The Graph Cases of the Riemannian Positive Mass and Penrose Inequalities in All Dimensions

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2011

Abstract
(Mathematics)

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Abstract

We consider complete asymptotically flat Riemannian manifolds that are the graphs of smooth functions over $\mathbb{R}^n$. By recognizing the scalar curvature of such manifolds as a divergence, we express the ADM mass as an integral of the product of the scalar curvature and a nonnegative potential function, thus proving the Riemannian positive mass theorem in this case. If the graph has convex horizons, we also prove the Riemannian Penrose inequality by giving a lower bound to the boundary integrals using the Aleksandrov-Fenchel inequality. We also prove the ZAS inequality for graphs in Minkowski space. Furthermore, we define a new quasi-local mass functional and show that it satisfies certain desirable properties.

Dedicated to the loving memories of my sister, Defi, and my father, Hok Po

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Introduction

The central object of study in this thesis is the concept of mass in general relativity. Suppose $(N^4, \bar{g})$ is a 4-dimensional spacetime satisfying Einstein's equation, and that it possesses a spacelike slice (M^3, g). Einstein's equation gives rise to the notion of local energy density at a point in (M^3, g), that is, how much mass there is at this point. Moreover, there is also a well-defined notion of the total mass, called the *ADM mass* [1], of the slice.

As we will see in Chapter 1, the ADM mass of a spacelike slice is in general not equal to the integral of the local energy density over the slice. Over the past several decades, much research has focused on understanding the relationship between the local energy density and the ADM mass of such a slice of a spacetime. The first fundamental result in this direction is the Riemannian positive mass theorem, first proved by Schoen and Yau in 1979 [22]. In a nutshell, this theorem states that if a totally geodesic spacelike slice has nonnegative local energy density everywhere, then its ADM mass must be nonnegative, and that the ADM mass is 0 only if the slice is flat. In [24], Schoen and Yau proved the general positive mass theorem for spacelike slices that are not necessarily totally geodesic.

Now suppose a totally geodesic spacelike slice in a spacetime has nonnegative local energy density and contains an outermost minimal surface. Physically, such a surface corresponds to the apparent horizon of a black hole. In 1973, Penrose [19] gave a heuristic argument that the ADM mass of the slice must be at least the mass of the black hole. His conjecture, now known as the Riemannian Penrose inequality, was proved independently by Huisken and Ilmanen [14] in 1997 using inverse mean curvature flow and by Bray [5] in 1999 using a conformal flow of metrics.

Furthermore, while the notions of local energy density at a point and the total mass of a spacelike slice are well understood, it is not clear how to define the mass of a given region. There have been numerous attempts [3, 4, 5, 9, 10, 13, 27] to define the mass of such a region. Assuming the underlying manifold has nonnegative local energy density, there are reasonable properties that such quasi-local mass functionals should satisfy, but to date, none of the proposed definitions of quasi-local mass satisfy all such desirable properties.

The major motivation that resulted in this thesis is to further our understanding of the relationship between local energy density and the ADM mass. Given a spacelike slice (M^3, g) that is totally geodesic in a spacetime, the dominant energy condition implies that it has nonnegative scalar curvature $R \geq 0$ everywhere. Thus in this context we can view the scalar curvature of the slice at each point as its local energy density. If we denote by m_{ADM} the ADM mass of (M^3, g), then our goal is to write down inequalities of the form

$$m_{ADM} \geq \frac{1}{16\pi} \int_{M^3} RQ \, dV_g, \tag{1}$$

where dV_g is the volume form on (M^3, g) and Q is a nonnegative 'potential function' on M^3 that goes to 1 at infinity. The discovery of such a potential function Q would imply the Riemannian positive mass theorem, for if $R \geq 0$, then (1) gives $m_{ADM} \geq 0$. Therefore (1) can be thought of as a generalization of the Riemannian positive mass theorem in the sense that it explicitly relates the ADM mass and local energy density.

If in addition, (M^3, g) contains an outermost minimal surface Σ of area A as in the context of the Riemannian Penrose inequality, then we also require Q to vanish along Σ and the ADM mass of (M^3, g) to satisfy

$$m_{ADM} \geq \sqrt{\frac{A}{16\pi}} + \frac{1}{16\pi} \int_{M^3} RQ \, dV_g. \tag{2}$$

While finding inequalities of the form (1) and (2) for an arbitrary complete, asymptotically flat manifold seems like a daunting task, we have succeeded in doing so for manifolds that are graphs of smooth functions over Euclidean spaces. Moreover, our argument is independent of the dimension of the manifold. Thus our result is valid for all dimensions $n \geq 3$.

In the process of our investigation, we have also discovered a new quasi-local mass functional. Suppose $\Sigma \subset M^3$ is a closed surface with Gauss curvature $K > 0$ and mean curvature $H > 0$, then the Weyl embedding theorem [17] implies that it can be isometrically embedded into $\mathbb{R}^3$. Moreover, the embedding is unique up to an isometry of $\mathbb{R}^3$. If H_0 is the mean curvature of Σ under such embedding, then we define the quasi-local mass of Σ to be

$$m_{QL}(\Sigma) = \frac{1}{16\pi} \int_{\Sigma} \frac{1}{H_0} (H_0^2 - H^2) dA.$$

and show that this definition has several useful properties.

In Chapter 1, we start with a brief overview of spacetimes and state the definitions of asymptotic flatness and the ADM mass. At this point, various examples of asymptotically flat manifolds are introduced. In particular, we study the 3-dimensional Schwarzschild manifold, which plays an important role in subsequent chapters. We conclude this chapter with discussions of the Riemannian positive mass theorem and the Riemannian Penrose inequality, which are two of the most important results regarding the ADM mass.

Chapter 2 gives a brief overview of the inverse mean curvature flow proof of the Riemannian Penrose inequality. In the process, the Hawking mass, which is an example of a quasi-local mass functional, is introduced. The main purpose of this chapter is to motivate inequalities of the form (1) and (2).

Chapter 3 starts by discussing the fact that a 3-dimensional Schwarzschild manifold can be isometrically embedded as a rotating parabola in $\mathbb{R}^4$, and that its exterior region can be expressed as the graph of a smooth function. More generally, we study properties of Riemannian manifolds that are graphs of smooth functions over Euclidean space and derive the formula for their scalar curvatures in local coordinates. We conclude the chapter by translating the notions of asymptotic flatness and the ADM mass for such manifolds.

We state and prove the first major result of this thesis in Chapter 4, namely, that the ADM mass of a complete, asymptotically graph over $\mathbb{R}^n$ can be expressed as an integral of the product of the scalar curvature and a nonnegative potential function. The key lemma we need is that the scalar curvature of a graph is precisely the divergence of a certain vector field. We also show in this chapter the surprising fact that the ADM mass of a spherically symmetric graph over $\mathbb{R}^n$ must be nonnegative, even without the nonnegative scalar curvature assumption. Finally, we express the ADM mass of a graph over any asymptotically flat manifold as an integral involving the scalar curvature of the base manifold.

We proceed to study graphs in the context of the Penrose inequality in Chapter 5. If a graph has a smooth minimal boundary whose connected components are in level sets of the graph function f, then we express the ADM mass of the graph as the sum of an interior integral over the manifold and a surface integral along the minimal boundary. Furthermore, with the additional assumption that each connected component of the boundary is convex, we prove the Penrose inequality via the Aleksandrov-Fenchel inequality. We also point out that most of the contents of Chapter 4 and 5 appear in [16].

In Chapter 6, we will apply the methods of Chapter 4 and Chapter 5 to study graphs in Minkowski space. It turns out Minkowski space is the natural setting in which zero area singularities (ZAS) arise. We begin this chapter by discussing the notion of ZAS, following [6]. Our goal in this chapter is to prove the ZAS inequality [6] for graphs in Minkowski space.

Finally in Chapter 7, we study in details the quasi-local mass functional that arises in Chapter 5. If (M^n, g) is the graph of an asymptotically flat smooth function over $\mathbb{R}^n$ with nonnegative scalar curvature, we show that our definition of quasi-local mass is nonnegative, monotonically nondecreasing and asymptotic to $m_{ADM}(g)$. Moreover, our definition is also valid for a closed surface in a general asymptotically flat 3-manifold (M^3, g) whenever the Brown-York mass is defined. We also show that a result of Shi and Tam [26] implies that our definition is nonnegative in this setting.

1

Technical Background

1.1 Spacetimes and Einstein's equation

A *spacetime* [18] $(N^4, \bar{g})$ is a connected, time-oriented 4-dimensional Lorentzian manifold of signature $(-,+,+,+)$. A tangent vector v on N^4 is called

1. *timelike* if $\bar{g}(v,v) < 0$,

2. *null* (or *light-like*) if $\bar{g}(v,v) = 0$, or

3. *spacelike* if $\bar{g}(v,v) > 0$.

A vector field X on a Lorentzian manifold $(N^4, \bar{g})$ is *timelike* (resp. *null* or *spacelike*) if at every point $p \in N^4$, $\bar{g}(X_p, X_p) < 0$ (resp. $\bar{g}(X_p, X_p) = 0$ or $\bar{g}(X_p, X_p) > 0$). If $(N^4, \bar{g})$ admits a smooth, timelike vector field X, then it is said to be *time-orientable* and the manifold with such a choice of X is called *time-oriented*. Note that these terminologies arise from the fact that X distinguishes timelike or null vectors v as *future-pointing* if $\bar{g}(X, v) < 0$ or *past-pointing* if $\bar{g}(X, v) > 0$.

Let $\bar{Ric}$ denote the Ricci curvature tensor and $\bar{R}$ the scalar curvature of $\bar{g}$. We assume that $(N^4, \bar{g})$ satisfies Einstein's equation,

$$\bar{Ric} - \frac{1}{2}\bar{R} \cdot \bar{g} = 8\pi T,$$

where T is called the *stress-energy tensor* of the spacetime. If T vanishes identically, then we say that the spacetime is *vacuum*. A reasonable assumption on $(N^4, \bar{g})$ is for it to have nonnegative energy density everywhere. This condition is called the *dominant energy condition* and it stipulates that

$$T(v, w) \geq 0$$

for all future timelike vectors v and w.

Furthermore, we assume that $(N^4, \bar{g})$ possesses a smooth 3-dimensional submanifold M^3 (i.e., a hypersurface) that is *spacelike*, which means every tangent vector of M^3 is spacelike when viewed as a vector in N^4. This implies that the restriction of $\bar{g}$ to M^3 is a Riemannian metric g on M^3. Such a hypersurface is called a *spacelike slice* of the spacetime.

Given a spacelike slice (M^3, g) of the spacetime, the stress-energy tensor gives the notions of local energy density and local current density at a point in the slice. Suppose n is a future-pointing unit normal vector to (M^3, g), then we define the *local energy density* to be $\mu = T(n, n)$ and the *local current density* to be the 1-form $J = T(n, \cdot)$. Thus Einstein's equation gives

$$\begin{aligned} \mu &= \frac{1}{16\pi}(R - (\mathrm{tr}_g h)^2 + \|h\|_g^2) \\ J &= \frac{1}{8\pi}\nabla_g \cdot (h - (\mathrm{tr}_g h) \cdot g), \end{aligned} \tag{1.1}$$

where R is the scalar curvature of (M^3, g), h is the second fundamental form of (M^3, g) in $(N^4, \bar{g})$, $\| \cdot \|_g$ is the norm with respect to g, and we use $\nabla_g \cdot$ to denote the divergence operator on (M^3, g). Moreover, the dominant energy condition implies that

$$\mu \geq \|J\|_g.$$

Now if we restrict our attention to spacelike slices (M^3, g) whose second fundamental form h in the spacetime is identically zero, then equation (1.1) becomes

$$\mu = \frac{R}{16\pi} \text{ and } J = 0.$$

In other words, the local energy density is proportional to the scalar curvature of g. In this case, the dominant energy condition implies

$$R \geq 0.$$

For the rest of this thesis, we will no longer discuss spacetimes. In fact, since the dominant energy condition on (M^3, g) consists solely of the scalar curvature of g, we can generalize our discussions to manifolds (M^n, g) of any dimensions $n \geq 3$.

1.2 Asymptotic flatness and the ADM mass

The main object of study in this thesis is the total mass (called the *ADM mass*) of an asymptotically flat manifold. Roughly speaking, an n-dimensional Riemannian manifold (M^n, g) is asymptotically flat if outside a compact subset K, M^n is diffeomorphic to the complement of the closed ball $\bar{B}_1 = \{|x| \leq 1\}$ in $\mathbb{R}^n$, and that the metric g decays sufficiently fast to the flat Euclidean metric at infinity. More precisely,

Definition 1. [21] A Riemannian manifold (M^n, g) of dimension n is said to be **asymptotically flat** if it satisfies the following two conditions:

1. there exists a compact subset $K \subset M^n$ and a diffeomorphism $\Phi : E = M^n \backslash K \to \mathbb{R}^n \backslash \bar{B}_1$,

2. in the coordinate chart $(x^1, x^2, \ldots, x^n)$ on E defined by Φ,

$$g = g_{ij}(x)dx^i dx^j,$$

where

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + O(|x|^{-p}) \\ |x||g_{ij,k}(x)| + |x|^2|g_{ij,kl}(x)| &= O(|x|^{-p}) \\ |R(x)| &= O(|x|^{-q}) \end{aligned} \tag{1.2}$$

for all $i, j, k, l = 1, 2, \ldots, n$, where $g_{ij,k} = \partial_k g_{ij}$ and $g_{ij,kl} = \partial_k \partial_l g_{ij}$ are the coordinate derivatives of the ij-component of the metric, $q > n$ and $p > (n - 2)/2$ are constants and $R(x)$ is the scalar curvature of g at a point x.

The set E is called the *end* of the asymptotically flat manifold. More generally, we say that (M^n, g) is an asymptotically flat manifold with *multiple ends* if $M^n \backslash K$ is the disjoint union of ends $\{E_k\}$ such that each end E_k is diffeomorphic to $\mathbb{R}^n \backslash \bar{B}_1$, and that the metric in each end satisfies the decay condition (1.2). See for example p.238 - 239 of [5] for the precise definition of an asymptotically flat manifold with multiple ends. For our purpose, an asymptotically flat manifold always has one end unless specified otherwise. In addition, we will make the extra assumption that an asymptotically flat manifold is always connected. We also point out that Bartnik explored the threshold values of $\frac{1}{2}$ and 3 for p and q in [2].

For an asymptotically flat manifold (M^n, g), we can define its ADM mass:

Definition 2. [21] The **ADM mass** $m_{ADM}(M^n, g)$ of an asymptotically flat manifold (M^n, g) is defined to be

$$m_{ADM}(M^n, g) = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \nu^j dS_r,$$

where ω_{n-1} is the volume of the $n-1$ unit sphere, S_r is the coordinate sphere of radius r, ν is the outward unit normal to S_r and dS_r is the area element of S_r in the coordinate chart.

The physicists Arnowitt, Deser and Misner [1] first proposed this definition in dimension $n = 3$ to describe the total mass in an isolated gravitational system. We generalize their definition of the ADM mass to any dimension $n \geq 3$ by choosing the correct constant in front of the integral. We also point out that in [2], Bartnik showed that the ADM mass is independent of the choice of asymptotically flat coordinates. We will often omit the argument M^n (or g) and write $m_{ADM}(M^n, g) = m_{ADM}(g) = m_{ADM}$ if the manifold or the metric is understood. Moreover, we will use the convention that repeated indices are automatically being summed over. Thus we will henceforth write

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu^j dS_r.$$

1.3 Examples of asymptotically flat manifolds

1. The simplest example of a complete asymptotically flat manifold is the Euclidean space $\mathbb{R}^n$ with the standard flat metric $g = \delta$. Since $\delta_{ij,k}$=0 for all i, j and k, its ADM mass is 0. We will see later that this is in fact the rigidity case of the Riemannian positive mass theorem.

2. If (M^n, g) is an asymptotically flat manifold, then one can easily construct a class of asymptotically flat manifolds using a conformal change of metrics. We say that a metric $\bar{g}$ is *conformal* to g if

   $$\bar{g} = u(x)^{\frac{4}{n-2}} g$$

   for some $u(x) \in C^\infty(M^n)$. Because of the exponent $\frac{4}{n-2}$, we can without loss of generality assume $u(x) \geq 0$. Note that the choice of the exponent is a convenient one since it simplifies the transformation of the scalar curvature: If R and $\bar{R}$ are the scalar curvatures of g and $\bar{g}$ respectively, then

$$\bar{R} = u(x)^{-(\frac{n+2}{n-2})}\left(-\frac{4(n-1)}{(n-2)}\Delta_g u + Ru.\right) \tag{1.3}$$

If (M^n, g) is asymptotically flat, then $(M^n, \bar{g})$ is also an asymptotically flat manifold given that $u(x)$ satisfies suitable decay conditions. In particular, $(M^n, \bar{g})$ is asymptotically flat if

$$\begin{aligned} u &\to 1 \text{ at } \infty \\ u_i &= O(|x|^{-p-1}) \\ u_{jk} &= O(|x|^{-p-2}) \\ \Delta_g u &= O(|x|^{-q}) \end{aligned}$$

for some constants $p > \frac{1}{2}$ and $q > 3$.

It is an easy calculation to show that the ADM masses of g and $\bar{g}$ and related by

$$m_{ADM}(\bar{g}) = m_{ADM}(g) - \lim_{r\to\infty} \frac{2}{(n-2)\omega_{n-1}} \int_{S_r} \frac{\partial u}{\partial \nu} dS_r, \tag{1.4}$$

where $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u along S_r.

