



THE GOMPERTZ EXTENDED GENERALIZED EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATIONS

J.T. EGHWERIDO, L.C. NZEI, I.J. DAVID, AND O.D. ADUBISI

ABSTRACT. In this article, a new class of distribution of the exponential family of distributions called the Gompertz extended generalized exponential (G-EGE) distribution for life time processes is proposed. The mathematical properties of the G-EGE distribution such as reliability, hazard rate function, reversed hazard, cumulative, odd functions, quantiles function, kurtosis, skewness and order statistics were derived. The parameters of the G-EGE distribution were estimated using the maximum likelihood method. The efficiency and flexibility of the G-EGE distribution were examined using a simulation study and a real life data application. The results revealed that the G-EGE distribution outperformed some existing distributions in terms of their test statistics.

1. INTRODUCTION

Modeling lifetime processes has received several attentions in recent years. However, the lifetime processes rely on the phenomena of distribution. Thus, developing a flexible distributions depends on how the researcher compound one or more distributions to form a more flexible distribution [1]. One of such distributions in modeling lifetime processes is exponential distribution. The exponential distribution is used to describing the time between events with a Poisson processes. Thus, the exponential distribution has been used to model processes with continuous memoryless random processes and constant failure rate. However, the occurrence of constant failure rate is almost impossible in real life. Hence, to account for this shortcoming in distribution theory, [2] modeled lifetime processes with inverted exponential (IE) distribution. The inverted exponential distribution was extensively studied in [3]; who applied it to various data from the field of engineering and medicine. [4] proposed the transmuted inverse exponential distribution and studied its statistical

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properties using data from medicine and engineering. [5] also examined the statistical properties of the exponentiated generalized inverted exponential distribution. [6] proposed the Kumaraswamy inverse exponential distribution. More so, [7] proposed the extended generalized exponential distribution. [8] proposed the Harris extended exponential distribution. [9] proposed the extended Poisson exponential distribution. [10] proposed fractional beta exponential distribution. [11] proposed the exponentiated generalized extended exponential distribution. [12] proposed the moments of the alpha power transformed generalized exponential distribution. [13] proposed the extended weighted exponential distribution. [14] proposed the type I general exponential class of distribution. [15] proposed the Gompertz alpha power inverted exponential distribution. [16] proposed extended new generalized exponential distribution. [17] proposed the alpha power Gompertz distribution. [18] proposed the odd exponentiated half logistic-G family of distribution. [19] proposed a new distribution using the tangent function. [20] proposed generalized exponential distribution. [21] proposed the alpha power inverted exponential distribution. [22] proposed the alpha power Weibull distribution. [23] proposed a new extension of generalized exponential distribution. [24] proposed transmuted exponentiated generalized-G family of distributions. [25] proposed exponentiated generalized-G Poisson distribution. [26] proposed exponentiated generalized class of distributions. [27] proposed a new method for generating distributions with an application to exponential distribution. [28] proposed a method for estimating the generalized inverted exponential distribution.

The cumulative distribution function (cdf) of the extended generalized distribution is given as

$$G(x; \gamma, \beta) = \frac{(\beta - e^{-x})^\gamma - (\beta - 1)^\gamma}{(\beta^\gamma - (\beta - 1)^\gamma)} \quad (\beta^\gamma - (\beta - 1)^\gamma) \neq 0 \quad x > 0, \quad \gamma > 1, \quad \beta > 1. \quad (1)$$

The corresponding probability density function (pdf) to Equation (1) is given as

$$g(x; \gamma, \beta) = \frac{\gamma(\beta - e^{-x})^{\gamma-1} e^{-x}}{(\beta^\gamma - (\beta - 1)^\gamma)} \quad (\beta^\gamma - (\beta - 1)^\gamma) \neq 0 \quad x > 0, \quad \gamma > 1, \quad \beta > 1, \quad (2)$$

where γ is shape parameter and β is the scale parameter.

Also, the Gompertz distribution is a continuous distribution used to describe the lifespan of stochastic processes. Hence, there exist a relationship between the exponential and the Gompertz distributions. A lot of researchers have developed different compound distributions using the exponential and Gompertz distributions. However, no knowledge of Gompertz extended generalized exponential distribution was found in existing literature. Hence, this study is motivated to bridge the gap in existing literature by proposing a lifetime distribution called Gompertz extended generalized exponential (G-EGE) distribution using the Gompertz-G characterization. This distribution is further applied to glass fibre to examine its efficiency and flexibility.

Let $G(x; \tau)$ and $g(x; \tau)$ be the baseline model with parameter vector τ . Then, the cdf of Gompertz-G family proposed in [29] is given as

$$F(x) = \int_0^{B[G(x; \tau)]} u(t) dt, \quad (3)$$

where $u(t)$ is the probability density function of the Gompertz distribution and $B[G(x; \tau)] = -\log[1 - G(x; \tau)]$ is the link function.

The cumulative distribution function in Equation (3) can be expressed as

$$F(x) = \int_0^{-\log[1-G(x; \tau)]} \theta e^{\lambda t - \frac{\theta}{\lambda}(e^{\lambda t} - 1)} dt = 1 - e^{\frac{\theta}{\lambda}(1 - (1 - G(x; \tau))^{-\lambda})} \text{ for } \theta > 0, \lambda > 0, \quad (4)$$

where λ and θ are additional two shape parameters.

