Math. Z. (2010) 265:321–342
DOI 10.1007/s00209-009-0518-7

Mathematische Zeitschrift

The Grothendieck ring of varieties and piecewise isomorphisms

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Received: 15 February 2008 / Accepted: 29 January 2009 / Published online: 17 April 2009

Abstract Let $K_0(\mathrm{Var}_k)$ be the Grothendieck ring of algebraic varieties over a field k. Let X, Y be two algebraic varieties over k which are piecewise isomorphic (i.e. X and Y admit finite partitions $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ into locally closed subvarieties such that X_i is isomorphic to Y_i for all $i \leq n$), then $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Larsen and Lunts ask whether the converse is true. For characteristic zero and algebraically closed field k, we answer positively this question when dim $X \leq 1$ or X is a smooth connected projective surface or if X contains only finitely many rational curves.

1 Introduction

The main topic of this article is to study the Grothendieck ring of algebraic varieties $K_0(\mathrm{Var}_k)$ over a field k. Appeared in a letter of Grothendieck to Serre ([5], letter of 16 Aug. 1964), this ring has been deeply used for developing the theorie(s) of motivic integration. But we know very little about this ring. For example, Poonen ([33], Theorem 1) and Kollár ([21], Example 6) show that this ring is not a domain when k has characteristic zero (see also Corollary 9), and Niko Naumann ([29], Theorem 22) provides zero divisors for $K_0(\mathrm{Var}_k)$ over finite fields k. On the opposite, one can construct infinite family of classes in $K_0(\mathrm{Var}_k)$ which are algebraically independent over $\mathbb{Z}$ (see references in Remark 8).

Let X, Y be algebraic varieties over k with same class $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. The aim of this work is to try to derive some consequences of this identity. It is clear by construction that if X and Y are *piecewise isomorphic* (i.e. X and Y admit finite partitions $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ into locally closed subvarieties such that X_i is isomorphic to Y_i for all $i \leq n$. See Definition 2 and Proposition 2), then $[X] = [Y]$. Conversely, the following question is raised by Larsen and Lunts [25], 1.2:

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Assertion 1 *Let k be a field. Let X, Y be k-varieties such that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Then X and Y are piecewise isomorphic.*

Over an algebraically closed field k of characteristic zero, we will prove this assertion in the following situations: (1) $\dim X \leq 1$ (Propositions 5 and 6); (2) X is a smooth connected projective surface (Theorem 4); (3) X contains only finitely many rational curves (Theorem 5). In the general case, we are quickly faced to the problem of zero divisors in $K_0(\mathrm{Var}_k)$, the motivic cancelation problem (i.e. is $\mathbb{L}$ a zero divisor ?) and the classification of varieties in higher dimensions. In positive characteristic, the answer does not seem to be obvious even for zero-dimensional varieties (see Sect. 5.1).

Convention Let k be a field. We denote by $\bar{k}$ an algebraic closure of k. An *algebraic k-variety* (or *k-variety*) is a separated reduced scheme of finite type over k. Any locally closed subset of a variety is endowed with the structure of reduced subvariety. A *k-curve* is a k-variety of pure dimension one. A *k-surface* is a k-variety of pure dimension two.

2 Preliminaries

2.1 Definitions and notations

Constructible topology. Let X be a scheme. The open subsets of X for the constructible topology are the ind-constructible subsets of X ([13], IV.1.9.13). When X is noetherian, the ind-constructible subsets of X are exactly the unions of locally closed subsets of X (op. cit., IV.1.9.6), and the family of locally closed subsets of X form a basis of this topology.

Cohomology theories. See for example [1, Sect. 3.4]. Let k be a field (of arbitrary characteristic) and let X be a k-variety.