3. As a special case of example 2, let $(M^3, g) = (\mathbb{R}^3\backslash\{0\}, u^4\delta)$ with $u = 1 + \frac{m}{2r}$, where m is a positive constant and $r = \sqrt{x^2+y^2+z^2}$ is the Euclidean distance from the origin. Then the resulting manifold,

$$(M^3, g) = \left(\mathbb{R}^3\backslash\{0\}, \left(1+\frac{m}{2r}\right)^4 \delta\right),$$

is a complete asymptotically flat manifold. This is called the 3-dimensional *Schwarzschild manifold*. As we will show in Chapter 3, it can be isometrically

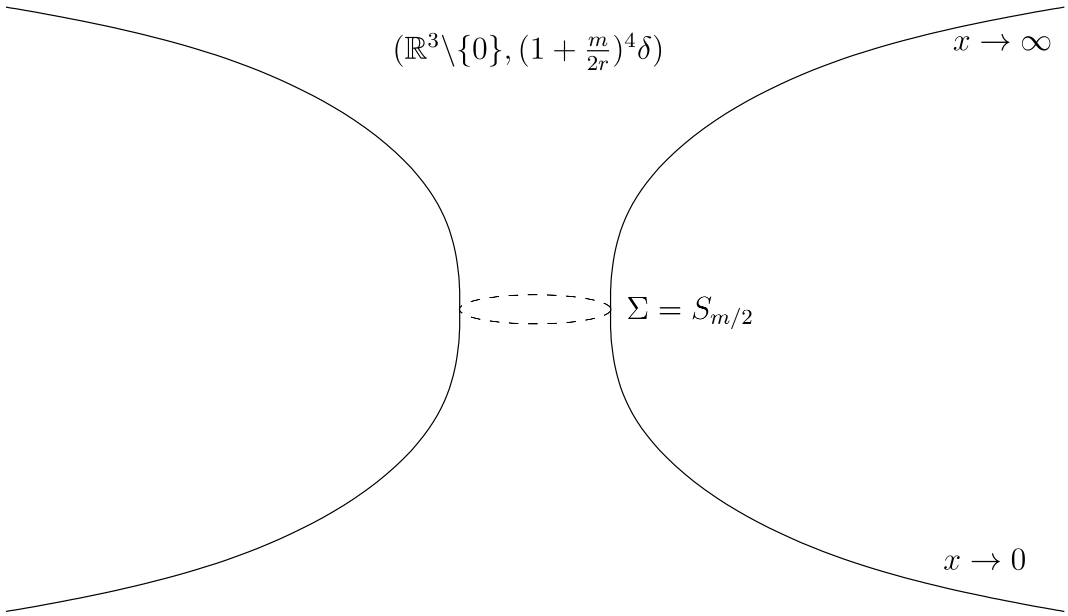

FIGURE 1.1: Visualization of the 3-dimensional Schwarzschild manifold.

embedded as a rotating parabola in $\mathbb{R}^4$. Hence we can visualize it in as in Figure 1.1.

Since $R = 0$ for the flat Euclidean metric and $\Delta u = 0$, (1.3) implies that the Schwarzschild manifold is scalar flat. Moreover, the hypersurface $S_{m/2} = \{|x| = m/2\}$ is a minimal surface in (M^3, g), for if $g = u(x)^4\delta$, then the mean curvature H of a sphere with respect to g satisfies

$$H = \frac{1}{u^2}\left(\frac{2}{r} + \frac{4}{u}\frac{\partial u}{\partial r}\right), \tag{1.5}$$

and $H = 0$ if $r = m/2$.

Furthermore, the 3-dimensional Schwarzschild manifold has two ends, with a reflection symmetry about the minimal sphere $\Sigma = S_{m/2}$. We will from now on refer to the outer end, given by

$$\left(\mathbb{R}^3\backslash B_{m/2}, \left(1 + \frac{m}{2r}\right)^4 \delta\right),$$

as the *exterior* Schwarzschild manifold. This is now a complete asymptotically flat manifold with one end that has a minimal boundary.

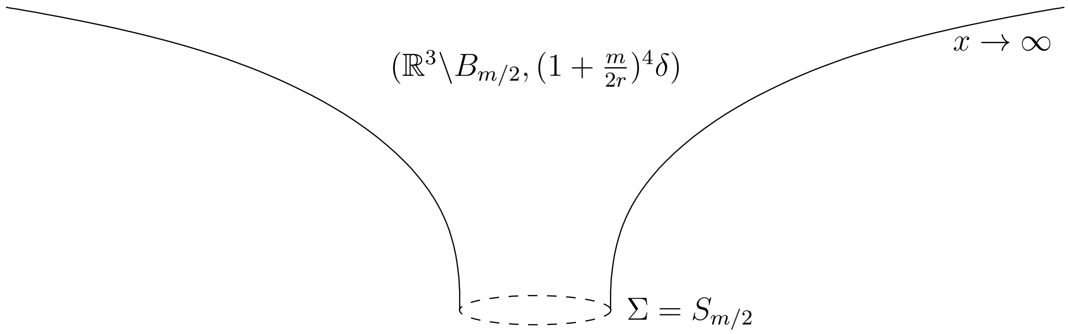

FIGURE 1.2: The exterior 3-dimensional Schwarzschild manifold.

Using (1.4), we can compute the ADM mass of the exterior Schwarzschild manifold $(M^3, g) = (\mathbb{R}^3 \backslash B_{m/2}, (1 + \frac{m}{2r})^4 \delta)$:

$$\begin{aligned} m_{ADM}(g) &= m_{ADM}(\delta) - \lim_{r\to\infty} \frac{1}{2\pi} \int_{S_r} \frac{\partial}{\partial r} \left(1 + \frac{m}{2r}\right)^4 dS_r \\ &= - \lim_{r\to\infty} \frac{1}{2\pi} \int_{S_r} 4 \left(1 + \frac{m}{2r}\right)^3 \left(-\frac{m}{2r^3}\right) dS_r \\ &= - \lim_{r\to\infty} \frac{1}{2\pi} \cdot 4\pi r^2 \cdot 4 \left(1 + \frac{m}{2r}\right)^3 \left(-\frac{m}{2r^3}\right) \\ &= m. \end{aligned}$$

Thus the ADM mass of the exterior Schwarzschild manifold is precisely the positive constant m.

1.4 Fundamental theorems on the ADM mass

Let us again focus on dimension $n = 3$ for the moment. Since the local energy density of (M^3, g) is $\mu = \frac{R}{16\pi}$, one would expect the total mass of (M^3, g) to equal the integral of the local energy density over the manifold, that is,

$$m_{ADM} = \frac{1}{16\pi} \int_{M^3} R \, dV_g.$$

But unfortunately this is not true in general as we can see from the last example. For the 3-dimensional exterior Schwarzschild manifold,

$$m_{ADM} = m > 0 = \frac{1}{16\pi} \int_{M^3} R \, dV_g$$

since $R = 0$ at every point. On the other hand, if we assume the dominant energy condition is satisfied, then nonnegative scalar curvature everywhere should tell us something about the ADM mass of the manifold. There are indeed two fundamental theorems relating nonnegative scalar curvature and the ADM mass. The first result is the following:

Theorem 3 (Riemannian Positive Mass Theorem)**.** *If (M^3, g) is a complete, asymptotically flat Riemannian 3-manifold with nonnegative scalar curvature and ADM mass m_{ADM}, then*

$$m_{ADM} \geq 0,$$

with $m_{ADM} = 0$ if and only if (M^3, g) is isometric to $\mathbb{R}^3$ with the standard flat metric.

Note that the adjective 'Riemannian' denotes the case in which the hypersurface in the spacetime has zero second fundamental form, since in this case the statement is purely one in Riemannian geometry about complete asymptotically manifolds with nonnegative scalar curvature. In 1979, Schoen and Yau [22] gave a proof of Theorem 3 using minimal surface techniques and a stability argument. In a follow-up paper shortly after [23], they showed that their argument is valid for all dimensions $n \leq 7$. In 1981, Witten [28] gave an alternate proof of the positive mass theorem using spinors and the Dirac operator in all dimensions with the assumption that the manifold is spin.

Once again considering the 3-dimensional exterior Schwarzschild metric, we see that the local energy density does not give any contribution to the mass. This phenomenon can be explained by the presence of the minimal boundary. In this case, the minimal boundary $S_{m/2}$ is *outermost*, where by an outermost minimal surface we mean a minimal surface that is not contained entirely inside another minimal surface. We also refer to a minimal surface as a *horizon*, since an outermost minimal surface corresponds to the apparent horizon of a blackhole. Penrose [19] gave a heuristic argument that the mass contribution of the black hole should be $\sqrt{A/16\pi}$, where A is the area of the horizon. More precisely, we have

Theorem 4 (Riemannian Penrose Inequality, version 1)**.** *Let (M^3, g) be a complete, asymptotically flat 3-manifold with a compact smooth boundary ∂M^3. If (M^3, g) has nonnegative scalar curvature and that ∂M^3 is an outermost minimal surface, then*

$$m_{ADM} \geq \sqrt{\frac{A}{16\pi}},$$

where A is the area of any connected component of ∂M^3. Moreover, equality holds if and only if (M^3, g) is isometric to an exterior Schwarzschild manifold.

In 1977, Jang and Wald [15], extending ideas proposed by Geroch [12], gave a heuristic proof of Theorem 4 by noticing that a certain quantity, called the *Hawking mass*, is monotonically nondecreasing and approaches the ADM mass at infinity under inverse mean curvature flow. However, their argument only works when inverse mean flow exists smoothly for all time. In 1997, Huisken and Ilmanen [14] formulated a weak version of inverse mean curvature and showed its existence, thereby proving Theorem 4.

The Riemannian Penrose inequality can also be rephrased for an asymptotically flat manifold without boundary that possesses an outermost minimal surface Σ. However, the case of equality now only implies that the region exterior to Σ in (M^3, g) is isometric to the Schwarzschild manifold.

In 1999, Bray [5] proved the following more general version of Theorem 4 using a different approach by defining a conformal flow of metrics. His proof generalizes in two ways. First, A is now the total area of the horizon. Second, 'outermost horizon' is replaced by 'outer minimizing horizon'. A horizon is (strictly) *outer minimizing* if every other surface which encloses it has (strictly) greater area. This is a slight generalization since outermost horizons are always strictly outer minimizing.

Theorem 5 (Riemannian Penrose Inequality, version 2)**.** *Let (M^3, g) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature and an outer minimizing horizon of total area A. Then*

$$m_{ADM} \geq \sqrt{\frac{A}{16\pi}},$$

with equality if and only if (M^3, g) is isometric to a Schwarzschild manifold outside their respective outer minimizing horizon.

In [8], Bray and Lee generalized Bray's proof of Theorem 5 to dimension $n \leq 7$, with the extra assumption that the manifold is spin for the equality case. In dimension n, the Riemannian Penrose inequality is

$$m_{ADM} \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

We point out that the Riemannian Penrose inequality remains widely open for dimension 8 or higher except in the spherically symmetric case, though recently Schwartz [25] proved a type of volumetric Penrose inequality for conformally flat manifolds in all dimensions.

2

The Inverse Mean Curvature Flow and Integral Penrose Inequality

In this chapter we give a brief overview of the inverse mean curvature flow proof of Theorem 4. Our main goal is to motivate inequalities of the form

$$m_{ADM} \geq \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} RQ\, dV_g \tag{2.1}$$

for a complete, asymptotically flat manifold (M^n, g) with ADM mass m_{ADM}, and

$$m_{ADM} \geq \frac{1}{2} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} RQ\, dV_g \tag{2.2}$$

if (M^n, g) possesses an outermost minimal surface Σ and $A = |\Sigma|$ is the $(n-1)$ volume of Σ. Note that (2.1) and (2.2) are the n-dimensional versions of (1) and (2). See [14, 5, 7] for more in-depth discussions of inverse mean curvature flow and the Riemannian Penrose inequality.

2.1 Inverse mean curvature flow and the Hawking mass

Let Σ be a 2-dimensional closed surface in (M^3, g). A *foliation* of Σ in M^3 in the normal direction with flow speed η is a smooth family $F : \Sigma \times [0, T] \to M$ of hypersurfaces $\Sigma_t := F(\Sigma, t)$ satisfying the evolution equation

$$\frac{\partial F}{\partial t} = \eta\nu, \;\; x \in \Sigma_t, \;\; 0 \leq t \leq T, \tag{2.3}$$

where η is a smooth function on Σ_t, ν is the outward unit normal to Σ_t and $\frac{\partial F}{\partial t}$ is the normal velocity vector field along the surface Σ_t. We also allow the possibility that $T = \infty$. If we choose $\eta = \frac{1}{H}$, where H is the mean curvature of Σ_t at a point $x \in \Sigma_t$, then we obtain the *inverse mean curvature flow*:

$$\frac{\partial F}{\partial t} = \frac{\nu}{H}. \tag{2.4}$$

Note that this is a parabolic evolution equation.

Now given a closed surface Σ in (M^3, g), with area $|\Sigma|$ with respect to the induced metric from g and mean curvature H, its *Hawking mass* is defined to be

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 dA\right).$$

Suppose (M^3, g) satisfies the hypotheses of Theorem 4. If Σ is a connected component of the outermost horizon ∂M^3 and that inverse mean curvature flow with Σ as the initial surface exists smoothly for all time $0 \leq t < \infty$, then m_H satisfies the following three properties:

1. If the initial surface Σ is minimal, then $m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}$. Note that this is the lower bound for the ADM mass in the Riemannian Penrose inequality.

2. The Hawking mass is monotonically nondecreasing under inverse mean curvature flow. Namely, if $\{\Sigma_t\}$ is a family of hypersurfaces satisfying (2.4) with

$\Sigma_0 = \Sigma$, then

$$\frac{d}{dt} m_H(\Sigma_t) \geq 0 \quad \text{for all } t. \tag{2.5}$$

(2.5) is known as the Geroch-Jang-Wald monotonicity formula.

3. The Hawking mass of sufficiently round spheres at infinity in the asymptotically flat end of (M^3, g) approaches m_{ADM}. Hence

$$\lim_{t \to \infty} m_H(\Sigma_t) = m_{ADM}(g)$$

if inverse mean curvature flow beginning with Σ eventually flows to sufficiently round spheres at infinity.

The above properties of m_H implies that

$$m_{ADM}(g) = \lim_{t \to \infty} m_H(\Sigma_t) \geq m_H(\Sigma_0) = \sqrt{\frac{|\Sigma|}{16\pi}},$$

which is the conclusion of Theorem 4. However, as noted by Jang and Wald, this argument only works when inverse mean curvature flow exists and is smooth, which in general is not the case. The main contribution of Huisken and Illmanen was that they defined a weak version of the inverse mean curvature via a level-set formulation and showed that this flow still exists and (2.5) remains valid.

To motivate the inequalities (2.1) and (2.2), we derive the monotonicity formula (2.5) in the next section.

2.2 Geroch-Jang-Wald monotonicity formula

To derive (2.5), we begin with the first and second variation formulas of area. Suppose $\{\Sigma_t\}$ is a 1-parameter family of surfaces in (M^3, g) satisfying (2.3), then the first variation formula of area is

$$\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} \eta H dA_t,$$

where dA_t is the area form on Σ_t induced by g. In other words,

$$\frac{d}{dt}dA_t = \eta H dA_t.$$

Since the mean curvature gives the first variation of area, the second variation formula of area can be thought of as the derivative of H:

$$\frac{d}{dt}H = -\Delta_{\Sigma_t}\eta - Ric(\nu,\nu)\eta - \|h\|^2\eta,$$

where Δ_{Σ_t} is the Laplacian of (M^3, g) restricted along Σ_t, Ric is the Ricci tensor of (M^3, g), ν is the outward unit normal to Σ_t and $\|h\|$ is the norm of the second fundamental form h of Σ_t in (M^3, g). Since $\eta = \frac{1}{H}$ under inverse mean curvature flow, the variation formulas become

$$\begin{aligned} \frac{d}{dt}dA_t &= dA_t \\ \frac{d}{dt}H &= -\Delta_{\Sigma_t}\frac{1}{H} - Ric(\nu,\nu)\frac{1}{H} - \|h\|^2\frac{1}{H}. \end{aligned}$$

We can now start computing

$$\begin{aligned} \frac{d}{dt}m_H(\Sigma_t) &= \frac{d}{dt}\left(\sqrt{\frac{|\Sigma_t|}{16\pi}}\right)\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H^2 dA_t\right) + \sqrt{\frac{|\Sigma_t|}{16\pi}}\frac{d}{dt}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H^2 dA_t\right) \\ &= \frac{1}{2}\sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H^2 dA_t\right) + \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(-\frac{1}{16\pi}\int_{\Sigma_t} 2H\frac{dH}{dt} + H^2 dA_t\right) \\ &= \sqrt{\frac{|\Sigma_t|}{16\pi}}\left[\frac{1}{2} - \frac{1}{16\pi}\int_{\Sigma_t} 2H\left(-\Delta_{\Sigma_t}\frac{1}{H} - Ric(\nu,\nu) - \|h\|^2\frac{1}{H}\right) + \frac{3}{2}H^2 dA_t\right] \\ &= \sqrt{\frac{|\Sigma_t|}{16\pi}}\left[\frac{1}{2} + \frac{1}{16\pi}\int_{\Sigma_t} 2H\Delta_{\Sigma_t}\frac{1}{H} + 2Ric(\nu,\nu) + 2\|h\|^2 - \frac{3}{2}H^2 dA_t\right]. \end{aligned}$$

To show that the derivative of the Hawking mass is nonnegative, we note the following facts:

- Integrating by parts,

$$\int_{\Sigma_t} 2H\Delta_{\Sigma_t}\frac{1}{H}dA_t = \int_{\Sigma_t} -\langle\nabla_{\Sigma_t}H, \nabla_{\Sigma_t}\frac{1}{H}\rangle dA_t = \int_{\Sigma_t} \frac{2|\nabla_{\Sigma_t}H|^2}{H^2}dA_t.$$

- The Gauss equation:

$$Ric(\nu,\nu) = \frac{1}{2}R - K + \frac{1}{2}H^2 - \frac{1}{2}\|h\|^2,$$

where R is the scalar curvature of (M^3, g) and K is the Gauss curvature of Σ_t.