The pdf that corresponds to the G-family of distribution is given as

$$f(x) = \left[\frac{d}{dx} B[G(x; \tau)] \right] u[B[G(x; \tau)]] = \theta g(x; \tau) [1 - G(x; \tau)]^{-\lambda - 1} e^{\frac{\theta}{\lambda}(1 - (1 - G(x; \tau))^{-\lambda})}. \quad (5)$$

A random variable X with pdf in Equation (5) is denoted by $X \sim \text{Gompertz} - G(\theta, \lambda, \tau)$.

The aim of this study is to propose a G-EGE class of the family of the exponential distribution and examining its statistical characteristics extensively.

This paper is unfolded as follows. In Section 2, we define the G-EGE distribution and a plot for its pdf, cdf and hazard rate function (hrf). Useful mixture representation of the pdf is derived in Section 3. In Section 4 derives some mathematical properties of the newly proposed class of distribution. In Section 5, the order statistics is obtained. The maximum likelihood estimates (MLEs) of the newly proposed class of distribution and simulation are performed in Section 6. The viability of the new class of distribution is examined in Section 7 by means of real life data sets. Section 7 is the concluding remarks.

2. THE GOMPERTZ EXTENDED GENERALIZED EXPONENTIAL DISTRIBUTION

In this section, we shall establish the pdf and the cdf of the newly proposed continuous distribution. Let X be a continuous random variable. Then, X follows an G-EGE distribution if its pdf is given as

$$\begin{aligned} f_{(G-EGE)}(x; \theta, \lambda, \beta, \gamma) &= \theta \gamma \exp(-x) \left(\beta^\gamma - (\beta - 1)^\gamma \right)^\lambda \left(\beta - \exp(-x) \right)^{\gamma - 1} \\ &\times \left[\beta^\gamma - (\beta - \exp(-x))^\gamma \right]^{-(\lambda + 1)} \\ &\times \exp \left(\frac{\theta}{\lambda} \left\{ 1 - \left(\frac{\beta^\gamma - (\beta - 1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma} \right)^\lambda \right\} \right) \end{aligned} \quad (6)$$

for $\theta > 0, \lambda > 0, x > 0, \gamma > 1, \beta > 1$.

The cdf that corresponds to the pdf is given as

$$F_{(G-EGE)}(x) = 1 - \exp\left(\frac{\theta}{\lambda} \left\{ 1 - \left(\frac{\beta^\gamma - (\beta - 1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma} \right)^\lambda \right\}\right) \tag{7}$$

for $\theta > 0, \lambda > 0, x > 0, \gamma > 1, \beta > 1$,

where γ is shape parameter and β is the scale parameter; λ and θ are additional two shape parameters.

Figure 1 shows the plots of the G-EGE density for some selected values of the parameters γ, β, λ and θ . The pdf plots indicate that the G-EGE distribution can be unimodal, left skewed, increasing and decreasing.

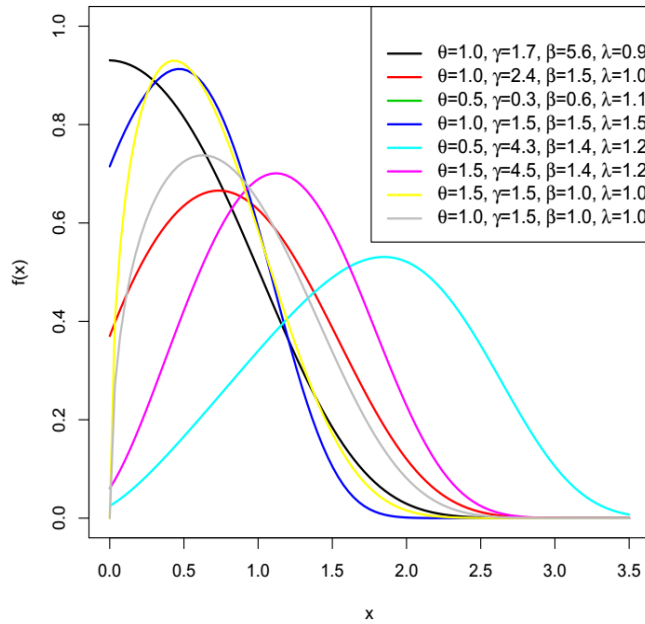


FIGURE 1. The plots of the G-EGE pdf for some parameter values.

The Hazard Rate function (hrf), reliability function (rf) and cumulative hazard rate function (chrf) of the random variable X are given respectively as

$$hrf(x) = \frac{f_{(G-EGE)}(x)}{1 - F_{(G-EGE)}(x)} = \theta \gamma \exp(-x) \left(\beta^\gamma - (\beta - 1)^\gamma \right)^\lambda \left(\beta - \exp(-x) \right)^{\gamma-1} \times \left[\beta^\gamma - (\beta - \exp(-x))^\gamma \right]^{-(\lambda+1)}. \tag{8}$$

Figure 2 shows the plots for the hazard rate function of the G-EGE distribution. The plots shows that the G-EGE density is increasing and bathtub depending on the values of the parameters γ, β, λ , and θ .