- *Étale cohomology*. Let ℓ be a prime number different from the characteristic exponent of k. We define the étale (ℓ-adic) cohomology with compact support $H_c^*(X_{\acute{e}t})$ of X to be the direct sum of the $\mathbb{Q}_\ell$-vector spaces $H_c^i(X_{\bar{k}}, \mathbb{Q}_\ell)$. We denote by $b_i(X) := \dim_{\mathbb{Q}_\ell} H_c^i(X_{\acute{e}t})$ the *ith Betti number* of X.
- *Singular cohomology*. If $k \subset \mathbb{C}$, we denote by $H^*(X) := H^*(X^{\mathrm{an}}, \mathbb{Q})$ the singular cohomology of X^{an} (the analytic space associated to X), with coefficients in $\mathbb{Q}$.
- *De Rham cohomology*. Let k be a field of characteristic zero. We denote by $H^*_{\mathrm{dR}}(X)$ the *algebraic de Rham cohomology*. Let X be a proper smooth k-variety. Then $\dim_k H^n_{\mathrm{dR}}(X) = \sum_{i+j=n} \dim_k H^i(X, \Omega^j_{X/k})$ (Hodge's decomposition theorem) for all n. The integer $h^{p,q}(X) := \dim_k H^q(X, \Omega^p_{X/k})$ is called the *(p, q)-Hodge number* of X.

Besides, if $k \subseteq \mathbb{C}$, by Grothendieck's results ([12], Theorem 1'), we know that the canonical map $H^*_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \simeq H^*(X) \otimes_{\mathbb{Q}} \mathbb{C}$ is an isomorphism.

Plurigenus and Kodaira dimension. Let k be a field, and let X be a projective connected smooth k-variety of dimension d. Let $M = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and let

$$T_M(X) := (\Omega^1_{X/k})^{\otimes m_1} \otimes \cdots \otimes (\Omega^n_{X/k})^{\otimes m_n}.$$

Note that if $m_j > 0$ for some $j > d$, then $T_M(X) = 0$. Besides, if $\tilde{M} = (m_1, \ldots, m_d, 0, \ldots, 0)$, then $T_{\tilde{M}}(X) = T_M(X)$. The *$M$-plurigenus* of X is the integer $p_M := \dim_k H^0(X, T_M(X))$.

Let $m \in \mathbb{N}$. The *m-plurigenus* of X is the integer $p_m(X) := p_M(X)$, with $M = (0, \dots, 0, m) \in \mathbb{N}^d$. The *Kodaira dimension* of X is defined by

$$\kappa(X) = \dim\left(\oplus_{m=0}^{+\infty} H^0(X, (\Omega^d_{X/k})^{\otimes m})\right) - 1$$

if the right-hand side is non-negative and $-\infty$ otherwise. Note that $\kappa(X)$ is a birational invariant, and that the condition $\kappa(X) \geq 0$ is equivalent to the existence of an integer $m \in \mathbb{N}^*$ such that $p_m(X) \geq 1$.

2.2 The Grothendieck ring of varieties

2.2.1 The usual definition

Definition 1 Let $\mathbb{Z}[\mathrm{Var}_k]$ be the free abelian group generated by the isomorphism classes of k-varieties. The Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ is the quotient of $\mathbb{Z}[\mathrm{Var}_k]$ by the following relations : whenever X is a k-variety, and Z is a closed subvariety of X, we impose that

$$[X] = [Z] + [X \setminus Z]. \tag{1}$$

We denote by $\mathbb{L}$ the class of $\mathbb{A}^1_k$, and by $\mathbb{I}$ the class of $\operatorname{Spec} k$. The multiplication on $K_0(\mathrm{Var}_k)$ is defined by

$$[X] \cdot [Y] := [(X \times_k Y)_{\mathrm{red}}].$$

for any pair of k-varieties X, Y. As the canonical homomorphism $\mathbb{Z} \to K_0(\mathrm{Var}_k)$, $n \mapsto n \cdot \mathbb{I}$, is injective (to see this property, just apply topological Euler characteristic, see Sect. 4.1), we will write n instead of $n \cdot \mathbb{I}$ when no confusion is possible.

Remark 1 It is equivalent to define $K_0(\mathrm{Var}_k)$ by starting from the k-schemes of finite type and imposing the same above relations. More generally, if S is an arbitrary scheme, one can also define a Grothendieck ring of S-schemes of finite type, by adapting these definitions to the relative setting (see [3, Sect. 5]).