- If λ_1 and λ_2 are the principal curvatures of Σ_t, then

$$\|h\|^2 - \frac{1}{2}H^2 = \frac{1}{2}(\lambda_1 - \lambda_2)^2,$$

since $H = \lambda_1 + \lambda_2$ and $\|h\|^2 = \lambda_1^2 + \lambda_2^2$ by definition.

Using the above three facts,

$$\begin{aligned}\frac{d}{dt}m_H(\Sigma_t) &= \sqrt{\frac{|\Sigma_t|}{16\pi}}\left[\frac{1}{2} + \frac{1}{16\pi}\int_{\Sigma_t}\frac{2|\nabla_{\Sigma_t}H|^2}{H^2} + R - 2K + \frac{1}{2}(\lambda_1-\lambda_2)^2\right] \\ &\geq \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(\frac{1}{2} - \frac{1}{8\pi}\int_{\Sigma_t} K\,dA_t\right) \\ &\geq 0\end{aligned}$$

since

$$\int_{\Sigma_t} K \leq 4\pi$$

by the Gauss-Bonnet Theorem.

2.3 Integral Penrose inequality and potential function Q

In fact, Bray and Khuri [7] noticed that the derivation of (2.5) reveals more. In particular, we actually have

$$\frac{d}{dt}m_H(\Sigma_t) \geq \sqrt{\frac{|\Sigma_t|}{16\pi}}\cdot\frac{1}{16\pi}\int_{\Sigma_t} R\,dA_t.$$

Now integrate this inequality in t from 0 to ∞,

$$\int_0^\infty \frac{d}{dt} m_H(\Sigma_t) dt \geq \frac{1}{16\pi} \int_0^\infty \int_{\Sigma_t} R \sqrt{\frac{|\Sigma_t|}{16\pi}} dA_t dt. \tag{2.6}$$

By the fundamental theorem of calculus, the left hand side of (2.6) is

$$\lim_{r\to\infty} m_H(\Sigma_r) - m_H(\Sigma_0) = m_{ADM}(g) - \sqrt{\frac{A}{16\pi}}$$

by the properties 1 and 3 of m_H. To rewrite the right hand side of (2.6), we use the co-area formula

$$dA_t dt = \frac{1}{\eta} dV_g = H dV_g.$$

Thus (2.6) becomes

$$\begin{aligned} m_{ADM} - \sqrt{\frac{A}{16\pi}} &\geq \frac{1}{16\pi} \int_{M^3} R \cdot H \sqrt{\frac{|\Sigma_t|}{16\pi}} dV_g \\ m_{ADM} &\geq \sqrt{\frac{A}{16\pi}} + \frac{1}{16\pi} \int_{M^3} RQ\, dV_g, \end{aligned} \tag{2.7}$$

for $Q = H\sqrt{\frac{|\Sigma_t|}{16\pi}}$. Notice that if inverse mean curvature flow exists smoothly for all time and eventually flows to sufficiently round spheres, then $Q \geq 0$ and $Q \to 1$ at infinity, and the above inequality is actually stronger statement than the Riemannian Penrose inequality since $R \geq 0$ implies

$$m_{ADM} \geq \sqrt{\frac{A}{16\pi}} + \frac{1}{16\pi} \int_{M^3} RQ\, dV_g \geq \sqrt{\frac{A}{16\pi}}.$$

In light of inequality (2.7), one could attempt to prove the Riemannian Penrose inequality by finding a nonnegative potential function Q such that

$$m_{ADM} \geq \sqrt{\frac{A}{16\pi}} + \frac{1}{16\pi} \int_{M^3} RQ\, dV_g.$$

$Q = H\sqrt{\frac{|\Sigma_t|}{16\pi}}$ from inverse mean curvature flow is one such example, but if the outermost horizon Σ is not connected, then A is only the area of one of the connected components of Σ. Moreover, this only works in dimension $n = 3$ as the Geroch-Jang-Wald monotonicity formula (2.5) relies critically on the Gauss-Bonnet Theorem. More generally, given a complete, asymptotically flat manifold (M^n, g) (without boundary) with $R \geq 0$ and ADM mass m_{ADM}, we would like to find such a 'potential function Q' such that

1. $Q \geq 0$ on M^n,
2. $Q \to 1$ at ∞, and
3. $m_{ADM} \geq \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} RQ \, dV_g$.

If (M^n, g) has an outermost (or more generally, outer minimizing) boundary $\Sigma = \partial M^n$ of area A, then Q should vanish on Σ and m_{ADM} should satisfy

$$m_{ADM} \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} RQ \, dV_g.$$

3

Graphs over Euclidean Space

In this chapter, we study complete, asymptotically flat manifolds that are the graphs of smooth functions over $\mathbb{R}^n$. We begin by discussing the fact that the 3-dimensional Schwarzschild manifold can be isometrically embedded into $\mathbb{R}^4$ as a rotating parabola, and that its exterior region can be expressed as the graph of a smooth function. We then proceed to study manifolds that are graphs of functions over Euclidean space in general and derive the formula of the scalar curvature of such manifolds in coordinates.

3.1 Motivation: Schwarzschild manifold as a graph

The equality cases of the Riemannian positive mass theorem and Penrose inequality are the distinguished cases, and we would hope that they can further our understanding of the relationship between local energy density and the ADM mass. The equality case of the Riemannian positive mass theorem is simply the flat Euclidean space $\mathbb{R}^n$, so that does not seem to give us much insights. On the other hand, the Schwarzschild manifold, which is the equality case of the Riemannian Penrose inequality, appears to be more interesting. In dimension $n = 3$, the Schwarzschild manifold is conformal to $\mathbb{R}^3 \backslash \{0\}$ and may be expressed as

$$\left(\mathbb{R}^3 \backslash \{0\}, \left(1 + \frac{m}{2r}\right)^4 \delta\right),$$

where m is a positive constant, $r = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean distance of the point (x, y, z) from the origin in $\mathbb{R}^3$ and δ is the flat Euclidean metric. Moreover, the 3-dimensional Schwarzschild manifold has another very interesting property, namely, that it can be isometrically embedded as a rotating parabola in $\mathbb{R}^4$ as the set of points $\{(x, y, z, w)\} \subset \mathbb{R}^4$ satisfying $|(x, y, z)| = \frac{w^2}{8m} + 2m$:

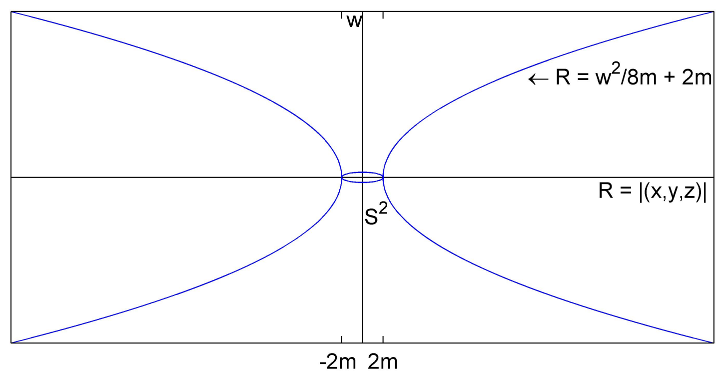

FIGURE 3.1: The three dimensional Schwarzschild metric of mass $m > 0$ (in blue) viewed as a spherically symmetric submanifold of four dimensional Euclidean space $\{(x, y, z, w)\}$ satisfying $R = w^2/8m + 2m$, where $R = \sqrt{x^2 + y^2 + z^2}$. Figure courtesy of Hubert Bray.

Explicitly, the inverse of the above isometric embedding is given by

$$\phi : \{r = \frac{w^2}{8m} + 2m\} \subset \mathbb{R}^4 \to \mathbb{R}^3 \backslash \{0\}$$

$$\phi(r, w) = \left(\left(1 + \frac{m}{2r}\right)^{-2} r\right)$$

Notice that under the isometric embedding, the image of the minimal surface $S_{m/2} = \{|x| = m/2\} \subset \mathbb{R}^3 \backslash \{0\}$ is $S_{2m} = \{(x, y, z, 0) : \sqrt{x^2 + y^2 + z^2} = 2m\} \subset \mathbb{R}^4$, and the exterior Schwarzschild manifold corresponds to the points such that $w \geq 0$.

In particular, solving for w, we see that the exterior 3-dimensional Schwarzschild metric is the graph of the spherically symmetric function $f : \mathbb{R}^3 \backslash B_{2m}(0) \to \mathbb{R}$ given by $f(r) = \sqrt{8m(r-2m)}$. In this case, one can check directly that the ADM mass of (M^3, g) is the positive constant m by computing a certain boundary integral at infinity involving the function f.

That an end of the three dimensional Schwarzschild metric can be isometrically embedded in $\mathbb{R}^4$ as the graph of a function over $\mathbb{R}^3 \backslash B_{2m}$ raises the following questions: if Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary $\Sigma = \partial\Omega$, and f is a smooth function on $\mathbb{R}^n \backslash \Omega$ such that the graph of f is an asymptotically flat manifold (M^n, g) with the induced metric g from $\mathbb{R}^{n+1}$, has nonnegative scalar curvature $R \geq 0$ and horizon $f(\Sigma)$, can we prove the Penrose inequality for (M^n, g) using elementary techniques in this setting? And if so, do can get a stronger statement than the standard Penrose inequality? We answer both questions in the affirmative, and we begin by proving a stronger version of the Riemannian positive mass theorem for manifolds that are graphs over $\mathbb{R}^n$ in Chapter 4 by expressing R as a divergence and applying the divergence theorem, giving the ADM mass as an integral over the manifold of the product of R and a nonnegative potential function. In the presence of a boundary whose connected components are convex, we prove a stronger Penrose inequality by giving lower bounds to the boundary integrals using the Aleksandrov-Fenchel inequality. Before proceeding further, we will present some basic properties of asymptotically flat manifolds that are graphs over $\mathbb{R}^n$.

3.2 Graphs over Euclidean space

If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then the graph of f is a hypersurface in $\mathbb{R}^{n+1}$. Moreover,

Proposition 6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let*

$$M^n = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in \mathbb{R}^n\}$$

be the graph of f. If M^n is equipped with the metric g induced from the flat metric on $\mathbb{R}^{n+1}$, then (M^n, g) is isometric to $(\mathbb{R}^n, \delta + df \otimes df)$, where δ is the flat metric on $\mathbb{R}^n$.

Proof. Let $x = (x_1, \ldots, x_n)$ be the standard coordinates on $\mathbb{R}^n$ and (x, x_{n+1}) the coordinates on $\mathbb{R}^{n+1}$. We show that the map

$$\begin{aligned} F : (\mathbb{R}^n, \delta + df \otimes df) &\to (M^n, g) \\ x &\mapsto (x, f(x)) \end{aligned}$$

is an isometry. Since f is smooth by assumption, F is clearly a smooth map. Moreover, F is a diffeomorphism whose smooth inverse is the projection map $\pi : (M^n, g) \to (\mathbb{R}^n, \delta + df \otimes df)$ defined by $\pi(x, f(x)) = x$.

Next, we check that the diffeomorphism F is an isometry, namely, $F^* g = \delta + df \otimes df$. By the definition of the pullback F^*,

$$(F^* g)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(F_* \frac{\partial}{\partial x^i}, F_* \frac{\partial}{\partial x^j}\right)$$

for all $i, j = 1, \ldots, n$. If $\phi \in C^\infty(M^n, g)$, then

$$\begin{aligned} \left(F_* \frac{\partial}{\partial x^i}\right) \phi = \frac{\partial}{\partial x^i}(\phi \circ F) &= \frac{\partial}{\partial x^i}(\phi(x, f(x))) \\ &= \left(\sum_{k=1}^{n+1} \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x^i}\right) \\ &= \frac{\partial \phi}{\partial x^{n+1}} \frac{\partial f}{\partial x^i} + \left(\sum_{k=1}^{n} \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x^i}\right) \\ &= \left(\frac{\partial}{\partial x^i} + \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^{n+1}}\right) \phi. \end{aligned}$$

Thus

$$F_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + f_i \frac{\partial}{\partial x^{n+1}},$$

where we have used the shorthand notation $f_i = \frac{\partial f}{\partial x^i}$ to denote the i-coordinate derivative of f. Since g is the induced metric from $\mathbb{R}^{n+1}$, we have

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} \quad \text{for} \quad 1 \le i, j \le n+1$$

and

$$\begin{aligned} g\left(F_* \frac{\partial}{\partial x^i}, F_* \frac{\partial}{\partial x^j}\right) &= g\left(\frac{\partial}{\partial x^i} + f_i \frac{\partial}{\partial x^{n+1}}, \frac{\partial}{\partial x^j} + f_j \frac{\partial}{\partial x^{n+1}}\right) \\ &= \delta_{ij} + f_i f_j. \end{aligned}$$

□

Remark 7. Because of Proposition 6, we will from now on refer to

$$(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$$

as the graph of the function f. More generally, if $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\Sigma = \partial\Omega$, then the graph of a smooth function f equipped with the induced metric g is a complete manifold with boundary $f(\Sigma)$, and it is isometric to $(\mathbb{R}^n \backslash \Omega, \delta + df \otimes df)$. We study such manifolds in Chapter 5

Since our goal is to understand the relationship between the ADM mass and the scalar curvature of a graph manifold, our next task is to compute the scalar curvature of a graph. Since we have a natural choice of global coordinates on the graph from the proof of the previous proposition, we will do our calculations in this coordinate chart.

Let $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ be the graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Since $g_{ij} = \delta_{ij} + f_i f_j$, the inverse of g_{ij} is

$$g^{ij} = \delta^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2},$$

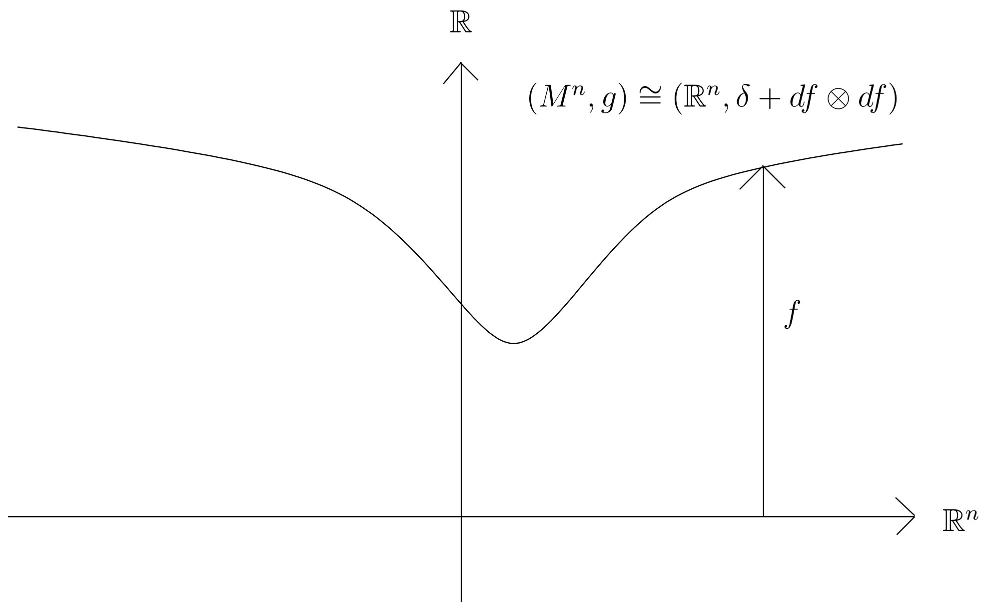

FIGURE 3.2: The graph of a smooth function f over $\mathbb{R}^n$

where the norm of ∇f is taken with respect to the flat metric δ on $\mathbb{R}^n$. We verify this with the following calculation:

$$\begin{aligned} g_{ij}g^{jk} &= (\delta_{ij} + f_i f_j)\left(\delta^{jk} - \frac{f^j f^k}{1+|\nabla f|^2}\right) \\ &= \delta_{ij}\delta^{jk} + f_i f_j \delta^{jk} - \delta_{ij}\frac{f^j f^k}{1+|\nabla f|^2} - \frac{f_i f_j f^j f^k}{1+|\nabla f|^2} \\ &= \delta_i^k + f_i f^k - \frac{f_i f^k(1+|\nabla f|^2)}{1+|\nabla f|^2} \\ &= \delta_i^k. \end{aligned}$$

Using the fact $g_{ij,k} = \partial_k(\delta_{ij} + f_i f_j) = f_{ik} f_j + f_i f_{jk}$, we compute the Christoffel symbols Γ_{ij}^k of (M^n, g):

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2}g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m}) \\
&= \frac{1}{2}\left(\delta^{km} - \frac{f^k f^m}{1+|\nabla f|^2}\right)(f_{ij}f_m + f_i f_{jm} + f_{ij}f_m + f_j f_{im} - f_{im}f_j - f_i f_{jm}) \\
&= \frac{1}{2}\left(\delta^{km} - \frac{f^k f^m}{1+|\nabla f|^2}\right)2f_{ij}f_m \\
&= f_{ij}f^k - \frac{f_{ij}f^k|\nabla f|^2}{1+|\nabla f|^2} \\
&= \frac{f_{ij}f^k}{1+|\nabla f|^2}.
\end{aligned}$$

Remark 8. Since the indices are raised and lowered using the flat metric on $\mathbb{R}^n$, it will be notationally more convenient from now on to write everything as lower indices, with the implicit assumption that any repeated indices are being summed over as usual.

