The Graphs of Planar Soap Bubbles

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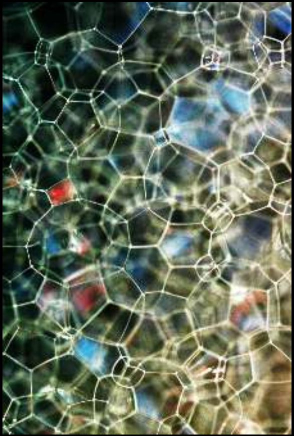
Soap bubbles and soap bubble foams



CC-BY photograph "cosmic soap bubbles (God takes a bath)" by woodleywonderworks from Flickr

Soap molecules form double layers separating thin films of water from pockets of air

A familiar physical system that produces complicated arrangements of curved surfaces, edges, and vertices

What can we say about the mathematics of these structures?

Plateau's laws

In every soap bubble cluster:

- Each surface has constant mean curvature
- Triples of surfaces meet along curves at 120° angles
- These curves meet in groups of four at equal angles

Observed in 19th c. by Joseph Plateau

Proved by JeanTaylor in 1976

1843 Daguerrotype of Joseph Plateau

Young–Laplace equation

Thomas Young

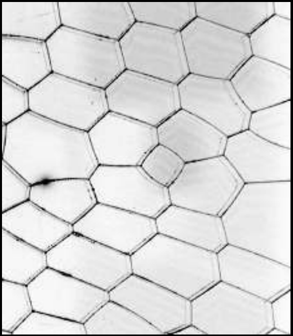
For each surface in a soap bubble cluster:

mean curvature
$$= 1/\text{pressure difference}$$
(with surface tension as constant of proportionality)

Formulated in 19th c., by Thomas Young and Pierre-Simon Laplace

Pierre-Simon Laplace

Planar soap bubbles



PD image "2-dimensional foam (colors inverted).jpg" by Klaus-Dieter Keller from Wikimedia commons

3d is too complicated, let's restrict to two dimensions

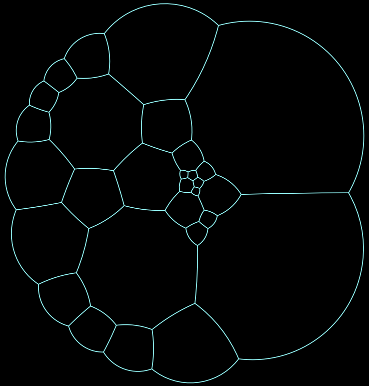
Equivalently, form 3d bubbles between parallel glass plates

Bubble surfaces are at right angles to the plates, so all 2d cross sections look the same as each other

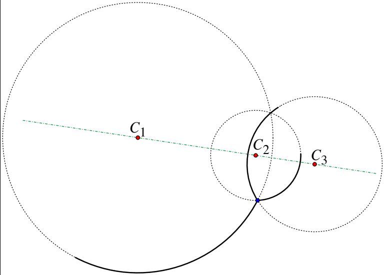
Plateau and Young–Laplace for planar bubbles

In every planar soap bubble cluster:

- Each curve is an arc of a circle or a line segment
- Each vertex is the endpoint of three curves at 120° angles
- It is possible to assign pressures to the bubbles so that curvature is inversely proportional to pressure difference



Geometric reformulation of the pressure condition



For arcs meeting at 120° angles, the following three conditions are equivalent:

- We can find pressures matching all curvatures
- Triples of circles have collinear centers
- Triples of circles form a "double bubble" with two triple crossing points

Möbius transformations

Fractional linear transformations

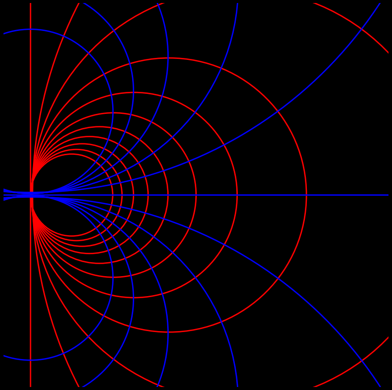
$$z \mapsto \frac{az + b}{cz + d}$$

in the plane of complex numbers

Take circles to circles and do not change angles between curves

Plateau's laws and the double bubble reformulation of Young–Laplace only involve circles and angles

so the Möbius transform of a bubble cluster is another valid bubble cluster



Theorem: Bubble clusters don't have bridges

Collapse of the Tacoma Narrows Bridge, 1940

Main ideas of proof:

- A bridge that is not straight violates the pressure condition
- A straight bridge can be transformed to a curved one that again violates the pressure condition

Theorem: Bridges are the only obstacle

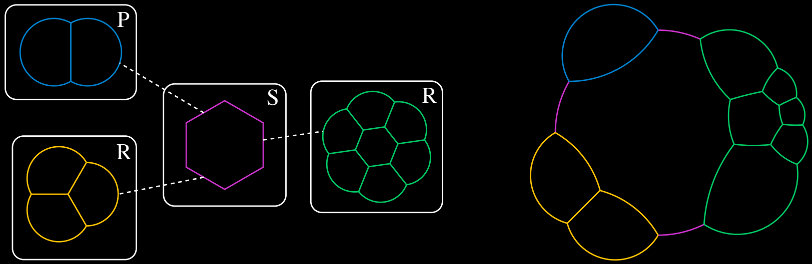
For planar graphs with three edges per vertex and no bridges, we can always find a valid bubble cluster realizing that graph