Remark 2 Consider $X := \mathbb{A}^1_k \sqcup \operatorname{Spec} k$ (disjoint union). Then X and $\mathbb{P}^1_k$ have the same class $\mathbb{L} + 1$ in $K_0(\mathrm{Var}_k)$. They are not isomorphic, and even not birational. But they are piecewise isomorphic (see Sect. 2.2.2).

Example 1 Let X be a projective k-variety, let Y be the affine cone of X, we have

$$[Y] = (\mathbb{L} - 1)[X] + 1$$

in $K_0(\mathrm{Var}_k)$. Consider a projective bundle $\mathbb{P}(\mathcal{E})$ of rank r over a projective k-variety Y. Then

$$\mathbb{P}(\mathcal{E}) = [Y][\mathbb{P}^{r-1}] = [Y] \cdot \sum_{i=0}^{r-1} \mathbb{L}^i$$

in $K_0(\mathrm{Var}_k)$.

Lemma 1 *Suppose* $\mathrm{char}(k) = 0$. *Let X be a k-variety of dimension d. Let $F_1, \dots, F_n$ be the irreducible components of dimension d of X. Then there exist smooth connected projective k-varieties $X_1, \dots, X_n$, and $C_1, \dots, C_m$ such that in $K_0(\mathrm{Var}_k)$, we have*

$$[X] = \sum_i [X_i] + \sum_j \epsilon_j [C_j],$$

with X_i *birational to* F_i, *and* $\dim\ C_j < \dim\ X$, $\epsilon_j = \pm 1$.

Proof This results from Hironaka's desingularization theorem, and from an induction on d.

2.2.2 The definition via the piecewise algebraic geometry

Let $\underline{Sch}$ be the category of schemes, and let $\underline{Sch}_0$ be the full subcategory of schemes whose local rings are fields (or equivalently, zero-dimensional reduced schemes).

Proposition 1 ([31,32]) *The canonical functor* $i : \underline{Sch}_0 \to \underline{Sch}$ *has a right adjoint, denoted by* $\cdot^{\mathrm{cons}}$.

The existence of X^{cons} as an object of $\underline{Sch}_0$ is announced in [13], IV.1.9.16. Let us sketch briefly the construction of X^{cons} as done in [31,32], Appendice.

First suppose $X = \mathrm{Spec}\ A$ is an affine scheme. Let

$$T(A) = A[T_a]_{a \in A}/(a^2 T_a - a, a T_a^2 - T_a)_{a \in A}.$$

Then $T(A)$ is the unique A-algebra satisfying the following universal property:

1. For every $a \in A$, there exists an element $t_a \in T(A)$ such that $at_a^2 = t_a$ and $a^2 t_a = a$. Notice that such an element t_a (called a *punctual inverse* of a) is automatically unique.
2. Let B be an A-algebra such that a has a punctual inverse in B for all $a \in A$, then there exists a unique homomorphism of A-algebra $T(A) \to B$.

Lemma 2 *Let* $\mathfrak{X} = \mathrm{Spec}\, T(A)$ *and let* $\pi : \mathfrak{X} \to X$ *be the canonical morphism. Then the following properties are true*:

1. *For all* $x \in \mathfrak{X}$, *the canonical homomorphism* $\mathcal{O}_{X,\pi(x)} \to \mathcal{O}_{\mathfrak{X},x}$ *induces an isomorphism* $k(\pi(x)) \to \mathcal{O}_{\mathfrak{X},x}$. *In particular,* $\mathfrak{X}$ *is reduced of dimension* 0.
2. *The map* $\pi : \mathfrak{X} \to X$ *is bijective*;
3. *The continuous map* π *is a homeomorphism from* $\mathfrak{X}$ *to* X *when the latter is endowed with the constructible topology* (*i.e. here the topology generated by the sets* $V(a)$ *and* $D(a)$ *for all* $a \in A$). *In particular,* $\mathfrak{X}$ *is a compact topological space, totally disconnected.*
4. *For every reduced zero-dimensional scheme* Z, *the canonical map*

$$\mathrm{Mor}(Z, \mathfrak{X}) \to \mathrm{Mor}(Z, X)$$

is a bijection. Therefore X^{cons} *exists and is isomorphic to* $\mathfrak{X}$.