With the above remark in mind, we have

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{f_{ij}f_k}{1+|\nabla f|^2} \\
\Gamma_{ij,k}^k &= \frac{f_{ijk}f_k}{1+|\nabla f|^2} + \frac{f_{ij}f_{kk}}{1+|\nabla f|^2} - \frac{2f_{ij}f_{kl}f_k f_l}{(1+|\nabla f|^2)^2}.
\end{aligned}$$

The scalar curvature R in local coordinates is [21]

$$R = g^{ij}(\Gamma_{ij,k}^k - \Gamma_{ik,j}^k + \Gamma_{ij}^l\Gamma_{kl}^k - \Gamma_{ik}^l\Gamma_{jl}^k).$$

For a graph, the terms involving the Christoffel symbols are

$$\Gamma^k_{ij,k} = \frac{f_{ijk}f_k}{1+|\nabla f|^2} + \frac{f_{ij}f_{kk}}{1+|\nabla f|^2} - \frac{2f_{ij}f_{kl}f_kf_l}{(1+|\nabla f|^2)^2}$$

$$\Gamma^k_{ik,j} = \frac{f_{ijk}f_k}{1+|\nabla f|^2} + \frac{f_{ik}f_{jk}}{1+|\nabla f|^2} - \frac{2f_{ik}f_{jl}f_kf_l}{(1+|\nabla f|^2)^2}$$

$$\Gamma^l_{ij}\Gamma^k_{kl} = \frac{f_{ij}f_{kl}f_kf_l}{(1+|\nabla f|^2)^2}$$

$$\Gamma^l_{ik}\Gamma^k_{jl} = \frac{f_{ik}f_{jl}f_kf_l}{(1+|\nabla f|^2)^2}$$

We now proceed to derive the formula of the scalar curvature of a graph:

$$\begin{aligned}
R &= g^{ij}(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}) \\
&= \left(\delta_{ij} - \frac{f_if_j}{1+|\nabla f|^2}\right)\left(\frac{f_{ijk}f_k}{1+|\nabla f|^2} + \frac{f_{ij}f_{kk}}{1+|\nabla f|^2} - \frac{2f_{ij}f_{kl}f_kf_l}{(1+|\nabla f|^2)^2} - \frac{f_{ijk}f_k}{1+|\nabla f|^2}\right. \\
&\quad \left. - \frac{f_{ik}f_{jk}}{1+|\nabla f|^2} + \frac{2f_{ik}f_{jl}f_kf_l}{(1+|\nabla f|^2)^2} + \frac{f_{ij}f_{kl}f_kf_l}{(1+|\nabla f|^2)^2} - \frac{f_{ik}f_{jl}f_kf_l}{(1+|\nabla f|^2)^2}\right) \\
&= \left(\delta_{ij} - \frac{f_if_j}{1+|\nabla f|^2}\right)\left(\frac{f_{ij}f_{kk}}{1+|\nabla f|^2} - \frac{f_{ik}f_{jk}}{1+|\nabla f|^2} - \frac{f_{ij}f_{kl}f_kf_l}{(1+|\nabla f|^2)^2} + \frac{f_{ik}f_{jl}f_kf_l}{(1+|\nabla f|^2)^2}\right) \\
&= \frac{1}{1+|\nabla f|^2}(f_{ii}f_{kk} - f_{ik}f_{ik}) - \frac{f_kf_l}{(1+|\nabla f|^2)^2}(f_{ii}f_{kl} - f_{ik}f_{il}) \\
&\quad - \frac{f_if_j}{(1+|\nabla f|^2)^2}(f_{ij}f_{kk} - f_{ik}f_{jk}) - \frac{f_if_jf_kf_l}{(1+|\nabla f|^2)^3}(f_{ij}f_{kl} - f_{ik}f_{jl}) \\
&= \frac{1}{1+|\nabla f|^2}(f_{ii}f_{kk} - f_{ik}f_{ik}) - \frac{f_kf_l}{(1+|\nabla f|^2)^2}(f_{ii}f_{kl} - f_{ik}f_{il}) \\
&\quad - \frac{f_if_j}{(1+|\nabla f|^2)^2}(f_{ij}f_{kk} - f_{ik}f_{jk})
\end{aligned}$$

since

$$\frac{f_if_jf_kf_l}{(1+|\nabla f|^2)^3}(f_{ij}f_{kl} - f_{ik}f_{jl})$$

by symmetry when we sum over i, j, k and l. If we relabel the indices by sending k

to j in the first term, l to j in the second term and switching i and k in the third term, we have

Proposition 9. *The scalar curvature R of a graph $(\mathbb{R}^n, \delta + df \otimes df)$ is given by*

$$R = \frac{1}{1+|\nabla f|^2}\left(f_{ii}f_{jj} - f_{ij}f_{ij} - \frac{2f_jf_k}{1+|\nabla f|^2}(f_{ii}f_{jk} - f_{ij}f_{ik})\right). \tag{3.1}$$

Alternately, we can also rewrite (3.1) using coordinate-free notations. Let Δf denote the Laplacian of f with respect to the flat metric and H^f the Hessian of f. Then

$$\begin{aligned} f_{ii}f_{jj} &= (\Delta f)^2 \\ f_{ij}f_{ij} &= \|H^f\|^2 \\ f_{ii}f_{jk}f_jf_k &= (\Delta f)H^f(\nabla f, \nabla f) \\ f_{ij}f_{ik}f_jf_k &= \|H^f(\nabla f, \cdot)\|^2, \end{aligned}$$

where by $H^f(\nabla f, \cdot)$ we mean the 1-form that takes a vector v to $H^f(\nabla, v)$. Thus the scalar curvature of a graph has the coordinate-free expression

$$R = \frac{1}{1+|\nabla f|^2}\left((\Delta f)^2 - \|H^f\|^2 - \frac{2\Delta f H^f(\nabla f, \nabla f) + 2\|H^f(\nabla f, \cdot)\|^2}{1+|\nabla f|^2}\right). \tag{3.2}$$

3.3 Asymptotically flat graphs and ADM mass

The calculations in the previous section is valid for graphs over $\mathbb{R}^n$, including ones that are not asymptotically flat. Our next step is to translate the notion of asymptotic flatness and the ADM mass to graphs.

Definition 10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let f_i denote the ith partial derivative of f. We say that f is *asymptotically flat* if

$$\begin{aligned} f_i(x) &= O(|x|^{-p/2}) \\ |x||f_{ij}(x)| + |x|^2|f_{ijk}(x)| &= O(|x|^{-p/2}) \end{aligned}$$

at infinity for some $p > (n-2)/2$.

Since $g_{ij} = \delta_{ij} + f_i f_j$, $g_{ij,k} = f_{ik} f_j + f_i f_{jk}$ for all i, j, k and

$$g_{ij,i} - g_{ii,j} = f_{ii} f_j + f_i f_{ij} - f_{ij} f_i - f_i f_{ij} = f_{ii} f_j - f_{ij} f_i.$$

Thus we can express the ADM mass of a graph in terms of f:

Definition 11. The *ADM mass* of a complete, asymptotically flat graph is defined to be

$$m_{ADM} = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (f_{ii} f_j - f_{ij} f_i) \nu^j dS_r, \tag{3.3}$$

where ω_{n-1} is the volume of the $n-1$ unit sphere, S_r is the coordinate sphere of radius r, ν is the outward unit normal to S_r and dS_r is the area element of S_r in the coordinate chart.

4

The Positive Mass Theorem for Graphs

In this chapter we derive an expression for the ADM mass of the graph (M^n, g) of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$ involving the scalar curvature of (M^n, g). More precisely, we prove

Theorem 12 (Positive mass theorem for graphs over $\mathbb{R}^n$)**.** *Let (M^n, g) be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$ with the induced metric from $\mathbb{R}^{n+1}$. Let R be the scalar curvature and m_{ADM} the ADM mass of (M^n, g). Let ∇f denote the gradient of f in the flat metric and $|\nabla f|$ its norm with respect to the flat metric. Let dV_g denote the volume form on (M^n, g). Then*

$$m_{ADM} = \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$

In particular, $R \geq 0$ implies $m \geq 0$.

Remark 13. Note that Theorem 12 gives an expression for the ADM mass regardless of the sign of the scalar curvature. Thus Theorem 12 also holds for graphs that have negative scalar curvature somewhere. On the other hand, we will see that Theorem

15 implies the scalar curvature of a graph in $\mathbb{R}^{n+1}$ cannot be too negative in a certain sense.

4.1 A naive approach via integration by parts

For motivation, we once again consider the 3-dimensional Schwarzschild manifold. As we noted in Chapter 3, it can be isometrically embedded into $\mathbb{R}^4$ as a rotating parabola, and that its exterior region outside the horizon may be expressed as the graph of the spherically symmetric function $f(x, y, z) = \sqrt{8m}(r-2m)^{1/2}$ on $\mathbb{R}^3 \backslash B_{2m}$, where $r = \sqrt{x^2 + y^2 + z^2}$. Recall also that $\Sigma = S_{2m}$ is the minimal boundary of the Schwarzschild manifold when isometrically embedded in $\mathbb{R}^4$. Moreover, we have expressed the ADM mass in terms of the graph function f in (3.3). Since the ADM mass is defined as a boundary integral of the dot product of the vector field

$$(f_{ii}f_j - f_{ij}f_i)\partial_j$$

with the outward unit normal at ∞ , and that our goal is to bound the ADM mass from below with an interior integral over the manifold and a surface integral along the minimal boundary , the most obvious thing to try is to use the divergence theorem and hope for the best. Since $|\nabla f(x)| \to \infty$ as $x \to \Sigma$, we cannot quite apply the divergence theorem to $\mathbb{R}^3 \backslash B_{2m}$ directly. Instead, we apply the divergence theorem to $\mathbb{R}^3 \backslash B_{2m+\epsilon}$ for some $\epsilon > 0$, hoping that the limit exists as $\epsilon \to 0$. Denote by dV_δ

the flat Euclidean volume form, then we have

$$
\begin{aligned}
m_{ADM} &= \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (f_{ii}f_j - f_{ij}f_i)\nu^j dS_r \\
&= \frac{1}{16\pi} \int_{\mathbb{R}^3\setminus B_{2m+\epsilon}} \nabla\cdot[(f_{ii}f_j - f_{ij}f_i)\partial_j] dV_\delta - \frac{1}{16\pi}\int_{S_{2m+\epsilon}} (f_{ii}f_j - f_{ij}f_i)\nu^j dA \\
&= \frac{1}{16\pi} \int_{\mathbb{R}^3\setminus B_{2m+\epsilon}} f_{iij}f_j + f_{ii}f_{jj} - f_{ijj}f_i - f_{ij}f_{ij} dV_\delta \\
&\quad - \frac{1}{16\pi}\int_{S_{2m+\epsilon}} (f_{ii}f_j - f_{ij}f_i)\nu^j dA \\
&= \frac{1}{16\pi} \int_{\mathbb{R}^3\setminus B_{2m+\epsilon}} f_{ii}f_{jj} - f_{ij}f_{ij} dV_\delta - \frac{1}{16\pi}\int_{S_{2m+\epsilon}} (f_{ii}f_j - f_{ij}f_i)\nu^j dA
\end{aligned}
$$

because $f_{iij}f_j = f_{ijj}f_i$ by switching i and j.

If this method were to work, we would hope that as $\epsilon \to 0$, the second integral in the last equation gives the mass of the minimal boundary. Because of this, let us naively define the mass of the minimal boundary $\Sigma = S_{2m}$ as

$$m_N(\Sigma) := \lim_{\epsilon\to 0} \frac{1}{16\pi}\int_{S_{2m+\epsilon}} (f_{ii}f_j - f_{ij}f_i)\nu^j dA. \tag{4.1}$$

Suppose $f = f(r)$ is a spherically symmetric function on $\mathbb{R}^3$ and let $f_r = \partial f/\partial r$ denote its radial derivative. By the chain rule, the coordinate derivatives of f satisfy

$$
\begin{aligned}
f_i &= f_r \frac{x_i}{r} \\
f_{ij} &= f_{rr}\frac{x_i x_j}{r^2} + f_r\left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right).
\end{aligned}
$$

Since the outward normal is $\nu^j = x^j/r$, we have

$$
\begin{aligned}
f_{ii}f_j\nu^j &= \left[f_{rr}\frac{x_i^2}{r^2} + f_r\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right] f_r \frac{x_j}{r}\frac{x^j}{r} \\
&= f_{rr}f_r\frac{x_i^2 x_j^2}{r^4} + f_r^2\left(\frac{x_j^2}{r^3} - \frac{x_i^2 x_j^2}{r^5}\right) \qquad (4.2) \\
&= f_{rr}f_r + \frac{2f_r^2}{r}
\end{aligned}
$$

and

$$
\begin{aligned}
f_{ij}f_i\nu^j &= \left[f_{rr}\frac{x_ix_j}{r^2} + f_r\left(\frac{\delta_{ij}}{r} - \frac{x_ix_j}{r^3}\right)\right] f_r\frac{x_i}{r}\frac{x^j}{r} \\
&= f_{rr}f_r\frac{x_i^2x_j^2}{r^4} + f_r^2\left(\frac{\delta_{ij}x_ix_j}{r^3} - \frac{x_i^2x_j^2}{r^5}\right) \\
&= f_{rr}f_r.
\end{aligned}
\tag{4.3}
$$

Plugging (4.2) and (4.3) into (4.1) gives

$$
m_N(\Sigma) = \lim_{\epsilon\to 0}\frac{1}{16\pi}\int_{S_{2m+\epsilon}} \frac{2}{r}f_r^2 dA.
$$

For the Schwarzschild manifold, $f = \sqrt{8m}(r-2m)^{1/2}$ and $f_r = \sqrt{\frac{2m}{r-2m}}$, thus

$$
\begin{aligned}
m_N(\Sigma) &= \lim_{\epsilon\to 0}\frac{1}{16\pi}\int_{S_{2m+\epsilon}} \frac{2}{r}\cdot\frac{2m}{r-2m}dA \\
&= \lim_{\epsilon\to 0}\frac{1}{16\pi}\cdot 4\pi r^2\cdot\left.\frac{4m}{r(r-2m)}\right|_{r=2m+\epsilon} \\
&= \lim_{\epsilon\to 0}\left.\frac{rm}{r-2m}\right|_{r=2m+\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{(2m+\epsilon)m}{\epsilon} \\
&= \infty,
\end{aligned}
$$

which is bad, since we know the mass of the minimal boundary of the Schwarzschild manifold should be $m = \sqrt{\frac{A}{16\pi}}$.

On the other hand, we can try to amend the situation with the following observation: since the ADM is a boundary integral at ∞, we can multiply the integrand in (3.3) by a function ϕ that goes to 1 at ∞ without changing the ADM mass. Thus

$$
m_{ADM} = \lim_{r\to\infty}\frac{1}{16\pi}\int_{S_r}\phi\cdot(f_{ii}f_j - f_{ij}f_i)\nu^j dS_r.
$$

If we start with this alternate definition of the ADM mass for the Schwarzschild manifold, then the divergence theorem implies that the surface integral along the minimal boundary is now

$$\lim_{\epsilon\to 0}\frac{1}{16\pi}\int_{S_{2m+\epsilon}}\phi\cdot\frac{4m}{r(r-2m)}dA.$$

Having in mind that the mass of the horizon should be the number m, we can choose $\phi=(r-2m)/r$ to get the correct mass, since

$$\lim_{\epsilon\to 0}\frac{1}{16\pi}\int_{S_{2m+\epsilon}}\frac{r-2m}{r}\cdot\frac{4m}{r(r-2m)}dA=m.$$

Moreover, the factor $\phi=(r-2m)/r$ happens to be precisely $1/(1+|\nabla f|^2)$ for the Schwarzschild manifold:

$$\frac{1}{1+|\nabla f|^2}=\frac{1}{1+f_r^2}=\left(1+\frac{2m}{r-2m}\right)^{-1}=\frac{r-2m}{r}.$$

Therefore if we choose $\phi=1/(1+|\nabla f|^2)$ and write the ADM mass of a graph as

$$m_{ADM}=\lim_{r\to\infty}\frac{1}{16\pi}\int_{S_r}\frac{1}{1+|\nabla f|^2}(f_{ii}f_j-f_{ij}f_i)\nu^j dA, \tag{4.4}$$

the divergence theorem may allow us to prove the positive mass theorem for graphs. In fact, this will work perfectly because of a key lemma in the next section.

4.2 The positive mass theorem

Equation (4.4) says that the ADM mass of a graph is a boundary integral at ∞ of the vector field

$$\frac{1}{1+|\nabla f|^2}(f_{ii}f_j-f_{ij}f_i)\partial_j. \tag{4.5}$$

Once again having in mind the divergence theorem, we would hope that the divergence of (4.5) is an expression involving the scalar curvature of the graph. Just as luck will have it, it turns out this divergence is precisely the scalar curvature.

