in \bar{V} \subseteq X$ is a parameter, where $\bar{V}$ is the closure of an open subset V of some Banach space X. We assume that the following conditions are satisfied for $F(u, \rho)$.

(i) $F: \mathbf{R}^N \times \bar{V} \to \mathbf{R}$ is C^1. Also, its derivative $D_1 F: \mathbf{R}^N \times \bar{V} \to \mathbf{R}^N$ with respect to the first argument $u \in \mathbf{R}^N$ is locally lipschitz in u.

(ii) For each $\rho \in \bar{V}$ there exists a nonempty set

$$\mathscr{U}(\rho) \subseteq \{2, 3, 4, ..., N\}$$

of indices such that

$$\frac{\partial F(u, \rho)}{\partial u_j} > 0, \qquad u \in \mathbf{R}^N, \qquad j \in \mathscr{U}(\rho)$$

$F(u, \rho)$ is independent of u_j if $2 \leqslant j \leqslant N$ and $j \notin \mathscr{U}(\rho)$

(iii) The quantities r_j, the so-called **shifts**, satisfy

$$r_1 = 0, \qquad r_j \neq r_k, \qquad 1 \leqslant j < k \leqslant N, \qquad N \geqslant 2$$

and hence $r_j \neq 0$ for $2 \leqslant j \leqslant N$.

(iv) Defining Φ: $\mathbf{R} \times \bar{V} \to \mathbf{R}$ by

$$\Phi(x, \rho) = F(x, x, ..., x, \rho)$$

we have for some quantity $q = q(\rho) \in [-1, 1]$ that

$$\Phi(x, \rho) > 0, \qquad x \in (-\infty, -1) \cup (q, 1)$$
$$\Phi(x, \rho) < 0, \qquad x \in (-1, q) \cup (1, \infty)$$
$$\Phi(-1, \rho) = \Phi(q, \rho) = \Phi(1, \rho) = 0$$

where $q(\rho) \in (-1, 1)$ if $\rho \in V$.

(v) We have for $q = q(\rho)$ that

$$D_1 \Phi(-1, \rho) < 0 \qquad \text{if} \quad q \neq -1$$
$$D_1 \Phi(q, \rho) > 0 \qquad \text{if} \quad q \in (-1, 1)$$
$$D_1 \Phi(1, \rho) < 0 \qquad \text{if} \quad q \neq 1$$

with D_1 denoting the derivative with respect to the first argument $x \in \mathbf{R}$.

As it is convenient to assume that the set of shifts r_j which appear nontrivially in Eq. (2.1) may depend on the parameter ρ, we employ the set $\mathscr{U}(\rho)$ in (ii). In general, the nonlinearity F will always depend on the variable u_1, although not generally in a monotone fashion. As a technical matter, we choose to exclude $j = 1$ from the set $\mathscr{U}(\rho)$. Let us denote

$$r_{\min}(\rho) = \min_{j \in \{1\} \cup \mathscr{U}(\rho)} r_j, \qquad r_{\max}(\rho) = \max_{j \in \{1\} \cup \mathscr{U}(\rho)} r_j$$

for each $\rho \in \bar{V}$, and observe that

$$r_{\min}(\rho) \leqslant 0 \leqslant r_{\max}(\rho), \qquad r_{\min}(\rho) < r_{\max}(\rho) \tag{2.2}$$

as $r_1 = 0$ and $\mathscr{U}(\rho) \neq \phi$. Note from (v) that $q(\rho)$ is C^1 in ρ whenever $q(\rho) \in (-1, 1)$, by an application of the implicit function theorem. For convenience let us denote

$$W = \{\rho \in \bar{V} \mid -1 < q(\rho) < 1\}$$

We see that $V \subseteq W \subseteq \bar{V}$ and that W is a relatively open subset of $\bar{V}$. Our primary interest is with $\rho \in W$, although as a technical convenience we sometimes allow ρ to take values in the closure $\bar{V}$. The cases $q(\rho) = \pm 1$, which occur for $\rho \in \bar{V} \backslash W$, can often be studied by various limiting arguments.

We shall take the above conditions (i)–(v) as standing assumptions throughout this section. They will also be taken as standing assumptions in some of the later sections, and this will be noted at the beginning of those sections. We shall let the parameter $c \in \mathbf{R}$ vary, along with ρ, although we keep the shifts r_j fixed.

By (iv), for each $\rho \in \bar{V}$ Eq. (2.1) has the constant (equilibrium) solutions $x = \pm 1$ and $x = q(\rho)$, and no others, and these are distinct when $\rho \in W$. We are interested in obtaining, and studying, heteroclinic solutions of (2.1) joining -1 to $+1$ for some c, that is, solutions $x: \mathbf{R} \to \mathbf{R}$ of (2.1) satisfying the boundary conditions (1.3). When $c \neq 0$ any such solution is smooth, and without loss we may translate ξ and assume that $x(0) = 0$. We shall therefore seek such solutions in the subspace

$$W_0^{1,\infty} = \{x \in W^{1,\infty} \mid x(0) = 0\}$$

of $W^{1,\infty} = W^{1,\infty}(\mathbf{R})$. In fact, by Proposition 6.3 below, any solution of (2.1) satisfying (1.3) with $c \neq 0$ is strictly increasing, so requiring $x \in W_0^{1,\infty}$ provides a unique normalization to the translation invariance of such solutions. When $c = 0$, we expect discontinuous solutions in general, so such a normalization is not possible, although we do restrict our attention to monotone solutions. In this case we seek $x \in L^\infty = L^\infty(\mathbf{R})$.

Throughout this paper we use **monotone increasing** as a synonym for nondecreasing, that is, when the monotonicity need not be strict, and **monotone decreasing** as a synonym for nonincreasing.

Following the above remarks, we state a main theorem of this paper.

Theorem 2.1. *For every $\rho \in W$, there exists $c \in \mathbf{R}$, and a monotone increasing solution $x = P(\xi)$ of (2.1) on $\mathbf{R}$ satisfying the boundary conditions*

(1.3). This $c = c(\rho) \in \mathbf{R}$ is unique, and depends continuously on $\rho \in W$, and C^1 smoothly on ρ when $c(\rho) \neq 0$. If $c(\rho) \neq 0$ then the solution P is unique up to translation among all (possibly nonmonotone) solutions satisfying (1.3), and also satisfies

$$P'(\xi) > 0, \qquad \xi \in \mathbf{R} \tag{2.3}$$

and thus there is a unique translate $P(\xi) = P(\xi, \rho)$ in the space $W_0^{1,\infty}$. The dependence of this normalized P on ρ, as an element of $W_0^{1,\infty}$, is C^1 when $c(\rho) \neq 0$.

Let us denote

$$U = \{\rho \in W \mid c(\rho) \neq 0\} \tag{2.4}$$

with $c(\rho)$ as in the statement of Theorem 2.1. Then the set U is relatively open as a subset of W. Let us also denote

$$\mathcal{M} = \{(c, P, \rho) \in \mathbf{R} \times W_0^{1,\infty} \times W \mid c \neq 0, \text{ and } x = P(\xi) \text{ satisfies (2.1) on } \mathbf{R}, \text{ with (1.3)}\} \tag{2.5}$$

or, equivalently,

$$\mathcal{M} = \{(c(\rho), P(\rho), \rho) \in (\mathbf{R} \backslash \{0\}) \times W_0^{1,\infty} \times W \mid \rho \in U\}$$

in the notation of Theorem 2.1. Then by this result,

$$c\colon U \to \mathbf{R} \backslash \{0\}, \qquad P\colon U \to W_0^{1,\infty} \tag{2.6}$$

are smooth (C^1) functions, and so the set $\mathcal{M}$ is a smooth manifold, in fact a graph over the subset U of the parameter space. [It will be immediate from our proof that if the nonlinearity F in (2.1) is C^k smooth, for $2 \leqslant k \leqslant \infty$, then the functions (2.6), hence the manifold $\mathcal{M}$, are also C^k.]

It is typical to have solutions with wave speed $c = 0$ over an open set of parameters, that is, it may happen in a robust or generic fashion that the set $W \backslash U$ of ρ where $c(\rho) = 0$ has nonempty interior. This phenomenon is known as **propagation failure**, as discussed in Section 1.

The next result gives an asymptotic description of the solution P obtained in Theorem 2.1. Additional information on c is also given.

Theorem 2.2. *Let $P\colon \mathbf{R} \to \mathbf{R}$ with $c = c(\rho)$ be a be a solution as in the statement of Theorem 2.1, for some $\rho \in W$. Then*

$$c \geqslant 0 \quad \text{if} \quad r_{\max}(\rho) = 0, \qquad c \leqslant 0 \quad \text{if} \quad r_{\min}(\rho) = 0 \tag{2.7}$$

If $c \neq 0$, then for some quantities $C_{\pm} > 0$ and $\varepsilon > 0$, we have that

$$P(\xi) = \begin{cases} -1 + C_- e^{\lambda^u_- \xi} + O(e^{(\lambda^u_- + \varepsilon)\xi}), & \xi \to -\infty \\ 1 - C_+ e^{\lambda^s_+ \xi} + O(e^{(\lambda^s_+ - \varepsilon)\xi}), & \xi \to \infty \end{cases} \tag{2.8}$$

where $\lambda^u_- \in (0, \infty)$ is the unique positive eigenvalue of the linearization of (2.1) about $x = -1$, and $\lambda^s_+ \in (-\infty, 0)$ is the unique negative eigenvalue of the linearization about $x = 1$. The formulas for $P'(\xi)$ obtained by formally differentiating (2.8) (including the remainder terms) also hold.

If $c = 0$, then

$$\begin{aligned} &\lim_{\xi \to -\infty} \frac{1}{\xi} \log(1 + P(\xi)) = \lambda^u_- \in (0, \infty), && \textit{if} \quad r_{\max}(\rho) > 0 \\ &\lim_{\xi \to \infty} \frac{1}{\xi} \log(1 - P(\xi)) = \lambda^s_+ \in (-\infty, 0), && \textit{if} \quad r_{\min}(\rho) < 0 \end{aligned} \tag{2.9}$$

and also

$$\begin{aligned} &P(\xi) \notin (-1, q(\rho)), \quad && \xi \in \mathbf{R}, \quad && \textit{if} \quad r_{\max}(\rho) = 0 \\ &P(\xi) \notin (q(\rho), 1), \quad && \xi \in \mathbf{R}, \quad && \textit{if} \quad r_{\min}(\rho) = 0 \end{aligned} \tag{2.10}$$

In particular, $P(\xi) = -1$ identically for all sufficiently negative ξ if $r_{\max}(\rho) = 0$, and $P(\xi) = 1$ identically for all large ξ if $r_{\min}(\rho) = 0$, when $c = 0$.

Remark. If $c(\rho) = 0$ and $r_{\max}(\rho) = 0$, then (2.10) forces the solution P to be discontinuous, jumping over the interval $[-1, q(\rho)]$, and similarly for the interval $[q(\rho), 1]$ if $c(\rho) = 0$ and $r_{\min}(\rho) = 0$.

Remark. As we shall see, there does not exist a positive eigenvalue λ^u_- for the linearization of (2.1) about $x = -1$ if and only if both $c \leqslant 0$ and $r_{\max}(\rho) = 0$ hold. Similarly λ^s_+ fails to exist if and only if $c \geqslant 0$ and $r_{\min}(\rho) = 0$. One easily sees that these facts, together with (2.7), are consistent with the existence of λ^u_- and λ^s_+ in the cases given in the statement of Theorem 2.2.

Remark. If $\rho \in \bar{V} \backslash W$, that is, if $q(\rho) = \pm 1$, then (2.7) can fail. In addition, there may be more than one value of c for which a monotone solution to (2.1), with (1.3), exists. Let $P: \mathbf{R} \to \mathbf{R}$ be C^1, with $P(\xi) = -1 + e^{\xi}$ for all sufficiently negative ξ and $P(\xi) = 1 - \xi^{-1}$ for all large ξ, and $P(\xi)$ linear in between so that $P'(\xi) > 0$ for all ξ. Denoting

$$y(\xi) = P'(\xi) + P(\xi + 1) - P(\xi)$$

we see that $y(\xi)>0$ for all ξ, with $y(\xi)=e^{\xi+1}$ near $-\infty$ and $y(\xi)=(2\xi+1)\,\xi^{-2}(\xi+1)^{-1}$ near $+\infty$, by a calculation. Therefore, there exists $f_1\colon \mathbf{R}\to\mathbf{R}$ with $f_1(P(\xi))=y(\xi)$ for all ξ, that is,

$$-cP'(\xi)=P(\xi+1)-P(\xi)-f_1(P(\xi)), \qquad \xi\in\mathbf{R}$$

with $c=1$. We moreover can ensure that $f_1(u)>0$ for $u\in(-1,1)\cup(1,\infty)$ and $f_1(u)<0$ for $u\in(-\infty,1)$, with $f_1(u)=e(u+1)$ near $u=-1$ and $f_1(u)=(u-1)^2\,(3-u)(2-u)^{-1}$ near $u=1$.

One can embed f_1 as part of a function $f(u,\rho)$, for $-1\leqslant\rho\leqslant 1$, with $f_1(u)=f(u,1)$, and where

$$-cx'(\xi)=x(\xi+1)-x(\xi)-f(x(\xi),\rho)$$

satisfies our standing hypotheses, in fact is a normal family (defined below). For this family, condition (2.7) fails for the solution P with $c=1$, at $\rho=1$. In addition, a second solution $P_0\colon\mathbf{R}\to\mathbf{R}$ exists with $c=0$ at the same parameter value $\rho=1$. Such P_0 is obtained by solving the difference equation $x_{n+1}=x_n+f_1(x_n)$ to obtain an increasing sequence with $x_n\to\pm1$ as $n\to\pm\infty$. That such a sequence exists is easily seen by graphing the function $u\to u+f_1(u)$, where one checks that $1+f_1'(u)>0$. One then sets $P_0(\xi)=x_n$ for $\xi\in[n,n+1)$.

The following theorem describes a continuity property of the solution P at points where $c(\rho)=0$. If $\rho_n\to\rho_0$, with $c_n=c(\rho_n)\to 0$, then after passing to a subsequence the limit $P_n(\xi)\to P_0(\xi)$ of the corresponding solutions exists, and satisfies appropriate boundary conditions. Obtaining the limiting solution P_0 is relatively simple. However, determining the boundary conditions, namely, the limits of $P_0(\xi)$ as $\xi\to\pm\infty$, is quite subtle and rather difficult.

Theorem 2.3. *Let $\rho_n\in W$ and $\rho_0\in\bar V$ with $\rho_n\to\rho_0$. Let $x=P_n(\xi)$ denote any monotone solution of (2.1) satisfying (1.3), with $\rho=\rho_n$ and $c=c_n$, where we suppose that $c_n\to 0$. Then after possibly passing to a subsequence, the limit*

$$\lim_{n\to\infty} P_n(\xi)=P_0(\xi) \tag{2.11}$$

exists pointwise, where $x=P_0(\xi)$ satisfies the limiting difference equation

$$0=F(x(\xi+r_1),x(\xi+r_2),\ldots,x(\xi+r_N),\rho_0) \tag{2.12}$$

at all but countably many points. The limiting function P_0 is monotone increasing, and either satisfies the boundary conditions (1.3), or else $P_0(\xi)=v$ identically for $\xi\in\mathbf{R}$, where $v\in\{-1,1\}$.

If in particular $c_n \neq 0$ *and* $P_n \in W_0^{1,\infty}$, *that is,* $P_n(0) = 0$ *so that* $P_n(\xi) = P_n(\xi, \rho_n)$ *is the normalized solution of Theorem 2.1, then (1.3) must hold for the limit* P_0.

Remark. By replacing $P_0(\xi)$ in Theorem 2.3 with its right-hand limit $P_0(\xi + 0)$ at every $\xi \in \mathbf{R}$, one obtains a function which satisfies Eq. (2.12) everywhere.

Remark. The solution P_0 in Theorem 2.3 quite generally is discontinuous, its discontinuities arising from transition layers in the solutions P_n in the singular perturbation limit $c_n \to 0$. In the case of shifts which are not rationally related, one expects such discontinuities on a dense set of values ξ. See in particular Ref. 15, and also the numerical studies in Refs. 34 and 35.

We next describe a monotonicity property of the function $c(\rho)$. Defining $M \subseteq (-1, 1)^N$ by

$$M = \{u \in (-1, 1)^N \mid u_j < u_k \text{ whenever } r_j < r_k\} \tag{2.13}$$

we have the following result.

Proposition 2.4. *If* $\rho\colon I \to U$ *is a* C^1 *function on some interval* $I \subseteq \mathbf{R}$, *and if*

$$\frac{d}{d\sigma} F(u, \rho(\sigma))\bigg|_{\sigma = \sigma_0} < 0, \qquad u \in M \tag{2.14}$$

at some $\sigma = \sigma_0 \in I$, *then*

$$\frac{d}{d\sigma} c(\rho(\sigma))\bigg|_{\sigma = \sigma_0} > 0 \tag{2.15}$$

The corresponding result with the signs of the above derivatives reversed also holds.

A special class of systems occurs when ρ is scalar, and F depends monotonically on ρ in the above sense. Let us define $\kappa\colon \mathbf{R} \to \mathbf{R}^N$ to be the diagonal map

$$\kappa(x) = (x, x, \ldots, x)$$

We say that (2.1) is a **normal family** in case the following hold in addition to (i)–(v).

(vi) The parameter space is $\bar{V}=[-1, 1]$, with $V=(-1, 1)$.

(vii) $q(\rho)=\rho$ for $\rho \in \bar{V}$, and so $W=V$.

(viii) We have that

$$\frac{\partial F(u, \rho)}{\partial \rho}<0, \qquad u \in M, \qquad \rho \in W$$

(ix) We have that

$$\left.\frac{\partial F(u, \rho)}{\partial u_j}\right|_{u=\kappa(\pm 1)}$$

is a C^1 function of $\rho \in W$, for $1 \leqslant j \leqslant N$.

(x) We have that

$$\pm \left.\frac{\partial^2 \Phi(x, \rho)}{\partial \rho\, \partial x}\right|_{x=\pm 1}>0, \qquad \rho \in W$$

The parameter ρ in a normal family is called the **detuning parameter**. The prototypical normal family is the system

$$-cx'(\xi)=\sum_{j=2}^{N} A_{j,0}(x(\xi+r_j)-x(\xi))-f(x(\xi), \rho)$$

where f is a cubic-like bistable nonlinearity, such as

$$f(x, \rho)=(x-\rho)(x^2-1)$$

and with positive constants $A_{j,0}>0$ for $2 \leqslant j \leqslant N$. Let us note that for any fixed $k>0$ the system (1.17) with the nonlinearity (1.16), modified as described, is a normal family with the parameter $\rho=q$. Indeed, we have that

$$F(u_1, u_2, \rho)=\gamma^{-1}(u_2-u_1)-f(u_1, \rho)$$

for this nonlinearity, and one readily verifies that conditions (i)–(x) hold.

The next result will follow from Proposition 2.4 once the continuity of $c(\rho)$ is established (we shall in fact define $c(\rho)$ and prove its continuity before giving the full proof of Theorem 2.1). Let us define

$$U_+=\{\rho \in U \mid c(\rho)>0\}, \qquad U_-=\{\rho \in U \mid c(\rho)<0\} \tag{2.16}$$

so both U_+ and U_- are relatively open in W, with $U=U_+ \cup U_-$.

***Corollary* 2.5.** *Let (2.1) be a normal family. Then there exist quantities*

$$-1 \leqslant \rho_- \leqslant \rho_+ \leqslant 1$$

such that $U_\pm \subseteq W = (-1, 1)$ *are the intervals*

$$U_+ = (\rho_+, 1), \qquad U_- = (-1, \rho_-)$$

and for the continuous function $c\colon (-1, 1) \to \mathbf{R}$ *we have that*

$$\frac{dc(\rho)}{d\rho} > 0, \qquad \rho \in U \tag{2.17}$$

Remark. It is possible above that either $\rho_+ = 1$, or $\rho_- = -1$, that is, $U_+ = \phi$ or $U_- = \phi$. Indeed, by Theorem 2.2, sufficient conditions for these are that $r_{\min}(\rho) = 0$ for all ρ near $+1$, and that $r_{\max}(\rho) = 0$ for all ρ near -1, respectively. On the other hand, the next result gives sufficient conditions for either $U_+ \neq \phi$, or for $U_- \neq \phi$, for a normal family.

We say that a normal family is **coercive at +1**, respectively **coercive at −1**, if the first, respectively the second, of the inequalities

$$\sum_{j=2}^{N} r_j \frac{\partial F(u, \rho)}{\partial u_j}\bigg|_{\substack{u=\kappa(1)\\ \rho=1}} < 0, \qquad \sum_{j=2}^{N} r_j \frac{\partial F(u, \rho)}{\partial u_j}\bigg|_{\substack{u=\kappa(-1)\\ \rho=-1}} > 0 \tag{2.18}$$

holds. We say a normal family is **weakly coercive at +1** (or **at +1**) if the relevant inequality in (2.18) holds, but not necessarily strictly (so with $\leqslant$ or $\geqslant$), and if also the function F at $\rho = \pm 1$ is affine in u_j for $2 \leqslant j \leqslant N$ in a neighborhood of $u = \kappa(\pm 1)$. In this case one sees that

$$F(u, \pm 1) = \sum_{j=2}^{N} A_{j\pm}(u_j - u_1) + \Phi(u_1, \pm 1), \qquad u \text{ near } \kappa(\pm 1)$$

for positive constants $A_{j\pm}$ satisfying also

$$\pm \sum_{j=2}^{N} r_j A_{j\pm} \leqslant 0$$

Let us note that the system (1.17), with (1.16), is coercive at $+1$, as $N = 2$, and $r_2 = -k < 0$ with $A_{2+} = \gamma^{-1} > 0$. This system will serve as the starting point for a homotopy which will play a key role in the proof of the following result.

Theorem 2.6. *Let (2.1) be a normal family which is either coercive or weakly coercive at* $+1$. *Then* $\rho_+ < 1$, *with* ρ_+ *as in the statement of Corollary 2.5, and so* $U_+ \neq \phi$. *Similarly, we have* $\rho_- > -1$ *and so* $U_- \neq \phi$ *for any normal family which is coercive or weakly coercive at* -1.