Proof First we notice that by the uniqueness of the punctual inverses, $t_a t_{a'} = t_{aa'}$ in $T(A)$ for all $a, a' \in A$. In particular, any $b \in T(A)$ can be written (in a non unique way) as $b = \sum_{i \in I} \alpha_i t_{a_i}$, with $\alpha_i, a_i \in A$, for all $i \in I$.

Let $\mathfrak{p} \in X$ be a fixed prime ideal. Then for any $b \in T(A)$, there exist $a' \in A \backslash \mathfrak{p}$, $a'' \in A$ and $\beta \in \mathfrak{p}T(A)$ such that

$$a'b = a'' + \beta.$$

This comes from the fact that $t_a \in \mathfrak{p}T(A)$ if $a \in \mathfrak{p}$ and $a^2 t_a = a \in A$ if $a \notin \mathfrak{p}$.

(1) Let $\mathfrak{q}$ be a prime ideal of $T(A)$ and let $\mathfrak{p} = A \cap \mathfrak{q}$. If $a \in \mathfrak{p}$, then $at_a - 1 \notin \mathfrak{q}$, so $at_a - 1 \in (T(A)_{\mathfrak{q}})^*$ and $a = t_a = 0$ in $T(A)_{\mathfrak{q}}$. Therefore $a'b = a''$ in $T(A)_{\mathfrak{q}}$. So b is either zero or invertible in $T(A)_{\mathfrak{q}}$. Thus $T(A)_{\mathfrak{q}}$ is a field, isomorphic to $k(\mathfrak{p})$.

(2) Let $\mathfrak{p} \in X$. The universal property gives a homomorphism of A-algebras $\pi_{\mathfrak{p}} : T(A) \to k(\mathfrak{p})$. Let $\mathfrak{q} = \ker \pi_{\mathfrak{p}}$. Then $\mathfrak{q} \cap A = \mathfrak{p}$, and any prime ideal $\mathfrak{q}'$ of $T(A)$ lying over $\mathfrak{p}$ is contained in $\mathfrak{q}$. As $\dim T(A) = 0$, we have $\mathfrak{q}' = \mathfrak{q}$. So π is bijective.

(3) From now on we identify $\mathfrak{X}$ to X (as sets) via π. Let us first show that locally closed subsets F of X are open in $\mathfrak{X}$. It is enough to consider the case $F = V_X(a)$ (principal closed subset of X associated to a) for some $a \in A$. We have $aT(A) = at_aT(A)$. As at_a is idempotent, $V_{\mathfrak{X}}(at_a)$ is open in $\mathfrak{X}$. So F is open in $\mathfrak{X}$.

Conversely, let $D_{\mathfrak{X}}(b)$ be a principal open subset of $\mathfrak{X}$. Let $\mathfrak{p} \in X$ belonging to $D_{\mathfrak{X}}(b)$. Write $a'b = a'' + \alpha$ as above with $a' \in A \backslash \mathfrak{p}$, $a'' \in A$, $\alpha \in \mathfrak{p}T(A)$ and $\alpha = \sum_{1 \le i \le n} a_i b_i$ with $a_i \in \mathfrak{p}$ and $b_i \in T(A)$. Then

$$\mathfrak{p} \in D_X(a') \cap D_X(a'') \cap (\cap_{1 \le i \le n} V_X(a_i)) \subseteq D_{\mathfrak{X}}(b).$$

Thus every open subset of $\mathfrak{X}$ is a union of locally closed subsets of X. The last part is a direct consequence of [13], IV.1.9.15, (i) and (ii).