Lemma 14. *The scalar curvature R of the graph $(\mathbb{R}^n, \delta + df \otimes df)$ satisfies*

$$R = \nabla \cdot \left(\frac{1}{1+|\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \partial_j \right).$$

Proof. Since we know what to look for, this is just a direct calculation:

$$\begin{aligned}
& \nabla \cdot \left(\frac{1}{1+|\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \partial_j \right) \\
= \;& \frac{1}{1+|\nabla f|^2} (f_{iij} f_j + f_{ii} f_{jj} - f_{ijj} f_i - f_{ij} f_{ij}) - \frac{2 f_{jk} f_k}{(1+|\nabla f|^2)^2} (f_{ii} f_j - f_{ij} f_i) \\
= \;& \frac{1}{1+|\nabla f|^2} \left(f_{ii} f_{jj} - f_{ij} f_{ij} - \frac{2 f_j f_k}{1+|\nabla f|^2} (f_{ii} f_{jk} - f_{ij} f_{ik}) \right) \\
= \;& R
\end{aligned}$$

by Proposition 3.1. □

We are now in the position to prove Theorem 12:

Proof of Theorem 12. By definition, the ADM mass of $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ is

$$\begin{aligned}
m_{ADM} &= \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu_j dS_r \\
&= \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (f_{ii} f_j + f_{ij} f_i - 2 f_{ij} f_i) \nu_j dS_r \\
&= \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (f_{ii} f_j - f_{ij} f_i) \nu_j dS_r.
\end{aligned}$$

By the asymptotic flatness assumption, the function $1/(1+|\nabla f|^2)$ goes to 1 at infinity. Hence we can alternately write the mass as

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1+|\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \nu_j dS_r.$$

Now apply the divergence theorem in $(\mathbb{R}^n, \delta)$ and use Lemma 14 to get

$$\begin{aligned} m_{ADM} &= \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} \nabla \cdot \left(\frac{1}{1+|\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \partial_j \right) dV_\delta \\ &= \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} R dV_\delta \\ &= \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g \end{aligned}$$

since

$$dV_g = \sqrt{\det g} dV_\delta = \sqrt{1+|\nabla f|^2} dV_\delta.$$

□

We also point out that we have not been able to prove the rigidity case of the positive mass theorem, which says that assuming nonnegative scalar curvature, $m_{ADM}(M^n, g) = 0$ implies that (M^n, g) is isometric to $(\mathbb{R}^n, \delta)$. If $m_{ADM} = 0$, then Theorem 12 gives

$$m_{ADM} = \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g = 0.$$

Assuming $R \geq 0$, then $1/\sqrt{1+|\nabla f|^2} \geq 0$ implies that $R \equiv 0$. In other words, the rigidity case of the positive mass theorem for graphs comes down to showing that if a complete, asymptotically flat graph defined on all of $\mathbb{R}^n$ has $R \equiv 0$, then the graph must be flat. Note the similarity of this statement with the Bernstein's theorem, which stipulates that if f is a smooth function on $\mathbb{R}^n$ such that the graph of f is a minimal surface in $\mathbb{R}^{n+1}$, then f must be a linear function. So

$$\nabla \cdot \left(\frac{\nabla f}{1+|\nabla f|^2} \right) = 0, \ f \text{ complete} \Rightarrow f \text{ is linear},$$

while to prove the rigidity case of the positive mass theorem for graphs, we need to show

$$\nabla \cdot \left(\frac{f_{ii} f_j - f_{ij} f_i}{1 + |\nabla f|^2} \partial_j \right) = 0, f \text{ complete and asymptotically flat} \Rightarrow f \text{ is constant.}$$

4.3 Spherically symmetric graphs

Note that equations (4.2) and (4.3) allow us to express the ADM mass of a spherically symmetric graph in terms of the radial derivatives of f. In fact, it turns out that if (M^n, g) is the graph of a smooth *spherically symmetric* function $f = f(r)$ on $\mathbb{R}^n$, then the ADM mass of (M^n, g) is nonnegative even without the nonnegative scalar curvature assumption.

Theorem 15. *Let (M^n, g) be the graph of a smooth, asymptotically flat and spherically symmetric function $f : \mathbb{R}^n \to \mathbb{R}$ with the induced metric from $\mathbb{R}^{n+1}$ and let f_r denote the radial derivative of f. Then the ADM mass of (M^n, g) satisfies*

$$m_{ADM} = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{2 f_r^2}{r} dS_r \geq 0.$$

Proof. This is immediate from Definition 3.3 and equations (4.2) and (4.3). □

Remark 16. A consequence of Theorem 15 and Theorem 12 is that there are no spherically symmetric asymptotically flat smooth functions on $\mathbb{R}^n$ whose graphs have negative scalar curvature everywhere.

4.4 Generalization to graphs over an arbitrary manifold

More generally, we can apply the same technique to a graph on any complete and asymptotically flat Riemannian manifold (M^n, g). In particular, we have

Theorem 17 (ADM mass of a graph over an arbitrary asymptotically flat manifold)**.** *Let (M^n, g) be a complete, smooth asymptotically flat Riemannian manifold. Let*

$(\tilde{M}^n, \tilde{g})$ be the graph of a smooth asymptotically flat function $f : M^n \to \mathbb{R}$ with the induced metric from $M^n \times \mathbb{R}$. Let Ric, R and m be the Ricci curvature, scalar curvature and total mass of (M^n, g). Let $\tilde{R}$ and $\tilde{m}$ be the scalar curvature and total mass of $(\tilde{M}^n, \tilde{g})$. Let $dV_{\tilde{g}}$ denote the volume form on $(\tilde{M}^n, \tilde{g})$. Then

$$\tilde{m} = m + \frac{1}{2(n-1)\omega_{n-1}} \int_{\tilde{M}^n} \left[\tilde{R} - R + \frac{1}{1+|\nabla f|_g^2} Ric(\nabla f, \nabla f) \right] \frac{1}{\sqrt{1+|\nabla f|_g^2}} dV_{\tilde{g}}.$$

Proof. Let $f \in C^\infty(M^n)$ and consider the graph $(\tilde{M}^n, \tilde{g}) = (M^n, g_{ij} + f_i f_j)$. The metric $\tilde{g}_{ij}$ has inverse

$$\tilde{g}^{ij} = g^{ij} - \frac{f^i f^j}{1+|\nabla f|_g^2}$$

To simplify the notations, we denote $\sigma = 1 + |\nabla f|_g^2$. The Christoffel symbols $\tilde{\Gamma}_{ij}^k$ of the metric $\tilde{g}$ are

$$\begin{aligned}
\tilde{\Gamma}_{ij}^k &= \frac{1}{2}\tilde{g}^{km}(\tilde{g}_{im,j} + \tilde{g}_{jm,i} - \tilde{g}_{ij,m}) \\
&= \frac{1}{2}\left(g^{km} - \frac{f^k f^m}{\sigma}\right)(g_{im,j} + g_{jm,i} - g_{ij,m} + 2 f_{ij} f_m) \\
&= \Gamma_{ij}^k + g^{km} f_{ij} f_m - \frac{1}{2\sigma} f^k f^m (g_{im,j} + g_{jm,i} - g_{ij,m}) - \frac{1}{\sigma} f^k f^m f_{ij} f_m \\
&= \Gamma_{ij}^k + f_{ij} f^k - \frac{|\nabla f|_g^2}{\sigma} f_{ij} f^k - \frac{1}{2\sigma} f^k f_l g^{lm} (g_{im,j} + g_{jm,i} - g_{ij,m}) \\
&= \Gamma_{ij}^k + \frac{f_{ij} f^k}{\sigma} - \frac{\Gamma_{ij}^l f_l f^k}{\sigma} \\
&= \Gamma_{ij}^k + \frac{(H^f)_{ij} f^k}{\sigma} \\
\tilde{\Gamma}_{ij,k}^k &= \Gamma_{ij,k}^k + \partial_k \left(\frac{(H^f)_{ij} f^k}{\sigma} \right) \\
&= \Gamma_{ij,k}^k + \frac{(H^f)_{ij,k} f^k}{\sigma} + \frac{(H^f)_{ij} \partial_k f^k}{\sigma} + \partial_k \left(\frac{1}{\sigma}\right)(H^f)_{ij} f^k
\end{aligned}$$

where $(H^f)_{ij} = f_{ij} - \Gamma^l_{ij} f_l$ is the Hessian of f. The scalar curvature $\tilde{R} = R(\tilde{g})$ is

$$\tilde{R} = \tilde{g}^{ij}(\tilde{\Gamma}^k_{ij,k} - \tilde{\Gamma}^k_{ik,j} + \tilde{\Gamma}^l_{ij}\tilde{\Gamma}^k_{kl} - \tilde{\Gamma}^l_{ik}\tilde{\Gamma}^k_{jl})$$

$$= \left(g^{ij} - \frac{f^i f^j}{\sigma}\right)\left(\Gamma^k_{ij,k} + \frac{(H^f)_{ij}\partial_k f^k}{\sigma} + \frac{(H^f)_{ij,k} f^k}{\sigma} + \partial_k(\frac{1}{\sigma})(H^f)_{ij} f^k - \Gamma^k_{ik,j}\right.$$

$$- \frac{(H^f)_{ik}\partial_j f^k}{\sigma} - \frac{(H^f)_{ik,j} f^k}{\sigma} - \partial_j(\frac{1}{\sigma})(H^f)_{ik} f^k + \Gamma^l_{ij}\Gamma^k_{kl} + \frac{(H^f)_{kl}\Gamma^l_{ij} f^k}{\sigma}$$

$$+ \frac{(H^f)_{ij}\Gamma^k_{kl} f^l}{\sigma} + \frac{(H^f)_{ij}(H^f)_{kl} f^k f^l}{\sigma^2} - \Gamma^l_{ik}\Gamma^k_{jl} - \frac{(H^f)_{jl}\Gamma^l_{ik} f^k}{\sigma} - \frac{(H^f)_{ik}\Gamma^k_{jl} f^l}{\sigma}$$

$$\left.- \frac{(H^f)_{ik}(H^f)_{jl} f^k f^l}{\sigma^2}\right)$$

$$= g^{ij}(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}) - \frac{1}{\sigma} f^i f^j (\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl})$$

$$+ g^{ij}\left[\partial_k\left(\frac{(H^f)_{ij} f^k}{\sigma}\right) - \frac{2}{\sigma}(H^f)_{jl}\Gamma^l_{ik} f^k + \frac{1}{\sigma}(H^f)_{ij}\Gamma^k_{kl} f^l\right]$$

$$- g^{ij}\left[\partial_j\left(\frac{(H^f)_{ik} f^k}{\sigma}\right) - \frac{1}{\sigma}(H^f)_{kl}\Gamma^l_{ij} f^k\right]$$

$$+ \frac{1}{\sigma^2} g^{ij} f^k f^l ((H^f)_{ij}(H^f)_{kl} - (H^f)_{ik}(H^f)_{jl})$$

$$- \frac{1}{\sigma^2} f^i f^j ((H^f)_{ij}\partial_k f^k - (H^f)_{ik}\partial_j f^k + (H^f)_{ij}\Gamma^k_{kl} f^l - (H^f)_{ik}\Gamma^k_{jl} f^l).$$

Note that

$$g^{ij}(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}) = R$$

$$\frac{1}{\sigma} f^i f^j (\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}) = \frac{1}{1 + |\nabla f|^2_g} Ric(\nabla f, \nabla f).$$

Let us now separate the rest of the calculations into a lemma:

Lemma 18.

$$g^{ij}\left[\partial_k\left(\frac{(H^f)_{ij} f^k}{\sigma}\right) - \frac{2}{\sigma}(H^f)_{jl}\Gamma^l_{ik} f^k + \frac{1}{\sigma}(H^f)_{ij}\Gamma^k_{kl} f^l\right] = \nabla_g \cdot \left(\frac{1}{\sigma} g^{ij}(H^f)_{ij} f^k \partial_k\right)$$

$$g^{ij}\left[\partial_j\left(\frac{(H^f)_{ik} f^k}{\sigma}\right) - \frac{1}{\sigma}(H^f)_{kl}\Gamma^l_{ij} f^k\right] = \nabla_g \cdot \left(\frac{1}{\sigma} g^{ij}(H^f)_{ik} f^k \partial_j\right)$$

$$\frac{1}{\sigma^2} g^{ij} f^k f^l ((H^f)_{ij}(H^f)_{kl} - (H^f)_{ik}(H^f)_{jl}) - \frac{1}{\sigma^2} f^i f^j ((H^f)_{ij} \partial_k f^k$$
$$- (H^f)_{ik} \partial_j f^k + (H^f)_{ij} \Gamma^k_{kl} f^l - (H^f)_{ik} \Gamma^k_{jl} f^l) = 0$$

Proof. By definition of the divergence,

$$\nabla_g \cdot \left(\frac{1}{\sigma} g^{ij} (H^f)_{ij} f^k \partial_k \right) = g^{ij} \partial_k \left(\frac{(H^f)_{ij} f^k}{\sigma} \right) + \frac{1}{\sigma} (\partial_k g^{ij})(H^f)_{ij} f^k + \frac{1}{\sigma} g^{ij} (H^f)_{ij} \Gamma^k_{kl} f^l$$

The point here is that

$$\frac{1}{\sigma} (\partial_k g^{ij})(H^f)_{ij} f^k = -\frac{2}{\sigma} g^{ij} (H^f)_{jl} \Gamma^l_{ik} f^k,$$

and we can see this from the following calculations:

$$\begin{aligned}
-\frac{2}{\sigma} g^{ij} (H^f)_{jl} \Gamma^l_{ik} f^k &= -\frac{2}{\sigma} g^{ij} (H^f)_{jl} f^k \frac{1}{2} g^{lm} (g_{im,k} + g_{km,i} - g_{ik,m}) \\
&= \frac{1}{\sigma} (H^f)_{jl} f^k ((\partial_k g^{ij}) g^{lm} g_{im} + (\partial_i g^{ij}) g^{lm} g_{km} - (\partial_k g^{lm}) g^{ij} g_{ik}) \\
&= \frac{1}{\sigma} (H^f)_{jl} f^k ((\partial_k g^{ij}) \delta^l_i + (\partial_i g^{ij}) \delta^l_k - (\partial_k g^{lm}) \delta^j_k) \\
&= \frac{1}{\sigma} (H^f)_{jl} f^k ((\partial_k g^{ij}) \delta^l_i + (\partial_i g^{ij}) \delta^l_k - (\partial_i g^{ij}) \delta^l_k) \\
&= \frac{1}{\sigma} (H^f)_{jl} f^k (\partial_k g^{ij}) \delta^l_i \\
&= \frac{1}{\sigma} (H^f)_{ij} f^k (\partial_k g^{ij}).
\end{aligned}$$

Putting everything together gives the first equation. For the second equation, we have

$$\nabla_g \cdot \left(\frac{1}{\sigma} g^{ij} (H^f)_{ik} f^k \partial_j \right) = g^{ij} \partial_j \left(\frac{(H^f)_{ik} f^k}{\sigma} \right) + \frac{1}{\sigma} (\partial_j g^{ij})(H^f)_{ik} f^k + \frac{1}{\sigma} g^{il} (H^f)_{ik} \Gamma^j_{jl} f^k.$$

The last term is

$$
\begin{aligned}
\frac{1}{\sigma} g^{il} (H^f)_{ik} \Gamma^j_{jl} f^k &= \frac{1}{\sigma} g^{il} (H^f)_{ik} f^k \frac{1}{2} g^{jm} (g_{jm,l} + g_{lm,j} - g_{jl,m}) \\
&= -\frac{1}{2\sigma} (H^f)_{ik} f^k (n \partial_l g^{il} + \delta^j_l \partial_j g^{il} - \delta^i_j \partial_m g^{jm}) \\
&= -\frac{1}{2\sigma} (H^f)_{ik} f^k (n \partial_l g^{il} + \partial_j g^{ij} - \partial_j g^{ij}) \\
&= -\frac{n}{2\sigma} (H^f)_{ik} f^k \partial_j g^{ij}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
-\frac{1}{\sigma} g^{ij} (H^f)_{kl} \Gamma^l_{ij} f^k &= -\frac{1}{\sigma} g^{ij} (H^f)_{kl} f^k \frac{1}{2} g^{lm} (g_{im,j} + g_{jm,i} - g_{ij,m}) \\
&= \frac{1}{2\sigma} (H^f)_{kl} f^k (\delta^l_i \partial_j g^{ij} + \delta^l_j \partial_i g^{ij} - n \partial_m g^{lm}) \\
&= \frac{1}{2\sigma} (H^f)_{kl} f^k (\partial_j g^{jl} + \partial_i g^{il} - n \partial_i g^{il}) \\
&= \frac{1}{\sigma} (H^f)_{ik} f^k \partial_j g^{ij} - \frac{n}{2\sigma} (H^f)_{jk} f^k \partial_i g^{ij}.
\end{aligned}
$$

Now put together to get the second equality. For the third equality, first note that

$$
\begin{aligned}
g^{ij} (H^f)_{ij} &= \Delta_g f \\
f^k f^l (H^f)_{kl} &= H^f(\nabla f, \nabla f).
\end{aligned}
$$

After regrouping, the left hand side becomes

$$
\frac{1}{\sigma^2} \left[H^f(\nabla f, \nabla f)(\Delta_g f - \partial_k f^k - \Gamma^k_{kl} f^l) + f^i f^j (H^f)_{ik} (-g^{kl} (H^f)_{jl} + \partial_j f^k + \Gamma^k_{jl} f^l) \right]
$$

Now $\Delta_g f - \partial_k f^k - \Gamma^k_{kl} f^l = 0$ since $\Delta_g f = \nabla_g \cdot (f^k \partial_k) = \partial_k f^k + \Gamma^k_{kl} f^k$. Next,

$$
\begin{aligned}
-g^{kl}(H^f)_{jl} + \partial_j f^k + \Gamma^k_{jl} f^l &= -g^{kl} f_{jl} + g^{kl}\Gamma^m_{jl} f_m + \partial_j f^k + \Gamma^k_{jl} f^l \\
&= f_l \partial_j g^{kl} + g^{kl}\Gamma^m_{jl} f_m + g^{lm}\Gamma^k_{jl} f_m \\
&= f_l \partial_j g^{kl} + \frac{1}{2} f_m g^{kl} g^{mp}(\partial_l g_{jp} + \partial_j g_{lp} - \partial_p g_{jl}) \\
&+ \frac{1}{2} f_m g^{lm} g^{kp}(\partial_l g_{jp} + \partial_j g_{lp} - \partial_p g_{jl}) \\
&= f_l \partial_j g^{kl} - \frac{1}{2} f_m \delta^m_j \partial_l g^{kl} - \frac{1}{2} f_m \delta^m_l \partial_j g^{kl} + \frac{1}{2} f_m \delta^k_j \partial_p g^{mp} \\
&- \frac{1}{2} f_m \delta^k_j \partial_l g^{lm} - \frac{1}{2} f_m \delta^k_l \partial_j g^{lm} + \frac{1}{2} f_m \delta^m_j \partial_p g^{kp} \\
&= f_l \partial_j g^{kl} - \frac{1}{2} f_j \partial_l g^{kl} - \frac{1}{2} f_l \partial_j g^{kl} - \frac{1}{2} f_m \partial_j g^{km} + \frac{1}{2} f_j \partial_p g^{kp} \\
&= f_l \partial_j g^{kl} - f_l \partial_j g^{kl} \\
&= 0.
\end{aligned}
$$

□

Thus it follows from the lemma that the scalar curvature of a graph over (M, g) is

$$\tilde{R} = R - \frac{1}{1 + |\nabla f|_g^2} Ric(\nabla f, \nabla f) + \nabla_g \cdot \left(\frac{g^{ij}(H^f)_{ij} f^k}{1 + |\nabla f|_g^2} \partial_k - \frac{g^{ij}(H^f)_{ik} f^k}{1 + |\nabla f|_g^2} \partial_j \right).$$