Our approach to proving Theorem 2.1 and Proposition 2.4 is to seek zeros of the map

$$\mathcal{G}: \mathbf{R} \times W_0^{1,\infty} \times W \to L^\infty$$

given by

$$\mathcal{G}(c, x, \rho)(\xi) = -cx'(\xi) - F(x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N), \rho), \qquad \xi \in \mathbf{R} \tag{2.19}$$

If $c \neq 0$ then x is a bounded solution of Eq. (2.1) on $\mathbf{R}$, with $x(0) = 0$, if and only if $x \in W_0^{1,\infty}$ and $\mathcal{G}(c, x, \rho) = 0$. The map $\mathcal{G}$ is C^1-Fréchet differentiable, and its derivative at $(c, x, \rho) \in \mathbf{R} \times W_0^{1,\infty} \times W$, with respect to its first two arguments, in the direction $(w, y) \in \mathbf{R} \times W_0^{1,\infty}$, is

$$D_{1,2}\mathcal{G}(c, x, \rho)(w, y)(\xi) = -wx'(\xi) + (A_{c,L}y)(xi), \qquad \xi \in \mathbf{R} \tag{2.20}$$

where $A_{c,L}: W_0^{1,\infty} \to L^\infty$ is the bounded linear operator

$$\begin{aligned}(A_{c,L}y)(\xi) &= -cy'(\xi) - L(\xi)\, y_\xi \\ &= -cy'(\xi) - \sum_{j=1}^{N} A_j(\xi)\, y(\xi + r_j)\end{aligned} \tag{2.21}$$

with coefficients

$$A_j(\xi) = \left.\frac{\partial F(u, \rho)}{\partial u_j}\right|_{u = \pi(x, \xi)} \tag{2.22}$$

where for convenience we write

$$\pi(x, \xi) = (x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N)) \tag{2.23}$$

We denote

$$L(\xi)\,\varphi = \sum_{j=1}^{N} A_j(\xi)\,\varphi(r_j), \qquad \varphi \in C[r_{\min}, r_{\max}] \tag{2.24}$$

with $y_\xi(\theta) = y(\xi + \theta)$ for $\theta \in [r_{\min}, r_{\max}]$, following the notation in Ref. 63, in the spirit of Ref. 46. Here $L(\xi)$, for each ξ, is a bounded linear functional on $C[r_{\min}, r_{\max}]$ with norm

$$\|L(\xi)\| = \sum_{j=1}^{N} |A_j(\xi)| \tag{2.25}$$

Very much as in Ref. 20, we shall show that for any solution x of (2.1), with (1.3), for $c \neq 0$ and $\rho_0 \in W$, the operator $\Lambda_{c,L}$ maps $W_0^{1,\infty}$ isomorphically onto a closed codimension-one subspace of L^∞ which does not contain x'. Thus $D_{1,2}\mathscr{G}(c, x, \rho)$ is an isomorphism from $\mathbf{R} \times W_0^{1,\infty}$ onto L^∞, and so an application of the implicit function theorem to $\mathscr{G}$ yields a local continuation, that is, gives the functions $c(\rho)$ and $x(\rho)$ for ρ near ρ_0. Denoting this continuation by $P(\rho) = x(\rho)$, we shall obtain, at least locally, the solution in the statement of Theorem 2.1.

In the proof of Theorem 2.6 we employ a global continuation argument to pass between two normal families, namely a "trivial" one (1.17) with (1.16), and the other of which is given. In this spirit, let us define a **homotopy of normal families**. By this we mean first a system (2.1) satisfying conditions (i)–(v) above, with $V = (-1, 1) \times (0, 1)$. Denoting $\rho = (\tilde{\rho}, \hat{\rho}) \in \bar{V}$, we further require that

(xi) $F(u, \tilde{\rho}, \hat{\rho})$ is a normal family with detuning parameter $\tilde{\rho} \in [-1, 1]$, for every fixed $\hat{\rho} \in [0, 1]$.

Necessarily, then, $W = (-1, 1) \times [0, 1]$ for a homotopy of normal families. We refer to $\hat{\rho}$ as the **homotopy parameter**, as it connects the normal family $F(u, \tilde{\rho}, 0)$ to the normal family $F(u, \tilde{\rho}, 1)$. It is easy to see that if (2.1) with $F_0(u, \rho)$ and with $F_1(u, \rho)$ each are normal families, then a homotopy between them can be given simply by

$$F(u, \tilde{\rho}, \hat{\rho}) = (1 - \hat{\rho})\, F_0(u, \tilde{\rho}) + \hat{\rho} F_1(u, \tilde{\rho}) \tag{2.26}$$

Thus any two normal families can be connected by a homotopy of normal families. Note that if normal families given by F_0 and by F_1 both are coercive at $+1$, then for each fixed $\hat{\rho} \in [0, 1]$ the normal family (2.26) is coercive at $+1$. The corresponding statement at -1 also holds, as do analogous statements for weak coercivity.

3. PRELIMINARY RESULTS

Here we establish some basic notation and terminology, and as well present several technical results of use later. Quite generally, let us consider equations of the form

$$-cx'(\xi) = G(\xi, x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N)) \tag{3.1}$$

which arise either as Eqs. (2.1) at specific values of ρ or as the linearization of (2.1) about particular solutions. (The nonautonomous case, with G depending on ξ, occurs for such linear equations.) Both the differential equation case $c \neq 0$, and the difference equation case $c = 0$, will be of interest. We assume that x is scalar, and take as a standing hypothesis throughout this section that the shifts r_j satisfy

$$r_1 = 0, \qquad r_j \neq r_k, \qquad 1 \leqslant j, k \leqslant N, \qquad N \geqslant 2 \tag{3.2}$$

namely, that condition (iii) of Section 2 holds. We will also make these assumptions in the next two sections, but will again note this at the beginning of those sections. With the parameter ρ absent in this and the next two sections we simply denote

$$r_{\min} = \min_{1 \leqslant j \leqslant N} r_j, \qquad r_{\max} = \max_{1 \leqslant j \leqslant N} r_j$$

noting that the analogs of (2.2) hold. The function G is assumed to satisfy the following conditions.

(i′) $G: \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$, written $G(\xi, u)$, is continuous, and is locally lipschitz in u.

(ii′) We have for every $\xi \in \mathbf{R}$ that

$$\frac{\partial G(\xi, u)}{\partial u_j} \geqslant 0, \qquad u \in \mathbf{R}^N, \qquad 2 \leqslant j \leqslant N$$

where here we mean $u = (u_1, u_2, ..., u_N)$.

We shall note these assumptions in the statements of our results, as needed.

When $c \neq 0$, by a solution of Eq. (3.1) on an interval J we mean a continuous function $x: J^\# \to \mathbf{R}$ (or sometimes complex-valued $x: J^\# \to \mathbf{C}$ in the case of a linear equation) on the set

$$J^\# = \{\xi + r_j \mid \xi \in J \text{ and } 1 \leqslant j \leqslant N\}$$

such that x is C^1 on J and satisfies Eq. (3.1) there. Observe that if J is large enough to contain a closed interval of length $r_{\max} - r_{\min}$, then the set $J^\#$ is also an interval. In any case, $J^\#$ always contains J, as $r_1 = 0$.

When $c = 0$, by a solution of

$$0 = G(\xi, x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N)) \tag{3.3}$$

on J we mean a (not necessarily continuous) function $x\colon J^\# \to \mathbf{R}$ for which the Eq. (3.3) holds at every $\xi \in J$. If all the ratios r_j/r_k for $r_k \neq 0$ are rational, then there is a unique minimal quantity $v > 0$ such that all shifts are integer multiples of v, in fact

$$\begin{aligned} &r_j = n_j v, \qquad 1 \leqslant j \leqslant N \\ &\gcd(n_2, n_3, ..., n_N) = 1 \end{aligned} \tag{3.4}$$

and we say the shifts are **rationally related**. In this case Eq. (3.3) can be regarded, in a general sense, as a finite-dimensional difference equation, and in many cases, can be interpreted as a map on a finite-dimensional space. If the shifts are not rationally related, so that r_j/r_k is irrational for some j and k, then Eq. (3.3) has no finite-dimensional interpretation, and indeed, the analysis of this equation can present formidable difficulties.

We begin with several comparison principles for classes of both functional differential ($c \neq 0$) and difference ($c = 0$) equations.

Lemma 3.1. *Assume that (i′) and (ii′) above hold. Let $x_j\colon \mathbf{R} \to \mathbf{R}$, for $j = 1, 2$, be two solutions of Eq. (3.1) at some nonzero parameter value $c \in \mathbf{R} \backslash \{0\}$. Assume that*

$$x_1(\xi) \geqslant x_2(\xi), \qquad \xi \in \mathbf{R} \tag{3.5}$$

Then if $x_1(\tau) = x_2(\tau)$ at some $\tau \in \mathbf{R}$, we have that $x_1(\xi) = x_2(\xi)$ for all $\xi \geqslant \tau$ in case $c > 0$, and that $x_1(\xi) = x_2(\xi)$ for all $\xi \leqslant \tau$ in case $c < 0$.

Proof. Let $y(\xi) = x_1(\xi) - x_2(\xi) \geqslant 0$, and assume that the inequality (3.5) is an equality at some $\tau \in \mathbf{R}$, that is, $y(\tau) = 0$. Assume for definiteness that $c > 0$, the proof when $c < 0$ being similar. Define the function

$$\begin{aligned} H(\xi, u) = -c^{-1}(&G(\xi, u + x_2(\xi + r_1), x_1(\xi + r_2), x_1(\xi + r_3), ..., x_1(\xi + r_N)) \\ &- G(\xi, x_2(\xi + r_1), x_2(\xi + r_2), x_2(\xi + r_3), ..., x_2(\xi + r_N))) \end{aligned}$$

and observe that $u = y(\xi)$ satisfies $u' = H(\xi, u)$, recalling that $r_1 = 0$. Also, from (3.5) and assumption (ii′), we have that $H(\xi, 0) \leqslant 0$ for every $\xi \in \mathbf{R}$, and so $y(\xi) \leqslant 0$ for all $\xi \geqslant \tau$ by a standard differential inequality. Thus $y(\xi) = 0$ for all $\xi \geqslant \tau$, as claimed. □

Lemma 3.2. *Assume that the conditions of Lemma 3.1 hold, except that the solutions x_j, for $j = 1, 2$, satisfy Eq. (3.1) at different values $c_1 > c_2$ of the parameter c, and where either $c_1 = 0$ or $c_2 = 0$ are permitted. Suppose that $x_1(\tau) = x_2(\tau)$ at some $\tau \in \mathbf{R}$. Then if*

$$x_2(\xi) \text{ is monotone increasing in } \xi \in \mathbf{R}, \quad \text{and} \quad c_1 > 0 \tag{3.6}$$

we have that $x_1(\xi) = x_2(\xi)$ is constant for all $\xi \geqslant \tau - r_{\min}$, while if

$$x_1(\xi) \text{ is monotone increasing in } \xi \in \mathbf{R}, \quad \text{and} \quad c_2 < 0 \tag{3.7}$$

we have that $x_1(\xi) = x_2(\xi)$ is constant for all $\xi \leqslant \tau - r_{\max}$.

Proof. The proof of this result is very similar to that of Lemma 3.1, except for the choice of the function H. Several cases must be considered, based on the signs of c_1 and c_2. First, suppose that (3.6) holds, and also that $c_2 \neq 0$, and let

$$\begin{aligned} H(\xi, u) = &-c_1^{-1} G(\xi, u + x_2(\xi + r_1), x_1(\xi + r_2), x_1(\xi + r_3), ..., x_1(\xi + r_N)) \\ &+ c_2^{-1} G(\xi, x_2(\xi + r_1), x_2(\xi + r_2), x_2(\xi + r_3), ..., x_2(\xi + r_N)) \end{aligned}$$

Then $u = x_1(\xi) - x_2(\xi)$ satisfies $u' = H(\xi, u)$. By replacing x_1 with x_2 in the formula for H, and using the resulting inequality from (3.5) and (ii'), one sees from (3.6) that $H(\xi, 0) \leqslant (c_2 c_1^{-1} - 1)\, x_2'(\xi) \leqslant 0$. As before, one concludes that $x_1(\xi) - x_2(\xi) \leqslant 0$ for $\xi \geqslant \tau$, and thus $x_1(\xi) = x_2(\xi)$ for all $\xi \geqslant \tau$. From the differential Eq. (3.1) we have also that $c_1 x_1'(\xi) = c_2 x_2'(\xi)$ for $\xi \geqslant \tau - r_{\min}$ and as $c_1 \neq c_2$ we conclude that $x_1'(\xi) = x_2'(\xi) = 0$ there.

Now suppose again that (3.6) holds, but that $c_2 = 0$. Set

$$H(\xi, u) = -c_1^{-1} G(\xi, u, x_1(\xi + r_2), x_1(\xi + r_3), ..., x_1(\xi + r_N))$$

$$\tilde{H}(\xi, u) = \begin{cases} H(\xi, x_2(\xi)), & u \geqslant x_2(\xi) \\ H(\xi, u), & u \leqslant x_2(\xi) \end{cases}$$

One easily checks that $H(\xi, x_2(\xi)) \leqslant 0$, and thus $\tilde{H}(\xi, u) \leqslant 0$ for all $u \geqslant x_2(\xi)$. From this inequality, from the fact that x_2 is monotone increasing, and from the fact that $\tilde{H}$ satisfies the standard Carathéodory and lipschitz conditions, we have that the unique solution $u = x_3(\xi)$ to the problem $u' = \tilde{H}(\xi, u)$ with $u(\tau) = x_2(\tau)$ satisfies $x_3(\xi) \leqslant x_2(\xi)$ for all $\xi \geqslant \tau$. Thus from the definition of $\tilde{H}$ we have that $u = x_3(\xi)$ in fact satisfies $u' = H(\xi, u)$ for

$\xi \geqslant \tau$. But one sees that $u = x_1(\xi)$ also satisfies $u' = H(\xi, u)$, and as $x_3(\tau) = x_2(\tau) = x_1(\tau)$, by uniqueness one has that $x_1(\xi) = x_3(\xi)$, hence $x_1(\xi) \leqslant x_2(\xi)$, for all $\xi \geqslant \tau$. One thus concludes that $x_1(\xi) = x_2(\xi)$ for all $\xi \geqslant \tau$. As before, these solutions are constant for $\xi \geqslant \tau - r_{\min}$.

The proof of the lemma when (3.7) holds is similar. □

The uniqueness result below will be useful when combined with Lemma 3.1.

Lemma 3.3. *Assume that (i') and (ii') above hold, and also that the inequality in the derivative in (ii') is everywhere strict. Let $x_j: J \to \mathbf{R}$, for $j = 1, 2$, be two solutions of Eq. (3.1), at some parameter value $c \in \mathbf{R}$, on some interval J. Assume that*

$$x_1(\xi) = x_2(\xi), \qquad \tau + r_{\min} \leqslant \xi \leqslant \tau + r_{\max} \tag{3.8}$$

for some $\tau \in J$ for which $[\tau + r_{\min}, \tau + r_{\max}] \subseteq J$. Then either

$$x_1(\xi) = x_2(\xi), \qquad \xi \in (-\infty, \tau + r_{\max}] \cap J^\# \tag{3.9}$$

or

$$r_{\min} = 0, \qquad c = 0 \tag{3.10}$$

both hold. The analogous result for solutions agreeing on $[\tau + r_{\min}, \infty) \cap J^\#$ also holds, with $r_{\max}$ replacing $r_{\min}$ in (3.10).

Proof. Note that the set $J^\#$ is an interval, as J contains a closed interval of length $r_{\max} - r_{\min}$. With $x_1(\xi)$ and $x_2(\xi)$ as in the statement of the lemma, assume that (3.9) fails. Then there exists $\tau_0 \in (-\infty, \tau + r_{\min}] \cap J^\#$ such that $x_1(\xi) = x_2(\xi)$ for all $\xi \in [\tau_0, \tau + r_{\max}]$, but that for every $\varepsilon > 0$ we have $x_1(\xi) \neq x_2(\xi)$ for some $\xi \in (\tau_0 - \varepsilon, \tau_0) \cap J^\#$. Note that $\tau_0 \in J^\#$ is not the left-hand endpoint of $J^\#$, and hence $\tau_0 - r_{\min} \in J$ is not the left-hand endpoint of J.

As we wish to prove (3.10), let us first assume that $r_{\min} < 0$, and obtain a contradiction. Let j_0 with $2 \leqslant j_0 \leqslant N$ be such that $r_{\min} = r_{j_0} < 0$. Consider Eq. (3.1) for $\tau_0 - r_{j_0} - \varepsilon < \xi < \tau_0 - r_{j_0}$, where ε is small enough that $\xi + r_j > \tau_0$ whenever $j \neq j_0$, and also that $\xi \in J$, for such ξ. Then $x_1(\xi + r_j) = x_2(\xi + r_j)$ for all such ξ whenever $j \neq j_0$. Also, $x_1'(\xi) = x_2'(\xi)$ for every such ξ if $c \neq 0$. Thus from (3.1) we have for such ξ that

$$\begin{aligned} &G(\xi, x_1(\xi + r_1), x_1(\xi + r_2), ..., x_1(\xi + r_{j_0}), ..., x_1(\xi + r_N)) \\ &\quad = G(\xi, x_1(\xi + r_1), x_1(\xi + r_2), ..., x_2(\xi + r_{j_0}), ..., x_1(\xi + r_N)) \end{aligned}$$

which implies, by the strict inequality (ii′), that $x_1(\xi + r_{j_0}) = x_2(\xi + r_{j_0})$ for such ξ. This contradicts the definition of τ_0, we conclude that $r_{\min} = 0$.

Now with $r_{\min} = 0$, assume that $c \neq 0$. We again seek a contradiction. We have $r_j \geqslant r_1 = 0$ for all j, and so Eq. (3.1) is an advanced functional differential equation (or equivalently, a delay differential equation upon making the change of variables $\xi \to -\xi$). Solutions of such equations are unique in the backward direction, and so regarding (3.8) as an initial condition for such a problem, we immediately conclude (3.9).

The proof of the analogous results for $\xi \geqslant \tau + r_{\min}$ follows similar lines. □

A special case of Eq. (3.1) is the linear equation

$$-cx'(\xi) = L(\xi)\, x_\xi + h(\xi) \tag{3.11}$$

with $L(\xi)$ as in (2.24), with bounded coefficients $A_j(\xi)$ which are continuous in ξ. A Fredholm alternative for (3.11), plus several results on asymptotic behavior of solutions, were obtained in Ref. 63 in the differential equation case $c \neq 0$. Let us recall some of those results which we shall need below. A particular case of (3.11) is the homogeneous constant coefficient equation

$$-cx'(\xi) = L_0 x_\xi = \sum_{j=1}^{N} A_{j,0}\, x(\xi + r_j) \tag{3.12}$$

with $A_{j,0} \in \mathbf{R}$, where L_0 is the linear functional

$$L_0 \varphi = \sum_{j=1}^{N} A_{j,0}\, \varphi(r_j), \qquad \varphi \in C[r_{\min}, r_{\max}]$$

In this case we have the characteristic equation $\Delta_{c, L_0}(s) = 0$, where we denote

$$\Delta_{c, L_0}(s) = -cs - \sum_{j=1}^{N} A_{j,0} e^{s r_j}$$

If $c \neq 0$ then in every strip $|\mathrm{Re}\, s| \leqslant K$ the characteristic equation has only finitely many roots, and to each root $s = \lambda$ there corresponds a finite-dimensional set of **eigensolutions** to (3.12) of the form $e^{\lambda \xi} p(\xi)$, where p is a polynomial. We say that Eq. (3.12) is **hyperbolic** in case the characteristic equation has no roots on the imaginary axis, that is, $\Delta_{c, L_0}(i\eta) \neq 0$ for all $\eta \in \mathbf{R}$. Hyperbolic systems were studied in general in Ref. 63 for $c \neq 0$, and we extend the definition of hyperbolicity here also to the case $c = 0$.

Remark. We caution the reader that hyperbolicity is defined solely in terms of the characteristic equation, and assumes nothing about the dynamics or the solutions of (3.12), which, when $c=0$, can be quite subtle. For example, when $c=0$ it is not generally the case that each vertical strip $|\mathrm{Re}\, s| \leqslant K$ contains finitely many eigenvalues, as is the case when $c \neq 0$. Indeed, if $c=0$ and the shifts r_j are not rationally related then the set of real parts of eigenvalues

$$R = \{\mathrm{Re}\,\lambda \mid \Delta_{0,L_0}(\lambda) = 0\}$$

can be dense in a nontrivial interval, that is, $\bar{R}$ can contain a nontrivial interval. Such a system can be hyperbolic according to our definition, yet possess a sequence of eigenvalues with $\mathrm{Re}\,\lambda_n \to 0$. It is for such reasons that the asymptotic formulas (2.9) when $c=0$ are not as sharp as those (2.8) for $c \neq 0$. Indeed, (2.9) is obtained only with significant effort when the shifts are not rationally related.

In the nonautonomous linear Eq. (3.11) the operator $L(\xi)$ is in some cases simply the sum $L(\xi) = L_0 + M(\xi)$ of a constant coefficient operator and a perturbation term

$$M(\xi)\,\varphi = \sum_{j=1}^{N} B_j(\xi)\,\varphi(r_j)$$

In particular, an **asymptotically autonomous** system is one for which we have such decompositions at both $\pm\infty$, namely,

$$\begin{aligned} &L(\xi) = L_+ + M_+(\xi) = L_- + M_-(\xi) \\ &L_\pm \varphi = \sum_{j=1}^{N} A_{j\pm}\,\varphi(r_j), \qquad M_\pm(\xi) = \sum_{j=1}^{N} B_{j\pm}(\xi)\,\varphi(r_j) \end{aligned} \tag{3.13}$$

with the limits

$$\lim_{\xi \to \infty} A_j(\xi) = A_{j+}, \qquad \lim_{\xi \to -\infty} A_j(\xi) = A_{j-}, \qquad 1 \leqslant j \leqslant N \tag{3.14}$$

of the coefficients in (2.24), or equivalently

$$\lim_{\xi \to \infty} \|M_+(\xi)\| = 0, \qquad \lim_{\xi \to -\infty} \|M_-(\xi)\| = 0 \tag{3.15}$$

following the notation (2.25). Such a system is called **asymptotically hyperbolic** if the limiting constant coefficient equations at both $\pm\infty$ are hyperbolic. With $\Lambda_{c,L}$ denoting the operator (2.21) from $W^{1,\infty}$ to L^∞ associated with Eq. (3.11), we have by Theorem A of Ref. 63 that if (3.11) is asymptotically hyperbolic and if $c \neq 0$, then $\Lambda_{c,L}$ is a Fredholm operator.

We shall denote the kernel and range of $\Lambda_{c,L}$ by

$$\mathscr{K}_{c,L} \subseteq W^{1,\infty}, \qquad \mathscr{R}_{c,L} \subseteq L^{\infty}$$

and the Fredholm index by

$$\operatorname{ind}(\Lambda_{c,L}) = \dim \mathscr{K}_{c,L} - \operatorname{codim} \mathscr{R}_{c,L} \tag{3.16}$$

It is convenient here to recall [63] the adjoint

$$cy'(\xi) = -L^*(\xi)\, y_\xi = \sum_{j=1}^{N} \overline{A_j(\xi - r_j)}\, y(\xi - r_j)$$

of Eq. (1.15), where by definition

$$L^*(\xi)\, \varphi = -\sum_{j=1}^{N} \overline{A_j(\xi - r_j)}\, \varphi(-r_j), \qquad \varphi \in C[-r_{\max}, -r_{\min}]$$

We have the associated operator Λ_{c,L^*}, and as above we denote the kernel and range of this operator by $\mathscr{K}_{c,L^*}$ and $\mathscr{R}_{c,L^*}$, respectively. Again by Theorem A of Ref. 63, we have the Fredholm alternative, namely

$$\mathscr{R}_{c,L} = \left\{ h \in L^{\infty} \,\middle|\, \int_{-\infty}^{\infty} y(\xi)\, h(\xi)\, d\xi = 0 \text{ for every } y \in \mathscr{K}_{c,L^*} \right\} \tag{3.17}$$

for asymptotically hyperbolic equations with $c \neq 0$. Here every $y \in \mathscr{K}_{c,L^*}$ decays exponentially at $\pm\infty$, so the above integral is well-defined.