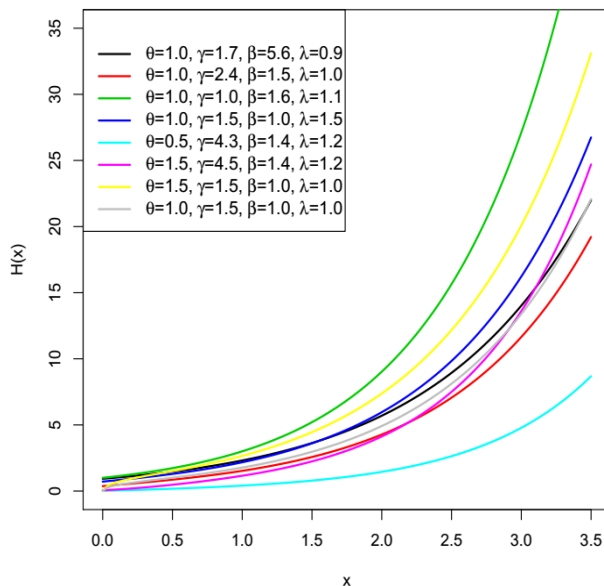


FIGURE 2. The plots of the G-EGE hrf for some parameter values.

$$R(x) = 1 - F_{(G-EGE)}(x) = \exp\left(\frac{\theta}{\lambda} \left\{ 1 - \left(\frac{\beta^\gamma - (\beta - 1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma} \right)^\lambda \right\}\right). \quad (9)$$

$$H(x) = -\ln R_{(G-EGE)}(x) = \left\{ \frac{\theta}{\lambda} \left(\frac{\beta^\gamma - (\beta - 1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma} \right)^\lambda \right\} - \frac{\theta}{\lambda}. \quad (10)$$

3. MIXTURE REPRESENTATION

The quantity $(\beta - \exp(-x))^\gamma$ can be expressed as

$$\sum_{k=0}^{\gamma} (-1)^k \binom{\gamma}{k} \beta^{\gamma-k} \exp(-xk).$$

More so, the quantity $\left(\beta^\gamma - (\beta - \exp(-x))^\gamma\right)^{\lambda+1}$ can be expressed as

$$\sum_{k=0}^{\gamma} \sum_{p=0}^{\lambda+1} (-1)^{p(k+1)} \binom{\lambda+1}{p} \binom{\gamma}{k}^p \beta^{\lambda(\gamma+1)+p(\gamma-\lambda-k)} \exp(-xkp).$$

Thus, inserting these expressions into Equation (6) and after some algebraic simplification we expanded Equation (6) as

$$\begin{aligned} f(x) &= \sum_{k=0}^{\gamma} \sum_{p=0}^{\lambda+1} \sum_{i=0}^{\gamma-1} \frac{(\gamma-1)!}{(\gamma-i-1)!i!} \theta^\gamma \exp(-x) \left(\beta^\gamma - (\beta-1)^\gamma\right)^\lambda (-1)^{i-p(k+1)} \\ &\quad \times \exp(-xi) a^{m+j} \beta^{\binom{\gamma-i-1}{\lambda(\gamma+1)+p(\gamma-\lambda-k)}} \exp(xkp) \\ &\quad \times \exp\left(\frac{\theta}{\lambda} \left\{ 1 - \left(\frac{\beta^\gamma - (\beta-1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma}\right)^\lambda \right\}\right), \end{aligned} \quad (11)$$

where

$$a^j = \left[\frac{(\lambda-p+1)!p!}{(\lambda+1)!}\right] \text{ and } a^m = \left[\frac{(\gamma-k)!k!}{\gamma!}\right]^p.$$

Expanding the binomial terms, we have

$$f(x) = \sum_{k=0}^{\gamma} \sum_{p=0}^{\lambda+1} \sum_{i=0}^{\gamma-1} v_{i,k,p} \exp\left(-x D_{ikp} - m \left(\beta^\gamma - (\beta - e^{-x})^\gamma\right)^{-\lambda}\right), \quad (12)$$

where

$$\begin{aligned} v_{i,k,p} &= \frac{(\gamma-1)!}{(\gamma-i-1)!i!} \theta^\gamma \left(\beta^\gamma - (\beta-1)^\gamma\right)^\lambda (-1)^{i-p(k+1)} a^{m+j} \\ &\quad \times \beta^{-\left(\lambda(\gamma+1)+p(\gamma-\lambda-k)\right)+(\gamma-i-1)} \exp\left(\frac{\theta}{\lambda}\right). \end{aligned}$$

$$D_{ikp} = (i - kp + 1),$$

$$m = \frac{\theta}{\lambda} \left(\frac{1}{\beta^\gamma - (\beta-1)^\gamma}\right)^{-\lambda}.$$

4. MATHEMATICAL PROPERTIES

This section investigates some statistical properties of the G-EGE distribution. This includes quantile and random number generation, Skewness, Kurtosis and order statistics. These structural properties of the G-EGE distribution can be computed efficiently by using programming softwares like R, Mathematical, Maple and Matlab.