This formula enables us to write down a proof of Theorem 17. By definition, the mass $\tilde{m}$ of $(\tilde{M}^n, \tilde{g})$ is

$$
\begin{aligned}
\tilde{m} &= \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j} (\tilde{g}_{ij,i} - \tilde{g}_{ii,j}) \nu_j dS_r \\
&= \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \nu_j dS_r \\
&+ \lim_{r\to\infty} \frac{1}{4\omega_{n-1}} \int_{S_r} \sum_{i,j} (f_{ii} f_j - f_{ij} f_i) \nu_j dS_r.
\end{aligned}
$$

The first limit is precisely m, the mass of (M^n, g). For the second limit, note that at infinity,

$$\frac{g^{ij}(H^f)_{ij}f^k}{1+|\nabla f|_g^2}\partial_k - \frac{g^{ij}(H^f)_{ik}f^k}{1+|\nabla f|_g^2}\partial_j = \frac{1}{1+|\nabla f|_g^2}(f_{ii}f_j - f_{ij}f_j)\partial_j$$

. Thus applying the divergence theorem gives

$$\begin{aligned}
\tilde{m} &= m + \frac{1}{2(n-1)\omega_{n-1}}\int_M \nabla_g \cdot \left(\frac{g^{ij}(H^f)_{ij}f^k}{1+|\nabla f|_g^2}\partial_k - \frac{g^{ij}(H^f)_{ik}f^k}{1+|\nabla f|_g^2}\partial_j\right) dV_g \\
&= m + \frac{1}{2(n-1)\omega_{n-1}}\int_M \tilde{R} - R + \frac{1}{1+|\nabla f|_g^2}\mathrm{Ric}(\nabla f, \nabla f) dV_g \\
&= m + \frac{1}{2(n-1)\omega_{n-1}}\int_{\tilde{M}} \left[\tilde{R} - R + \frac{1}{1+|\nabla f|_g^2}\mathrm{Ric}(\nabla f, \nabla f)\right]\frac{1}{\sqrt{1+|\nabla f|_g^2}} dV_{\tilde{g}}
\end{aligned}$$

□

5

The Penrose Inequality for Graphs

As in the context of the Riemannian Penrose inequality, we consider graphs with nonnegative scalar curvature that have minimal boundaries in the chapter. Using the techniques developed in Chapter 4, we express the ADM mass of a graph with a minimal boundary as the sum of an integral over the graph and a surface integral along the boundary. In particular, we prove the following theorem in this chapter:

Theorem 19. *Let Ω be a bounded and open (but not necessarily connected) set in $\mathbb{R}^n$ with smooth boundary $\Sigma = \partial\Omega$. Let $f : \mathbb{R}^n \backslash \Omega \to \mathbb{R}$ be a smooth asymptotically flat function such that each connected component of $f(\Sigma)$ is in a level set of f and $|\nabla f(x)| \to \infty$ as $x \to \Sigma$. Let (M^n, g) be the graph of f with the induced metric from $\mathbb{R}^n \backslash \Omega \times \mathbb{R}$ and ADM mass m_{ADM}. Let H_0 be the mean curvature of Σ in $(\mathbb{R}^n \backslash \Omega, \delta)$. Then*

$$m_{ADM} = \frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_0 dA + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$

Furthermore, suppose Ω_i are the connected components of Ω, $i = 1, \ldots, k$, and let $\Sigma_i = \partial\Omega_i$. If we in addition assume that each Ω_i is convex, then we have the following Penrose inequality:

Corollary 20 (Penrose inequality for graphs on $\mathbb{R}^n$ with convex boundaries)**.** *With the same hypotheses as in Theorem 19, and the additional assumption that each connected component Ω_i of Ω is convex and $\Sigma_i = \partial\Omega_i$, then*

$$m_{ADM} \geq \sum_{i=1}^{k} \frac{1}{2}\left(\frac{|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g.$$

In particular,

$$R \geq 0 \quad \textit{implies} \quad m_{ADM} \geq \sum_{i=1}^{k} \frac{1}{2}\left(\frac{|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Remark 21. Since

$$\sum_{i=1}^{k} \frac{1}{2}\left(\frac{|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \geq \frac{1}{2}\left(\frac{\sum_{i=1}^{k}|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} = \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

with A being the total area of the minimal boundary Σ, Corollary 20 is a stronger statement than the standard Riemannian Penrose inequality on top of the fact that the lower bound for the ADM mass involves a nonnegative integral when the scalar curvature is nonnegative. In this case, the masses of the connected components of the black hole are additive, but this seems to be too strong of a conclusion to hold in general.

5.1 Graphs with smooth minimal boundaries

Let Ω be a bounded (but not necessarily connected) open set in $\mathbb{R}^n$ with smooth boundary $\Sigma = \partial\Omega$. If $f : \mathbb{R}^n \backslash \Omega \to \mathbb{R}$ is a smooth asymptotically flat function, then the graph of f is a complete asymptotically flat manifold with boundary $f(\Sigma)$. By Proposition 6 and Remark 7, we will refer to $(M^n, g) = (\mathbb{R}^n \backslash \Omega, \delta + df \otimes df)$ as the graph of f.

If $f(\Sigma)$ lies entirely in a level set $\{x : f(x) = c\}$ of f, then $f(\Sigma)$ is in fact an outer minimizing surface of (M^n, g): Suppose $f(\Sigma')$ is another surface on the graph with $\Sigma' \subset \mathbb{R}^n$ and $\Sigma' = \partial\Omega'$, then $\Omega \subset \Omega'$ and $f(\Sigma')$ is larger than $f(\Sigma)$. Moreover, that $f(\Sigma)$ is in a level set of f implies that the area form on $f(\Sigma)$ is bounded above by the area form on $f(\Sigma')$.

If $f(\Sigma)$ consists of multiple connected components, then we will assume that each connected component lies in a level set of f. In this case we can view a connected component of $f(\Sigma)$ as a surface both in (M^n, g) and in the slice of $\mathbb{R}^n$ it sits in.

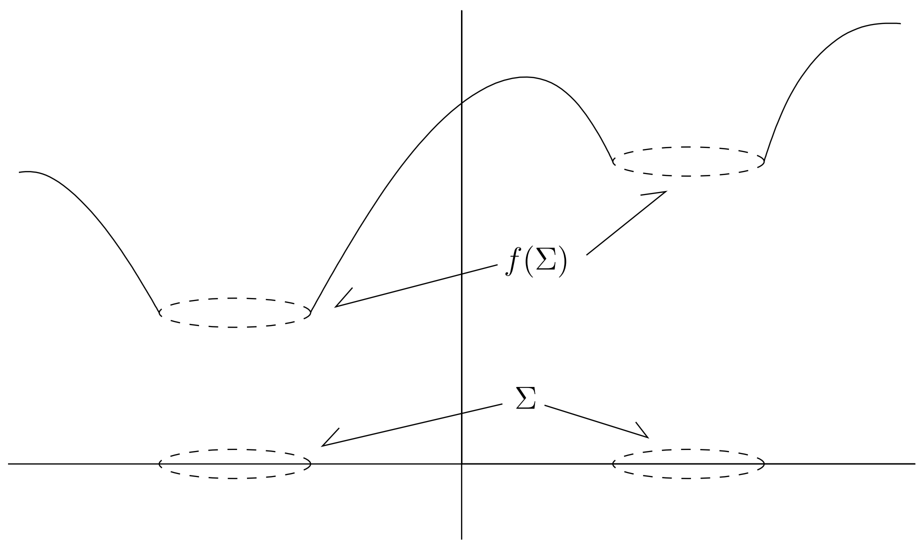

FIGURE 5.1: The graph of a smooth function f over $\mathbb{R}^n \backslash \Omega$

Let H and H_0 denote the mean curvature of $f(\Sigma)$ in (M^n, g) and in $\mathbb{R}^n$ respectively. Since the metric $g = \delta + df \otimes df$ does not change the metric on the level sets of f, and it stretches lengths perpendicular to the level sets of f by a factor of $\sqrt{1 + |\nabla f|^2}$, the two mean curvatures H and H_0 are related by

$$H = \frac{1}{\sqrt{1 + |\nabla f|^2}} H_0. \tag{5.1}$$

Note that H_0 is also the mean curvature of Σ. Equation (5.1) implies that if

$|\nabla f(x)| \to \infty$ as $x \to \Sigma$, then $f(\Sigma)$ is an outer minimizing horizon of (M^n, g) since $H \equiv 0$ on Σ. Graphically, this means f is 'vertical' along the boundary.

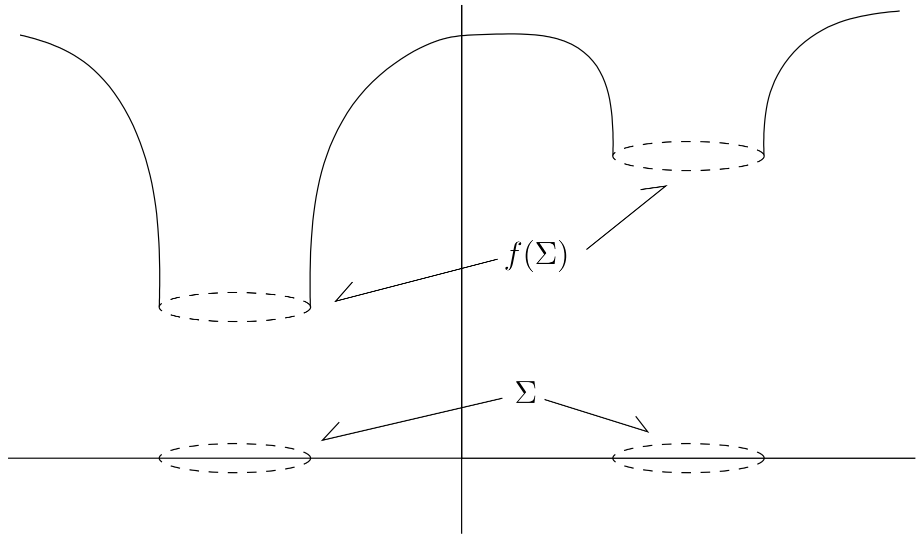

FIGURE 5.2: The graph of a smooth function f over $\mathbb{R}^n \backslash \Omega$ with minimal boundary

In this setting, proving Theorem 19 is a matter of keeping track of the extra boundary term when we apply the divergence theorem in the proof of Theorem 12. As before, any repeated indices are being summed over.

Proof of Theorem 19. As in the proof of Theorem 12, we can write the mass of (M^n, g) as

$$m_{ADM} = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1 + |\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \nu^j dS_r.$$

The difference here is that when we apply the divergence theorem, we get an extra boundary integral:

$$
\begin{aligned}
m_{ADM} &= \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\nu^j dS_r \\
&= -\frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} \frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\nu^j dA \\
&\quad + \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n\setminus\Omega} \nabla\cdot\left(\frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\partial_j\right) dV_\delta,
\end{aligned}
$$

where technically we cannot apply the divergence theorem to all of $\mathbb{R}^n\setminus\Omega$ since $|\nabla f(x)| \to \infty$ as $x \to \partial\Omega = \Sigma$. However, we will slightly abuse our notations here since we will show later that the improper integrals do in fact converge. Thus

$$
\begin{aligned}
m_{ADM} = &-\frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} \frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\nu^j dA \\
&+ \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R\frac{1}{\sqrt{1+|\nabla f|^2}} dV_g,
\end{aligned}
$$

where we have rewritten the second integral using Lemma 14. The outward normal to M^n along Σ is $\nu = \nabla f/|\nabla f|$. Viewing Σ as a closed surface in $(\mathbb{R}^n, \delta)$, we denote by Δf the Laplacian of f with respect to the flat metric and $\Delta_\Sigma f$ the Laplacian of f restricted along Σ. Let H^f denote the Hessian of f and H_0 the mean curvature of Σ with respect to the flat metric. We will use the following well known formula to relate the two Laplacians:

$$
\begin{aligned}
\Delta f &= \Delta_\Sigma f + H^f(\nu,\nu) + H_0 \cdot \nu(f) \\
&= \frac{1}{|\nabla f|} H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right) + H_0|\nabla f|,
\end{aligned}
\tag{5.2}
$$

where $\Delta_\Sigma f = 0$ since f is constant on Σ. Now

$$\begin{aligned}
&-\frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\nu_j \\
&= \frac{1}{1+|\nabla f|^2}\left[(\Delta f)|\nabla f| - H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right)\right] \\
&= \frac{1}{1+|\nabla f|^2}\left[\left(\frac{1}{|\nabla f|}H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right) + H_0|\nabla f|\right)|\nabla f| - H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right)\right] \\
&= \frac{|\nabla f|^2}{1+|\nabla f|^2}H_0.
\end{aligned} \tag{5.3}$$

Therefore,

$$\begin{aligned}
m_{ADM} &= \frac{1}{2(n-1)\omega_{n-1}}\int_\Sigma \frac{|\nabla f|^2}{1+|\nabla f|^2}H_0 dA + \frac{1}{2(n-1)\omega_{n-1}}\int_{M^n} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g \\
&= \frac{1}{2(n-1)\omega_{n-1}}\int_\Sigma H_0 dA + \frac{1}{2(n-1)\omega_{n-1}}\int_{M^n} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g.
\end{aligned}$$

□

Remark 22. Since $f(\Sigma)$ is in a level set of f, it is the same surface as Σ translated vertically. Thus we can equivalently express the ADM mass as

$$m_{ADM} = \frac{1}{2(n-1)\omega_{n-1}}\int_{f(\Sigma)} H_0 dA + \frac{1}{2(n-1)\omega_{n-1}}\int_{M^n} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g.$$

5.2 Aleksandrov-Fenchel inequality

Since the Penrose inequality bounds the ADM mass below by the mass of the black hole, we would like to apply Theorem 19 to this setting. In dimension $n = 3$, Theorem 19 gives

$$m_{ADM} = \frac{1}{16\pi}\int_\Sigma H_0 dA + \frac{1}{16\pi}\int_{M^3} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g.$$

Suppose for now that Σ is connected. If in addition we assume that Σ is the boundary of a *convex* region, then a classical inequality in convex geometry states that

$$\frac{1}{16\pi}\int_{\Sigma} H_0 dA \geq \sqrt{\frac{A}{16\pi}},$$

where $A = |\Sigma|$ as usual. If this is the case, then Theorem 19 implies

$$m_{ADM} \geq \sqrt{\frac{A}{16\pi}} + \frac{1}{16\pi}\int_{M^3} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g \geq \sqrt{\frac{A}{16\pi}}$$

assuming $R \geq 0$, which is precisely the Riemannian Penrose inequality. In fact, the above inequality is true in higher dimensions for a boundary whose connected components are convex. The tool we need is the following lemma, which is a consequence of the Aleksandrov-Fenchel inequality [20].

Lemma 23. *If Σ is a connected convex surface in $\mathbb{R}^n$ with mean curvature H_0 and area $|\Sigma|$, then*

$$\frac{1}{2(n-1)\omega_{n-1}}\int_{\Sigma} H_0 dA \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Proof. Let $\Sigma \subset \mathbb{R}^n$ be a convex surface with principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$. Let

$$\sigma_j(\kappa_1, \ldots, \kappa_{n-1}) = \binom{n-1}{j}^{-1} \sum_{1\leq i_1<\cdots<i_k\leq n-1} \kappa_{i_1}\cdots\kappa_{i_j}$$

be the jth normalized elementary symmetric functions in $\kappa_1, \ldots, \kappa_{n-1}$ for $j = 1, \ldots, n-1$. In particular,

$$\sigma_0(\kappa_1, \ldots, \kappa_{n-1}) = 1$$

$$\sigma_1(\kappa_1, \ldots, \kappa_{n-1}) = \frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i = \frac{1}{n-1}H_0$$

$$\sigma_{n-1}(\kappa_1, \ldots, \kappa_{n-1}) = \prod_{i=1}^{n-1}\kappa_i.$$

The kth quermassintegral V_k of Σ is defined to be

$$V_k = \int_\Sigma \sigma_k(\kappa_1, \ldots, \kappa_{n-1}).$$

A special case of the Aleksandrov-Fenchel inequality states that, for $0 \leq i < j < k \leq n-1$,

$$V_j^{k-1} \geq V_i^{k-j} V_k^{j-i}.$$

Taking $i = 0$, $j = 1$, $k = n-1$,

$$V_1^{n-1} \geq V_0^{n-2} V_{n-1}. \tag{5.4}$$

Now

$$\begin{aligned} V_0 &= \int_\Sigma \sigma_0(\kappa_1, \ldots, \kappa_{n-1}) = |\Sigma| \\ V_1 &= \int_\Sigma \sigma_1(\kappa_1, \ldots, \kappa_{n-1}) = \frac{1}{n-1} \int_\Sigma H_0 \\ V_{n-1} &= \int_\Sigma \sigma_{n-1}(\kappa_1, \ldots, \kappa_{n-1}) = \omega_{n-1}. \end{aligned}$$

Thus (5.4) becomes

$$\begin{aligned} \left(\frac{1}{n-1} \int_\Sigma H_0 \right)^{n-1} &\geq |\Sigma|^{n-2} \omega_{n-1} \\ \frac{1}{n-1} \int_\Sigma H_0 &\geq |\Sigma|^{\frac{n-2}{n-1}} \omega_{n-1}^{\frac{1}{n-1}} \\ \frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_0 &\geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \end{aligned}$$

as claimed. □

We can prove Corollary 20 using Theorem 19 and Lemma 23:

Proof of Corollary 20. Let us denote by Ω_i, $i = 1, \ldots, k$ the connected, convex components of the bounded open set Ω and let $\Sigma_i = \partial\Omega_i$. The ADM mass of (M^n, g) is

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\nu^j dS_r$$

$$= \sum_{i=1}^{k} \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma_i} H_0 dA + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g$$

$$\geq \sum_{i=1}^{k} \frac{1}{2}\left(\frac{|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g$$

□

6

Graphs in Minkowski Space and the Mass of ZAS

Up to this point, we have assumed that the parameter m in the 3-dimensional Schwarzschild manifold $(\mathbb{R}^3\backslash\{0\}, (1+\frac{m}{2r})^4\delta)$ is positive. If we consider the same metric but with $m < 0$, then this gives a Riemannian metric on $\mathbb{R}^3$ minus a closed ball of radius $|m|/2$ about the origin that approaches zero near its inner boundary. In this case, one may think of the 'point' $r = |m|/2$ as a black hole of negative mass. In fact, this metric has a naked singularity, that is, a singularity not enclosed by any apparent horizon. Such singularity is called a *zero area singularity* (ZAS), and the Schwarzschild manifold with negative mass is known as the *Schwarzschild ZAS manifold*. We use our techniques from earlier chapters to study such manifolds that are graphs in Minkowski space and we begin by defining the notion of ZAS, following the approach of Bray and Jauregui [6].