We now present a technical result on constant coefficient systems which we need later.

Lemma 3.4. *Consider a real-valued function x: $[\tau, \infty) \to \mathbf{R}$ of the form*

$$x(\xi) = y(\xi) + O(e^{-(b+\varepsilon)\xi}), \qquad \xi \to \infty$$

for some $b \in \mathbf{R}$ and $\varepsilon > 0$, where y is a nontrivial solution of a constant coefficient system (3.12), with $c \neq 0$, given by a finite sum of eigensolutions corresponding to a set F of eigenvalues λ, all of which satisfy $\operatorname{Re} \lambda = -b$. If $\operatorname{Im} \lambda \neq 0$ for all $\lambda \in F$, then there exist arbitrarily large ξ for which $x(\xi) > 0$, and arbitrarily large ξ for which $x(\xi) < 0$. On the other hand, if $F = \{-b\}$, then $x(\xi) \neq 0$ for all large ξ.

The analogous result for $\xi \to -\infty$ holds.

Proof. First suppose that $\operatorname{Im}\lambda \neq 0$ for all $\lambda \in F$. Then we may write y as a linear combination of functions $\xi \to e^{-b\xi}\sin(\eta\xi+\kappa)\,p(\xi)$, for various $\eta > 0$ and $\kappa \in \mathbf{R}$, and real polynomials p. Thus, for some integer $J \geqslant 0$, we have that $y(\xi) = \xi^J e^{-b\xi}(q(\xi) + O(\xi^{-1}))$ as $\xi \to \infty$ where q is a nontrivial quasiperiodic function of mean value zero. In particular,

$$\liminf_{\xi \to \infty} q(\xi) < 0 < \limsup_{\xi \to \infty} q(\xi)$$

and it follows that $\xi^{-J}e^{b\xi}x(\xi) = q(\xi) + O(\xi^{-1})$ must assume both positive and negative values for arbitrarily large values of ξ, as claimed.

Now suppose that $F = \{-b\}$. Then $y(\xi) = e^{-b\xi}p(\xi)$ for some polynomial p, hence the limit

$$\lim_{\xi \to \infty} \xi^{-J}e^{b\xi}x(\xi) = \alpha \neq 0$$

exists and is nonzero for some integers $J \geqslant 0$. □

We shall later need the following technical result about Laplace transforms.

Lemma 3.5. *Suppose that* $f\colon [0,\infty) \to [0,\infty)$ *is a nonnegative measurable function, and let* $a \in [-\infty, \infty]$ *be the unique (possibly infinite) value such that*

$$\begin{aligned} \tilde{f}(s) &< \infty, \qquad s > a \\ \tilde{f}(s) &= \infty, \qquad s < a \end{aligned} \tag{3.18}$$

where $\tilde{f}$ *denotes the Laplace transform*

$$\tilde{f}(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \tag{3.19}$$

Then $\tilde{f}$ *is homomorphic in the right-half plane* $\operatorname{Re} s > a$. *If, further,* $|a| < \infty$, *then* $\tilde{f}$ *cannot be extended as a homomorphic function to any neighborhood of* $s = a$.

Proof. The fact that $\tilde{f}(s)$ is holomorphic for $\operatorname{Re} s > a$ is standard. We therefore prove only the final sentence in the statement of the lemma. Assume therefore that $|a| < \infty$ and that $\tilde{f}$ can be extended as a holomorphic function to some region containing the disc $|s - a| \leqslant r$ for some

$r > 0$. For each $n \geqslant 0$, the derivative $\tilde{f}^{(n)}(a)$ is the limit of $\tilde{f}^{(n)}(s)$ for $s > a$ approaching a. Upon differentiating (3.19) and taking this limit we obtain

$$\tilde{f}^{(n)}(a) = (-1)^n \int_0^\infty e^{-a\xi} \xi^n f(\xi)\, d\xi$$

by an application of the monotone convergence theorem, using the nonnegativity of f. As the radius of convergence of the power series for $\tilde{f}$ about a is greater than r, we have that $|\tilde{f}^{(n)}(a)| \leqslant K r^{-n} n!$ for some $K > 0$. Thus if $s \in (a - r, a)$, we have again by the monotone convergence theorem that

$$\begin{aligned} \int_0^\infty e^{-s\xi} f(\xi)\, d\xi &= \int_0^\infty e^{(a-s)\xi} e^{-a\xi} f(\xi)\, d\xi \\ &= \sum_{n=0}^\infty \frac{(a-s)^n}{n!} \int_0^\infty e^{-a\xi} \xi^n f(\xi)\, d\xi \\ &\leqslant K \sum_{n=0}^\infty \left(\frac{a-s}{r}\right)^n < \infty \end{aligned}$$

However, the finiteness of the above integral contradicts the definition of a. This completes the proof. □

4. LINEAR DIFFERENTIAL EQUATIONS: A ONE-DIMENSIONAL KERNEL

In this section we consider scalar equations of the form (3.11), where $L(\xi)$ in (2.24) has continuous coefficients $A_j(\xi)$. Our main result, Theorem 4.1, is that for a particular class of such equations with $c \neq 0$ the kernel of $\Lambda_{c,L}$ is one-dimensional, and the Fredholm index of $\Lambda_{c,L}$ is zero. The equations in this class include those that arise as linearizations of our traveling wave problem, and our results here will allow the use of the implicit function theorem.

Throughout this section we continue to assume (3.2) as a standing hypothesis on the shifts r_j.

Theorem 4.1. *Consider an operator $\Lambda_{c,L}$ as in (2.21) associated to the differential Eq. (1.15), where $c \neq 0$ is real, x is scalar, and where there exist quantities*

$$\begin{aligned} &\alpha_j, \beta_j \in \mathbf{R}, \qquad 1 \leqslant j \leqslant N \\ &\alpha_j > 0, \qquad 2 \leqslant j \leqslant N \end{aligned} \tag{4.1}$$

such that

$$\alpha_j \leqslant A_j(\xi) \leqslant \beta_j, \qquad \xi \in \mathbf{R}, \qquad 1 \leqslant j \leqslant N \tag{4.2}$$

Assume that Eq. (1.15) is asymptotically autonomous and that, in addition, the limits (3.14) are approached at an exponential rate, say

$$\|M_{\pm}(\xi)\| = O(e^{-k|\xi|}), \qquad \xi \to \pm\infty \tag{4.3}$$

for some $k > 0$ with the notation (3.13), (3.15). Also, assume that each of the sums $A_{\Sigma\pm}$ given below, of the limiting coefficients at $\pm\infty$, is negative, namely,

$$A_{\Sigma\pm} = \sum_{j=1}^{N} A_{j\pm} < 0 \tag{4.4}$$

with $A_{j\pm}$ as in (3.14). Finally, assume that there exists a nontrivial solution $x = p(\xi)$ to Eq. (1.15) which is nonnegative and bounded on $\mathbf{R}$,

$$p \in L^{\infty}, \qquad p(\xi) \geqslant 0, \qquad \xi \in \mathbf{R}$$

Then Eq. (1.15) is asymptotically hyperbolic, and $\Lambda_{c,L}\colon W^{1,\infty} \to L^{\infty}$ is a Fredholm operator, with

$$\dim \mathscr{K}_{c,L} = \dim \mathscr{K}_{c,L^*} = \operatorname{codim} \mathscr{R}_{c,L} = 1, \qquad \operatorname{ind}(\Lambda_{c,L}) = 0 \tag{4.5}$$

The element $p \in \mathscr{K}_{c,L}$ is strictly positive,

$$p(\xi) > 0, \qquad \xi \in \mathbf{R}$$

and there exists an element $p^* \in \mathscr{K}_{c,L^*}$ which is strictly positive,

$$p^*(\xi) > 0, \qquad \xi \in \mathbf{R} \tag{4.6}$$

Note above that we require a positive bound $A_j(\xi) \geqslant \alpha_j > 0$ when $j \neq 1$, that is, when $r_j \neq 0$. This is not required for A_1, and indeed this coefficient may take negative values.

We begin by developing some properties of the constant coefficient Eq. (3.12) which will hold for the limiting equations in Theorem 4.1. First note that if in (3.12) we have that

$$A_{j,0} > 0, \qquad 2 \leqslant j \leqslant N \tag{4.7}$$

then we have the strict concavity property

$$\Lambda''_{c,L_0}(s) < 0, \qquad s \in \mathbf{R} \tag{4.8}$$

which implies that there are at most two real eigenvalues of Eq. (3.12), counting multiplicity. The following three results give further information about the eigenvalues of this problem. In these results c can be either zero or nonzero. Recall the quantity $\nu > 0$ in (3.4), when the r_j are rationally related. We define a set $\Gamma \subseteq \mathbf{R}$ by

$$\Gamma = \begin{cases} \{0\} & \text{if the shifts } r_j \text{ are not rationally related} \\ \{2\pi n \nu^{-1} \mid n \in \mathbf{Z}\} & \text{if the shifts } r_j \text{ are rationally related} \end{cases} \tag{4.9}$$

which will play a role below.

Lemma 4.2. *Assume that (4.7) holds for the constant coefficient Eq. (3.12), and suppose that $a \in \mathbf{R}$ is such that $\Delta_{c, L_0}(a) > 0$. Then there do not exist any eigenvalues $\lambda \in \mathbf{C}$ such that* Re $\lambda = a$.

Now suppose that $\Delta_{c, L_0}(a) = 0$ for some $a \in \mathbf{R}$. If $c \neq 0$, then there do not exist any eigenvalues $\lambda \in \mathbf{C}$ such that Re $\lambda = a$, *except for $\lambda = a$ itself.*

Finally, if $\Delta_{c, L_0}(a) = 0$ for some $a \in \mathbf{R}$, and $c = 0$, then $\lambda \in \mathbf{C}$ with Re $\lambda = a$ *is an eigenvalue if and only if $\lambda = a + i\eta$ for some $\eta \in \Gamma$.*

Proof. Suppose that $\lambda = a + i\eta$ satisfies $\Delta_{c, L_0}(\lambda) = 0$ for some real η and that $\Delta_{c, L_0}(a) \geqslant 0$. Then

$$|ca + A_{1,0}| \leqslant |c\lambda + A_{1,0}| = \left| \sum_{j=2}^{N} A_{j,0} e^{\lambda r_j} \right| \leqslant \sum_{j=2}^{N} A_{j,0} e^{a r_j} \leqslant -(ca + A_{1,0}) \tag{4.10}$$

By examining the first and last terms in (4.10), we see that the three inequalities there are equalities. In particular, the third inequality implies (as an equality) that $\Delta_{c, L_0}(a) = 0$, and so the first claim in the lemma holds.

Now assume that $\Delta_{c, L_0}(a) = 0$. The first inequality in (4.10) becomes $|ca + A_{1,0}| = |c\lambda + A_{1,0}|$, which implies that $c\eta = 0$. If $c \neq 0$ then $\eta = 0$, so $\lambda = a$, establishing the second claim in the lemma.

Finally, suppose that $c = 0$, still with $\Delta_{c, L_0}(a) = 0$. Then the second inequality (as an equality) implies, from the positivity of $A_{j,0}$ for $2 \leqslant j \leqslant N$, that $e^{\eta r_j} = 1$ for such j. Necessarily the r_j are rationally related, and $\eta = 2\pi n \nu^{-1}$ for some integer n, from (3.4), hence $\eta \in \Gamma$. Lastly, one sees these conditions are sufficient, namely that $\Delta_{c, L_0}(a + 2\pi i n \nu^{-1}) = 0$ if $\Delta_{c, L_0}(a) = 0$ and $c = 0$. □

For convenience, let us denote the quantity

$$A_\Sigma = -\Delta_{c, L_0}(0) = \sum_{j=1}^{N} A_{j,0}$$

associated with Eq. (3.12). Observe that the corresponding quantities $A_{\Sigma\pm}$ for the limiting equations of (1.15) are participants in Theorem 4.1. We also define sets $M_\pm \subseteq (0, \infty)^N$ and $M_\pm^* \subseteq \mathbf{R}^N$ by

$$M_\pm = \{v \in (0, \infty)^N \mid \pm(v_k - v_j) > 0 \text{ whenever } r_j < r_k\}$$

$$M_\pm^* = \left\{ w \in \mathbf{R}^N \,\middle|\, \sum_{j=1}^{N} w_j v_j > 0 \text{ for every } v \in M_\pm \right\}$$

Compare these with the definition (2.13) of the set M.

Proposition 4.3. *Assume that (4.7) holds for the constant coefficient Eq. (3.12), and that $A_\Sigma < 0$. Then Eq. (3.12) is hyperbolic. There exist at most one real positive eigenvalue $\lambda^u \in (0, \infty)$, and one real negative eigenvalue $\lambda^s \in (-\infty, 0)$, and each of these eigenvalues is simple. The eigenvalues λ^u and λ^s depend C^1 smoothly on c and on the coefficients $A_{j,0}$, and we have that*

$$\frac{\partial \lambda^u}{\partial c} < 0, \qquad \frac{\lambda \lambda^s}{\partial c} < 0 \tag{4.11}$$

and also that

$$\begin{aligned} \sum_{j=1}^{N} w_j \frac{\partial \lambda^u}{\partial A_{j,0}} < 0, \qquad & w \in M_+^* \\ \sum_{j=1}^{N} w_j \frac{\partial \lambda^s}{\partial A_{j,0}} > 0, \qquad & w \in M_-^* \end{aligned} \tag{4.12}$$

and in addition, the inequality

$$\Delta_{c, L_0}(s) > 0, \qquad s \in (\lambda^s, \lambda^u) \tag{4.13}$$

The quantities $s = \lambda^s + i\eta$ and $s = \lambda^u + i\eta$, for $\eta \in \Gamma$, are eigenvalues if $c = 0$. For any $c \in \mathbf{R}$, all remaining eigenvalues satisfy

$$\operatorname{Re} \lambda \in (-\infty, \lambda^s) \cup (\lambda^u, \infty), \qquad \operatorname{Im} \lambda \neq 0 \tag{4.14}$$

where we interpret $(-\infty, \lambda^s) = \phi$ if λ^s does not exist, and similarly with (λ^u, ∞).

Proof. Hyperbolicity follows from Lemma 4.2, along with the observation that $\Delta_{c, L_0}(0) = -A_\Sigma > 0$. With the concavity condition (4.8) we conclude the existence of at most one $\lambda^u > 0$, and at most one $\lambda^s < 0$, and as well the inequality (4.13). In addition, we see that $\Delta'_{c, L_0}(\lambda^s) > 0$ and $\Delta'_{c, L_0}(\lambda^u) < 0$, and so these eigenvalues are simple and depend C^1 smoothly on c and on the coefficients $A_{j,0}$.

The inequalities (4.11) follow from implicit differentiation, as do the inequalities (4.12). To verify in particular the first inequality in (4.12), we calculate the derivative

$$\sum_{j=1}^{N} w_j \left. \frac{\partial \Delta_{c, L_0}(s)}{\partial A_{j,0}} \right|_{s=\lambda^u} = -\sum_{j=1}^{N} w_j e^{\lambda^u r_j} < 0 \tag{4.15}$$

for any $w \in M_+^*$, where the sign in (4.15) follows from the fact that the vector $v \in (0, \infty)^N$, with coordinates $v_j = e^{\lambda^u r_j}$, belongs to M_+. As $\Delta'_{c, L_0}(\lambda^u) < 0$, we conclude the first inequality in (4.12). The other inequality in (4.12), and also those in (4.11), follow by similar arguments.

The final claims about the eigenvalues $\lambda^{s,u} + i\eta$, for $\eta \in \Gamma$, and (4.14), follow directly from Lemma 4.2. □

Remark. One easily sees in the above proof that λ^u does not exist precisely when both $r_{\max} = 0$ and $c \leqslant 0$, for then $\Delta_{c, L_0}(s) > 0$ for all $s \geqslant 0$. Similarly λ^s does not exist precisely when $r_{\min} = 0$ and $c \geqslant 0$. As a convention, we may set $\lambda^u = \infty$ and $\lambda^s = -\infty$, respectively, in these two cases, and thus

$$\begin{array}{llll} \lambda^u = \infty & \text{if and only if } r_{\max} = 0 & \text{and} & c \leqslant 0 \\ \lambda^s = -\infty & \text{if and only if } r_{\min} = 0 & \text{and} & c \geqslant 0 \end{array} \tag{4.16}$$

We see that by regarding these eigenvalues as taking values in the extended real line, so $\lambda^u \in (0, \infty]$ and $\lambda^s \in [-\infty, 0)$, both λ^u and λ^s depend continuously on c and on the coefficients $A_{j,0}$.

Proposition 4.4. *Assume that (4.7) holds for Eq. (3.12) and that* $A_\Sigma > 0$. *Then either all real eigenvalues of (3.12) lie in* $(0, \infty)$ *or they all lie in* $(-\infty, 0)$.

Proof. We have that $\Delta_{c, L_0}(0) = -A_\Sigma < 0$, and $\Delta''_{c, L_0}(s) < 0$ for all real s. The result follows directly from this, using Lemma 4.2. □

The following result precludes the possibility of superexponentially decaying solutions of Eq. (1.15), when $c \neq 0$, at least among the class of nonnegative solutions. In particular, this result applies to Eq. (1.15) as in the statement as in the statement of Theorem 4.1.

Proposition 4.5. *Let $x: J^{\#} \to \mathbf{R}$ be a solution of (1.15) on $J = [\tau, \infty)$ for some $\tau \in \mathbf{R}$, with $c \neq 0$, and with conditions (4.1) and (4.2) on the coefficients holding, but only for $\xi \in J$ in (4.2). Assume also that $x(\xi) \geqslant 0$ for all $\xi \in J^{\#}$. Then there exist constants $a, k \in \mathbf{R}$, and $R > 0$, such that*

$$ax(\xi) \leqslant x'(\xi) \leqslant kx(\xi), \qquad \xi \geqslant \tau + R \tag{4.17}$$

In particular,

$$\lim_{\xi \to \infty} e^{b\xi} x(\xi) = 0 \tag{4.18}$$

cannot hold for every $b \in \mathbf{R}$ unless $x(\xi) = 0$ identically for $\xi \geqslant \tau + R$.

The analogous result for $J = (-\infty, \tau]$ also holds.

Proof. Let us observe first, that the lower bound in (4.17) precludes superexponential decay of x. In particular, (4.18) is false for $b = -a$, unless $x(\xi)$ vanishes identically for $\xi \geqslant \tau + R$, since $e^{-a\xi} x(\xi)$ is monotone increasing in ξ.

Without loss take $J = [\tau, \infty)$, as the case of $J = (-\infty, \tau]$ can be handled by a change of variables $\xi \to -\xi$. There are now two cases to consider, namely, $c > 0$ and $c < 0$. We shall only prove the result in the case that $c > 0$. Even though the case of $c < 0$ cannot be reduced to the case of positive c, the proof for negative c is very similar, so we omit it.

To begin, we have that

$$x'(\xi) = -c^{-1} \sum_{j=1}^{N} A_j(\xi)\, x(\xi + r_j) \leqslant -c^{-1} A_1(\xi)\, x(\xi) \leqslant -c^{-1} \alpha_1 x(\xi)$$

for $\xi \geqslant \tau$, which establishes the right-hand inequality of (4.17) with $k = -c^{-1}\alpha_1$.

We now prove the left-hand inequality in (4.17). We first consider the case in which $r_{\min} < 0$. Let $y(\xi) = e^{-k\xi} x(\xi)$ and note that $y'(\xi) \leqslant 0$ for $\xi \geqslant \tau$. Now fix $\xi_1 \geqslant \tau - r_{\min}$, let $\varepsilon > 0$ be such that $2\varepsilon = \min\{|r_j| \mid r_j < 0\}$, and observe for $\tau - r_{\min} \leqslant \xi \leqslant \xi_1$ that we have

$$\begin{aligned} y'(\xi) &= -(c^{-1} A_1(\xi) + k)\, y(\xi) - c^{-1} \sum_{j=2}^{N} A_j(\xi)\, e^{kr_j} y(\xi + r_j) \\ &\leqslant -c^{-1} \sum_{r_j < 0} \alpha_j e^{kr_j} y(\xi + r_j) \\ &\leqslant -c^{-1} \left(\sum_{r_j < 0} \alpha_j e^{kr_j} \right) y(\xi_1 - 2\varepsilon) \end{aligned} \tag{4.19}$$

where we have used the fact that $c^{-1}A_1(\xi)+k \geqslant c^{-1}\alpha_1+k=0$, from the definition of k above. Assuming further that $\xi_1-\varepsilon \geqslant \tau - r_{\min}$, we integrate (4.19) with respect to ξ, from $\xi_1-\varepsilon$ to ξ_1, and discard the term $y(\xi_1)\geqslant 0$ which appears in the left-hand side, to obtain

$$y(\xi_1-\varepsilon)\geqslant C^{-1}y(\xi_1-2\varepsilon), \qquad C=\varepsilon^{-1}c\left(\sum_{r_j<0}\alpha_j e^{kr_j}\right)^{-1}$$

and hence

$$y(\xi-\delta)\leqslant y(\xi-\varepsilon)\leqslant Cy(\xi) \tag{4.20}$$

provided that $\xi \geqslant \tau - r_{\min}$ and $0\leqslant\delta\leqslant\varepsilon$. Now by repeatedly using (4.20), we conclude that if $r<0$ and $\xi+r>\tau-r_{\min}$, then

$$y(\xi+r)\leqslant C^{1+[|r|/\varepsilon]}y(\xi) \tag{4.21}$$

where $[\gamma]$ denotes the greatest integer less that or equal to γ. Therefore, from the first line of (4.19), from (4.21), and from the fact that y is monotone decreasing, we have for $\xi \geqslant \tau - 2r_{\min}$ that

$$\begin{aligned} y'(\xi) &\geqslant -(c^{-1}\beta_1+k)\,y(\xi)-c^{-1}\left(\sum_{r_j>0}\beta_j e^{kr_j}+\sum_{r_j<0}\beta_j e^{kr_j}C^{1+[|r_j|/\varepsilon]}\right)y(\xi)\\ &=(a-k)\,y(\xi) \end{aligned} \tag{4.22}$$

where (4.22) defines a. From (4.22) we conclude that $x'(\xi)\geqslant ax(\xi)$, as claimed.