4.1. Quantile function and random number generation. Let X be a random variable such that $X \sim G - EGE(\theta, \beta, \gamma, \lambda)$. Then, the quantile function of X for $p \in (0, 1)$ is obtained by inverting Equation (7) as

$$x_p = -\log \left[\beta - \left(\beta^\gamma - (\beta^\gamma - (\beta - 1)^\gamma) \left(1 - \frac{\lambda}{\theta} \log(1 - p) \right)^{-\frac{1}{\lambda}} \right)^{\frac{1}{\gamma}} \right]. \quad (13)$$

Setting $p = 0.5$ in Equation (13) gives the median M of X as

$$x_{0.5} = -\log \left[\beta - \left(\beta^\gamma - (\beta^\gamma - (\beta - 1)^\gamma) \left(1 - \frac{\lambda}{\theta} \log(0.5) \right)^{-\frac{1}{\lambda}} \right)^{\frac{1}{\gamma}} \right] \quad 0 < p < 1. \quad (14)$$

Simulating the G-EGE random variable is flexible. If U is a uniform variates on the interval $(0, 1)$, then the random variable $X = x_p$ at $p = U$ follows the $x_p \sim G - EGE(\theta, \beta, \gamma, \lambda)$ of Equation (6).

However, the 25th and 75th percentile for the random variable X are obtained as

$$x_{0.25} = -\log \left[\beta - \left(\beta^\gamma - (\beta^\gamma - (\beta - 1)^\gamma) \left(1 - \frac{\lambda}{\theta} \log(0.75) \right)^{-\frac{1}{\lambda}} \right)^{\frac{1}{\gamma}} \right], \quad (15)$$

$$x_{0.75} = -\log \left[\beta - \left(\beta^\gamma - (\beta^\gamma - (\beta - 1)^\gamma) \left(1 - \frac{\lambda}{\theta} \log(0.25) \right)^{-\frac{1}{\lambda}} \right)^{\frac{1}{\gamma}} \right]. \quad (16)$$

4.2. Skewness and Kurtosis. The Bowleys formula for coefficient of skewness is given as

$$Sk = \frac{x_{0.75} - 2x_{0.5} + x_{0.25}}{x_{0.75} - x_{0.25}}.$$

However, the Moors formula for coefficient of kurtosis is given as

$$Ks = \frac{x_{0.875} - x_{0.625} - x_{0.375} + x_{0.125}}{x_{0.75} - x_{0.25}}.$$

4.3. Order statistics. Let X_1, X_2, \dots, X_n be a random sample of size n of the $f_{APEGE}(x)$ distribution and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the corresponding order statistics. Then, probability density function of the i th order statistics X_k , say $f_k(x)$ is expressed as

$$g_k(y_k) = \frac{n!}{(k-1)! (nk)!} \left[F_{G-EGE}(y_k) \right]^{k-1} f_{G-EGE}(y_k) \left[1 - F_{G-EGE}(y_k) \right]^{k-1}. \quad (17)$$

We can write

$$\begin{aligned}
 g_k(y_k) &= \frac{n!}{(k-1)!(nk)!} \\
 &\times \left[1 - \exp\left(\frac{\theta}{\lambda} \left\{ 1 - \left(\frac{\beta^\gamma - (\beta-1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma} \right)^\lambda \right\} \right) \right]^{k-1} \\
 &\times \sum_{k=0}^{\gamma} \sum_{p=0}^{\lambda+1} \sum_{i=0}^{\gamma-1} v_{i,k,p} \exp\left(-x(i-kp+1) + \frac{\theta}{\lambda} \left\{ 1 - \left(\frac{\beta^\gamma - (\beta-1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma} \right)^\lambda \right\} \right) \\
 &\times \left[\exp\left(\frac{\theta}{\lambda} \left\{ 1 - \left(\frac{\beta^\gamma - (\beta-1)^\gamma}{\beta^\gamma - (\beta - \exp(-x))^\gamma} \right)^\lambda \right\} \right) \right]^{n-k}.
 \end{aligned} \tag{18}$$

The order statistics for the G-EGE distribution can be obtained as follows:

- The minimum order statistics is obtained for $k = 1$.
- The median is obtained when $k = m = 1$, given n is odd expressed as $n = 2m + 1$.
- The maximum order statistics is obtained for $k = n$ for even n expressed as $n = 2m$.

5. PARAMETER ESTIMATION

Several approaches have been employed for parameter estimation in literature. In this article, the maximum likelihood method was adopted to obtain the parameters of the G-EGE. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a random sample of the G-EGE model with unknown parameter vector $\theta = (\theta, \beta, \gamma, \lambda)^T$. Then, the log-likelihood function ℓ of the G-EGE distribution can be expressed as

$$\ell = n \log \theta + n \log \gamma - \sum_{i=1}^n x_i + n \lambda \log z - (\lambda + 1) \sum_{i=1}^n s_i + \sum_{i=1}^n \frac{\theta}{\lambda} \left\{ 1 - \left(\frac{z}{s_i} \right)^\lambda \right\}, \tag{19}$$

where

$$z = \beta^\gamma - (\beta - 1)^\gamma \quad \text{and} \quad s = \beta^\gamma - (\beta - \exp(-x))^\gamma$$

However, the partial derivative of the ℓ with respect to each parameter is given as