6.1 The notion of zero area singularities

A zero area singularity is characterized by the property that nearby surfaces have arbitrarily small area. More precisely,

Definition 24. Let M^3 be a smooth 3-manifold with a compact, smooth, nonempty boundary ∂M^3 and let g be an asymptotically flat Riemannian metric on $M^3 \backslash \partial M^3$. A connected component Σ^0 of ∂M^3 is a **zero area singularity** of g if for every sequence of surfaces $\{S_n\}$ converging in C^1 to Σ^0, the areas of S_n measured with respect to g converge to zero.

Topologically, a ZAS is a boundary surface in M^3 and not a point. However, in terms of the metric, it is often convenient to think of a ZAS as a point formed by shrinking the metric to zero.

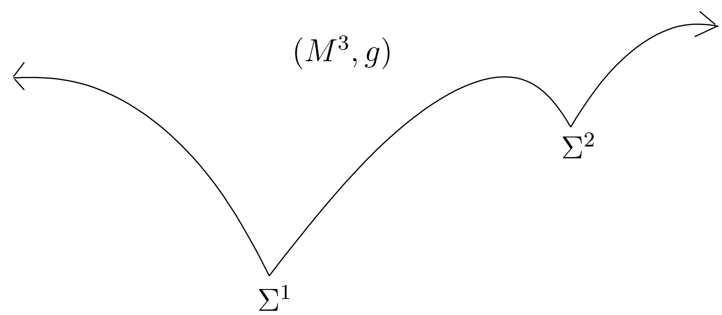

FIGURE 6.1: A manifold with two ZAS Σ^1 and Σ^2.

We denote by Σ the ZAS of g if Σ is the union of all ZAS Σ^i of g. An important special case of ZAS, called a *regular ZAS*, occurs when a smooth metric on M^3 is deformed by a conformal factor that vanishes on the boundary:

Definition 25. Let Σ^0 be a ZAS of g. Then Σ^0 is **regular** if there exists a smooth, nonnegative function $\bar{\varphi}$ and a smooth metric $\bar{g}$, both defined on a neighborhood U of Σ^0, such that

1. $\bar{\varphi}$ vanishes precisely on Σ^0,

2. $\bar{\nu}(\bar{\varphi}) > 0$ on Σ^0, where $\bar{\nu}$ is the unit normal to Σ^0 (taken with respect to $\bar{g}$ and pointing into the manifold), and

3. $g = \bar{\varphi}^4 \bar{g}$ on $U \backslash \Sigma^0$.

If such a pair $(\bar{g}, \bar{\varphi})$ exists, it is called a **local resolution** of Σ^0.

An example of a regular ZAS is that of the Schwarzschild ZAS manifold

$$(M^3, g) = \left(\mathbb{R}^3 \backslash B_{|m|/2}, (1 + \frac{m}{2r})^4 \delta\right) \quad \text{for} \quad m < 0.$$

The boundary of M^3 consists of a single connected component $\Sigma = \Sigma^0 = S_{|m|/2}$. It is easy to check that $\bar{\varphi} = (1 + \frac{m}{2r})$ and $\bar{g} = \delta$ satisfy the conditions of Definition 25. Moreover, this is in fact an example of a *globally harmonic resolution* of Σ, in the sense that the factor $\bar{\varphi}$ is defined on all of $M^3 \backslash \Sigma$ and that $\bar{\varphi}$ is harmonic with respect to $\bar{g}$.

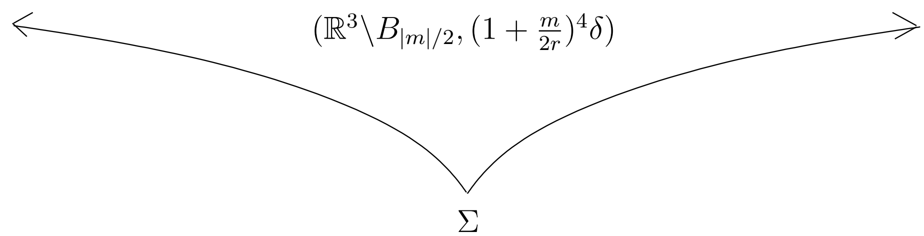

FIGURE 6.2: The 3-dimensional Schwarzschild ZAS manifold.

We can now define the mass of a regular ZAS:

Definition 26. Let $(\bar{g}, \bar{\varphi})$ be a local resolution of a ZAS Σ^0 of g. Then the **regular mass** of Σ^0 is defined by the flux integral

$$m_{reg}(\Sigma^0) = -\frac{1}{4}\left(\frac{1}{\pi}\int_{\Sigma^0} \bar{\nu}(\bar{\varphi})^{4/3} d\bar{A}\right)^{3/2},$$

where $\bar{\nu}$ is the unit normal to Σ^0 (pointing into the manifold) and $d\bar{A}$ is the area form induced by $\bar{g}$.

As shown in [6], this definition

1. is independent of the choice of local resolution,

2. depends only on the local geometry of (M^3, g) near Σ^0,

3. is related to the Hawking masses of nearby surfaces,

4. arises naturally in the proof of the Riemannian ZAS inequality, and

5. equals m for the Schwarzschild ZAS metric of ADM mass $m < 0$.

And finally, we can define the mass of an arbitrary ZAS using regular mass:

Definition 27. Let $\Sigma = \partial M^3$ be zero area singularities of (M^3, g). The **mass** of Σ is

$$m_{ZAS}(\Sigma) := \sup_{\{\Sigma_n\}} \left(\limsup_{n\to\infty} m_{reg}(\Sigma_n) \right),$$

where the supremum is taken over all sequences $\{\Sigma_n\}$ converging in C^1 to Σ.

6.2 Graphs in Minkowski space

As we discussed in Chapter 3, the 3-dimensional Schwarzschild manifold with positive mass can be isometrically embedded as a rotating parabola

$$r = \frac{w^2}{8m} + 2m$$

in flat $\mathbb{R}^4$, where $r = \sqrt{x^2 + y^2 + z^2}$ and (x, y, z, w) are the standard coordinates on $\mathbb{R}^4$. Now if $m < 0$, then solving for w^2 gives

$$w^2 = 8m(r - 2m).$$

Note that since $m < 0$, we cannot solve for w like we did before to get a graph in Euclidean space. However, if we send $w \mapsto iw$, then

$$(iw)^2 = 8m(r - 2m)$$
$$-w^2 = 8m(r - 2m)$$
$$w = \sqrt{8|m|}(r - 2m)^{1/2}$$

is once again a function $w = f(r)$ on $\mathbb{R}^3$, where f is a smooth function for $r > 0$, that is, when it is away from the singularity $r = 0$. Note that under the transformation $w \mapsto iw$, the underlying metric $dx^2+dy^2+dz^2+dw^2$ on $\mathbb{R}^4$ becomes $dx^2+dy^2+dz^2-dw^2$. In other words, we can embed the Schwarzschild ZAS manifold into Minkowski space $\mathbb{R}^{3,1}$. Moreover, since

$$|\nabla f|^2 = \frac{2|m|}{r-2m} < 1 \quad \text{for} \quad r > 0,$$

a tangent vector to the Schwarzschild ZAS manifold has length greater than 0 in $\mathbb{R}^{3,1}$ and the Schwarzschild ZAS manifold is a spacelike slice of Minkowski space.

Motivated by this example, we can more generally consider functions on $\mathbb{R}^n$ whose graph is a spacelike slice in Minkowski space $\mathbb{R}^{n,1}$ with the induced metric. We will assume that such functions are smooth outside any singularities. By an argument similar to that in the proof of Proposition 6, the graph manifold (M^n, g) with the induced metric g from $\mathbb{R}^{n,1}$ is isometric to $(\mathbb{R}^n, \delta - df \otimes df)$. Thus we will henceforth refer to $(\mathbb{R}^n, \delta - df \otimes df)$ as the graph of f (in Minkowski space).

We now proceed to compute the scalar curvature of such a graph (M^n, g) in Minkowski space and hope that just like before, it is a divergence. First, the scalar curvature is given by the following Proposition.

Proposition 28. *The scalar curvature R of a spacelike graph $(\mathbb{R}^n, \delta - df \otimes df)$ in $\mathbb{R}^{n,1}$ is given by*

$$R = \frac{1}{1-|\nabla f|^2}\left(f_{ij}f_{ij} - f_{ii}f_{jj} + \frac{2f_jf_k}{1+|\nabla f|^2}(f_{ij}f_{ik} - f_{ii}f_{jk})\right).$$

Proof. If $g_{ij} = \delta_{ij} - f_if_j$, then

$$g^{ij} = \delta^{ij} + \frac{f^if^j}{1-|\nabla f|^2},$$

since

$$g_{ij}g^{jk} = (\delta_{ij} - f_i f_j)\left(\delta^{jk} + \frac{f^j f^k}{1 - |\nabla f|^2}\right)$$

$$= \delta_{ij}\delta^{jk} - f_i f_j \delta^{jk} + \delta_{ij}\frac{f^j f^k}{1 - |\nabla f|^2} - f_i f_j \frac{f^j f^k}{1 - |\nabla f|^2}$$

$$= \delta_i^k - f_i f^k + \frac{f_i f^k (1 - |\nabla f|^2)}{1 - |\nabla f|^2}$$

$$= \delta_i^k.$$

The Christoffel symbols of g are

$$\Gamma_{ij}^k = \frac{1}{2} g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m})$$

$$= \frac{1}{2}\left(\delta^{km} + \frac{f^k f^m}{1 - |\nabla f|^2}\right)(-f_{ij}f_m - f_i f_{jm} - f_{ij} f_m - f_j f_{im} + f_{im} f_j + f_i f_{jm})$$

$$= \frac{1}{2}\left(\delta^{km} + \frac{f^k f^m}{1 - |\nabla f|^2}\right)(-2 f_{ij} f_m)$$

$$= -f_{ij} f^k - \frac{f_{ij} f^k |\nabla f|^2}{1 - |\nabla f|^2}$$

$$= -\frac{f_{ij} f^k}{1 - |\nabla f|^2}.$$

Just like before, we will write all indices as lower indices and sum over all repeated ones. Thus

$$\Gamma_{ij}^k = -\frac{f_{ij} f_k}{1 - |\nabla f|^2}$$

$$\Gamma_{ij,k}^k = -\frac{f_{ijk} f_k}{1 - |\nabla f|^2} - \frac{f_{ij} f_{kk}}{1 - |\nabla f|^2} - \frac{2 f_{ij} f_{kl} f_k f_l}{(1 - |\nabla f|^2)^2}.$$

The scalar curvature R can now be computed as follows:

$$R = g^{ij}(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl})$$
$$= \left(\delta_{ij} + \frac{f_i f_j}{1-|\nabla f|^2}\right)\left(-\frac{f_{ijk}f_k}{1-|\nabla f|^2} - \frac{f_{ij}f_{kk}}{1-|\nabla f|^2} - \frac{2f_{ij}f_{kl}f_k f_l}{(1-|\nabla f|^2)^2} + \frac{f_{ijk}f_k}{1-|\nabla f|^2}\right.$$
$$\left. + \frac{f_{ik}f_{jk}}{1-|\nabla f|^2} + \frac{2f_{ik}f_{jl}f_k f_l}{(1-|\nabla f|^2)^2} + \frac{f_{ij}f_{kl}f_k f_l}{(1-|\nabla f|^2)^2} - \frac{f_{ik}f_{jl}f_k f_l}{(1-|\nabla f|^2)^2}\right)$$
$$= \left(\delta_{ij} + \frac{f_i f_j}{1-|\nabla f|^2}\right)\left(-\frac{f_{ij}f_{kk}}{1-|\nabla f|^2} + \frac{f_{ik}f_{jk}}{1-|\nabla f|^2} - \frac{f_{ij}f_{kl}f_k f_l}{(1-|\nabla f|^2)^2} + \frac{f_{ik}f_{jl}f_k f_l}{(1-|\nabla f|^2)^2}\right)$$
$$= \frac{1}{1-|\nabla f|^2}(f_{ik}f_{ik} - f_{ii}f_{kk}) + \frac{f_k f_l}{(1-|\nabla f|^2)^2}(f_{ik}f_{il} - f_{ii}f_{kl})$$
$$+ \frac{f_i f_j}{(1-|\nabla f|^2)^2}(f_{ik}f_{jk} - f_{ij}f_{kk}) - \frac{f_i f_j f_k f_l}{(1-|\nabla f|^2)^3}(f_{ij}f_{kl} - f_{ik}f_{jl}).$$

Once again, the last term is 0 by symmetry, and relabeling the indices gives

$$R = \frac{1}{1-|\nabla f|^2}\left(f_{ij}f_{ij} - f_{ii}f_{jj} + \frac{2f_j f_k}{1+|\nabla f|^2}(f_{ij}f_{ik} - f_{ii}f_{jk})\right).$$

□

Now since $g_{ij} = \delta_{ij} - f_i f_j$, we have

$$g_{ij,i} - g_{ii,j} = -f_{ii}f_j - f_i f_{ij} + 2f_{ij}f_i = f_{ij}f_i - f_{ii}f_j.$$

Motivated by our discussion of the ADM mass of a graph in Euclidean space, we give an alternate definition of the ADM mass for a spacelike graph in Minkowski space by multiplying the integrand in the ADM mass by the factor $1/(1-|\nabla f|^2)$.

Definition 29. If (M^n, g) is the graph of an asymptotically flat function that is a spacelike slice in $\mathbb{R}^{n,1}$ with the induced metric, then the ADM mass of (M^n, g) is

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1-|\nabla f|^2}(f_{ij}f_i - f_{ii}f_j)\nu^j dS_r. \tag{6.1}$$

Fortunately, the scalar curvature in this case is once again the divergence of the above integrand:

Lemma 30. *The scalar curvature R of the graph $(\mathbb{R}^n, \delta - df \otimes df)$ satisfies*

$$R = \nabla \cdot \left(\frac{1}{1 - |\nabla f|^2} (f_{ij} f_i - f_{ii} f_j) \partial_j \right)$$

Proof. Now that we know what to look for, this lemma is once again a direct calculation:

$$\begin{aligned}
& \nabla \cdot \left(\frac{1}{1 - |\nabla f|^2} (f_{ij} f_i - f_{ii} f_j) \partial_j \right) \\
= \quad & \frac{1}{1 - |\nabla f|^2} (f_{ijj} f_i + f_{ij} f_{ij} - f_{iij} f_j - f_{ii} f_{jj}) + \frac{2 f_{jk} f_k}{(1 - |\nabla f|^2)^2} (f_{ij} f_i - f_{ii} f_j) \\
= \quad & \frac{1}{1 - |\nabla f|^2} \left(f_{ij} f_{ij} - f_{ii} f_{jj} + \frac{2 f_j f_k}{1 - |\nabla f|^2} (f_{ij} f_{ik} - f_{ii} f_{jk}) \right) \\
= \quad & R.
\end{aligned}$$

□

6.3 ZAS inequality for graphs in Minkowski space

We begin this section by giving an alternate definition of the mass of a ZAS for spacelike graphs in Minkowski space.

Definition 31. Let (M^n, g) be the graph of a function f on $\mathbb{R}^n$ that is a spacelike slice in $\mathbb{R}^{n,1}$ with the induced metric. We say that $\Sigma^0 \in M^n$ is a **zero area singularity (ZAS)** of (M^n, g) if the areas of a sequence of surfaces $\{\Sigma_k\}$, each Σ_k being in a level set of f, converge to 0 and $\{\Sigma_k\}$ converges to Σ^0.

If Σ^i, $i = 1, \ldots, m$ are all the zero area singularities of (M^n, g), then we denote by Σ the (disjoint) union of all the Σ^i.

To motivate our definition of the mass of a ZAS, let us for now assume that the ZAS of (M^n, g) consists of only one connected component Σ^0 (that is, a single point). Let $\{\Sigma_k\}$ be a sequence of surfaces, each of which is in a level set of f, such that $\{\Sigma_k\}$ converges to Σ^0 and let $\Sigma_k = \partial\Omega_k$. If we apply the divergence theorem to (6.1) and use Lemma 30, then for a given k,

$$\begin{aligned}
m_{ADM} &= \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1-|\nabla f|^2}(f_{ij}f_i - f_{ii}f_j)\nu^j dS_r. \\
&= \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n\setminus\bar{\Omega}_k} \nabla\cdot\left(\frac{1}{1-|\nabla f|^2}(f_{ij}f_i - f_{ii}f_j)\partial_j\right) dV_\delta \\
&\quad - \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma_k} \frac{1}{1-|\nabla f|^2}(f_{ij}f_i - f_{ii}f_j)\nu^j dA \\
&= \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n\setminus f(\bar{\Omega}_k)} R\frac{1}{\sqrt{1-|\nabla f|^2}} dV_g \\
&\quad - \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma_k} \frac{|\nabla f|^2}{1-|\nabla f|^2} H_0 dA,
\end{aligned}$$

where the last equality follows from the fact that $\nu^j = \nabla f/|\nabla f|$ and

$$\begin{aligned}
(f_{ij}f_i - f_{ii}f_j)\nu^j &= H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right) - (\Delta f)\nabla f\cdot\frac{\nabla f}{|\nabla f|} \\
&= H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right) - \Delta f|\nabla f| \\
&= H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right) - \left[H^f\left(\nabla f, \frac{\nabla f}{|\nabla f|}\right) + H_0|\nabla f|\right]|\nabla f| \\
&= H_0|\nabla f|^2
\end{aligned}$$

using (5.2).