Now suppose that $r_{\min}=0$. With $y(\xi)=e^{-k\xi}x(\xi)$ as before, we have that $y(\xi+r_j)\leqslant y(\xi)$ for $\xi\geqslant\tau$, and hence from the first line of (4.19), we have that

$$\begin{aligned} y'(\xi) &\geqslant -\left(k+c^{-1}\sum_{j=1}^{N}A_j(\xi)\,e^{kr_j}\right)y(\xi)\\ &\geqslant -\left(k+c^{-1}\sum_{j=1}^{N}\beta_j e^{kr_j}\right)y(\xi)=(a-k)\,y(\xi) \end{aligned}$$

in fact with the same a obtained from (4.22). Again, $x'(\xi)\geqslant ax(\xi)$. □

The next proposition is a basic result that will help establish a number of the claims of Theorem 4.1.

Proposition 4.6. *Assume that Eq. (1.15) satisfies all the conditions in the statement of Theorem .1, except possibly for the existence of the solution* $x = p(\xi)$. *Then Eq. (1.15) is asymptotically hyperbolic. There exist four (possibly infinite) quantities* λ^s_+, λ^s_-, λ^u_+, *and* λ^u_-, *satisfying*

$$-\infty \leqslant \lambda^s_{\pm} < 0 < \lambda^u_{\pm} \leqslant \infty$$

such that $\lambda^s_{\pm}$ *and* $\lambda^u_{\pm}$ *(if finite) are real eigenvalues of the limiting equations at* $\pm\infty$. *These are the only such real eigenvalues, so are uniquely determined. When finite they are simple eigenvalues.*

If we only assume the asymptotic conditions of Theorem 4.1 at $+\infty$, *and if* $x(\xi)$ *satisfies Eq. (1.15) on some interval* $[\tau, \infty)$ *and is bounded as* $\xi \to \infty$, *then*

$$x(\xi) = C_+ e^{\lambda^s_+ \xi} + O(e^{(\lambda^s_+ - \varepsilon)\xi}), \qquad \xi \to \infty \tag{4.23}$$

for some constant $C_+ \in \mathbf{R}$ *and some* $\varepsilon > 0$ *if* λ^s_+ *is finite, while* $x(\xi) = 0$ *identically for* $\xi \geqslant \tau$ *if* $\lambda^s_+ = -\infty$. *If* λ^s_+ *is finite, then the asymptotic formula for* $x'(\xi)$ *obtained by formally differentiating (4.23) (including the remainder term) holds. If, moreover,* $x(\xi) \geqslant 0$ *for all large* ξ, *and if* $x(\xi)$ *does not vanish identically for large* ξ, *then* $C_+ > 0$.

The analogous results, in particular,

$$x(\xi) = C_- e^{\lambda^u_- \xi} + O(e^{(\lambda^u_- + \varepsilon)\xi}), \qquad \xi \to -\infty \tag{4.24}$$

when λ^u_- *is finite, hold for solutions which are bounded at* $-\infty$.

Proof. The characteristic equations of the limiting equations

$$-cx'(\xi) = \sum_{j=1}^{N} A_{j\pm} x(\xi + r_j) \tag{4.25}$$

are $\Delta_{c, L_{\pm}}(s) = 0$, where

$$\Delta_{c, L_{\pm}}(s) = -cs - \sum_{j=1}^{N} A_{j\pm} e^{sr_j}$$

By Proposition 4.3, and (4.4), both limiting Eqs. (4.25) are hyperbolic, and the claims about the eigenvalues $\lambda^s_{\pm}$, $\lambda^u_{\pm}$ follow from this, as does the fact that all other eigenvalues satisfy

$$\operatorname{Re} \lambda \in (-\infty, \lambda^s_{\pm}) \cup (\lambda^u_{\pm}, \infty), \qquad \operatorname{Im} \lambda \neq 0$$

where we interpret $(-\infty, \lambda^s_\pm) = \phi$ and $(\lambda^u_\pm, \infty) = \phi$ if $\lambda^s_\pm = -\infty$, respectively, $\lambda^u_\pm = \infty$.

Now suppose that x is a solution of (1.15) on some interval $[\tau, \infty)$, as in the statement of the proposition. First, if $\lambda^s_+ = -\infty$, then $r_{\min} = 0$ by (4.16), and $\operatorname{Re} \lambda \geqslant \lambda^u_+ > 0$ for all eigenvalues of the limiting equation at $+\infty$. The transformation $\xi \to -\xi$ converts (1.15) into a delay differential equation which is asymptotically hyperbolic at $-\infty$ with $\operatorname{Re} \lambda \leqslant -\lambda^u_+ < 0$ for all limiting eigenvalues there. Denoting $z(\xi) = x(-\xi)$ as a solution of this equation, one argues in a standard fashion, using a variation of constants formula [46], to show that $\|z_{\xi_1}\| \leqslant K e^{-b(\xi_1 - \xi_2)} \|z_{\xi_2}\|$ for some $K, b > 0$, for $\xi_2 \leqslant \xi_1 \leqslant -\tau$, where $\|z_\xi\|$ denotes the supremum of $|z(\xi + \theta)|$ for $\theta \in [-r_{\max}, 0]$. Letting $\xi_2 \to -\infty$ implies, as z is bounded, that $z_{\xi_1} = 0$, and thus $x(\xi) = 0$ identically for $\xi \geqslant \tau$.

Now suppose for the remainder of the proof that λ^s_+ is finite. Then Proposition 7.2 of Ref. 63 and the above properties of the eigenvalues immediately imply the claim (4.23). Indeed, this result implies either that

$$x(\xi) = y(\xi) + O(e^{-(b+\varepsilon)\xi}), \qquad \xi \to \infty \tag{4.26}$$

where y is a nontrivial eigensolution corresponding to a set of eigenvalues with $\operatorname{Re} \lambda = -b \leqslant 0$, or that (4.18) holds for all $b \in \mathbf{R}$. In the former case (4.26), either $-b = \lambda^s_+$, in which case $y(\xi) = C_+ e^{\lambda^s_+ \xi}$ for some $C_+ \neq 0$, or $-b < \lambda^s_+$, in which case we have (4.23) with $C_+ = 0$. And if (4.18) holds for all b, then, again, (4.23) holds with $C_+ = 0$.

The asymptotic expression for $x'(\xi)$ now follows by substituting the expression (4.23) into the right-hand side of the differential Eq. (1.15), and by noting that the leading term $C_+ e^{\lambda^s_+ \xi}$ is a solution of the limiting differential equation.

Suppose further that $x(\xi) \geqslant 0$, but is not identically zero, for all large ξ. In (4.23) we have $C_+ \geqslant 0$, and as we wish to prove $C_+ > 0$, we assume, to the contrary, that $C_+ = 0$. Proposition 4.5 implies that superexponential decay, namely, (4.18) for all b, is impossible, and so (4.26) holds with $-b < \lambda^s_+$. All eigenvalues λ with $\operatorname{Re} \lambda = -b$ have nonzero imaginary part, so by Lemma 3.4 there exist arbitrarily large ξ for which $x(\xi) > 0$, and arbitrarily large ξ with $x(\xi) < 0$. This is a contradiction, and so we conclude that $C_+ > 0$, as desired.

The proofs of the corresponding claims at $-\infty$ follow similar lines.

□

With the next result, we establish all but one of the claims of Theorem 4.1.

Proposition 4.7. *Assume the conditions of Theorem 4.1. Then the conclusions of Theorem 4.1 all hold, except possibly for the positivity (4.6) of some element $p^* \in \mathscr{K}_{c, L^*}$. We also have the asymptotic expressions*

$$p(\xi) = \begin{cases} C_- e^{\lambda^u_- \xi} + O(e^{(\lambda^u_- + \varepsilon)\xi}), & \xi \to -\infty \\ C_+ e^{\lambda^s_+ \xi} + O(e^{(\lambda^s_+ - \varepsilon)\xi}), & \xi \to \infty \end{cases} \tag{4.27}$$

for some $\varepsilon > 0$, where both $C_\pm > 0$, with finite exponents

$$\lambda^s_+ \in (-\infty, 0), \qquad \lambda^u_- \in (0, \infty) \tag{4.28}$$

and, as well, the asymptotic expressions for $p'(\xi)$ obtained by formally differentiating (4.27), including the remainder terms.

Proof. Proposition 4.6 gives the asymptotic hyperbolicity of (1.15), and Theorem A of Ref. 63 ensures that $\Lambda_{c, L}$ is a Fredholm operator. Lemmas 3.1 and 3.3 ensure the strict positivity $p(\xi) > 0$ for all $\xi \in \mathbf{R}$. Indeed, if $p(\tau) = 0$ at some τ then $p(\xi) = 0$ identically either for $\xi \geqslant \tau$ or for $\xi \leqslant \tau$, by Lemma 3.1. Then Lemma 3.3 would force $p(\xi) = 0$ identically for all $\xi \in \mathbf{R}$, a contradiction. The asymptotic expressions (4.27), with the positivity $C_\pm > 0$ of the coefficients and the finiteness (4.28) of the eigenvalues now follow from Proposition 4.6, as do the asymptotic expressions for $p'(\xi)$.

What remains is to establish the claims (4.5) about the kernel, the range, and the Fredholm index. Let us consider the kernel $\mathscr{K}_{c, L}$ of $\Lambda_{c, L}$. Suppose that $\dim \mathscr{K}_{c, L} > 1$, and take any $x \in \mathscr{K}_{c, L}$ which is linearly independent of $p \in \mathscr{K}_{c, L}$. By Proposition 4.6, again, the solution x enjoys estimates as in (4.27), but with generally different constants $C_\pm$ which need not be positive. By adding an appropriate multiple of p to x, we may assume without loss that the coefficient of $e^{\lambda^s_+ \xi}$ in the asymptotic expression for $x(\xi)$ as $\xi \to \infty$ vanishes. That is, we have that

$$x(\xi) = \begin{cases} C_0 e^{\lambda^u_- \xi} + O(e^{(\lambda^u_- + \varepsilon)\xi}), & \xi \to -\infty \\ O(e^{(\lambda^s_+ - \varepsilon)\xi}), & \xi \to \infty \end{cases} \tag{4.29}$$

for some $C_0 \in \mathbf{R}$. By replacing x with $-x$ if necessary, we may assume that $C_0 \leqslant 0$.

By Lemma 3.3, since x is not identically zero, there exist arbitrarily large ξ for which $x(\xi) \neq 0$. We claim further that there exist arbitrarily large ξ for which $x(\xi) > 0$. If not, and so $x(\xi) \leqslant 0$ for all large ξ, then by Proposition 4.6 applied to $-x$ we have the asymptotic formula (4.23) with coefficient C_+ (different from C_+ above) which is strictly negative, contradicting (4.29).

Now consider the family $p - \mu x \in \mathcal{K}_{c, L}$, for $\mu \geqslant 0$. There exists $\mu_0 > 0$ such that $p(\xi) - \mu_0 x(\xi) < 0$ for some $\xi \in \mathbf{R}$; this is simply because $x(\xi) > 0$ at some ξ, as noted above. The asymptotic formulas (4.27), (4.29), and the signs of $C_\pm > 0$ and $C_0 \leqslant 0$, ensure that there exists $\tau > 0$ such that

$$p(\xi) - \mu x(\xi) > 0, \qquad |\xi| \geqslant \tau, \qquad 0 \leqslant \mu \leqslant \mu_0 \tag{4.30}$$

It follows by setting

$$\mu_* = \sup\{\mu \in [0, \mu_0] \mid p(\xi) - \mu x(\xi) \geqslant 0 \text{ for all } \xi \in \mathbf{R}\}$$

that we have $\mu_* x(\xi) \leqslant p(\xi)$ for all $\xi \in \mathbf{R}$, with equality $\mu_* x(\xi_0) = p(\xi_0)$ at some $\xi_0 \in \mathbf{R}$ (in fact, with $\xi_0 \in [-\tau, \tau]$). However, Lemma 3.1 implies that $p(\xi) - \mu_* x(\xi) = 0$ identically either on $[\xi_0, \infty)$ or on $(-\infty, \xi_0]$, and this contradicts (4.30). The proof that $\mathcal{K}_{c, L}$ is one-dimensional is thus complete.

To complete the proof of the proposition, we show that $\operatorname{ind}(\Lambda_{c, L}) = 0$. Indeed, from this and from the definition (3.16) of index, it follows that $\operatorname{codim} \mathcal{R}_{c, L} = 1$, and from the Fredholm alternative (3.17) we have that $\dim \mathcal{K}_{c, L^*} = 1$, establishing (4.5). By Theorem B of Ref. 63, the quantity $\operatorname{ind}(\Lambda_{c, L})$ depends only on the limiting operators $L_\pm$ as in (3.13), (3.14), and on c. Moreover, by the so-called spectral flow formula, Theorem C of Ref. 63, we have that $\operatorname{ind}(\Lambda_{c, L}) = -\operatorname{cross}(L^\rho)$. Here L^ρ is any generic homotopy of constant coefficient operators joining L_- at $\rho = -1$ to L_+ at $\rho = 1$, and $\operatorname{cross}(L^\rho)$ denotes the net number of roots $s = \lambda$ of the characteristic equation $\Delta_{c, L^\rho}(s) = 0$ which cross the imaginary axis along this homotopy (we keep c fixed here). For the homotopy $L^\rho = ((1 - \rho) L_- + (1 + \rho) L_+)/2$ one sees that the corresponding constant coefficient Eq. (3.12) is hyperbolic for $-1 \leqslant \rho \leqslant 1$, by Proposition 4.3, using (4.4) in particular. No eigenvalues cross the imaginary axis, thus $\operatorname{ind}(\Lambda_{c, L}) = -\operatorname{cross}(L^\rho) = 0$. □

We now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. All that remains is to establish the positivity (4.6) of some $p^* \in \mathcal{K}_{c, L^*}$. It is in fact enough to show that $p^*(\xi) \geqslant 0$ for all $\xi \in \mathbf{R}$, for some nontrivial $p^* \in \mathcal{K}_{c, L^*}$, for strict positivity (4.6) of this solution follows by an application of Lemmas 3.1 and 3.3. Let us therefore suppose that there exists $p^* \in \mathcal{K}_{c, L^*}$ which assumes both signs, say $p^*(\xi_1) > 0$ and $p^*(\xi_2) < 0$, for some $\xi_1, \xi_2 \in \mathbf{R}$. We seek a contradiction.

First, we have that $p^*(\xi)$ does not vanish identically on any interval of length $r_{\max} - r_{\min}$, for otherwise, by Lemma 3.3 the solution p^* would be identically zero. Thus without loss we may assume $|\xi_1 - \xi_2| < r_{\max} - r_{\min}$.

It follows next that there exists a continuous function $h\colon \mathbf{R} \to \mathbf{R}$ such that

$$\int_{-\infty}^{\infty} p^*(\xi)\, h(\xi)\, dx = 0$$

where $h(\xi) \geqslant 0$ for all $\xi \in \mathbf{R}$, and where h has compact support, in fact $h(\xi) = 0$ for all $\xi \in \mathbf{R} \backslash [\tau_1, \tau_2]$ where $\tau_1 < \tau_2$ can be chosen to satisfy $\tau_2 - \tau_1 < r_{\max} - r_{\min}$. By the Fredholm alternative (3.17), and as $\dim \mathscr{K}_{c,L^*} = 1$, we have that $h \in \mathscr{R}_{c,L}$ and so there exists $x \in W^{1,\infty}$ such that $\Lambda_{c,L} x = h$. That is, $x\colon \mathbf{R} \to \mathbf{R}$ is a bounded solution of the inhomogeneous Eq. (3.11). We note that adding a multiple of $p \in \mathscr{K}_{c,L}$ to x yields any other such solution $x + \mu p$.

As h has compact support, $x(\xi)$ satisfies the homogeneous Eq. (1.15) for large $|\xi|$ and, so by Proposition 4.6, enjoys asymptotic estimates of the form (4.23) and (4.24), although with generally constants $C_{\pm}$ different from those (4.27) for p. As the constants in the formulas (4.27) for p are both positive, $C_{\pm} > 0$, we see that if μ is sufficiently large, then $x(\xi) + \mu p(\xi) > 0$ for all $\xi \in \mathbf{R}$, and that

$$\mu_* = \inf\{\mu \in \mathbf{R} \mid x(\xi) + \mu p(\xi) \geqslant 0 \text{ for all } \xi \in \mathbf{R}\}$$

is a finite quantity. Let us denote $y(\xi) = x(\xi) + \mu_* p(\xi)$. Of course, $y(\xi) \geqslant 0$ for all $\xi \in \mathbf{R}$.

By Proposition 4.6 again, and the nonnegativity of y, we have either that $y(\xi) = \tilde{C}_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \varepsilon)\xi})$ as $\xi \to \infty$, for some strictly positive $\tilde{C}_+ > 0$ and $\varepsilon > 0$, in which case $y(\xi) > 0$ for large ξ, or that $y(\xi) = 0$ identically for all large ξ. The corresponding statements for $\xi \to -\infty$ also hold. We claim that it is not the case that $y(\xi) = 0$ both for $\xi \to \infty$ and for $\xi \to -\infty$. Suppose to the contrary that $y(\xi) = 0$ for all large $|\xi|$. Now y satisfies the homogeneous Eq. (1.15) on the interval $J = [\tau_2, \infty)$, as $h(\xi) = 0$ there. One concludes from this, by Lemma 3.3, that $y(\xi) = 0$ identically on $J^\# = [\tau_2 + r_{\min}, \infty)$. Similarly, $y(\xi) = 0$ identically on $(-\infty, \tau_1]^\# = (-\infty, \tau_1 + r_{\max}]$. But $(-\infty, \tau_1 + r_{\max}] \cup [\tau_2 + r_{\min}, \infty) = \mathbf{R}$ as $\tau_2 - \tau_1 < r_{\max} - r_{\min}$, hence y is identically zero on $\mathbf{R}$. This contradicts the fact that $h = \Lambda_{c,L} y$ is not the zero function.

For definiteness, let us therefore assume that $y(\xi) > 0$ for all large ξ. It is also the case that $y(\xi) = 0$ for some $\xi \in \mathbf{R}$. Indeed, the proof of this uses arguments similar to those in the proof of Proposition 4.7, so we omit them. Now denote

$$\xi_0 = \sup\{\xi \in \mathbf{R} \mid y(\xi) = 0\}$$

Certainly both $y(\xi_0)=0$ and $y'(\xi_0)=0$, and also $y(\xi)>0$ for all $\xi>\xi_0$. If it is the case that $r_{\max}>0$ then we immediately have a contradiction. Indeed, if $2\leqslant j_0\leqslant N$ is such that $r_{j_0}=r_{\max}$, then by Eq. (3.11), as $A_j(\xi_0)>0$ for $2\leqslant j\leqslant N$, and as $y(\xi_0+r_{j_0})>0$ and $h(\xi_0)\geqslant 0$, we have that

$$-cy'(\xi_0)=\sum_{j=1}^{N} A_j(\xi_0)\, y(\xi_0+r_j)+h(\xi_0)\geqslant A_{j_0}(\xi_0)\, y(\xi_0+r_{j_0})>0$$

which is false.

We may also obtain a contradiction if $r_{\max}=0$. In this case, as $\lambda^u_-<\infty$ holds (4.28), necessarily $c>0$ by (4.16). If it is the case that $y(\xi)>0$ as $\xi\to-\infty$, then as $r_{\min}<0$, an argument analogous to the one in the above paragraph, but at the point $\xi_{00}=\inf\{\xi\in\mathbf{R}\mid y(\xi)=0\}$, yields a contradiction. Suppose therefore that $y(\xi)=0$ identically as $\xi\to-\infty$. Denote

$$\xi_*=\sup\{\xi\in\mathbf{R}\mid y(\zeta)=0 \text{ for all } \zeta\leqslant\xi\}$$

and consider Eq. (3.11) on an interval $[\xi_*,\xi_*+\varepsilon]$ for sufficiently small $\varepsilon>0$, namely, $\varepsilon\leqslant|r_j|$ for $2\leqslant j\leqslant N$. Then $y(\xi+r_j)=0$ for $2\leqslant j\leqslant N$ and $\xi\in[\xi_*,\xi_*+\varepsilon]$, so (3.11) takes the form of an ordinary differential equation $-cy'(\xi)=A_1(\xi)\,y(\xi)+h(\xi)$ on this interval, with initial condition $y(\xi_*)=0$. As both $c>0$ and $h(\xi)\geqslant 0$, We conclude by a standard differential inequality that $y(\xi)\leqslant 0$ in this interval, hence $y(\xi)=0$ there. However, this contradicts the definition of ξ_*. With this, the proof of the theorem is complete. □

5. LINEAR DIFFERENCE EQUATIONS

Our focus in this section is the linear difference equation obtained by setting $c=0$ in (3.11), that is,

$$L(\xi)\,x_\xi+h(\xi)=0 \tag{5.1}$$

with $L(\xi)$ as in (2.24). We continue to assume that x is scalar, the shifts r_j satisfy (3.2), and the coefficients $A_j(\xi)$ are continuous and bounded. We recall that generally, solutions of (5.1) need not be continuous, and indeed, one often expects discontinuities.

We begin with the following lemma, which provides an exponential bounds on solutions.

Lemma 5.1. *Let $x\colon J^\# \to(0,\infty)$ be a positive, monotone decreasing solution of the homogeneous equation*

$$L(\xi)\,x_\xi=0 \tag{5.2}$$

either on $J=[\tau,\infty)$ for some $\tau\in\mathbf{R}$, or on $J=\mathbf{R}$. Assume either that (5.2) is actually a constant coefficient system

$$L_0 x_\xi = 0 \tag{5.3}$$

or that $r_{\min}<0$, where, in the latter case, assume that the coefficients A_j satisfy the bounds (4.1) and (4.2), but only for $\xi\in J$ in (4.2). Then there exist constants $K>0$ and $b>0$ such that

$$x(\xi_1)\leqslant Ke^{b(\xi_2-\xi_1)}x(\xi_2), \qquad \xi_1\leqslant\xi_2, \qquad \xi_1,\xi_2\in J^\#\tag{5.4}$$

Proof. Letting j_0, with $1\leqslant j_0\leqslant N$, be such that $r_{\min}=r_{j_0}$, we have that

$$x(\xi+r_{\min}) = -\sum_{\substack{j=1\\ j\neq j_0}}^{N} A_j(\xi)\,A_{j_0}(\xi)^{-1}\,x(\xi+r_j), \qquad \xi\in J \tag{5.5}$$

in either case (5.2) or (5.3). If the coefficients A_j are not constant, then $r_{\min}<0$, so the uniform lower bound $A_{j_0}(\xi)\geqslant\alpha_{j_0}>0$ implies that $A_{j_0}(\xi)^{-1}$ is bounded. From the bounds (4.1), (4.2), and the monotonicity of x, we have from (5.5) that

$$x(\xi)\leqslant Kx(\xi+\varepsilon), \qquad \xi\in J^\#$$

for some $K>0$ and some $\varepsilon>0$, indeed with $\varepsilon=\min\{r_j-r_{\min}\mid 1\leqslant j\leqslant N$ and $j\neq j_0\}$. The result (5.4) now follows easily, with b such that $e^{b\varepsilon}=K$. □

The following three results very broadly relate the rate of decay of monotone solutions of the homogeneous Eqs. (5.2) or (5.3) to their real eigenvalues. As noted in an earlier remark, our results here are not as strong as those for the differential equation in Section 4, due to possible pathologies of the spectrum when the shifts are not rationally related.