(4) We can suppose that $Z = \operatorname{Spec} B$ is affine. Let $A \to B$ be a ring homomorphism. We have to prove that it factorizes into $A \to T(A) \to B$. By the above, this comes down to show that any element $b \in B$ has a punctual inverse in B. The latter can be seen to be equivalent to $bB = b^2B$ (if $b = b^2x$, then bx^2 is the punctual inverse of b). Now b is either invertible or 0 in every local ring of Z. Therefore bB/b^2B equals to 0 locally at every point of $\operatorname{Spec} B$ and we have $bB = b^2B$.

For any open subset U of $X = \operatorname{Spec} A$, the canonical morphism $\pi : X^{\text{cons}} \to X$ induces a bijection $\operatorname{Mor}(Z, \pi^{-1}(U)) \to \operatorname{Mor}(Z, U)$. So $\pi^{-1}(U) = U^{\text{cons}}$. This then allows us to define X^{cons} in general case by glueing.

By construction, the topological space X^{cons} can be identified with X endowed with the constructible topology, and $\mathcal{O}_{X^{\text{cons}}}$ is the unique sheaf on X^{cons} such that, for all $x \in X$, the local ring $\mathcal{O}_{X^{\text{cons}},x}$ is exactly the residue field $k(x)$ of (the scheme) X at x.

Example 2 If X is reduced, zero-dimensional and noetherian, then $X^{\text{cons}} = X$. This holds in particular when $X = \operatorname{Spec} k$, with k a field.

Remark 3 As X^{cons} is a zero-dimensional scheme, all points $x \in X$ are closed in X^{cons}. On the other hand, if X is noetherian and if $x \in X$ is open in X^{cons}, then x is open in $\overline{\{x\}}$ for the topology induced by that of X, and it is easy to see that $\overline{\{x\}}$ then has only finitely many points ([13], IV.1.9.6). Note also that for any reduced zero-dimensional scheme Z (e.g. X^{cons}), the identity map $Z \to Z$ factorizes into $Z \to Z^{\text{cons}}$ followed by the canonical map $Z^{\text{cons}} \to Z$. The latter is thus an isomorphism.

Lemma 3 *Let X be a noetherian scheme. Then we have the following properties.*

1. $(X_{\text{red}})^{\text{cons}} \simeq X^{\text{cons}}$.
2. $X^{\text{cons}} \simeq (X \backslash Y)^{\text{cons}} \sqcup Y^{\text{cons}}$ *(where $\sqcup$ stands for disjoint union) for any subscheme Y of X.*

Proof (1) This comes from the fact that $\operatorname{Mor}(Z, X_{\text{red}}) = \operatorname{Mor}(Z, X)$ for any scheme Z in $\underline{Sch}_0$.

(2) As Y is constructible (because X is noetherian), hence open in X^{cons}, the canonical morphism

$$(X \backslash Y)^{\text{cons}} \sqcup Y^{\text{cons}} \to X^{\text{cons}}$$

is clearly homeomorphic. It is an isomorphism of schemes because the homomorphism on the local ring at every point is an isomorphism.

Let k be a field. As $(\operatorname{Spec} k)^{\text{cons}} = \operatorname{Spec} k$, the functor $.^{\text{cons}}$ can be restricted to the category of k-schemes. Let X be a k-variety, X^{cons} is a k-scheme which is not locally noetherian when $\dim X \geq 1$. Let X, Y be k-varieties. The canonical map

$$\operatorname{Mor}_k(X, Y) \to \operatorname{Mor}_k(X^{\text{cons}}, Y^{\text{cons}}) \simeq \operatorname{Mor}_k(X^{\text{cons}}, Y)$$

is in general not injective nor surjective.