Once again, we can view the ADM mass as the sum of the mass contribution from the local energy density over M^n and the mass of the surface Σ_k. Because of this, and motivated by the standard definition of a ZAS, we make the following definition:

Definition 32 (Mass of ZAS, single component)**.** Let $\{\Sigma_k\}$ be a sequence of surfaces in M^n, each being in a level set of f, such that Σ_k converges to Σ^0. Then we define

$$m_{ZAS}(\Sigma^0) = \lim_{k\to\infty} -\frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma_k} \frac{|\nabla f|^2}{1-|\nabla f|^2} H_0 dA.$$

If the ZAS Σ of (M^n, g) consists of multiple connected components Σ^i, $i = 1, \ldots, m$, then we define the mass of the ZAS as the sum of the masses of the components.

Definition 33 (Mass of ZAS)**.**

$$m_{ZAS}(\Sigma) = \sum_{i=1}^{k} m_{ZAS}(\Sigma^i)$$

We can now state and prove the ZAS inequality for graphs in Minkowski space.

Theorem 34 (ZAS inequality for graphs)**.** *Let (M^n, g) be the graph of an asymptotically flat function in Minkowski space $\mathbb{R}^{n,1}$ with the induced metric. Let R be the scalar curvature of g and m_{ADM} its ADM mass. Suppose Σ^i, $i = 1, \ldots, m$ are the connected components of the ZAS Σ of (M^n, g) such that $\{\Sigma_k^i\}$ are the surfaces converging to Σ^i. Let $\Sigma_k^i = \partial\Omega_k^i$ and denote by $\Omega_k = \cup_{i=1}^m \Omega_k^i$. Then the ADM mass of (M^n, g) satisfies*

$$m_{ADM} = \lim_{k\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n \setminus f(\bar{\Omega}_k)} R \frac{1}{\sqrt{1-|\nabla f|^2}} dV_g + m_{ZAS}(\Sigma).$$

In particular, if $R \geq 0$, then we have the usual ZAS inequality

$$m_{ADM} \geq m_{ZAS}(\Sigma).$$

Proof of Theorem 34. Applying the divergence theorem to (6.1),

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1-|\nabla f|^2}(f_{ij}f_i - f_{ii}f_j)\nu^j dS_r.$$

$$= \lim_{k\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n\setminus f(\bar{\Omega}_k)} R\frac{1}{\sqrt{1-|\nabla f|^2}}dV_g$$

$$- \lim_{k\to\infty} -\frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma_k} \frac{|\nabla f|^2}{1-|\nabla f|^2}H_0 dA$$

$$= \lim_{k\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n\setminus f(\bar{\Omega}_k)} R\frac{1}{\sqrt{1-|\nabla f|^2}}dV_g + m_{ZAS}(\Sigma).$$

□

7

A New Quasi-Local Mass

Let (M^n, g) be the graph of a smooth function on $\mathbb{R}^n$ and let $\Sigma \subset \mathbb{R}^n$ be a closed surface such that $f(\Sigma) \subset M^n$ is in a level set of f. Let H be the mean curvature of Σ in (M^n, g) and let H_0 be the mean curvature of Σ in $(\mathbb{R}^n, \delta)$. We define the quasi-local mass of the surface $f(\Sigma)$ to be

$$m_{QL}(f(\Sigma)) = \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Sigma)} \frac{1}{H_0}(H_0^2 - H^2)dA$$

and show that this definition of mass is nonnegative, monotonically nondecreasing and approaches the ADM mass of (M^n, g) at ∞.

In dimension $n = 3$, we can define the masses of closed surfaces in more general manifolds besides those that are graphs. If (M^3, g) is a complete, asymptotically flat manifold with nonnegative scalar curvature and Σ is a closed surface in (M^3, g) with Gauss curvature $K > 0$, then the Weyl embedding theorem [17] implies that Σ can be isometrically embedded into $\mathbb{R}^3$. Moreover, the embedding is unique up to an isometry of $\mathbb{R}^3$. Let H be the mean curvature of Σ in (M^3, g) and let H_0 be the mean curvature of Σ in $\mathbb{R}^3$ under the isometric embedding. We define a new quasi-local mass functional of Σ by

$$m_{QL}(\Sigma) = \frac{1}{16\pi}\int_\Sigma \frac{1}{H_0}(H_0^2 - H^2)dA.$$

We give the motivation for our definition of quasi-local mass and study some of its properties in this chapter.

7.1 On quasi-local mass functionals

As we noted in Chapter 1, the 3-dimensional Schwarzschild manifold is the equality case of the Riemannian Penrose inequality with

$$m_{ADM} = \sqrt{\frac{A}{16\pi}},$$

where A is the area of the outermost minimal surface corresponding to the apparent horizon of a black hole. Since the Schwarzschild manifold is scalar flat, it has zero local energy density everywhere and one can view the mass contribution as coming solely from the black hole. Therefore it is natural to consider the quantity $\sqrt{A/16\pi}$ as the mass of a black hole of area A. More generally, given any bounded region Ω with smooth boundary $\Sigma = \partial\Omega$ in a complete, asymptotically flat manifold (M^3, g), we would like to define how much mass Ω (or equivalently, Σ) possesses. Assuming (M^3, g) has nonnegative scalar curvature (local energy density) everywhere, there are certain reasonable properties one would expect from such *quasi-local mass* functional $m(\Sigma)$, most notably [4, 11],

1. **(nonnegativity)** $m(\Sigma) \geq 0$;

2. **(rigidity/strict positivity)** $m(\Sigma) = 0$ if and only if Σ is flat;

3. **(monotonicity)** $m(\Sigma_1) \leq m(\Sigma_2)$ if $\Sigma_1 = \partial\Omega_1$ and $\Sigma_2 = \partial\Omega_2$ with $\Omega_1 \subset \Omega_2$;

4. **(ADM limit)** $m(\Sigma)$ should be asymptotic to the ADM mass, that is, if Σ_k is a sequence of surfaces that exhaust (M^3, g), then

$$\lim_{k\to\infty} m(\Sigma_k) = m_{ADM}(g);$$

5. **(black hole limit)** $m(\Sigma)$ should agree with the black hole mass, that is, if Σ is a horizon in (M^3, g), then

$$m(\Sigma) = \sqrt{\frac{A}{16\pi}},$$

where A is the area of Σ.

There have been numerous attempts to define such quasi-local mass functionals [3, 4, 5, 9, 10, 13, 27], but to date, none of them satisfy all of the properties listed above. For example, as discussed in Chapter 2, the Hawking mass

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 dA\right)$$

is monotonically nondecreasing under inverse mean curvature flow, and that its asymptotic limit coincides with the ADM mass. Moreover, it gives the correct black hole mass if Σ is a horizon, as $H \equiv 0$ along Σ. Thus the Hawking mass satisfies the last three properties listed above, with monotonicity in the sense of inverse mean curvature flow.

Unfortunately, the Hawking mass does not satisfy the first two properties. In fact, $m_H(\Sigma) \leq 0$ for any closed surface Σ in $\mathbb{R}^3$. Nevertheless, it is known that $m_H \geq 0$ for a stable constant mean curvature 2-sphere in a 3-manifold of nonnegative scalar curvature [11].

Another notable example of a quasi-local mass functional is the Brown-York mass [9, 10]. If $\Sigma \subset M^3$ is a closed surface with positive Gauss curvature, then by the

Weyl embedding theorem [17], it can be isometrically embedded into $\mathbb{R}^3$. Moreover, the embedding is unique up to an isometry of $\mathbb{R}^3$. Let H be the mean curvature of Σ in (M^3, g) and let H_0 be the mean curvature in $\mathbb{R}^3$ under the isometric embedding. The Brown-York mass of Σ is defined to be

$$m_{BY}(\Sigma) = \frac{1}{8\pi}\int_\Sigma H_0 - H dA,$$

where dA is the area element on Σ induced by g. In 2002, Shi and Tam [26] proved that the Brown-York mass is nonnegative using a gluing argument by solving a parabolic partial differential equation of a certain foliation and applying the positive mass theorem.

While the Brown-York mass is nonnegative, it tends to overestimate the mass of a given region. For example, the isometric image in $\mathbb{R}^3$ of the horizon in the 3-dimensional Schwarzschild manifold $(M^3, g) = (\mathbb{R}^3\backslash\{0\}, (1 + \frac{m}{2r})^4)$ is the sphere $\Sigma = S_{2m}$ of radius $2m$. Thus $H_0 = 2/2m = 1/m$ and $H = 0$ and

$$m_{BY}(\Sigma) = \frac{1}{8\pi}\int_\Sigma H_0 - H dA = \frac{1}{8\pi}\cdot 4\pi(2m)^2 \cdot \frac{1}{m} = 2m,$$

but the mass of the horizon should be m in this case. More generally, for a sphere S_r in (M^3, g), its isometric image in $\mathbb{R}^3$ is S_{ru^2} with $u = 1 + \frac{m}{2r}$. Thus

$$H_0 = \frac{2}{ru^2} \quad \text{and} \quad H = \frac{1}{u^2}\left(\frac{2}{r} + \frac{4}{u}\frac{\partial u}{\partial r}\right).$$

The Brown-York mass of S_r is then

$$\begin{aligned}
m_{BY}(S_r) &= \frac{1}{8\pi}\int_{S_r} H_0 - H dA \\
&= \frac{1}{8\pi}\cdot 4\pi r^2 u^4 \cdot \left[\frac{2}{ru^2} - \frac{1}{u}\left(\frac{2}{r} + \frac{4}{u}\frac{\partial u}{\partial r}\right)\right] \\
&= -2r^2 u \frac{\partial u}{\partial r} \\
&= -2r^2 u\left(-\frac{m}{2r^2}\right) \\
&= m\left(1 + \frac{m}{2r}\right).
\end{aligned}$$

Thus $m_{BY}(S_r)$ is actually monotonically *decreasing* to the ADM mass as $r \to \infty$ in the Schwarzschild manifold.

7.2 Motivation for the definition of m_{QL}

In Chapter 5, we considered the graph (M^n, g) of a smooth function $f : \mathbb{R}^n \backslash \Omega \to \mathbb{R}$, where Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary $\Sigma = \partial\Omega$. If we in addition assume that $f(\Sigma)$ is a minimal boundary of (M^n, g) that is contained in a level set of f, then Theorem 19 states that

$$m_{ADM}(g) = \frac{1}{2(n-1)\omega_{n-1}}\int_{f(\Sigma)} H_0 dA + \frac{1}{2(n-1)\omega_{n-1}}\int_{M^n} R\frac{1}{\sqrt{1+|\nabla f|^2}} dV_g,$$

where H_0 is the mean curvature with respect to the flat metric. One can view the above as an expression of the ADM mass of the manifold as the sum of the mass of the boundary and the mass contribution from the local energy density over the manifold.

If we remove the assumption that $f(\Sigma)$ is minimal, equation (5.3) implies that the boundary integral is

$$\frac{1}{2(n-1)\omega_{n-1}}\int_{f(\Sigma)} \frac{|\nabla f|^2}{1+|\nabla f|^2} H_0 dA. \tag{7.1}$$

Recall that equation (5.1) relates the mean curvatures H and H_0 by

$$H = \frac{1}{\sqrt{1+|\nabla f|^2}} H_0,$$

which we can rewrite as

$$\left(\frac{H}{H_0}\right)^2 = \frac{1}{1+|\nabla f|^2}$$

$$1 - \left(\frac{H}{H_0}\right)^2 = \frac{|\nabla f|^2}{1+|\nabla f|^2}.$$

Thus equation (7.1) can be expressed as

$$\begin{aligned}\frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Sigma)} \frac{|\nabla f|^2}{1+|\nabla f|^2} H_0 dA &= \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Sigma)} \left[1 - \left(\frac{H}{H_0}\right)^2\right] H_0 dA \\ &= \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Sigma)} \frac{1}{H_0}(H_0^2 - H^2) dA.\end{aligned} \tag{7.2}$$

Because of (7.2), we make the following definition:

Definition 35. Let (M^n, g) be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$ with the induced metric g from $\mathbb{R}^{n+1}$. If $\Sigma \subset \mathbb{R}^n$ is a closed surface such that $f(\Sigma) \subset M^n$ is in a level set of f, then we define the quasi-local mass of Σ to be

$$m_{QL}(f(\Sigma)) = \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Sigma)} \frac{1}{H_0}(H_0^2 - H^2) dA.$$

Moreover, if Ω is a bounded set in $\mathbb{R}^n$ such that $\partial\Omega = \Sigma$, then the divergence theorem implies that

$$\begin{aligned} m_{QL}(f(\Sigma)) &= \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Sigma)} \frac{1}{H_0}(H_0^2 - H^2) dA \\ &= \frac{1}{2(n-1)\omega_n} \int_{f(\Omega)} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g. \end{aligned} \tag{7.3}$$

We can view (7.3) as an alternate characterization of the mass of $f(\Sigma)$.

7.3 Properties of m_{QL}

We collect the properties of m_{QL} in the following theorem:

Theorem 36. *Let (M^n, g) be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$ with nonnegative scalar curvature. Let $\Sigma \subset \mathbb{R}^n$ be a closed surface such that $f(\Sigma) \subset M^n$ is in a level set of f. Let H and H_0 denote its mean curvature with respect to g and the flat metric respectively. The quasi-local mass functional*

$$m_{QL}(f(\Sigma)) = \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Sigma)} \frac{1}{H_0}(H_0^2 - H^2)dA$$

satisfies

1. *(**Nonnegativity**) $m_{QL}(f(\Sigma)) \geq 0$,*

2. *(**Monotonicity**) If $\Sigma_1 = \partial\Omega_1$ and $\Sigma_2 = \partial\Omega_2$ are such that $\Omega_1 \subset \Omega_2$, then $m_{QL}(f(\Sigma_1)) \leq m_{QL}(f(\Sigma_2))$.*

3. *(**ADM limit**) If $\{\Sigma_k\}$ is a sequence that exhausts $\mathbb{R}^n$, then*

$$\lim_{k\to\infty} m_{QL}(f(\Sigma_k)) = m_{ADM}(g).$$

Proof. 1. is an immediate consequence of (7.3) since

$$m_{QL}(f(\Sigma)) = \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Omega)} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g > 0$$

and $R \geq 0$ by assumption. Thus $m_{QL}(f(\Sigma))$ satisfies nonnegativity.

If Σ_1 and Σ_2 are such that $\Sigma_1 = \partial\Omega_1$, $\Sigma_2 = \partial\Omega_2$ with $\Omega_1 \subset \Omega_2$, then

$$\begin{aligned} m_{QL}(f(\Sigma_1)) &= \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Omega_1)} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g \\ &\leq \frac{1}{2(n-1)\omega_{n-1}} \int_{f(\Omega_2)} R\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g = m_{QL}(f(\Sigma_2)) \end{aligned}$$

using $R \geq 0$.

Finally, if $\{\Sigma_k\}$ is a sequence of surfaces that exhaust $\mathbb{R}^n$, then we can express the ADM mass of (M^n, g) as

$$m_{ADM} = m_{QL}(f(\Sigma_k)) + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n \setminus f(\Omega_k)} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$

For each k. As $k \to \infty$, the second integral approaches 0 and thus m_{QL} is asymptotic to the ADM mass. □

7.4 m_{QL} for more general manifolds

In the previous section we showed that m_{QL} is nonnegative, monotonically nondecreasing and asymptotic to the ADM mass for closed surfaces in complete graph manifolds. We point out in this section that m_{QL} can in fact be defined in more general settings, namely, that whenever the Brown-York mass is defined. Suppose (M^3, g) is a complete, asymptotically flat manifold with nonnegative scalar curvature and let Σ be a closed surface in (M^3, g). If each connected component of Σ has positive Gauss curvature, then Σ can be isometrically embedded in $\mathbb{R}^3$ by the Weyl's embedding theorem, and that the embedding is unique up to an isometry of $\mathbb{R}^3$. Let H be the mean curvature of Σ in (M^3, g) and let H_0 be the mean curvature of Σ when isometrically embedded into $\mathbb{R}^3$, then we define the quasi-local mass of Σ to be

$$m_{QL}(\Sigma) = \frac{1}{16\pi} \int_\Sigma \frac{1}{H_0} (H_0^2 - H^2) dA.$$

Since

$$m_{QL}(\Sigma) = \frac{1}{16\pi} \int_\Sigma \frac{1}{H_0} (H_0^2 - H^2) dA = \frac{1}{16\pi} \int_\Sigma H_0 - H \frac{H}{H_0} dA,$$

m_{QL} looks strikingly similar to the Brown-York mass m_{BY}. In fact,

$$\begin{aligned}
m_{BY}(\Sigma) - m_{QL}(\Sigma) &= \frac{1}{8\pi}\int_{\Sigma} H_0 - H dA - \frac{1}{16\pi}\int_{\Sigma} H_0 - \frac{H^2}{H_0} \\
&= \frac{1}{16\pi}\int_{\Sigma} 2H_0 - 2H - H_0 + \frac{H^2}{H_0} dA \\
&= \frac{1}{16\pi}\int_{\Sigma} \frac{1}{H_0}(H - H_0)^2 dA \\
&\geq 0.
\end{aligned}$$

Therefore, by [26], we have

$$m_{BY}(\Sigma) \geq m_{QL}(\Sigma) \geq 0,$$

and that equality holds if and only if Ω is isometric to a domain in $\mathbb{R}^3$.

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Biography

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