Lemma 5.2. *Suppose that $x\colon J^\#\to(0,\infty)$ is a positive, monotone decreasing solution of the homogeneous constant coefficient Eq. (5.3) on $J=[\tau,\infty)$, for some $\tau\in\mathbf{R}$. Then there exists $\lambda\in(-\infty,0]$ with $\Delta_{0,L_0}(\lambda)=0$.*

Proof. Without loss assume that $\tau=0$. Taking the Laplace transform of (5.3) now yields, in a standard fashion,

$$\tilde{x}(s)=\Delta_{0,L_0}(s)^{-1}\,\psi(s), \qquad \psi(s)=-\sum_{j=1}^{N} A_{j,0}\int_{-r_j}^{0} e^{-s\xi}(\xi+r_j)\,d\xi \tag{5.6}$$

where the meromorphic function $\tilde{x}(s)$ is holomorphic for $\operatorname{Re} s > 0$, from the boundedness of $x(\xi)$. Let $a \in [-\infty, \infty]$ be as in (3.18) for the Laplace transform $\tilde{x}$ of x. We have that $a \leqslant 0$ as x is bounded, and that $a \geqslant -b > -\infty$ with b as in Lemma 5.1. Thus $|a| < \infty$, and as ψ is entire it follows that $\Delta_{0, L_0}(a) = 0$. This proves the lemma, with $\lambda = a$. □

Recall the **order** of an entire function f, namely, the quantity $\omega(f)$ defined by

$$\omega(f) = \limsup_{R \to \infty} \frac{\log \log M(R, f)}{\log R}, \qquad M(R, f) = \sup_{|s| \leqslant R} |f(s)|$$

Clearly, the order is either finite $\omega(f) \in [0, \infty)$, or infinite $\omega(f) = \infty$. It is known [56] that if f is the quotient $f(s) = f_1(s)/f_2(s)$ of two entire functions of finite order, and if f is also an entire function, then f has finite order.

Lemma 5.3. *Suppose that $x\colon \mathbf{R} \to (0, \infty)$ is a positive, monotone decreasing solution of (5.3). Assume that there exists a unique $\lambda \in (-\infty, 0]$ at which $\Delta_{0, L_0}(\lambda) = 0$, and assume further that*

$$\begin{aligned} &\Delta'_{0, L_0}(\lambda) \neq 0 \\ &\Delta_{0, L_0}(\lambda + i\eta) \neq 0, \qquad \eta \in \mathbf{R} \backslash \Gamma \end{aligned} \tag{5.7}$$

where Γ is as in (4.9). Then

$$x(\xi) = K(\xi)\, e^{\lambda \xi}, \qquad \xi \in \mathbf{R} \tag{5.8}$$

where the function K has period v (as in (3.4)) if the shifts are rationally related, and K is constant if the shifts are not rationally related.

Proof. We first modify the solution x in the case that the r_j are rationally related. With v as in (3.4) and λ as in the statement of the lemma, fix any $\zeta \in [0, v)$ and let

$$x^*(\xi) = x(\zeta + nv)\, e^{\lambda(\xi - \zeta - nv)}, \qquad nv \leqslant \xi < (n+1)\, v \tag{5.9}$$

for every $n \in \mathbf{Z}$. One easily sees that x^* satisfies Eq. (5.3) everywhere, as all the shifts in that equation are integer multiples of v. We shall show below that $x^*(\xi) = x^*(0)\, e^{\lambda \xi}$ for all $\xi \in \mathbf{R}$, and therefore that

$$x(\zeta + nv) = x^*(\zeta + nv) = x^*(0)\, e^{\lambda(\zeta + nv)} \tag{5.10}$$

By (5.9) the quantity $x^*(0)$ depends only on $\zeta \in [0, \nu)$, and denoting $K(\zeta) = x^*(0)$ and extending K as a ν-periodic function, we obtain (5.8) from (5.10).

For ease of notation, in the case of rationally related shifts we denote the modified function x^* simply by x in the argument below. In this case we have

$$x(\xi) = x(n\nu)\, e^{\lambda(\xi - n\nu)}, \qquad n\nu \leqslant \xi < (n+1)\,\nu \tag{5.11}$$

for every $n \in \mathbf{Z}$, by (5.9).

We take the Laplace transform of (5.3), to obtain (5.6) as before. Let us also take the Laplace transform $\tilde{y}(s)$ of the function $y(\xi) = x(-\xi)$, where Lemma 5.1 ensures that $y(\xi)$ has at most exponential growth, so the transform exists. A simple calculation reveals that

$$\tilde{y}(s) = -\Delta_{0,\, L_0}(-s)^{-1}\, \psi(-s) \tag{5.12}$$

for $\operatorname{Re} s$ sufficiently large, with the same function ψ as in (5.6). Since $\Delta_{0,\, L_0}(s)^{-1}\, \psi(s)$ is holomorphic for $s \in \mathbf{R}$, except possibly at the point $s = \lambda$, we conclude from Lemma 3.5 that $\tilde{x}(s)$ is holomorphic for $\operatorname{Re} s > \lambda$. Similarly, we have that $\tilde{y}(s)$ is holomorphic for $\operatorname{Re} s > -\lambda$, but as $\tilde{x}(s) = -\tilde{y}(-s)$ identically by (5.6) and (5.12), we have that $\tilde{x}(s)$ is holomorphic also for $\operatorname{Re} s < \lambda$. These facts, together with the assumptions (5.7), now imply that

$$\tilde{x}(s) = \Delta_{0,\, L_0}(s)^{-1}\, \psi(s) \qquad \text{is holomorphic in} \quad \mathbf{C} \setminus \{\lambda + i\eta \mid \eta \in \Gamma\} \tag{5.13}$$

with at most a simple pole at $s = \lambda$.

Let us also note here the bound

$$|\tilde{x}(s)| = |\Delta_{0,\, L_0}(s)^{-1}\, \psi(s)| \leqslant K_1, \qquad |\operatorname{Re} s| \geqslant K_2 \tag{5.14}$$

for some K_1 and K_2. Indeed, if j_0 is such that $r_{j_0} = r_{\max}$, then $|\Delta_{0,\, L_0}(s)| \sim |A_{j_0,\, 0}|\, e^{(\operatorname{Re} s)\, r_{\max}}$ uniformly as $\operatorname{Re} s \to \infty$. In addition we have from (5.6) that $|\psi(s)| = O(e^{(\operatorname{Re} s)\, r_{\max}})$ uniformly as $\operatorname{Re} s \to \infty$, and this gives (5.14) for $\operatorname{Re} s \geqslant K_2$. A similar argument gives (5.14) for $\operatorname{Re} s \leqslant -K_2$.

We next claim that

$$\Psi(s) = (s - \lambda)\, \tilde{x}(s) \qquad \text{is an entire function} \tag{5.15}$$

Certainly Ψ is holomorphic at $s = \lambda$, as this is a simple pole of $\tilde{x}$. Thus (5.15) holds if the r_j are not rationally related, by (5.13), as $\Gamma = \{0\}$. Suppose, on the other hand, that the r_j are rationally related. Then $x(\xi)\, e^{-\lambda\xi}$

is constant on each interval $[nv, (n+1)v)$ by (5.11). One easily shows directly from the formula (3.19) for the Laplace transform that the function Ψ is a (generally infinite) linear combination of functions $s \to e^{(\lambda - s)nv}$ for $n \in \mathbf{Z}$, and hence $\Psi(s + 2\pi i v^{-1}) = \Psi(s)$ for $s \in \mathbf{C}$. Since $\Psi(s)$ is holomorphic at $s = \lambda$, it is holomorphic at $s = \lambda + i\eta$ for each $\eta \in \Gamma$, so by (5.13) is entire.

As Ψ is a ratio of two entire functions $(s - \lambda)\psi(s)$ and $\Delta_{c, L_0}(s)$ of at most exponential growth, and hence of finite order, we have that $\omega(\Psi) < \infty$ for the order of Ψ, and similarly $\omega(\Omega) < \infty$ for the order of the entire function $\Omega(s) = s^{-1}(\Psi(s) - \Psi(0))$. In addition, from the bound (5.14) we have that

$$|\Omega(s)| = \left| \frac{\Psi(s) - \Psi(0)}{s} \right| \leqslant K_3, \qquad |\mathrm{Re}\, s| \geqslant K_2$$

for some K_3. By a theorem of Phragmén–Lindelöf type (see, for example, Rudin [78]), we conclude that Ω is in fact constant, and so $\Psi(s) = \Psi(0) + \gamma s$ for some γ, and hence

$$\tilde{x}(s) = \frac{\Psi(0) + \gamma s}{s - \lambda}, \qquad s \in \mathbf{C}$$

We see directly from (3.19) that $\tilde{x}(s) \to 0$ as $s \to \infty$, and so $\gamma = 0$. From here it is immediate that $x(\xi) = x(0)\, e^{\lambda \xi}$ for $\xi \geqslant 0$, with $x(0) = \Psi(0)$. To obtain the same formula for $\xi \leqslant 0$, we see that $\tilde{y}(s) = -\tilde{x}(-s) = \Psi(0)(s + \lambda)^{-1}$, hence $x(\xi) = y(-\xi) = \Psi(0)\, e^{\lambda \xi}$ for $\xi \leqslant 0$. With this, (5.8) is established. □

Proposition 5.4. *Suppose that $x\colon J^{\#} \to (0, \infty)$ is a positive, monotone decreasing solution of the the homogeneous Eq. (5.2) on $J = [\tau, \infty)$, for some $\tau \in \mathbf{R}$. Assume also that conditions (4.1) and (4.2) hold, but only for $\xi \in J$ in (4.2), and that $r_{\min} < 0$. Finally, assume that (5.2) is asymptotically autonomous at $+\infty$, in the sense that the limits in (3.14) hold, but only for $\xi \to \infty$. Then there exists $\lambda \in (-\infty, 0]$ with $\Delta_{0, L_+}(\lambda) = 0$. Also, if $A_{\Sigma +} < 0$ then $\lambda = \lambda^s_+ \in (-\infty, 0)$ and*

$$\lim_{\xi \to \infty} \frac{1}{\xi} \log x(\xi) = \lambda^s_+ \qquad (5.16)$$

The analogous results at $-\infty$ hold when $r_{\max} > 0$, with λ^u_- replacing λ^s_+.

Remark. The notation λ^s_+ and λ^u_- follows that of Proposition 4.6. The uniqueness of these eigenvalues is guaranteed by Proposition 4.3.

Proof. Take any sequence $\xi_n \to \infty$, and let

$$y_n(\zeta) = \frac{x(\xi_n + \zeta)}{x(\xi_n)}$$

for each ζ for which $\xi_n + \zeta \in J^{\#}$. By passing to a subsequence, and using the estimate (5.4) of Lemma 5.1, we may assume the limit $y_n(\zeta) \to y(\zeta)$ exists at every $\zeta \in \mathbf{R}$. The limiting function $y\colon \mathbf{R} \to [0, \infty)$ is monotone decreasing, with $y(0) = 1$, satisfies the limiting equation $L_+ y_\zeta = 0$ at every $\zeta \in \mathbf{R}$, and inherits the estimate (5.4) so in particular is positive. It follows from Lemma 5.2 that there exists $\lambda \in (-\infty, 0]$ with $\Delta_{0, L_+}(\lambda) = 0$, as claimed.

Assume further that $A_{\Sigma +} < 0$. Then by Proposition 4.3 we have $\lambda = \lambda^s_+ < 0$. By Lemma 5.3 we conclude that $y(\zeta) = K(\zeta)\, e^{\lambda^s_+ \zeta}$ for all $\zeta \in \mathbf{R}$, with K either ν-periodic or constant, as in the statement of that result, and with $K(0) = 1$ in any case. Note that the properties of the roots of $\Delta_{0, L_+}(s)$ as described in Lemma 4.2 are used here.

Now fix $\zeta > 0$ so that $K(\zeta) = 1$, taking, for example, $\zeta \nu$ for rationally related shifts, or any positive ζ if the shifts are not rationally related. As the sequence ξ_n is arbitrary, we in fact have that

$$\lim_{\xi \to \infty} \frac{x(\xi + \zeta)}{x(\xi)} = e^{\lambda^s_+ \zeta} \tag{5.17}$$

It is now easy to conclude (5.16). Writing any large ξ as $\xi = \xi_0 + n\zeta$ for some $\xi_0 \in [\tau, \tau + \zeta)$ and integer n, we have

$$\begin{aligned} \log x(\xi) &= \log x(\xi_0) + \sum_{k=1}^{n} (\log x(\xi_0 + k\zeta) - \log x(\xi_0 + (k-1)\,\zeta)) \\ &= \log x(\xi_0) + n\lambda^s_+ \zeta + \sum_{k=1}^{n} \varepsilon_k \end{aligned} \tag{5.18}$$

by (5.17), where $\varepsilon_k = \varepsilon_k(\xi_0) \to 0$ as $k \to \infty$, uniformly in ξ_0. Dividing both sides of (5.18) by $n\zeta$ and letting $n \to \infty$ yields (5.16), as desired. □

The following lemma prepares for Lemma 5.6, which will be instrumental in proving Lemma 8.1, and then Theorem 2.6.

Lemma 5.5. *Let $x\colon [\tau, \infty) \to \mathbf{R}$ be a nonnegative, monotone decreasing function with $x(\xi) \to 0$ as $\xi \to \infty$. Then for any $r \in \mathbf{R}$, we have that*

$$\int_{\tau_0}^{\infty} x(\xi + r) - x(\xi)\, d\xi \geqslant -r x(\tau_0) \tag{5.19}$$

provided that $\tau_0 + \min\{0, r\} \geqslant \tau$.

Remark. We allow the function x to have discontinuities. If x is discontinuous at $\xi = \tau_0$, then we may redefine x at this point to be either the left- or right-hand limit $x(\tau_0 - 0)$ or $x(\tau_0 + 0)$, without changing the value of the integral. We obtain the strongest conclusion by maximizing the right-hand side of (5.19), that is, by taking $x(\tau_0 - 0)$ when $r < 0$ and $x(\tau_0 + 0)$ when $r > 0$.

Proof. Without loss we may take $\tau_0 = 0$. First suppose that $r < 0$, and let

$$S = \{(\xi, y) \mid \xi > 0 \text{ and } x(\xi) < y < x(\xi + r)\}$$

Then the integral in (5.19) is simply the two-dimensional measure of the set S. Let

$$S_0 = S \cap \{(\xi, y) \mid 0 < y < x(0)\} \tag{5.20}$$

We shall show that the measure of S_0 equals $-rx(0)$, thereby establishing (5.19).

As x is monotone decreasing, we have that for all but countably many $y \in (0, x(0))$, there exists a unique $\xi_0 = \xi_0(y)$ such that $x(\xi_0 + 0) \leqslant y \leqslant x(\xi_0 - 0)$. Note that $\xi_0 \geqslant 0$. For such y, we have

$$(\xi_0, \xi_0 - r) \subseteq \{\xi \mid (\xi, r) \in S\} \subseteq [\xi_0, \xi_0 - r]$$

from the definition of S. Thus, almost every horizontal line in the range $O < y < x(0)$ intersects the set S_0 in an interval of length exactly $-r$, and so the measure of S_0 is $-rx(0)$, as claimed.

The proof of the case $r > 0$ follows similar lines, but is not completely analogous. Let

$$S = \{(\xi, y) \in \mathbf{R}^2 \mid \xi > 0 \text{ and } x(\xi + r) < y < x(\xi)\}$$

and let

$$S_0 = \{(\xi, y) \in \mathbf{R}^2 \mid \xi > -r \text{ and } x(\xi + r) < y < x(\xi)\}$$
$$\cap \{(\xi, y) \in \mathbf{R}^2 \mid 0 < y < x(0)\} \tag{5.21}$$

The measure of S equals the negative of the integral in (5.19). Also, the same argument used above to calculate the measure of the set (5.20), shows that S_0 in (5.21) has measure $rx(0)$. But one also sees that $S \subseteq S_0$, which implies (5.19). □

Lemma 5.6. *Let* $x\colon J^{\#}\to\mathbf{R}$ *be a nonnegative, monotone decreasing function which satisfies Eq. (5.1) on* $J=[\tau,\infty)$ *for some* $\tau\in\mathbf{R}$, *and where* $x(\xi)\to 0$ *as* $\xi\to\infty$. *Assume that*

$$\sum_{j=1}^{N} A_j(\xi)=0, \qquad h(\xi)\geqslant 0, \qquad \xi\geqslant\tau \tag{5.22}$$

and also that the associated homogeneous Eq. (5.2) is asymptotically autonomous at $+\infty$ *with limiting coefficients satisfying* $A_{j+}>0$ *for* $2\leqslant j\leqslant N$, *and*

$$\sum_{j=2}^{N} r_j A_{j+}<0 \tag{5.23}$$

Then we have that

$$h(\xi)=0 \qquad \text{for almost every} \quad \xi\geqslant\tau_0 \tag{5.24}$$

for some sufficiently large τ_0. *If (5.23) is replaced with*

$$\begin{gathered}\sum_{j=2}^{N} r_j A_{j+}=0\\ A_j(\xi)=A_{j+} \qquad \text{identically for all large } \xi, \qquad 1\leqslant j\leqslant N\end{gathered} \tag{5.25}$$

then the same conclusion (5.24) holds.

The analogous result for nonnegative, monotone increasing solutions of (5.1) on $(-\infty,\tau]$, *where (5.22) is assumed for* $\xi\leqslant\tau$, *with* $A_{j-}>0$ *for* $2\leqslant j\leqslant N$ *and (5.23) replaced with*

$$\sum_{j=2}^{N} r_j A_{j-}>0$$

and an analogous condition replacing (5.25), also holds.

Proof. If either (5.23) or (5.25) is assumedly then for some $\tau_0\geqslant\tau$ there exist quantities $\gamma_j>0$ such that

$$\begin{gathered} r_j A_j(\xi)\leqslant r_j\gamma_j, \qquad \xi\geqslant\tau_0, \qquad 2\leqslant j\leqslant N\\ \sum_{j=2}^{N} r_j\gamma_j\leqslant 0\end{gathered}$$

Thus for $\xi \geqslant \tau_0$ we have that

$$0 = L(\xi)\,x_\xi + h(\xi) = \sum_{j=1}^{N} A_j(\xi)\,x(\xi + r_j) + h(\xi)$$

$$= \sum_{j=2}^{N} A_j(\xi)(x(\xi + r_j) - x(\xi)) + h(\xi)$$

$$\geqslant \sum_{j=2}^{N} \gamma_j(x(\xi + r_j) - x(\xi)) + h(\xi) \tag{5.26}$$

where the equation in (5.22) and the monotonicity of x are used. Assuming that $\tau_0 \geqslant \tau - r_{\min}$, we integrate (5.26) and use Lemma 5.5 to obtain

$$0 \geqslant \sum_{j=2}^{N} \gamma_j \int_{\tau_0}^{\infty} x(\xi + r_j) - x(\xi)\,d\xi + \int_{\tau_0}^{\infty} h(\xi)\,d\xi$$

$$\geqslant -\left(\sum_{j=2}^{N} r_j\gamma_j\right) x(\tau_0) + \int_{\tau_0}^{\infty} h(\xi)\,d\xi$$

$$\geqslant \int_{\tau_0}^{\infty} h(\xi)\,d\xi \tag{5.27}$$

and hence the final integral in (5.27) is zero by the nonnegativity of h. This now implies the result.

The results for solutions on $(-\infty, \tau]$ are obtained after a change of variable $\xi \to -\xi$. □

6. THE GLOBAL STRUCTURE OF $\mathcal{M}$

We return now to the analysis of the nonlinear Eq. (2.1), with the remainder of the paper devoted to proving the results stated in Section 2. Throughout, we assume conditions (i)–(v), given in Section 2, as standing hypotheses. Recall also from Section 2 the additional conditions (vi)–(x) for a normal family, and condition (xi) for a homotopy of normal families. These will be used below where noted, although we do not take them as standing hypotheses.

The results in Section 2 will not be proved in the order in which they are stated, and in particular the proofs of the various parts of Theorem 2.1 are interspersed as propositions throughout this and the following two sections. We therefore provide, for the benefit of the reader, a rough outline of the main points of these sections. The strict monotonicity condition

$P'(\xi)>0$ of every solution of Eq. (2.1) satisfying (1.3), when $c\neq 0$, is proved in Proposition 6.3. In Proposition 6.5 we establish for each $\rho\in W$ that there is at most one value $c\in\mathbf{R}$ for which (2.1), with (1.3), possesses a monotone increasing solution $x=P(\xi)$. We may thus denote this unique value by $c=c(\rho)$, for every $\rho\in W$ for which such a solution exists. Proposition 6.5 also proves the uniqueness of the solution up to translation whenever $c(\rho)\neq 0$. Observe that with this, all the uniqueness claims (but not yet the results on existence) of Theorem 2.1 are established. Also, whenever $c(\rho)\neq 0$ there is a unique translate of the solution such that $P(0)=0$, that is, $P\in W_0^{1,\infty}$. We denote this translate by $P(\xi,\rho)$, or simply by $P(\rho)\in W_0^{1,\infty}$ when the argument ξ is suppressed. The smooth dependence of $c(\rho)$ and $P(\rho)$ on ρ, whenever $c(\rho)\neq 0$, is a consequence of an implicit function theorem argument provided by Proposition 6.4. This establishes the relative openness of the set $U\subseteq W$ in (2.4), and shows that the set $\mathscr{M}$ in (2.5) is a graph over U, as claimed. The present section also contains the proof of Theorem 2.2, which describes the asymptotic behavior of solutions, and it closes with the proof of Proposition 2.4, which gives a monotonicity property for $c(\rho)$ under a related condition on the nonlinearity F.

Section 7 is largely concerned with the behavior of $c(\rho)$ and $P(\rho)$ at the relative boundary $(\bar{U}\backslash U)\cap W$ of U in W. It is shown in Proposition 7.2 that $c(\rho)\to 0$ as ρ approaches this boundary. Theorem 2.3, which involves related continuity properties of the solution $P(\rho)$, is also proved there. In particular, this establishes the existence of a limiting solution $x=P(\xi)$ for each $\rho\in(\bar{U}\backslash U)\cap W$, with $c=c(\rho)=0$, thereby proving the continuity of $c(\rho)$ at such points. The proof of Corollary 2.5 is also given in this section.

In Section 8 the existence of a solution to (2.1), with (1.3), for every $\rho\in W$, is established, thereby completing the proof of Theorem 2.1. Existence is first proved for normal families which are coercive (or weakly coercive) at ± 1. First, Theorem 2.6 is proved by means of a homotopy argument, thereby giving $U=(-1,\rho_-)\cup(\rho_+,1)\subseteq W=(-1,1)$ for some $-1<\rho_-\leqslant\rho_+<1$. This provides solutions for every $\rho\in\bar{U}\cap W=(-1,\rho_-]\cap[\rho_+,1)$. Next, the existence of a solution with $c=0$ for every $\rho\in W\backslash\bar{U}=(\rho_-,\rho_+)$ is obtained by means of comparison arguments, using $P(\rho_\pm)$ as sub- and supersolutions. It is then a simple matter to prove existence in general, and thus complete the proof of Theorem 2.1, by showing that any fixed system (2.1), but with the parameter ρ absent, can be embedded in a normal family which is coercive.