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \frac{1}{\lambda} \sum_{i=1}^n \left\{ 1 - \left(\frac{z}{s_i} \right)^\lambda \right\}, \tag{20}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} + \frac{n \lambda z'_\gamma}{z} - (\lambda + 1) \sum_{i=1}^n s'_\gamma - \sum_{i=1}^n \theta \left\{ z^{\lambda-1} z'_\gamma s_i^{-\lambda} - z^\lambda s_i^{-\lambda-1} s'_\gamma \right\}, \tag{21}$$

$$\frac{\partial \ell}{\partial \lambda} = n \log z - \sum_{i=1}^n s_i - \frac{n\theta}{\lambda^2} + \sum_{i=1}^n \left\{ \frac{\theta}{\lambda^2} \left(\frac{z}{s_i} \right)^\lambda - \frac{\theta}{\lambda} \left(\frac{z}{s_i} \right)^\lambda \ln \left(\frac{z}{s_i} \right) \right\}, \quad (22)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{n\lambda\gamma \left(\beta^{\gamma-1} - (\beta-1)^{\gamma-1} \right)}{\beta^\gamma - (\beta-1)^\gamma} - \gamma(\lambda+1) \sum_{i=1}^n \left(\beta^{\gamma-1} - (\beta - \exp(-x)) \right)^{\gamma-1} \\ &\quad - \sum_{i=1}^n \gamma\theta \left(z^{\lambda-1} s_i^{-\lambda} \left(\beta^{\gamma-1} - (\beta-1)^{\gamma-1} \right) - z^\lambda s_i^{(\lambda+1)} \left(\beta^\gamma - (\beta - \exp(-x))^{\gamma-1} \right) \right), \end{aligned} \quad (23)$$

where

$$z'_\gamma = \frac{\partial z}{\partial \gamma}; s'_\gamma = \frac{\partial s}{\partial \gamma}.$$

The solution to the vector is obtained analytically using Newton-Raphson algorithm. Software like MATLAB, R, MAPLE, and so on could be used to obtain the estimates.

5.1. Simulations study. A simulation is carried out to test the flexibility and efficiency of the G-EGE distribution. Table 1 shows the simulation for different values of parameters for the G-EGE distribution. The simulation is performed as follows:

- Data are generated using

$$x = -\log \left[\beta - \left(\beta^\gamma - (\beta^\gamma - (\beta-1)^\gamma) \left(1 - \frac{\lambda}{\theta} \log(1-p) \right)^{-\frac{1}{\lambda}} \right)^\frac{1}{\gamma} \right], \quad 0 < p < 1$$

- The values of the parameters are set as follows: $\gamma = 1.5$, $\theta = 1.3$, $\lambda = 1.5$, and $\beta = 2.0$
- The sample sizes are taken as $n = 50, 100, 150, 250$ and 350 .
- Each sample size is replicated 1000 times.

The bias is calculated by (for $S = \hat{a}, \hat{b}, \hat{\alpha}, \hat{\lambda}$)

$$\hat{Bias}_S = \frac{1}{1000} \sum_{i=1}^{1000} \left(\hat{S}_i - S \right).$$

Also, the MSE is obtained as

$$\hat{MSE}_S = \frac{1}{1000} \sum_{i=1}^{1000} \left(\hat{S}_i - S \right)^2.$$

The simulation study investigates the average estimates (MEs), biases, variance, means squared errors and roots means squared errors. The results are shown in Table 1. The results of the Monte Carlo study show that the MSEs and RMSEs decay towards zero as the sample size increases. This corroborates the first-order asymptotic theory. The mean estimates of the parameters tend to the true parameter

TABLE 1. A simulation Study of the G-EGE Distribution

Sample size	Parameter	Average estimate	Bias	Variance	MSE	RMSE
50	γ	1.4859	-0.0141	0.1470	0.1472	0.3836
	θ	1.2792	-0.0208	0.0299	0.0303	0.1741
	λ	1.5913	0.0913	0.1078	0.1162	0.3408
	β	2.1554	0.1554	0.3791	0.4032	0.6350
100	γ	1.5168	0.0168	0.0739	0.0742	0.2723
	θ	1.3097	0.0097	0.0146	0.0147	0.1211
	λ	1.5657	0.0657	0.0775	0.0818	0.2860
	β	2.0793	0.0793	0.1920	0.1983	0.4453
150	γ	1.5143	0.0143	0.0549	0.0551	0.2348
	θ	1.3153	0.0153	0.0105	0.0107	0.1036
	λ	1.5752	0.0752	0.0716	0.0773	0.2779
	β	2.0492	0.0492	0.1212	0.1237	0.3517
250	γ	1.5187	0.0187	0.0340	0.0343	0.1853
	θ	1.3325	0.0325	0.0065	0.0076	0.0869
	λ	1.5665	0.0665	0.0491	0.0535	0.2314
	β	2.0381	0.0381	0.0729	0.0743	0.2726
350	γ	1.5135	0.0135	0.0228	0.0229	0.1515
	θ	1.3317	0.0317	0.0040	0.0050	0.0706
	λ	1.5792	0.0792	0.0416	0.0478	0.2187
	β	2.0248	0.0248	0.0501	0.0508	0.2253

values as the sample size increases. This corroborates the fact that the asymptotic normal distribution provides an adequate approximation of the estimates.