Main ideas of proof:

1. Partition into 3-connected components and handle each component independently
2. Use Koebe–Andreev–Thurston circle packing to find a system of circles whose tangencies represent the dual graph
3. Construct a novel type of Möbius-invariant *power diagram* of these circles, defined using 3d hyperbolic geometry
4. Use symmetry and Möbius invariance to show that cell boundaries are circular arcs satisfying the angle and pressure conditions that define soap bubbles

Step 1: Partition into 3-connected components

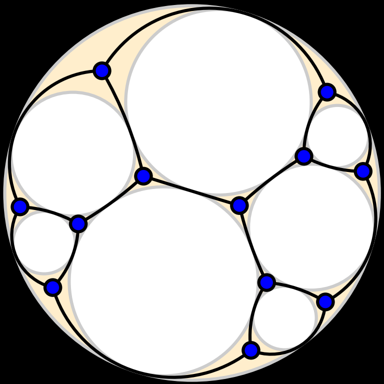
For graphs that are not 3-regular or 3-connected, decompose into smaller subgraphs, draw them separately, and glue them together



The decomposition uses *SPQR trees*, standard in graph drawing

Use Möbius transformations in the gluing step to change relative sizes of arcs so that the subgraphs fit together without overlaps

Step 2: Circle packing



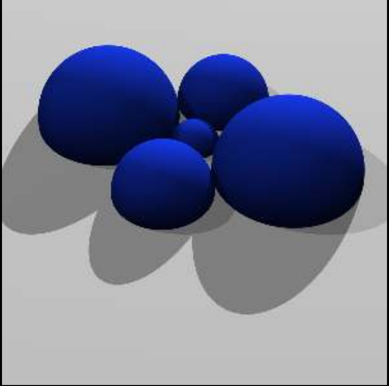
After the previous step we have a 3-connected 3-regular graph

Koebe–Andreev–Thurston circle packing theorem guarantees the existence of a circle for each face, so circles of adjacent faces are tangent, other circles are disjoint

Can be constructed by efficient numerical algorithms

Step 3a: Hyperbolic Voronoi diagram

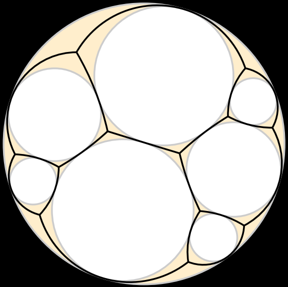
Embed the plane in 3d, with a hemisphere above each face circle



Use the space above the plane as a model of *hyperbolic geometry*, and partition it into subsets nearer to one hemisphere than another

Step 3b: Möbius-invariant power diagram

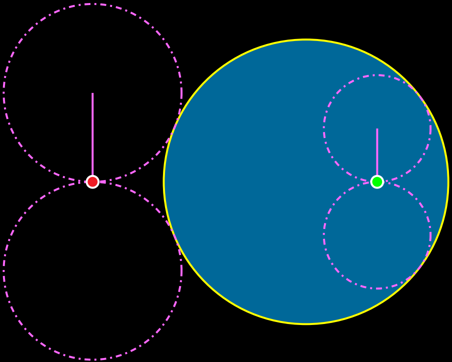
Restrict the 3d Voronoi diagram to the plane containing the circles (the plane at infinity of the hyperbolic space).



Symmetries of hyperbolic space restrict to Möbius transformations of the plane ⇒ diagram is invariant under Möbius transformations

2d Euclidean description of same power diagram

To find distance from point q to circle O:

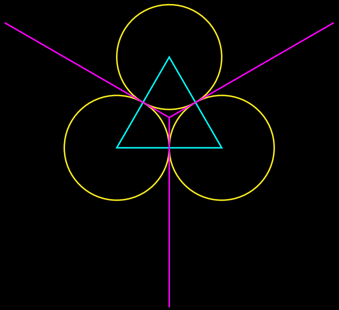


Draw equal circles tangent to each other at q, both tangent to O

Distance is their radius (if q outside O) or $-$radius (if inside)

Our diagram is the minimization diagram of this distance

Step 4: By symmetry, these are soap bubbles



Each three mutually tangent circles can be transformed to have equal radii, centered at the vertices of an equilateral triangle.

By symmetry, the power diagram boundaries are straight rays (limiting case of circular arcs with infinite radius), meeting at 120° angles (Plateau's laws)

Setting all pressures equal fulfils the Young–Laplace equation on pressure and curvature

Conclusions and future work

Precise characterization of 2d soap bubble clusters

Closely related to the author's earlier work on *Lombardi drawing* of graphs

How stable are our clusters? Only partial results so far

What about 3d?

Do there exist stable clusters with surfaces that do not separate two volumes?

CC-SA image "world of soap" by Martin Fisch on Flickr