We now proceed with the above program, beginning with some basic remarks about linearized systems. If $x\colon\mathbf{R}\to\mathbf{R}$ is any solution of (2.1) for some $\rho\in\bar{V}$, then we have the **linearization about** $\boldsymbol{x}$, namely, Eq. (1.15) with coefficients (2.22), (2.23). If $c \neq 0$, then $x'(\xi)$ satisfies this equation. The linearization about any of the three equilibrium solutions $x = \pm 1$ and $x = q(\rho)$ is Eq. (3.12) with constant coefficients

$$A_{j,0}(\rho) = \left. \frac{\partial F(u, \rho)}{\partial u_j} \right|_{u=\kappa(x)} \tag{6.1}$$

We shall distinguish these three sets of coefficients by denoting

$$A_{j\pm}(\rho) = \left. \frac{\partial F(u, \rho)}{\partial u_j} \right|_{u=\kappa(\pm 1)}, \qquad A_{j\diamond}(\rho) = \left. \frac{\partial F(u, \rho)}{\partial u_j} \right|_{u=\kappa(q(\rho))} \tag{6.2}$$

Observe that condition (ix) for a normal family says precisely that $A_{j\pm}(\rho)$ is a C^1 function of $\rho \in W$.

Remark. Note here a departure from the notation of the previous sections, in particular (3.14), where we used $A_{j\pm}$ to denote the limits of coefficients $A_j(\xi)$ as $\xi \to \pm\infty$. If we linearize about a solution of (2.1) connecting $x = -1$ to $x = 1$, that is (1.3), then our old notation is still valid. However, we shall also need to consider solutions joining -1 to $q(\rho)$, or joining $q(\rho)$ to 1, in which case (3.14) no longer holds. Henceforth, we shall use the notation (6.2) exclusively, and will abandon the notation (3.14).

We shall also denote, in the spirit of the above remark, the constant coefficient operators $L_{\cdot}(\rho)$, and the quantities

$$A_{\Sigma\cdot}(\rho) = \sum_{j=1}^{N} A_{j\cdot}(\rho)$$

where $\cdot$ denotes one of the symbols $+$, $-$, or $\diamond$. Let us note here that

$$A_{\Sigma\pm}(\rho) = D_1 \Phi(\pm 1, \rho), \qquad A_{\Sigma\diamond}(\rho) = D_1 \Phi(q(\rho), \rho) \tag{6.3}$$

where Φ is as in (iv) of Section 2, and note by (v) that $A_{\Sigma\pm}(\rho) < 0$ and $A_{\Sigma\diamond}(\rho) > 0$, as long as $\rho \in W$. We observe by (ii), that for any choice of $\cdot$ we have $A_{j\cdot}(\rho) > 0$ for those $j \in \{1\} \cup \mathscr{U}(\rho)$, with of course $A_{j\cdot}(\rho) = 0$ when $j \notin \{1\} \cup \mathscr{U}(\rho)$, and so from Proposition 4.3 the linearizations about the equilibria $x = \pm 1$ are both hyperbolic. Following the notation of Proposition 4.6, we let $\lambda^s_\pm$ and $\lambda^u_\pm$ denote the unique real [or infinite, following the convention (4.16)] stable and unstable eigenvalues of these linearizations.

Let $x_1, x_2: \mathbf{R} \to \mathbf{R}$ be any two solutions of (2.1) for some c. Then the difference $y(\xi) = x_1(\xi) - x_2(\xi)$ satisfies the linear Eq. (1.15) with coefficients

$$A_j(\xi) = \int_0^1 \left. \frac{\partial F(u, \rho)}{\partial u_j} \right|_{u = t\pi(x_1, \xi) + (1-t)\pi(x_2, \xi)} dt \tag{6.4}$$

Indeed, one sees this from the general formula

$$\begin{aligned} F(v, \rho) - F(w, \rho) &= \int_0^1 \frac{dF(tv + (1-t) w, \rho)}{dt} dt \\ &= \sum_{j=1}^{N} \left(\int_0^1 \frac{\partial F(tv + (1-t) w, \rho)}{\partial u_j} dt \right) (v_j - w_j) \end{aligned}$$

for any $v, w \in \mathbf{R}^N$.

Suppose now that $P: \mathbf{R} \to \mathbf{R}$ is a solution to (2.1) for some $\rho \in W$, with $c \neq 0$, and with the boundary conditions (1.3) holding. That is, P (or a translate of P) is the solution described in the statement of Theorem 2.1, and which occurs in the definition of $\mathscr{M}$ which follows that theorem. Then the operator $\Lambda_{c,L}$ associated to the linearization (1.15) about P satisfies all the conditions of Theorem 4.1 except possibly the condition (4.3) of exponential decay, and the existence of a nonnegative element p in the kernel. In the following results, we show that in fact any such P is strictly increasing, and approaches the limits ± 1 exponentially fast as $\xi \to \pm\infty$, and conclude that this $\Lambda_{c,L}$ does indeed satisfy the conditions of Theorem 4.1.

We begin with a technical lemma.

Lemma 6.1. *Let $x: \mathbf{R} \to \mathbf{R}$ be a solution of (2.1) for some $\rho \in \bar{V}$ with $c \neq 0$, let*

$$\mu_- = \inf_{\xi \in \mathbf{R}} x(\xi), \qquad \mu_+ = \sup_{\xi \in \mathbf{R}} x(\xi) \tag{6.5}$$

and assume that both $\mu_\pm$ are finite. Then

$$\mu_- \in [-1, q(\rho)] \cup \{1\}, \qquad \mu_+ \in \{-1\} \cup [q(\rho), 1] \tag{6.6}$$

The same conclusion (6.6) holds for

$$\mu_- = \liminf_{\xi \to \infty} x(\xi), \qquad \mu_+ = \limsup_{\xi \to \infty} x(\xi) \tag{6.7}$$

and similarly for the lim inf *and* lim sup *at* $-\infty$.

Proof. With $\mu_\pm$ first as in (6.5), assuming that both $\mu_\pm$ are finite, we shall prove that

$$\Phi(\mu_-, \rho) \leqslant 0 \leqslant \Phi(\mu_+, \rho) \tag{6.8}$$

which, by (iv), is equivalent to

$$\mu_- \in [-1, q(\rho)] \cup [1, \infty), \qquad \mu_+ \in (-\infty, -1] \cup [q(\rho), 1] \tag{6.9}$$

As $\mu_- \leqslant \mu_+$, we see that (6.9) implies (6.6).

We shall prove only the second inequality in (6.8), as the proof of the first is similar. Let $\xi_n \in \mathbf{R}$ be a sequence such that $x(\xi_n) \to \mu_+$ as $n \to \infty$. Then the fact that x, and hence x', are uniformly continuous (from the differential Eq. (2.1)) implies that $x'(\xi_n) \to 0$. Upon passing to a subsequence, we may assume for some μ_j that $x(\xi_n + r_j) \to \mu_j \leqslant \mu_+$ as $n \to \infty$, for each j, and upon inserting $\xi = \xi_n$ into (2.1) and taking the limit, we have (noting that $\mu_1 = \mu_+$) that

$$0 = F(\mu_+, \mu_2, \mu_3, ..., \mu_N, \rho) \leqslant F(\mu_+, \mu_+, \mu_+, ..., \mu_+, \rho) = \Phi(\mu_+, \rho)$$

where condition (ii) is used. This now proves the inequality in (6.8), as desired.

Assume now that $\mu_\pm$ are given by (67). Let $\xi_n \to \infty$ be such that $x(\xi_n) \to \mu_+$ as $n \to \infty$, and set $y_n(\xi) = x(\xi + \xi_n)$. On each interval $[\tau, \infty)$ the sequence y_n is uniformly bounded and equicontinuous, so by passing to a subsequence we may assume the limit $y_n(\xi) \to y(\xi)$ holds uniformly on compact intervals. The function y: $\mathbf{R} \to \mathbf{R}$ is a solution to Eq. (2.1), and satisfies $\mu_- \leqslant y(\xi) \leqslant \mu_+$ for all $\xi \in \mathbf{R}$, and in fact $\mu_+ = \sup_{\xi \in \mathbf{R}} y(\xi)$. Therefore, from the first part of this proof, we have that μ_+ is in the set as indicated by (6.9), as claimed. The proof for μ_- is similar, as is the proof for $\xi \to -\infty$. □

Corollary 6.2. *Let P: $\mathbf{R} \to \mathbf{R}$ be as solution of (2.1), with the boundary conditions (1.3), for some $\rho \in \bar{V}$ with $c \neq 0$. Then*

$$-1 < P(\xi) < 1, \qquad \xi \in \mathbf{R} \tag{6.10}$$

Proof. Lemma 6.1 implies that $-1 \leqslant P(\xi) \leqslant 1$ for all $\xi \in \mathbf{R}$. The strict inequalities (6.10) follow from applications of Lemmas 3.1 and 3.3. □

We now prove Theorem 2.2. Even though Theorem 2.1 is not yet proved, below we establish the claims of Theorem 2.2 for any solution as in the statement of Theorem 2.1.

Proof of Theorem 2.2. We consider only the case of $\xi \to \infty$, as the proofs of the results for $\xi \to -\infty$ are similar.

First suppose that $c \neq 0$. Then $y(\xi) = 1 - P(\xi)$ satisfies Eq. (1.15) with coefficients (6.4), where $x_1(\xi) = 1$ and $x_2(\xi) = P(\xi)$ in (6.4). This Eq. (1.15) is asymptotically hyperbolic at $+\infty$, so $y(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$, for some $a > 0$, by Proposition 5.3 of Ref. 63. It follows from this exponential decay, from the formula (6.4), and from the Lipschitz condition (i) on the derivative of F, that $A_j(\xi)$ approaches the limiting coefficients $A_{j+} = A_{j+}(\rho)$ in (6.2) exponentially fast as $\xi \to \infty$. Recalling also that $A_{\Sigma+}(\rho) < 0$ by (6.3) and condition (v), we see that this Eq. (1.15) satisfies the conditions of Proposition 4.6, at least for $\xi \to \infty$. As $y(\xi) > 0$ by Corollary 6.2, we conclude by Proposition 4.6 the asymptotic formula (2.8) at $+\infty$ for some $C_+ > 0$ and with $\lambda^s_+ > -\infty$. In addition, the differentiated version of that formula holds. Also, the finiteness of λ^s_+ implies that $c < 0$ if $r_{\min}(\rho) = 0$, by (4.16), so we have (2.7).

Now suppose that $c = 0$ and $r_{\min}(\rho) < 0$. The proof of (2.9) follows similar lines to the proof of (2.8), except that Proposition 5.4 is used in place of Proposition 4.6. We note, in particular, that Lemma 3.3 ensures the strict inequality $P(\xi) < 1$ for all $\xi \in \mathbf{R}$, as (3.10) there does not hold.

Lastly suppose that $c = 0$ and $r_{\min}(\rho) = 0$. Then $P(\xi + r_j) \geqslant P(\xi)$ for every j, as P is monotone increasing. Thus for every $\xi \in \mathbf{R}$ we have by (ii) that

$$\begin{aligned} 0 &= F(P(\xi + r_1), P(\xi + r_2), \ldots, P(\xi + r_N), \rho) \\ &\geqslant F(P(\xi + r_1), P(\xi + r_1), \ldots, P(\xi + r_1), \rho) \\ &= \Phi(P(\xi), \rho) \end{aligned}$$

hence $P(\xi) \in [-1, q(\rho)] \cup [1, \infty)$ by (iv), and we have (2.10). □

We now prove strict monotonicity of solutions joining the equilibria ± 1, when $c \neq 0$.

Proposition 6.3. *Let P: $\mathbf{R} \to \mathbf{R}$ be a solution of (2.1), with the boundary conditions (1.3), for some $\rho \in W$ with $c \neq 0$. Then $P'(\xi) > 0$ for all $\xi \in \mathbf{R}$.*

Proof. By (2.8) of Theorem 2.2 there exists $\tau > 0$ such that $P'(\xi) > 0$ whenever $|\xi| \geqslant \tau$, and such that $P(-\tau) < P(\xi) < P(\tau)$, whenever $|\xi| < \tau$. From this we have $P(\xi + k) > P(\xi)$ for all $\xi \in \mathbf{R}$, provided that $k \geqslant 2\tau$. Now suppose that $P'(\xi) < 0$ for some ξ, and set

$$k_0 = \inf\{k > 0 \mid P(\xi + k) > P(\xi) \text{ for all } \xi \in \mathbf{R}\}$$

Certainly $k_0 > 0$. Also, $k_0 \leqslant 2\tau$, and $P(\xi + k_0) \geqslant P(\xi)$ for all $\xi \in \mathbf{R}$. If $0 < k < k_0$ then $P(\xi + k) \leqslant P(\xi)$ for some ξ, where necessarily $|\xi| \leqslant \tau$. Therefore, there exists some ξ_0, with $|\xi_0| \leqslant \tau$, for which $P(\xi_0 + k_0) = P(\xi_0)$. Lemma 3.1 applied to the two solutions $x_1(\xi) = P(\xi + k_0)$ and $x_2(\xi) = P(\xi)$ of (2.1) implies that $P(\xi + k_0) = P(\xi)$ either for all large ξ, or all large $-\xi$, which is a contradiction. We conclude that $P'(\xi) < 0$ is impossible, and so $P'(\xi) \geqslant 0$ for all $\xi \in \mathbf{R}$.

The strict inequality $P'(\xi) > 0$ now follows from another application of Lemma 3.1, this time to the linearization about P, where we take the two solutions $x_1(\xi) = P'(\xi)$ and $x_2(\xi) = 0$, knowing that $P'(\xi) > 0$ for large $|\xi|$. □

We now see that if $P: \mathbf{R} \to \mathbf{R}$ satisfies Eq. (2.1), with the boundary conditions (1.3), for some $\rho \in W$ with $c \neq 0$, then the operator $\Lambda_{c,L}$ associated to the linearization about P satisfies all the conditions of Theorem 4.1. With this we may use the implicit function theorem and establish some of the claims of Theorem 2.1. Recall (2.19) the map $\mathscr{G}$.

Proposition 6.4. *Let $(c_0, P_0, \rho_0) \in \mathscr{M}$, with $\mathscr{M}$ as in (2.5). Then the derivative of $\mathscr{G}$,*

$$D_{1,2}\mathscr{G}(c_0, P_0, \rho_0): \mathbf{R} \times W_0^{1,\infty} \to L^\infty \tag{6.11}$$

at this point, with respect to the first two arguments, is an isomorphism from $\mathbf{R} \times W_0^{1,\infty}$ onto L^∞. Thus, by the implicit function theorem, there exists for each ρ near ρ_0 a unique point $(c, P) = (c(\rho), P(\rho)) \in (\mathbf{R} \backslash \{0\}) \times W_0^{1,\infty}$ near (c_0, P_0), depending C^1 smoothly on ρ, for which

$$\mathscr{G}(c(\rho), P(\rho), \rho) = 0 \tag{6.12}$$

and with $(c(\rho_0), P(\rho_0)) = (c_0, P_0)$. For each such ρ, the solution $P(\rho)$ to (2.1) satisfies the boundary conditions (1.3), hence $(c(\rho), P(\rho), \rho) \in \mathscr{M}$. This, moreover, accounts for all points of $\mathscr{M}$ near (c_0, P_0, ρ_0).

Proof. Consider the linearization (1.15) of Eq. (2.1) about P_0, and let $\Lambda_{c_0,L}$ as in (2.21) denote the associated linear operator from $W^{1,\infty}$ to L^∞. We see that this operator satisfies all the conditions of Theorem 4.1. In particular, the exponential condition (4.3) follows from the exponential approach of $P(\xi)$ to ± 1, by Theorem 2.2. Certainly $x = P_0'(\xi)$ satisfies the linear Eq. (1.15), so by Proposition 6.3 this gives the nonnegative element $p = P_0' \in \mathscr{K}_{c_0,L}$ in the statement of Theorem 4.1. One also sees (4.4), following from condition (v) and (6.3). Thus by Theorem 4.1, the kernel $\mathscr{K}_{c_0,L}$ of $\Lambda_{c_0,L}$ is precisely the one-dimensional span of P_0'.

The strict positivity $P_0'(0)>0$ implies that $P_0'\notin W_0^{1,\infty}$, hence the restriction of $\Lambda_{c_0,L}$ to $W_0^{1,\infty}\subseteq W^{1,\infty}$ is one-to-one, and in fact an isomorphism from $W_0^{1,\infty}$ onto its range $\mathscr{R}_{c_0,L}\subseteq L^\infty$, which has codimension one. Also by Theorem 4.1, there exists $p^*\in\mathscr{K}_{c_0,L^*}$, with $p^*(\xi)>0$ for all $\xi\in\mathbf{R}$, so

$$\int_{-\infty}^{\infty} p^*(\xi)\, P_0'(\xi)\, d\xi>0$$

implying that $P_0'\notin\mathscr{R}_{c_0,L}$ by the Fredholm alternative (3.17). We conclude from this, and from the formula (2.20) for $D_{1,2}\mathscr{G}$, that the derivative (6.11) is an isomorphism from $\mathbf{R}\times W_0^{1,\infty}$ onto L^∞, as claimed. With this, the implicit function theorem yields $c(\rho)$ and $P(\rho)$ satisfying (6.12).

The proof of the limits (1.3) for $P(\rho)$ follows easily from Lemma 6.1. Since $P(\rho)$ varies continuously with ρ in L^∞, the quantities $\mu_\pm=\mu_\pm(\rho)$ in (6.7) for this solution also vary continuously. As $\mu_+(\rho_0)=\mu_-(\rho_0)=1$, we conclude from (6.6) that $\mu_+(\rho)=\mu_-(\rho)=1$ for ρ near ρ_0. With a similar argument for the limit at $-\infty$, we conclude (1.3) for $x=P(\rho)$, for all $\rho\in W$ near ρ_0. □

We have now proved that $\mathscr{M}$ in (2.5) is a smooth manifold, which is locally the graph of a function from a subset of V into $\mathbf{R}\times W_0^{1,\infty}$. Let us now prove that $\mathscr{M}$ is globally a graph, that is, for each $\rho\in W$ there exists at most one $(c,P,\rho)\in\mathscr{M}$. In fact, we prove all the uniqueness claims of Theorem 2.1.

Proposition 6.5. *For each $\rho\in W$, there exists at most one value $c\in\mathbf{R}$ such that Eq. (2.1) possesses a monotone increasing solution $x=P(\xi)$ satisfying (1.3). For each $c\in\mathbf{R}\backslash\{0\}$ and $\rho\in W$, there exists at most one solution $x=P(\xi)$ of (2.1), up to translation, satisfying (1.3).*

Remark. Following Proposition 6.5 we may define a relatively open set $U\subseteq W$ by

$$U=\{\rho\in W\mid \text{there exists } c\in\mathbf{R}\backslash\{0\} \text{ and } P\in W_0^{1,\infty} \text{ such that } (c,P,\rho)\in\mathscr{M}\} \tag{6.13}$$

with $\mathscr{M}$ as in (2.5). By Propositions 6.3 and 6.5 we have that the point $(c,P,\rho)\in\mathscr{M}$, normalized so that $P(0)=0$, is unique for each $\rho\in U$. Thus we have well-defined functions

$$c\colon U\to\mathbf{R}\backslash\{0\},\qquad P\colon U\to W_0^{1,\infty}$$

which by Proposition 6.4 are smooth. Also, the sets $U_\pm \subseteq U$ in (2.16) are well-defined and relatively open in U.

Remark. It remains to prove that (2.1), with (1.3), possesses a monotone solution with $c = 0$ when $\rho \in W \backslash U$, as claimed by Theorem 2.1. This will be done in the next two sections. There we shall set $c(\rho) = 0$ for such ρ, thereby defining c: $W \to \mathbf{R}$ on all of W. Of course, then one sees that the set U in (6.13) is identical to the set U given in (2.4).

In addition, we show there that the function c: $W \to \mathbf{R}$ so defined is continuous on all of W, or equivalently, that $c(\rho) \to 0$ as $\rho \in U$ approaches the boundary $(\bar{U} \backslash U) \cap W$ of U in W. We shall also prove Theorem 2.3, namely that $P(\rho)$ also approaches a limit satisfying the boundary conditions (1.3).

Proof. Suppose for some $\rho \in W$ that there exist $c_1 > c_2$, and monotone solutions $x = P_j(\xi)$ of (2.1), satisfying (1.3), with $c = c_j$ for $j = 1, 2$. To begin, assume without loss that $P_1(\xi) < P_2(\xi)$ for some ξ; this may be accomplished by replacing P_2 with some translate of itself, namely by shifting the graph of P_2 to the left. Also, by (4.11) of Proposition 4.3 we have that

$$\lambda_-^u(c_1, \rho) < \lambda_-^u(c_2, \rho), \qquad \lambda_+^s(c_1, \rho) < \lambda_+^s(c_2, \rho) \tag{6.14}$$

for the real eigenvalues of the linearized equations at $x = \pm 1$. [Recall the convention (4.16), which one uses here when (2.10) applies. Also note that $\lambda_-^u(c_1, \rho) = \lambda_-^u(c_2, \rho) = \infty$ is impossible, since then $r_{\max}(\rho) = 0$ and $c_2 < c_1 \leqslant 0$ by (4.16), which would contradict (2.7) of Theorem 2.2. Similarly, $\lambda_+^s(c_1, \rho) = \lambda_+^s(c_2, \rho) = -\infty$ is impossible.] From (6.14) and from Theorem 2.2, it follows that $P_1(\xi) > P_2(\xi)$ for all large $|\xi|$. Let

$$k_0 = \inf\{k > 0 \mid P_1(\xi + k) > P_2(\xi) \text{ for all } \xi \in \mathbf{R}\}$$

If both $c_1 \neq 0$ and $c_2 \neq 0$, then much as in the proof of Proposition 6.3, one shows that $P_1(\xi + k_0) \geqslant P_2(\xi)$ for all $\xi \in \mathbf{R}$, with equality at some $\xi = \xi_0$. But then Lemma 3.2 implies these solutions are equal and constant either as $\xi \to \infty$ or $\xi \to -\infty$, contradicting Proposition 6.3.

If either $c_1 = 0$ or $c_2 = 0$, then the above reasoning also yields a contradiction, provided that the appropriate one-sided continuity (right or left) of the solution is taken. To be specific, assume that $c_1 > c_2 = 0$. Then without loss we may assume that P_2 is right-continuous everywhere in $\mathbf{R}$, by replacing it with its right-hand limit if necessary. From the continuity of P_1 one has that $P_1(\xi + k_0) \geqslant P_2(\xi)$ for every $\xi \in \mathbf{R}$, so certainly $k_0 > 0$.