6. DATA ANALYSIS

In this section, the flexibility of the newly developed G-EGE model is proven by means of a real life datasets. The fits of G-EGE model is compared with Weibull Frechét (WFr), extended generalized exponential (EGE), Weibull alpha power inverted exponential (WAPIE), Kumaraswamy Frechét (KFr), transmuted Frechét (TFr), transmuted Marshall-Olkin Frechét (TMOFr), Kumaraswamy alpha power inverted exponential (KAPIE), Kumaraswamy inverted exponential (KIE), beta Lomax (BL), alpha power inverted exponential (APIE) and exponential(E) distributions. However, these models were chosen base on their relationship that enables us make effective and efficient conclusion about their test statistics.

The following criteria were used to determine the best fit: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC). The test statistics are given as follows: $AIC = -2\hat{\ell} + 2k$, $BIC = -2\hat{\ell} + k \log(n)$,

$CAIC = -2\hat{\ell} + \frac{2kn}{n-k-1}$, $HQIC = -2\hat{\ell} + 2k \log(\log(n))$, where n is the sample size, k is the number of model parameters and $\hat{\ell}$ is minus twice the maximized log-likelihood. The model with the lowest values test statistics is chosen as the best model to fit the datasets.

The first set of data on 1.5 cm strengths of glass fibres were obtained by workers at the UK National Physical Laboratory was used to compare the performance of the G-EGE distribution as used by [30], [31], [32], [33], [34], [35] and [36].

The performance of a model is determined by the value that corresponds to the lowest Akaike Information Criteria (AIC) as the *best* model. In the real life cases considered in Table 2, the *G-EGE* distribution has the lowest AIC value with 37.6.

Figure 3 shows the plots of the estimated densities together with the estimated cdfs of the models under consideration. These plots show that the G-EGE distribution produces a better fit than others models.

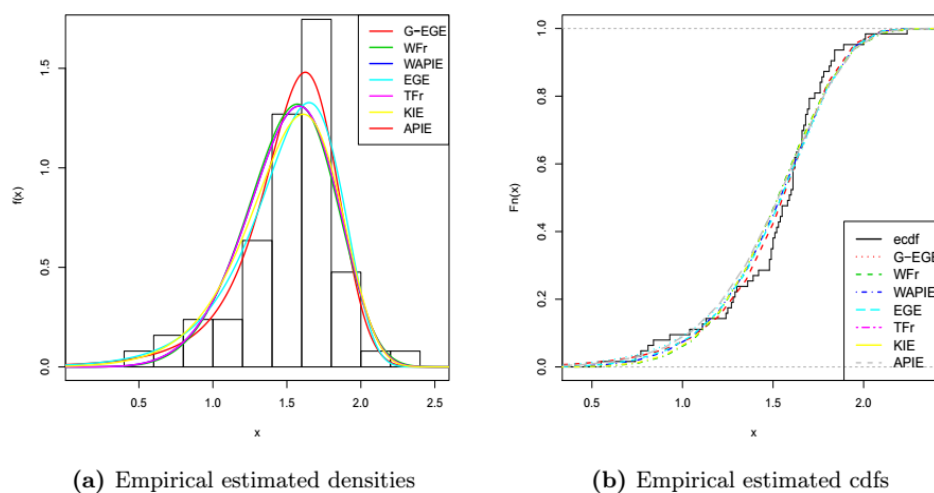


FIGURE 3. The plots of empirical estimated pdfs and cdfs of the G-EGE model

7. CONCLUSION

The G-EGE distribution has been successfully derived. The basic statistical properties of the G-EGE distribution such as the order statistics, cumulative hazard function, reversed hazard function, quantile, median, hazard function, odds function have been successfully established. The G-EGE distribution was also explicitly expressed as a linear function of the exponential distribution. The order statistics of the proposed distribution was also derived. A simulation study of the proposed model was also illustrated. The simulation shows that the shape of the proposed