Lemma 4 *Let X, Y be k-varieties.*

1. *Let Z be a reduced k-scheme of dimension* 0*, let $z_0 \in Z$. Then the canonical map*

$$\varinjlim_U \operatorname{Mor}_k(U, Y^{\text{cons}}) \to \operatorname{Mor}_k(\operatorname{Spec} \mathcal{O}_{Z,z_0}, Y^{\text{cons}}),$$

   where U runs the open neighborhoods of z_0, is bijective.
2. *Let $\pi : X^{\text{cons}} \to Y^{\text{cons}}$ be a morphism of k-schemes. Then there exists a stratification $X = \sqcup_i X_i$ of X into locally closed subsets such that $\pi|_{X_i^{\text{cons}}}$ is induced by a morphism of k-varieties $X_i \to Y$.*

Proof (1) The property is true if we replace Y^{cons} by Y because the latter is of finite type over k. Now we can go back to Y^{cons} because the canonical map $\operatorname{Mor}_k(W, Y^{\text{cons}}) \to \operatorname{Mor}_k(W, Y)$ is bijective for all zero-dimensional reduced k-schemes W.

(2) Let $\xi \in X$ be a generic point. Then the set of U^{cons}, for open neighborhoods U of ξ in X, is cofinal in the set of open neighborhoods of ξ in X^{cons}. Applying (1) to $Z = X^{\text{cons}}$, we see that there exists an open neighborhood U of ξ in X such that $\pi|_{U^{\text{cons}}}$ is induced by a morphism $U \to Y$. We can continue by noetherian induction using Lemma 3.2.

Definition 2 Let X, Y be k-varieties. We say that they are *piecewise isomorphic* if X^{cons} is isomorphic to Y^{cons} as k-schemes.

Notice that if $f : X \to Y$ induces an isomorphism $f^{\text{cons}} : X^{\text{cons}} \to Y^{\text{cons}}$, f is not necessarily an isomorphism. For instance, if f is the normalization of an irreducible unibranched curve Y over algebraically closed field k, then f^{cons} is an isomorphism.

Proposition 2 *Let k be a field. Let X and Y be k-varieties. Then X is piecewise isomorphic to Y if and only if there is a stratification of X into locally closed subsets $(X_i)_{i\in I}$ and a stratification of Y into locally closed subsets $(Y_i)_{i\in I}$, such that $(X_i)_{\text{red}} \simeq (Y_i)_{\text{red}}$ for all $i \in I$.*

Proof The if part comes from Lemma 3.2. Conversely, suppose that we have an k-isomorphism of schemes $\pi : X^{\text{cons}} \to Y^{\text{cons}}$. According to Lemma 4.2, $\pi|_{U^{\text{cons}}}$ is induced by a morphism $f : U \to Y$ for some dense open subset U of X. Similarly we get a morphism $g : V \to X$ inducing $\pi^{-1}|_{V^{\text{cons}}}$. Applying Lemma 4.1 to $g \circ f$ and $f \circ g$ (over suitable open subsets of X and Y), we see that f is an isomorphism $U' \to V'$ between dense open subsets of X and Y. Moreover, $f^{\text{cons}} = \pi|_{U'^{\text{cons}}}$. Therefore, π induces an isomorphism $(X \setminus U')^{\text{cons}} \to (Y \setminus V')^{\text{cons}}$ (Lemma 3.2). As X, Y are noetherian, we can deduce the result by a noetherian induction.

Remark 4 Let X, Y be k-varieties. Then we have a canonical morphism

$$(X \times_k Y)^{\text{cons}} \to X^{\text{cons}} \times_k Y^{\text{cons}}$$

coming from the projections $(X \times_k Y)^{\text{cons}}$ to X^{cons} and Y^{cons}. One should be aware that this morphism is not an isomorphism in general, for the right-hand side is not zero-dimensional when $\dim X, \dim Y \geq 1$.