We claim that again $P_1(\xi_0+k_0)=P_2(\xi_0)$ for some ξ_0, necessarily with $|\xi_0|\leqslant\tau$, where τ is such that $P_1(\xi)>P_2(\xi)$ for all $|\xi|\geqslant\tau$. Suppose such ξ_0 does not exist, and so $P_1(\xi+k_0)>P_2(\xi)$ for all $\xi\in[-\tau,\tau]$. Then for any fixed $\xi_*\in[-\tau,\tau]$, we observe from the continuity of P_1 and from the right-continuity and monotonicity of P_2, that there exists $\varepsilon>0$ such that $P_1(\xi+k)>P_2(\xi)$ for $|\xi-\xi_*|<\varepsilon$ and $|k-k_0|<\varepsilon$. By a simple compactness argument, which entails taking a finite cover of $[-\tau,\tau]$ by intervals $(\xi_*-\varepsilon,\xi_*+\varepsilon)$, we conclude that there exists $k_*\in(0,k_0)$ such that $P_1(\xi+k_*)>P_2(\xi)$ for all $\xi\in[-\tau,\tau]$. However, as $P_1(\xi+k_*)>P_2(\xi)$ holds also for $|\xi|\geqslant\tau$, we obtain a contradiction to the definition of k_0. Having now established the existence of ξ_0, one obtains from Lemma 3.2 a contradiction as before. With this, the uniqueness of $c\in\mathbf{R}$ is established.

Now suppose, for some $c\neq 0$, that there exist two solutions $x=P_j(\xi)$, for $j=1,2$, to (2.1), satisfying (1.3), where P_1 and P_2 are not translates of one another. With the same quantities $\lambda^u_-=\lambda^u_-(c,\rho)$ and $\lambda^s_+=\lambda^s_+(c,\rho)$ for both solutions, we have from Theorem 2.2 that

$$P_j(\xi)=\begin{cases}-1+C_{j-}\,e^{\lambda^u_-\xi}+O(e^{(\lambda^u_-+\varepsilon)\xi}), & \xi\to-\infty\\ 1-C_{j+}\,e^{\lambda^s_+\xi}+O(e^{(\lambda^s_+-\varepsilon)\xi}), & \xi\to\infty\end{cases}\tag{6.15}$$

for (generally different) constants $C_{j\pm}>0$ and $\varepsilon>0$. By replacing $P_1(\xi)$ with $P_1(\xi+k)$, with some $k>0$, if necessary, we may assume without loss that $C_{1-}>C_2$ and $C_{2+}>C_{1+}$, and hence that $P_1(\xi)>P_2(\xi)$ for all $|\xi|$. By further shifting the graph of P_1 to the left, we may assume without loss that $P_1(\xi)>P_2(\xi)$ for all $\xi\in\mathbf{R}$. Assuming this, set

$$k_0=\sup\{k\geqslant 0\mid P_1(\xi)\geqslant P_2(\xi+k)\text{ for all }\xi\in\mathbf{R}\}$$

Then certainly $k_0<\infty$ and $P_1(\xi)\geqslant P_2(\xi+k_0)$ for all $\xi\in\mathbf{R}$, and hence $C_{1-}\geqslant C_{2-}\,e^{\lambda^u_-k_0}$ and $C_{2+}\,e^{\lambda^s_+k_0}\geqslant C_{1+}$. We claim that we in fact have the strict inequalities

$$C_{1-}>C_{2-}\,e^{\lambda^u_-k_0},\qquad C_{2+}\,e^{\lambda^s_+k_0}>C_{1+}\tag{6.16}$$

and therefore that $P_1(\xi)>P_2(\xi+k_0)$ for all large $|\xi|$. If this is true, then as before, we have that $P_1(\xi_0)=P_2(\xi_0+k_0)$ for some $\xi_0\in\mathbf{R}$, which with Lemma 3.1 yields a contradiction.

To prove the strict inequalities (6.16), let $y(\xi)=P_1(\xi)-P_2(\xi+k_0)$. Then

$$y(\xi)=(C_{2+}\,e^{\lambda^s_+k_0}-C_{1+})\,e^{\lambda^s_+\xi}+O(e^{(\lambda^s_+-\varepsilon)\xi}),\qquad \xi\to\infty\tag{6.17}$$

by (6.15), and $y(\xi) \geqslant 0$ for all $\xi \in \mathbf{R}$. Also, y is a solution of the linear Eq. (1.15) with coefficients (6.4), where $\xi_1(\xi) = P_1(\xi)$ and $x_2(\xi) = P_2(\xi + k_0)$. Moreover, y does not vanish identically on any interval $[\tau, \infty)$, otherwise by Lemma 3.3 it would vanish identically on $\mathbf{R}$, implying the solutions P_1 and P_2 were translates of one another. Proposition 4.6 now implies that the coefficient $C_{2+} e^{\lambda^s_+ k_0} - C_{1+}$ in (6.17) is positive, and this establishes the second inequality in (6.16). The first inequality in (6.16) follows from a similar argument. This proves the uniqueness of P, for each $c \neq 0$. □

Let us conclude this section with a proof of Proposition 2.4, which establishes a monotone relation (2.15) between the wave speed c and the parameter ρ, under condition (2.14).

Proof of Proposition 2.4. With $\rho = \rho(\sigma)$ as in the statement of the proposition, we differentiate the identity (6.12) with respect to σ at $\sigma = \sigma_0$. Recalling the definition (2.19) of $\mathscr{G}$, we have

$$-wP_0'(\xi) + (\Lambda_{c_0, L} x)(\xi) - Q(\xi) = 0 \tag{6.18}$$

where $P_0 = P(\rho(\sigma_0))$, and where

$$w = \frac{d}{d\sigma} c(\rho(\sigma))\bigg|_{\sigma = \sigma_0}, \qquad x = \frac{d}{d\sigma} P(\rho(\sigma))\bigg|_{\sigma = \sigma_0},$$

$$Q(\xi) = \frac{d}{d\sigma} F(u, \rho(\sigma))\bigg|_{\substack{u = \pi(P_0, \xi) \\ \sigma = \sigma_0}}$$

The operator $\Lambda_{c_0, L}$ corresponds to the linearization of (2.1) about P_0, and satisfies all the conditions of Theorem 4.1, as in the proof of Proposition 6.4. Also, $\pi(P_0, \xi) \in M$ as P_0 is strictly increasing, and so $Q(\xi) < 0$, for each $\xi \in \mathbf{R}$, by (2.14). Upon multiplying Eq. (6.18) by the strictly positive element $p^* \in \mathscr{K}_{c_0, L^*}$ of the adjoint kernel, and then integrating from $-\infty$ to $+\infty$, we obtain

$$-w \int_{-\infty}^{\infty} p^*(\xi)\, P_0'(\xi)\, d\xi - \int_{-\infty}^{\infty} p^*(\xi)\, Q(\xi)\, dx = 0 \tag{6.19}$$

the term $\Lambda_{c_0, L} x \in \mathscr{R}_{c_0, L}$ having been annihilated. We conclude from the signs of the integrands in (6.19) that $w > 0$, that is, (2.15) holds as desired. □

7. THE SINGULAR PERTURBATION LIMIT $c \to 0$

In this section we examine how solutions of (2.1) with small $c \neq 0$ converge to solutions with $c = 0$. In particular, we prove Theorem 2.3 and Corollary 2.5. We maintain the standing hypotheses (i)–(v) on Eq. (2.1), as in the previous section.

Recall (6.13) the subset $U \subseteq W$ consisting of those parameters $\rho \in W$ for which there exists a solution of Eq. (2.1), with boundary conditions (1.3), with $c \neq 0$. We prove in Proposition 7.2 that $c = c(\rho) \to 0$ as $\rho \in U$ approaches the boundary of U in W. We first need the following technical lemma.

Lemma 7.1. *Suppose for some $\rho \in W$ and $c \in \mathbf{R}$ that there exists a monotone increasing solution $x_-: \mathbf{R} \to \mathbf{R}$ of Eq. (2.1) such that*

$$\lim_{\xi \to -\infty} x_-(\xi) = -1, \qquad \lim_{\xi \to \infty} x_-(\xi) = q(\rho) \tag{7.1}$$

Then either there exists a real, negative eigenvalue $\lambda^s_\diamond < 0$ of the linearization of (2.1) about $x = q(\rho)$ or

$$r_{\min}(\rho) = 0, \qquad c = 0 \tag{7.2}$$

both hold. Similarly, if there exists a monotone increasing solution x_+: $\mathbf{R} \to \mathbf{R}$ with

$$\lim_{\xi \to -\infty} x_+(\xi) = q(\rho), \qquad \lim_{\xi \to \infty} x_+(\xi) = 1 \tag{7.3}$$

then either the linearization about $x = q(\rho)$ possesses a real, positive eigenvalue $\lambda^u_\diamond > 0$ or

$$r_{\max}(\rho) = 0, \qquad c = 0 \tag{7.4}$$

both hold.

In any case, for each $\rho \in W$ and $c \in \mathbf{R}$ there do not simultaneously exist monotone increasing solutions x_- and x_+ on $\mathbf{R}$ satisfying (7.1) and (7.3).

Proof. Consider the solution x_-, and assume that (7.2) is false. Then by Lemma 3.3 and the monotonicity of x_- we have the strict inequality $x_-(\xi) < q(\rho)$ for all $\xi \in \mathbf{R}$. Let $y(\xi) = q(\rho) - x_-(\xi)$. Then y is a strictly positive, monotone decreasing solution of the linear Eq. (1.15), with coefficients (6.4), where $x_1(\xi) = q(\rho)$ and $x_2(\xi) = x_-(\xi)$. If $c = 0$ then $r_{\min}(\rho) < 0$, and by Proposition 5.4 we have the existence of an eigenvalue $\lambda^s_\diamond \leqslant 0$ of the linearized equation, namely, $\Delta_{0, L_\diamond(\rho)}(\lambda^s_\diamond) = 0$. As

$$\Delta_{0, L_\diamond(\rho)}(0) = -A_{\Sigma\diamond}(\rho) = -D_1\Phi(q(\rho), \rho) < 0 \tag{7.5}$$

we have in fact $\lambda^s_\diamond < 0$, as desired.

Suppose, then, that $c \neq 0$. We would like to apply Proposition 7.2 of Ref. 63 directly to $y(\xi)$ as in the proof of Proposition 4.6, however, we have no assurance that $\|M(\xi)\|$ in the statement of the first proposition decays exponentially, since the equilibrium $x = q(\rho)$ need not be hyperbolic. We instead take a different approach. By Proposition 4.5 we have the inequality $y'(\xi) \geqslant ay(\xi)$ in (4.17), for all $\xi \in \mathbf{R}$, for some $a \in \mathbf{R}$. Now take any sequence $\xi_n \to \infty$, and let $z_n(\xi) = y(\xi + \xi_n)/y(\xi_n)$. Then each z_n also satisfies the left-hand inequality in (4.17), for the same a, on $\mathbf{R}$. As $z_n(0) = 1$, we conclude that the sequence of functions z_n is uniformly bounded and equicontinuous on each compact interval, and so without loss we have that $z_n(\xi) \to z(\xi)$ uniformly on compact intervals. One easily sees that z satisfies the autonomous limiting Eq. (3.12), (6.1), at $x = q(\rho)$ that is, the linearization of (2.1) about $x = q(\rho)$. Moreover, $az(\xi) \leqslant z'(\xi) \leqslant 0$ for all $\xi \in \mathbf{R}$, with $z(0) = 1$, so $z(\xi) > 0$, and z does not decay faster than exponentially.

We may now apply Proposition 7.2 of Ref. 63 to the above solution z. We conclude that $z(\xi) = w(\xi) + O(e^{-(b+\varepsilon)\xi})$ as $\xi \to \infty$, where w is a nontrivial sum of eigensolution corresponding to a set of eigenvalues with $\operatorname{Re}\lambda = -b \leqslant 0$. The positivity of z, together with Lemma 3.4, implies that the linearization about $x = q(\rho)$ possesses a nonpositive eigenvalue $\lambda^s_\diamond \leqslant 0$. As before, $\Delta_{c, L_\diamond(\rho)}(0) < 0$, so in fact $\lambda^s_\diamond < 0$.

The proof of the statements about x_+ are similar.

To prove the final statement of the lemma, first note from Proposition 4.4, using (7.5), that there do not simultaneously exist eigenvalues $\lambda^s_\diamond < 0 < \lambda^u_\diamond$ as above. Therefore either (7.2) or (7.4) holds; to be specific, assume that (7.2) holds. But this now implies that

$$\Delta'_{0, L_\diamond(\rho)}(s) = -\sum_{j=2}^{N} A_{j\diamond}(\rho)\, r_j e^{sr_j} < 0 \tag{7.6}$$

as $r_j > 0$ for each $j \in \mathscr{U}(\rho)$. Together with (7.5) we see that (7.6) implies that there does not exist $\lambda^u_\diamond > 0$ with $\Delta_{0, L_\diamond(\rho)}(\lambda^u_\diamond) = 0$. Thus (7.4) also holds, however, this contradicts the strict inequality in (2.2). We conclude that there do not Simultaneously exist solutions $x_\pm$ as in (7.1), (7.3). □

***Proposition** 7.2. Let $\rho_n \in U$ with $\rho_n \to \rho_0 \in W$, and suppose that $\rho_0 \notin U$. Then $c(\rho_n) \to 0$.*

Proof. Let us denote $c_n = c(\rho_n) \neq 0$ and $P_n = P(\rho_n) \in W_0^{1,\infty}$ for simplicity, and also $q_n = q(\rho_n)$, namely, the zero of Φ at $\rho = \rho_n$, in condition (iv). We denote $q_0 = q(\rho_0)$, which is the limit $q_n \to q_0$, and as $\rho_0 \in W$, we have that $q_0 \in (-1, 1)$. We assume for definiteness that $c_n > 0$ for all n, the proof for negative c_n being similar. We finally assume, without loss, that the limit $c_n \to c_0$ does exist, although it is possibly infinite, and so $0 \leqslant c_0 \leqslant \infty$.

We first eliminate the possibility that $c_0 = \infty$. Assume that $c_n \to \infty$, fix a point

$$q_* \in (q_0, 1) \tag{7.7}$$

and let $x_n(\xi) = P_n(c_n \xi + \xi_n)$, where $\xi_n \in \mathbf{R}$ is such that $P_n(\xi_n) = q_*$. Then from (2.1),

$$-x_n'(\xi) = F(x_n(\xi + c_n^{-1} r_1), x_n(\xi + c_n^{-1} r_2), ..., x_n(\xi + c_n^{-1} r_N), \rho_n) \tag{7.8}$$

for $\xi \in \mathbf{R}$. The functions x_n are uniformly bounded and equicontinuous, as solutions of (7.8), and without loss we have $x_n(\xi) \to x(\xi)$ uniformly on compact intervals, after possibly passing to a subsequence. The limiting function x satisfies $-x'(\xi) = \Phi(x(\xi), \rho_0)$, obtained by taking the limit of (7.8). Now $x_n(0) = q_*$, hence $x(0) = q_*$, and so $x'(0) = -\Phi(q_*, \rho_0) < 0$. On the other hand, $x_n'(\xi) > 0$, hence $x'(\xi) \geqslant 0$, for all ξ. This contradiction implies that $c_0 = \infty$ is impossible.

As we wish to show that $c_0 = 0$, let us assume that $c_0 \in (0, \infty)$. We shall prove that $\rho_0 \in U$, which is a contradiction. Now fix, in addition to the point q_* satisfying (7.7), a second point $q_{**} \in (-1, q_0)$, let $\xi_n, \zeta_n \in \mathbf{R}$ be such that

$$P_n(\xi_n) = q_*, \qquad P_n(\zeta_n) = q_{**} \tag{7.9}$$

and set

$$x_{n+}(\xi) = P_n(\xi + \xi_n), \qquad x_{n-}(\xi) = P_n(\xi + \zeta_n) \tag{7.10}$$

Again, we may take limits $x_{n+}(\xi) \to x_+(\xi)$ and $x_{n-}(\xi) \to x_-(\xi)$, where both $x_\pm(\xi)$ satisfy the Eq. (2.1) at $c = c_0$ and $\rho = \rho_0$.

Both x_+ and x_- are monotone increasing bounded functions, and so possess limits at $\pm\infty$, which we denote by $x_+(\pm\infty)$ and $x_-(\pm\infty)$. These limits are equilibria of the differential Eq. (2.1), and so belong to the set $\{-1, q_0, 1\}$. From (7.9) and (7.10), we have $x_+(0) = q_* \in (q_0, 1)$ and $x_-(0) = q_{**} \in (-1, q_0)$, so necessarily

$$x_+(-\infty) \in \{-1, q_0\}, \qquad x_+(\infty) = 1,$$
$$x_-(-\infty) = -1, \qquad x_-(\infty) \in \{q_0, 1\}$$

By Lemma 7.1, it is impossible that $x_+(-\infty) = x_-(\infty) = q_0$, and so either $x_+(-\infty) = -1$, or else $x_-(\infty) = 1$. But this means that either x_+ or x_- satisfies the boundary conditions (1.3), and so provides a point $(c_0, P_0, \rho_0) \in \mathcal{M}$, where P_0 is a translate of either x_+ or x_-. Thus $\rho_0 \in U$, as desired, completing the proof. □

Let us now set

$$c(\rho) = 0, \qquad \rho \in W \backslash U$$

From the above result we have that $c\colon W \to \mathbf{R}$ is continuous on all of W. Of course, we hare not yet shown the existence of a solution $P(\rho)$ as in the statement of Theorem 2.1 when $c(\rho) = 0$. Theorem 2.3, which we now prove, gives such a solution in particular for $\rho \in (\bar{U} \backslash U) \cap W$ on the boundary of U in W. Following this, we prove Corollary 2.5.

Proof of Theorem 2.3. In a standard fashion, we may assume upon passing to a subsequence, that the limit (2.11) of the monotone functions $x = P_n(\xi)$ holds at every $\xi \in \mathbf{R}$. Taking this limit in the integrated form

$$-c_n(P_n(\xi) - P_n(0)) = \int_0^\xi F(P_n(s + r_1), P_n(s + r_2), ..., P_n(s + r_N), \rho_n)\, ds$$

of (2.1) yields, for each ξ,

$$0 = \int_0^\xi F(P_0(s + r_1), P_0(s + r_2), ..., P_0(s + r_N), \rho_0)\, ds$$

which when differentiated gives (2.12) almost everywhere for $x = P_0(\xi)$. In fact (2.12) holds at every ξ for which $\xi + r_j$ is a point of continuity of P_0 for each j, and hence at all but countably many ξ.

Upon taking either limit $\xi \to \pm\infty$ in (2.12), with $x = P_0(\xi)$ we obtain

$$0 = F(P_0(\pm\infty), P_0(\pm\infty), ..., P_0(\pm\infty), \rho_0) = \Phi(P_0(\pm\infty), \rho_0)$$

which implies that

$$P_0(\pm\infty) \in \{-1, q(\rho_0), 1\} \tag{7.11}$$

If $q(\rho_0) \in \{-1, 1\}$, then we are done, so assume that $q(\rho_0) \in (-1, 1)$, that is, $\rho_0 \in W$. Fix any points q_* and q_{**} satisfying $-1 < q_{**} < q(\rho_0) < q_* < 1$, and much as in the proof of Proposition 7.2, let $\xi_n, \zeta_n \in \mathbf{R}$ be such that

$$\begin{aligned} &P_n(\xi) \leqslant q_{**}, && \xi < \zeta_n \\ &q_{**} \leqslant P_n(\xi) \leqslant q_*, && \zeta_n < \xi < \xi_n \\ &P_n(\xi) \geqslant q_*, && \xi > \xi_n \end{aligned}$$

Without loss we may assume the limits $\xi_n \to \xi_0$ and $\zeta_n \to \zeta_0$ both exist, although possibly are infinite. It is enough to show that the difference $\xi_n - \zeta_n$ is bounded. Indeed, if this is the case, and if ξ_n (and hence also ζ_n) are themselves bounded, so that ξ_0 and ζ_0 both are finite, then $P_0(\xi) \leqslant q_{**}$ for all $\xi < \zeta_0$ and $P_0(\xi) \geqslant q_*$ for all $\xi > \xi_0$, which with (7.11) implies (1.3). If, on the other hand, $\xi_0 = \zeta_0 = \infty$, then $P_0(\xi) \leqslant q_{**}$, hence $P_0(\xi) = -1$, for all $\xi \in \mathbf{R}$, and if $\xi_0 = \zeta_0 = -\infty$, then $P_0(\xi) = 1$ for all $\xi \in \mathbf{R}$.

To prove that $\xi_n - \zeta_n$ is bounded, assume that $\xi_n - \zeta_n \to \infty$, and define $x_{n\pm}(\xi)$ by (7.10). Upon passing to a subsequence and taking limits $x_{n\pm}(\xi) \to x_{\pm}(\xi)$, as above, we obtain solutions of (2.12) which satisfy the four boundary conditions in (7.1), (7.3), with $q(\rho_0)$ replacing $q(\rho)$. However, this is impossible by Lemma 7.1.

The final claim in the theorem is established by taking q_* and q_{**} in the above argument to satisfy also $q_{**} < P_n(0) = 0 < q_*$. In this case $\zeta_n < 0 < \xi_n$ holds, and it follows that both ξ_n and ζ_n are themselves bounded. From here the boundary conditions (1.3) follows easily. □

Proof of Corollary 2.5. The strict monotonicity (2.17) follows from Proposition 2.4 using condition (viii) in the definition of a normal family. The remainder of the corollary follows from the continuity of $c(\rho)$. □

8. THE EXISTENCE OF SOLUTIONS

In this section we prove Theorem 2.6 by globally continuing a solution along a homotopy between two normal families. As the families are coercive at ± 1, we are able to maintain $c \neq 0$ throughout the homotopy. We then complete the proof of Theorem 2.1 by constructing solutions for $\rho \in W \backslash U$, using an argument involving sub- and supersolutions.

Lemma 8.1. *Suppose (2.1) is a normal family which is either coercive or weakly coercive at* $+1$. *Then there does not exist a solution* $P\colon \mathbf{R} \to \mathbf{R}$ *which is monotone increasing on* $\mathbf{R}$ *and Which satisfies the boundary conditions (1.3), with* $c = 0$ *and* $\rho = 1$. *The analogous result for a family which is coercive or weakly coercive at* -1 *holds, with* $\rho = -1$.