TABLE 2. Performance rating of the G-EGE distribution with glass fibers dataset

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
G-EGE	$\hat{\theta} = 0.0085$ $\hat{\lambda} = 3.5696$	37.6	38.3	46.2	41.0	0.14	0.84
Weibull Frechét	$\hat{\beta} = 8.6251$ $\hat{\gamma} = 0.1765$ $\hat{\alpha} = 0.0207$ $\hat{\beta} = 10.0442$	39.3	39.7	47.6	42.4	0.26	1.42
Weibull Alpha Power Inverted Exponential	$\hat{a} = 0.4430$ $\hat{b} = 0.3690$ $\hat{\alpha} = 0.0058$ $\hat{\beta} = 4.9797$	39.6	40.2	48.1	42.9	0.27	1.46
Kumaraswamy Frechét	$\hat{\lambda} = 0.3655$ $\hat{\gamma} = 2.0357$ $\hat{\alpha} = 2.1160$ $\hat{\beta} = 0.7401$	47.6	48.3	56.2	51.0	0.26	1.42
Kumaraswamy Alpha Power Inverted Exponential	$\hat{a} = 5.5043$ $\hat{b} = 857.3434$ $\hat{\alpha} = 1.04428$ $\hat{\beta} = 19.3039$	52.7	53.4	61.3	56.1	0.51	2.77
Kumaraswamy Inverted Exponential	$\hat{c} = 7.4277$ $\hat{\alpha} = 0.0021$ $\hat{\alpha} = 3.0232$ $\hat{\lambda} = 163.2152$ $\hat{\beta} = 2.6961$	53.4	53.8	59.9	56.0	0.51	2.83
Transmuted Marshall-Olkin Frechét	$\hat{\alpha} = 0.6524$ $\hat{\beta} = 6.8744$	56.5	57.2	65.1	59.9	2.50	3.10
Beta Lomax	$\hat{\lambda} = 376.2684$ $\hat{\gamma} = 0.1499$ $\hat{\alpha} = 18.1737$ $\hat{\beta} = 26.7645$	56.8	57.5	65.4	60.2	2.54	3.20
Transmuted Frechét	$\hat{a} = 10.8769$ $\hat{b} = 0.0329$ $\hat{\alpha} = 1.3068$ $\hat{\beta} = 2.7898$ $\hat{\lambda} = 0.1298$	100.1	100.5	106.6	102.7	0.99	4.28
G	$\hat{\alpha} = 0.5128$ $\hat{\beta} = 0.5009$	141.4	141.6	145.6	143.1	2.02	3.42
Extended Generalized Exponential	$\hat{\alpha} = 144.0791$ $\hat{\beta} = 0.0550$	145.3	145.9	153.8	148.6	0.99	4.25
Exponential	$\hat{\lambda} = 137.8711$ $\hat{\gamma} = 7.994$	179.6	181.8	185.9	179.7	1.00	4.29
Alpha Power Inverted Exponential	$\hat{\alpha} = 53.5634$ $\hat{\lambda} = 0.3509$	196.3	196.5	200.6	198.0	0.78	4.24

distribution could be inverted bathtub or decreasing (depending on the value of the parameters). The new distribution was applied to a real life data. It shows that the G-EGE distribution performed better than some existing models in literature.

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REFERENCES

- [1] Lee, C., Famoye, F. and Alzaatreh A. Y. Methods for generating families of univariate continuous distributions in the recent decades, *Wiley Interdisciplinary Reviews: Computational Statistics*, 5(3) (2013), 219 - 238.
- [2] Keller, A. Z., Kamath, A. R. R. and Perera, U. D. Reliability analysis of CNC machine tools, *Reliability Engineering*, 3(6) (1982), 449 - 473.
- [3] Lin, C. T., Duran, B. S. and Lewis, T. O. Inverted gamma as a life distribution, *Microelectronics Reliability*, 29(4) (1989), 619 - 626.
- [4] Oguntunde, P. E. and Adejumo, A. The transmuted inverse exponential distribution, *International Journal of Advanced Statistics and Probability*, 3 (1), (2014a) 1 - 7.
- [5] Oguntunde, P. E., Adejumo, A., and Balogun, O. S. Statistical properties of the exponentiated generalized inverted exponential distribution, *Applied Mathematics*, 4 (2) (2014), 47 - 55.
- [6] Oguntunde, P. E., Babatunde, O. S., Ogunmola, A. O. Theoretical analysis of the Kumaraswamy-inverse exponential distribution, *International Journal of Statistics and Applications*, 4 (2), (2014c), 113 - 116.
- [7] Olapade, A. K. On Extended generalized exponential distribution, *British Journal of Mathematics and Computer Science*, 4(9) (2014), 1280-1289.
- [8] Pinho, L. G. B., Cordeiro, G. M. and Nobre, J. S. The Harris extended exponential distribution, *Communications in Statistics -Theory and Methods*, 44(16) (2015), 3486 - 3502.
- [9] Fatima, A. and Roohi, A. Extended Poisson exponential distribution, *Pakistan Journal of Statistics and Operation Research*, 11(3), (2015) 361-375.
- [10] Anake, T. A., Oguntunde, F. E. and Odetunmbi, A. O. On a fractional beta-exponential distribution, *International Journal of Mathematics and Computations*, 26(1) (2015), 26 - 34.
- [11] Thiago, A. N. de-Andrade, M. Bourguignon, M. B. and Cordeiro, G. M. The exponentiated generalized extended exponential distribution, *Journal of Data Science*, 14, (2016), 393-414
- [12] Nadarajah, S. and Okorie, I. E. On the moments of the alpha power transformed generalized exponential distribution, *Ozone: Science and Engineering*, (2017), 1 - 6.
- [13] Mahdavi, A. and Jabari, L. An Extended Weighted Exponential Distribution, *Journal of Modern Applied Statistical Methods*, 16(1) (2017), 296-307.
- [14] Hamedani, G. G., Yousof, H. M., Rasekhi, M., Alizadeh, M. and Najibi, M. S. Type I general exponential class of distributions, *Pakistan Journal of Statistics and Operation Research*, 14(1), (2018), 39-55.
- [15] Eghwerido, J. T., Zelibe, S. C and Efe-Eyefia, E. Gompertz-Alpha power inverted exponential distribution: properties and applications. *Journal of Thailand Statisticians*, (2019). Article in the press.
- [16] Eghwerido, J. T., Zelibe, S. C., Ekuma-Okereke, E. and Efe-Eyefia, E. On the Extended New Generalized Exponential Distribution: Properties and Applications. *FUPRE Journal of Scientific and Industrial Research*, 3 (1) (2019a), 112-122.
- [17] Eghwerido, J. T., Nzei, L. C. and Agu, F. I. Alpha power Gompertz distribution: properties and applications, *Sankhya A - The Indian Journal of Statistics* 2020. Article in the press.
- [18] Afify, A. Z., Altun, E., Yousof, H. M., Alizadeh, M., Ozel, G. and Hamedani, G. G. The odd exponentiated half-logistic-G family: properties, characterizations and applications, *Chilean Journal of Statistics*, 8(2) (2017), 65 - 91.
- [19] Al-Moffel, H. On generating a new family of distributions using the tangent function, *Journal of Statistics and operation Research*, 3 (2018), 471 - 499.
- [20] Gupta, R. C. and Kundu, D. Generalized exponential distribution. *Australian and New Zealand Journal of Statistics*, 41 (2001), 173 - 188.
- [21] Unal, C., Cakmakyapan, S., and Ozel, G. Alpha power inverted exponential distribution: Properties and application, *Gazi University Journal of Science*, 31 (3) (2018), 954 - 965.

- [22] Nassar, M., Alzaatreh, A., Mead, A. and Abo-Kasem, O. Alpha power Weibull distribution: Properties and applications, *Communications in Statistics-Theory and Methods*, 46 (2017), 10236 - 10252.
- [23] Dey, S., Alzaatreh, A., Zhang, C. and Kumar, D. A new extension of generalized exponential distribution with application to ozone data, *Ozone: Science and Engineering*, 39(4) (2017), 273 - 285.
- [24] Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., Hamedani, G. G. and Ali M. M. The transmuted exponentiated generalized-G family of distributions, *Pakistan Journal of Statistics and operation Research*, 11(4) (2015), 441 - 464.
- [25] Aryal, G. R. and Yousof, H. M. The exponentiated generalized-G Poisson family of distributions, *Economic Quality Control*, 32(1) (2017), 1 - 17.
- [26] Cordeiro, G. M., Ortega, E. M. and da-Cunha, D. C. C. The exponentiated generalized class of distributions, *Journal of Data Science*, 11 (2013), 1 - 27.
- [27] Mahdavi, A. and Kundu, D. A new method for generating distributions with an application to exponential distribution, *Communications in Statistics - Theory and Methods*, 46(13) (2017), 6543 - 6557.
- [28] Abouammoh, A. M. and Alshingiti, A. M. Reliability estimation of generalized inverted exponential distribution, *Journal of Statistical Computation and Simulation*, 79(11) (2009), 1301 - 1315.
- [29] Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B. and Ghosh, I. The Gompertz-G family of distributions, *Journal of Statistical Theory and Practice*, 11(1) (2017), 179 - 207.
- [30] Smith, R. L., and Naylor, J. C. A comparison of maximum likelihood and bayesian estimators for the three-parameter Weibull distribution, *Applied Statistics*, 36, (1987), 258 - 369.
- [31] Haq, M. A., Butt, N. S., Usman, R. M. and Fattah, A. A. Transmuted power function distribution, *Gazi University Journal of Science*, 29(1) (2016), 177 - 185.
- [32] Bourguignon, M. B., Silva, R. and Cordeiro, G. M. The Weibull-G family of probability distributions, *Journal of Data Science*, 12 (2014), 53 - 68.
- [33] Merovci, F., Khaleel, M. A., Ibrahim, N. A. and Shitan, M. The beta type-X distribution: properties with application, *Springer-Plus*, 5 (2016), 697.
- [34] Rastogi, M. K. and Oguntunde, P. E. Classical and bayes estimation of reliability characteristics of the Kumaraswamy-inverse exponential distribution. *International Journal of System Assurance Engineering and Management*, 2018.
- [35] Efe-Eyefia, E., Eghwerido, J. T. and Zelibe, S. C. Weibull-Alpha power inverted exponential distribution: properties and applications, *Gazi University Journal of Science*, (2019), forthcoming.
- [36] Zelibe, S. C., Eghwerido, J. T. and Efe-Eyefia, E. Kumaraswamy-Alpha power inverted exponential distribution: properties and applications, *Istatistik Journal of the Turkish Statistical Association*, 12(2), (2019), 35 -48.

Current address: Joseph Thomas Eghwerido: Department of Mathematics, Federal University of Petroleum Resources Effurun, Delta State Nigeria

E-mail address: eghwerido.joseph@fupre.edu.ng

ORCID Address: <http://orcid.org/0000-0001-8986-753X>

Current address: Lawrence Chukwudumebi Nzei: Department of Statistics, University of Benin, Edo State, Nigeria.

E-mail address: nzeilawrence@ymail.com

ORCID Address: <http://orcid.org/0000-0002-4441-805X>

Current address: Ikwoche John David: Department of Mathematics and Statistics, Federal University Wukari, Taraba, Nigeria.

E-mail address: davidij@fuwukari.edu.ng

ORCID Address: <http://orcid.org/0000-0002-7100-5357>

Current address: Obinna Damian Adubisi: Department of Mathematics and Statistics, Federal University Wukari, Taraba, Nigeria.

E-mail address: adubisiobinna@fuwukari.edu.ng

ORCID Address: <http://orcid.org/0000-0001-8575>