Proof. Assume (2.1) is coercive at $+1$, and that P is monotone increasing and satisfies (2.1), with (1.3), for $c = 0$ and $\rho = 1$. Let us write (2.1) for this solution as

$$\begin{aligned} 0 &= F(P(\xi + r_1), P(\xi + r_2), ..., P(\xi + r_N), 1) \\ &\quad - F(P(\xi), P(\xi), ..., P(\xi), 1) + \Phi(P(\xi), 1) \\ &= \sum_{j=2}^{N} A_j(\xi)(P(\xi + r_j) - P(\xi)) + \Phi(P(\xi), 1) \\ &= -\sum_{j=2}^{N} A_j(\xi)(x(\xi + r_j) - x(\xi)) - h(\xi) \\ &= -\sum_{j=1}^{N} A_j(\xi)\, x(\xi + r_j) - h(\xi) \end{aligned}$$

where here

$$A_j(\xi) = \int_0^1 \left. \frac{\partial F(u, \rho)}{\partial u_j} \right|_{\substack{u = t\pi(P,\xi) + (1-t)\kappa(P(\xi)) \\ \rho = 1}} dt, \qquad 2 \leqslant j \leqslant N$$

$$A_1(\xi) = -\sum_{j=2}^{N} A_j(\xi), \qquad h(\xi) = -\Phi(P(\xi), 1)$$

and where we denote $x(\xi) = 1 - P(\xi)$. We note that $h(\xi) \geqslant 0$ for all ξ and, also, that the limits as $\xi \to \infty$ in (3.14) hold, with

$$\sum_{j=2}^{N} r_j A_{j+} < 0 \tag{8.1}$$

in the case of coercivity (2.18). In the case of weak coercivity the inequality in (8.1) may not be strict, although $A_j(\xi) = A_{j+}$ identically for large ξ. In any case, note that $r_{\min}(1) < 0$, since $r_{\min}(1) = 0$ would imply that the left-hand side of (8.1) is positive. Thus $h(\xi) = 0$ for almost every large ξ, by Lemma 5.6, and so $x(\xi) = 0$, that is, $P(\xi) = 1$ for all large ξ. But this contradicts Lemma 3.3 since $r_{\min}(1) < 0$, and so completes the proof. □

Recall the system (1.17), with the nonlinearity (1.16), described in the Introduction. Also recall that $P(\xi) = \tanh \xi$ is a solution to this system for

$c=1$ and $q=0$. As noted in Section 2, this system is a normal family with the parameter $\rho=q$, and moreover it is coercive at $+1$. This system plays a key role in the following proof.

Proof of Theorem 2.6. Assume (2.1) is a normal family which is either coercive or weakly coercive at $+1$, and denote the nonlinearity F there by $F(u, \rho)=F_1(u, \rho)$. Similarly consider the system (1.17), with (1.16), with the nonlinearity there denoted by $F_0(u, \rho)$. Form the homotopy (2.26), that is,

$$\begin{aligned}-cx'(\xi) &= (1-\hat{\rho})(\gamma(x(\xi-k)-x(\xi))-f(x(\xi), \tilde{\rho}))\\ &\quad +\hat{\rho}F_1(x(\xi+r_1), x(\xi+r_2), ..., x(\xi+r_N), \tilde{\rho}) \qquad (8.2)\end{aligned}$$

with f as in (1.16). Certainly, the system (8.2), which is a homotopy between normal families, is coercive at $+1$ for each fixed $\hat{\rho}\in[0, 1)$, and either coercive or weakly coercive at $+1$ when $\hat{\rho}=1$. For each $\hat{\rho}$ let us denote by $\tilde{\rho}_+=\tilde{\rho}_+(\hat{\rho})$ the value ρ_+ of the detuning parameter as in the statement of Corollary 2.5. Then $U_+\subseteq W=(-1, 1)\times[0, 1]$ for the system (8.2) is given by

$$U_+=\{(\tilde{\rho}, \hat{\rho})\in(-1, 1)\times[0, 1] \mid \tilde{\rho}_+(\hat{\rho})<\tilde{\rho}<1\}$$

Moreover, since the system (8.2) at $\hat{\rho}=0$, with $\tilde{\rho}=0$ and $c=1$, possesses the solution $x=\tanh\xi$, we have that $(0, 0)\in U_+$, that is, $\tilde{\rho}_+(0)<0$.

We claim for all sufficiently small $\varepsilon>0$, that

$$\{1-\varepsilon\}\times[0, 1]\subseteq U_+ \qquad (8.3)$$

or equivalently that $\tilde{\rho}_+(\hat{\rho})<1-\varepsilon$ for all $\hat{\rho}\in[0, 1]$. Suppose that (8.3) is false for all $\varepsilon>0$. Then taking a sequence $\varepsilon_m\to 0$, we have quantities $\sigma_m\in(0, 1]$ such that

$$\{1-\varepsilon_m\}\times[0, \sigma_m)\subseteq U_+, \qquad (1-\varepsilon_m, \sigma_m)\notin U_+$$

as $U_+\subseteq W$ is relatively open. Also, $c(1-\varepsilon_m, \hat{\rho})\to 0$ as $\hat{\rho}\to\sigma_m$ from the left, for each m, by Proposition 7.2, hence there exists $\zeta_m\in[0, \sigma_m)$ such that $c(1-\varepsilon_m, \zeta_m)\to 0$ as $m\to\infty$. Denoting $\rho_m=(1-\varepsilon_m, \zeta_m)\in U_+$, we have upon passing to a subsequence that

$$\rho_m\to\rho_0=(1, \zeta_0)\in\bar{V}$$

for some $\zeta_0 \in [0, 1]$. Now letting $P_m(\xi) = P(\xi, \rho_m)$ denote the solution $P_m \in W_0^{1,\infty}$ to Eq. (8.2), with $c = c(\rho_m)$, upon passing to a further subsequence, we obtain by Theorem 2.3 a monotone solution $x = P_0(\xi)$ to the equation at $\rho_0 = (1, \zeta_0)$, with $c = 0$, which satisfies the boundary conditions (1.3). However, the normal family (8.2) at $\hat{\rho} = \zeta_0$ is coercive or weakly coercive at $+1$, so by Lemma 8.1 does not possess such a solution P_0. With this contradiction, the proof is complete. □

Proposition 8.2. *Consider a normal family (2.1), for which $\rho_+ < 1$. Suppose also that $\rho_+ > -1$. Then there exists, for $\rho = \rho_+$, with $c = 0$, a monotone increasing solution $x = P(\xi)$ on $\mathbf{R}$ which satisfies the boundary conditions (1.3). The same conclusion holds for ρ_- if $-1 < \rho_- < 1$.*

Proof. This is a straightforward application of Theorem 2.3, with $\rho_n \in (\rho_+, 1)$ any sequence with $\rho_n \to \rho_+$. The assumption $\rho_+ > -1$ ensures that $\rho_+ \in W = (-1, 1)$, and so $c_n = c(\rho_n) \to 0$ by Proposition 7.2. □

The next two results concern values $\rho \in W \backslash U$. Using sub- and supersolutions, we shall obtain a monotone solution $x = P(\xi)$ to (2.1), satisfying (1.3), for $c = 0$. By a **subsolution** of (2.1) with $c = 0$, for some $\rho \in \bar{V}$, we mean a function $x\colon \mathbf{R} \to \mathbf{R}$, not necessarily continuous, for which

$$F(x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N), \rho) \geqslant 0, \qquad \xi \in \mathbf{R}$$

and by a **supersolution** with $c = 0$ we mean such a function satisfying the opposite inequality

$$F(x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N), \rho) \leqslant 0, \qquad \xi \in \mathbf{R}$$

We note that we shall only be concerned here with sub- and supersolutions with $c = 0$, as above.

Lemma 8.3. *Suppose for some $\rho \in \bar{V}$, that $\mathscr{B}$ is a nonempty set of subsolutions to (2.1), and that there exists a pointwise upper bound*

$$x(\xi) \leqslant K(\xi), \qquad \xi \in \mathbf{R}, \qquad x \in \mathscr{B}$$

for some $K\colon \mathbf{R} \to \mathbf{R}$. Then the function $x_0\colon \mathbf{R} \to \mathbf{R}$ given by

$$x_0(\xi) = \sup_{x \in \mathscr{B}} x(\xi) \tag{8.4}$$

is also a subsolution.

Proof. Fix $\xi \in \mathbf{R}$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\begin{aligned}|F(u_1, x_0(\xi + r_2), ..., x_0(\xi + r_N), \rho)\\ - F(x_0(\xi + r_1), x_0(\xi + r_2), ..., x_0(\xi + r_N), \rho)| \leqslant \varepsilon\\ \text{whenever} \quad |u_1 - x_0(\xi + r_1)| \leqslant \delta\end{aligned} \tag{8.5}$$

There exists $x \in \mathscr{B}$ such that $x(\xi) \leqslant x_0(\xi) \leqslant x(\xi) + \delta$. We therefore have

$$\begin{aligned}F(x(\xi + r_1), x_0(\xi + r_2), ..., x_0(\xi + r_N), \rho)\\ \geqslant F(x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N), \rho) \geqslant 0\end{aligned}$$

as $x_0(\xi + r_j) \geqslant x(\xi + r_j)$ for every j, and with (8.5) we conclude that

$$F(x_0(\xi + r_1), x_0(\xi + r_2), ..., x_0(\xi + r_N), \rho) \geqslant -\varepsilon$$

Since ε is arbitrary, as is ξ, we conclude that x_0 is a subsolution. □

Proposition 8.4. *Suppose for some $\rho \in \bar{V}$ that there exist $x_-: \mathbf{R} \to \mathbf{R}$ and $x_+: \mathbf{R} \to \mathbf{R}$, which are sub- and supersolutions respectively, and which satisfy*

$$x_-(\xi) \leqslant x_+(\xi), \qquad \xi \in \mathbf{R} \tag{8.6}$$

Then there exists $P: \mathbf{R} \to \mathbf{R}$, with $x_-(\xi) \leqslant P(\xi) \leqslant x_+(\xi)$ for all $\xi \in \mathbf{R}$, which is a solution to (2.1) with $c = 0$. If in addition x_+ is monotone increasing, then P can be chosen monotone increasing.

Proof. Let

$$\begin{aligned}\mathscr{B} = \{x: \mathbf{R} \to \mathbf{R} \mid x \text{ is a subsolution, and}\\ x_-(\xi) \leqslant x(\xi) \leqslant x_+(\xi) \text{ for all } \xi \in \mathbf{R}\}\end{aligned}$$

and let

$$P(\xi) = \sup_{x \in \mathscr{B}} x(\xi)$$

Then P is a subsolution by Lemma 8.3, and we claim that it in fact is a solution. Suppose, to the contrary, that

$$F(P(\xi_0 + r_1), P(\xi_0 + r_2), ..., P(\xi_0 + rN), \rho) > 0$$

at some $\xi_0 \in \mathbf{R}$. Let us first observe the strict inequality $P(\xi_0) < x_+(\xi_0)$ at this point. If this were false then $P(\xi_0) = x_+(\xi_0)$, and one obtains

$$\begin{aligned} &F(x_+(\xi_0 + r_1), x_+(\xi_0 + r_2), ..., x_+(\xi_0 + r_N), \rho) \\ &\quad \geqslant F(P(\xi_0 + r_1), P(\xi_0 + r_2), ..., P(\xi_0 + r_N), \rho) > 0 \end{aligned}$$

a contradiction.

Next define $\bar{P}: \mathbf{R} \to \mathbf{R}$ by

$$\bar{P}(\xi) = \begin{cases} P(\xi), & \xi \neq \xi_0 \\ P(\xi) + \varepsilon, & \xi = \xi_0 \end{cases}$$

where $\varepsilon > 0$ is sufficiently small that

$$F(\bar{P}(\xi_0 + r_1), \bar{P}(\xi_0 + r_2), ..., \bar{P}(\xi_0 + r_N), \rho) \geqslant 0$$

holds and so that also $\bar{P}(\xi_0) \leqslant x_+(\xi_0)$. At any $\xi \neq \xi_0$ we also have that

$$\begin{aligned} &F(\bar{P}(\xi + r_1), \bar{P}(\xi + r_2), ..., \bar{P}(\xi + r_N), \rho) \\ &\quad \geqslant F(P(\xi + r_1), P(\xi + r_2), ..., P(\xi + r_N), \rho) \geqslant 0 \end{aligned}$$

as $\bar{P}(\xi + r_1) = P(\xi + r_1)$, while $\bar{P}(\xi + r_j) \geqslant P(\xi + r_j)$ for $2 \leqslant j \leqslant N$. Thus $\bar{P}$ is a subsolution, and so $\bar{P} \in \mathscr{B}$. However, $\bar{P}(\xi_0) > P(\xi_0)$ contradicts the definition of P as the supremum of elements in $\mathscr{B}$. From this contradiction we conclude that P is a solution.

Suppose, in addition, that x_+ is monotone increasing. We shall show that for any $x \in \mathscr{B}$, the function

$$\tilde{x}(\xi) = \sup_{s \leqslant \xi} x(s) \tag{8.7}$$

also belongs to $\mathscr{B}$. As $x(\xi) \leqslant \tilde{x}(\xi)$ for all $\xi \in \mathbf{R}$, it follows that the function P given by (8.4) is monotone increasing, as claimed.

One easily checks from the monotonicity of x_+ that $x_-(\xi) \leqslant \tilde{x}(\xi) \leqslant x_+(\xi)$ for all ξ, so all that must be shown is that $\tilde{x}$ is a subsolution. Fix any $\xi \in \mathbf{R}$, and let $s_n \leqslant \xi$ be such that $x(s_n) \to \tilde{x}(\xi)$, using the definition (8.7). Then

$$\begin{aligned} 0 &\leqslant F(x(s_n + r_1), x(s_n + r_2), ..., x(s_n + r_N), \rho) \\ &\leqslant F(x(s_n + r_1), \tilde{x}(\xi + r_2), ..., \tilde{x}(\xi + r_N), \rho) \\ &\to F(\tilde{x}(\xi + r_1), \tilde{x}(\xi + r_2), ..., \tilde{x}(\xi + r_N), \rho) \end{aligned}$$

which implies that $\tilde{x}$ is a subsolution. □

The next result shows how the real eigenvalues λ_-^u and λ_+^s of the linearization about $x = \pm 1$ in a normal family vary with ρ.

Proposition 8.8. *For a normal family we have that*

$$\frac{\partial \lambda_-^u}{\partial \rho} > 0, \qquad \frac{\partial \lambda_+^s}{\partial \rho} > 0$$

for any $c \in \mathbf{R}$ and $\rho \in (-1, 1)$ for which $\lambda_-^u(c, \rho)$ or $\lambda_+^s(c, \rho)$ is finite.

Proof. These results follow directly from Proposition 4.3 once we show, for each ρ, that

$$-w_- \in M_+^*, \qquad w_+ \in M_-^* \tag{8.8}$$

for the two vectors $w_\pm \in \mathbf{R}^N$ defined componentwise by

$$w_{j\pm} = A'_{j\pm}(\rho), \qquad 1 \leqslant j \leqslant N$$

where $A_{j\pm}(\rho)$ are the coefficients as in (6.2). These coefficients are C^1 in ρ by condition (ix). Moreover, from condition (viii) and the fact that $\Phi(\pm 1, \rho) = 0$ we see directly that

$$\pm \sum_{j=1}^{N} v_j A'_{j\pm}(\rho) = \pm \sum_{j=1}^{N} v_j \frac{\partial^2 F(u, \rho)}{\partial \rho\, \partial u_j}\bigg|_{u=\kappa(\pm 1)} \geqslant 0, \qquad v \in M_\mp \tag{8.9}$$

and so (8.8) will follow once we show the inequality in (8.9) is strict. Now $M_\mp \subseteq (0, \infty)^N$ is an open set, and so if (8.9) is an equality for some $v \in M_\mp$, necessarily

$$A'_{j\pm}(\rho) = 0 \qquad \text{for each } j \tag{8.10}$$

However, condition (x) is simply the inequality

$$\pm \sum_{j=1}^{N} A'_{j\pm}\rho) > 0$$

and so (8.10) is impossible. □

We will be able to complete the proof Theorem 2.1 once we have the following lemma, which shows that any system (2.1), but without a parameter ρ, can be embedded in a normal family. In fact, by possibly introducing two additional shifts r_{N+1} and r_{N+2}, the normal family can be made coercive at ± 1.

Lemma 8.6. *Consider a system*

$$-cx'(\xi) = F_0(x(\xi + r_1), x(\xi + r_2), ..., x(\xi + r_N)) \qquad (8.11)$$

as in (2.1), satisfying (i)–(v), but for which the parameter ρ is absent. Assume that $q = q_0 \in (-1, 1)$ for the quantity in condition (v). Then there exists a normal family (2.1), possibly with N replaced by $N+1$ or by $N+2$, which is coercive at ± 1, and which reduces to (8.11) at $\rho = q_0$.

Proof. Denoting $\Phi_0(x) = F_0(x, x, ..., x)$, we see that it is straightforward to find a C^1 smooth function $\Phi: \mathbf{R} \times [-1, 1] \to \mathbf{R}$ such that $\Phi(x, \rho)$ satisfies conditions (iv) and (v) for all $\rho \in [-1, 1]$, with $q(\rho) = \rho$ identically (thus $\Phi(\rho, \rho) = 0$), satisfies condition (x), and such that also

$$\Phi(x, q_0) = \Phi_0(x), \qquad x \in \mathbf{R}$$

$$\frac{\partial \Phi(x, \rho)}{\partial \rho} < 0, \qquad x \in (-1, 1), \quad \rho \in (-1, 1)$$

[We note here that the given function Φ_0 may have multiple local maxima and minima in $(-1, q_0)$ and in $(q_0, 1)$, so some care is needed in constructing Φ.] For such Φ, let $\tilde{F}: \mathbf{R}^N \times (-1, 1) \to \mathbf{R}$ be given by

$$\tilde{F}(u_1, u_2, ..., u_N, \rho) = F_0(u_1, u_2, ..., u_N) + \Phi(u_1, \rho) - \Phi_0(u_1)$$

It is now easy to see that the system (2.1) with this $\tilde{F}$ is a normal family, which of course reduces to (8.11) when $\rho = q_0$.

A final modification is needed to produce a normal family which is coercive at ± 1. This is accomplished by introducing two new variables u_{N+1} and u_{N+2}, with shifts r_{N+1} and r_{N+2} satisfying

$$r_{N+1} < 0 < r_{N+2}$$

We define $F: \mathbf{R}^{N+2} \times [-1, 1] \to \mathbf{R}$ by

$$F(u_1, u_2, ..., u_{N+2}, \rho)$$
$$= \tilde{F}(u_1, u_2, ..., u_N, \rho) + K_+(\rho)(u_{N+1} - u_1) + K_-(\rho)(u_{N+2} - u_1)$$

with coefficients $K_\pm(\rho)$ to be determined. We require first that $K_\pm(\rho)$ be C^1 in ρ and satisfy

$$K_\pm(\rho) \geq 0, \qquad \pm K'_\pm(\rho) \geq 0, \qquad \rho \in [-1, 1]$$

With this, all conditions (i)–(x) for a normal family hold. Note in particular that $K_{\pm}(\rho) \geqslant 0$ implies condition (ii), with $N+1 \in \mathscr{U}(\rho)$ if and only if $K_{+}(\rho) > 0$, and $N+2 \in \mathscr{U}(\rho)$ if and only if $K_{-}(\rho) > 0$. Also, $\pm K'_{\pm}(\rho) \geqslant 0$ implies condition (viii). Next, our modified system must reduce to (8.11) at $\rho = q_0$, and this requires that $K_{\pm}(\rho_0) = 0$. Finally, coercivity is achieved by choosing r_{N+1} and r_{N+2} to have sufficiently large norm. Indeed, our system is coercive at $+1$ if

$$\sum_{j=1}^{N} r_j \left. \frac{\partial \tilde{F}(u, \rho)}{\partial u_j} \right|_{\substack{u=\kappa(1) \\ \rho=1}} + r_{N+1} K_{+}(1) < 0$$

with a similar condition for coercivity at -1. One easily sees that it is possible to choose $K_{\pm}(\rho)$ satisfying all the required conditions. □

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. With the exception of existence of a monotone solution for each $\rho \in W \backslash U$, with $c = 0$, all the claims of Theorem 2.1 have been proved. [Here the set $U \subseteq W$ is defined by (6.13), with $\mathscr{M}$ as in (2.5).] In particular, the monotonicity condition (2.3), when $c \neq 0$, follows from Proposition 6.3. The uniqueness of c, and of P when $c \neq 0$, is given in Proposition 6.5. The smooth dependence of $c = c(\rho)$ and of $P = P(\rho)$ on $\rho \in U$, that is when $c(\rho) \neq 0$, is given in Proposition 6.4. Finally, if we set

$$c(\rho) = 0, \qquad \rho \in W \backslash U$$

then Proposition 7.2 implies that $c\colon W \to \mathbf{R}$ so defined is continuous.

We consider then any given system (8.11) as in the statement of Lemma 8.6. By this lemma, the system (8.11) can be embedded in a normal family (2.1) which moreover is coercive at ± 1. By Theorem 2.6 we have $-1 < \rho_{-} \leqslant \rho_{+} < 1$, and so a solution (c, ρ), as required, exists for $\rho \in U = (-1, \rho_{-}) \cup (\rho_{+}, 1)$. Also, by Proposition 8.2 there exist monotone solutions of (2.1) satisfying (1.3) for $c = 0$, at $\rho = \rho_{\pm}$. Let us denote these solutions by

$$x = x_{+}(\xi) \quad \text{for } \rho = \rho_{-}, \qquad x = x_{-}(\xi) \quad \text{for } \rho = \rho_{+} \tag{8.12}$$

[Observe the reversal of signs $\pm$ in the above definition (8.12) of the solutions $x_{\pm}$. We choose this notation so as later to conform with the notation of Proposition 8.4.]

Now fix any $\rho_0 \in (\rho_{-}, \rho_{+})$. It is enough to obtain a monotone solution $x = P(\xi)$ of (2.1) satisfying (1.3), for $c = 0$, at this ρ_0. In fact, the solution P

will be obtained from Proposition 8.4 by showing for $\rho = \rho_0$ that x_- and x_+ are sub- and supersolutions, respectively, which satisfy (8.6).

We first establish (8.6). In fact, it is enough to show that

$$x_-(\xi) \leqslant x_+(\xi), \qquad |\xi| \geqslant \tau \tag{8.13}$$

for some τ, since the inequality (8.6) on all of $\mathbf{R}$ can then be achieved by replacing either x_- or x_+ with a translate of itself. By Proposition 8.5 we have that either

$$-\infty \leqslant \lambda_+^s(0, \rho_-) < \lambda_+^s(0, \rho_+) \tag{8.14}$$

or else that

$$-\infty = \lambda_+^s(0, \rho_-) = \lambda_+^s(0, \rho_+) \tag{8.15}$$

for the stable eigenvalues of the linearization about $x = 1$. In particular, $\lambda_+^s(0, \rho_\pm) = -\infty$ holds if and only if $r_{\min}(\rho_\pm) = 0$, by (4.16). From (2.9) and (2.10) of Theorem 2.2 we thus conclude in either case (8.14) or (8.15) that

$$x_-(\xi) \leqslant x_+(\xi), \qquad \xi \geqslant \tau$$

for some τ. With a similar argument at $-\infty$, we obtain (8.13).

We now show that x_- is a subsolution, omitting the similar proof that x_+ is a supersolution. We have from the monotonicity of x_- that $\pi(x_-, \xi) \in \bar{M}$, where $\bar{M} \subseteq [-1, 1]^N$ denotes the closure of the set M in (2.13). With condition (viii) in the definition of normal family, it follows that

$$0 = F(\pi(x_-, \xi), \rho_+) \leqslant F(\pi(x_-, \xi), \rho_0)$$

as $\rho_+ > \rho_0$. This implies that x_- is a subsolution for $\rho = \rho_0$, and completes the proof of Theorem 2.1. □

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