Consider the free abelian group $\mathbb{Z}[\mathrm{Var}_k^{\mathrm{cons}}]$ generated by the isomorphism classes of k-schemes X^{cons} when X runs the k-varieties. Let $K_0^{\mathrm{cons}}(\mathrm{Var}_k)$ be the quotient of $\mathbb{Z}[\mathrm{Var}_k^{\mathrm{cons}}]$ by the subgroup generated by the relations $[(X \sqcup Y)^{\mathrm{cons}}] = [X^{\mathrm{cons}}] + [Y^{\mathrm{cons}}]$, whenever X and Y are two k-varieties. By Lemma 3.2, we have a canonical surjective homomorphism of groups

$$\theta \; : \; K_0(\mathrm{Var}_k) \to K_0^{\mathrm{cons}}(\mathrm{Var}_k), \quad [X] \mapsto [X^{\mathrm{cons}}].$$

Proposition 3 *The homomorphism θ is an isomorphism of groups.*

Proof By Proposition 2, the map $\mathbb{Z}[\mathrm{Var}_k^{\mathrm{cons}}] \to K_0(\mathrm{Var}_k)$ which send the isomorphism class of X^{cons} to $[X]$ is well defined. Clearly this map induces a map $K_0^{\mathrm{cons}}(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$ which is the inverse of θ.

Notice that this isomorphism defines a ring structure on $K_0^{\mathrm{cons}}(\mathrm{Var}_k)$, and the product is given by

$$[X^{\mathrm{cons}}] \cdot [Y^{\mathrm{cons}}] = [(X \times_k Y)^{\mathrm{cons}}].$$

3 Birational geometry and cancelation problem

The classical cancelation problem is the following. Let k be a field and let X, Y be integral k-varieties of the same dimension such that there are integral k-varieties W, Z and an isomorphism

$$f : X \times_k W \simeq Y \times_k Z.$$

Under which conditions (on X, Y, W and Z) does f induce an isomorphism $X \simeq Y$? Here we consider a slightly different problem. Suppose that W, Z are geometrically integral and that there is a birational map

$$f : X \times_k W \dashrightarrow Y \times_k Z.$$

We will establish below some sufficient conditions on X, Y, W and Z for f to induce a birational map $X \dashrightarrow Y$.

Definition 3 We will say that an integral k-variety X is *uniruled* if there exists an integral algebraic k-variety Y and a dominant, generically finite rational map $Y \times_k \mathbb{P}^1_k \dashrightarrow X$.

It is easy to see that X is uniruled if and only if all irreducible components of $X_{\overline{k}}$ are uniruled.

Theorem 2 *Let k be a field, let X, Y be integral k-varieties of the same dimension. Assume that there are geometrically integral k-varieties W, Z such that one of the following conditions is satisfied*:

1. *X or Y is non-uniruled, and W, Z are rationally chain connected*;
2. $\mathrm{char}(k) = 0$, *X, Y, W, Z are projective, smooth and*

$$\kappa(X) \geq 0 \text{ or } \kappa(Y) \geq 0, \quad \text{and} \quad \kappa(W) = \kappa(Z) = -\infty,$$

and that there exists a birational map

$$f : X \times_k W \dashrightarrow Y \times_k Z.$$

Then there exists a unique birational map $g : X \dashrightarrow Y$ *making the diagram*

$$\begin{tikzcd} X \times_k W \arrow[r, dashed, "f"] \arrow[d, "p_1"] & Y \times_k Z \arrow[d, "q_1"] \\ X \arrow[r, dashed, "g"] & Y \end{tikzcd}$$

commutative (*where the vertical arrows are the projections*).

Proof We use the method of [19, Sect. 3] that we will refine in the birational context. We can suppose that Y is non-uniruled in (1) and $\kappa(Y) \geq 0$ in (2). Let us first notice that the uniqueness of g is obvious because p_1 is faithfully flat.

Suppose that the existence of g is proved for separable closed base field. Let k be an arbitrary field and let k^s be a separable closure of k. Then X_{k^s} and Y_{k^s} are disjoint unions of integral varieties, and it is easy to see that f_{k^s} induces a birational map $g^s : X_{k^s} \dashrightarrow Y_{k^s}$. As g^s is unique, we see by Galois descent that g^s is defined over k.

So suppose that k is separably closed. Let Ω be a dense open subset of $X \times_k W$ such that f is defined on Ω and induces an isomorphism from Ω to its image. Then $p_1(\Omega)$ is a dense open subset of X. We first show that there exists a set-theoretical map $g : p_1(\Omega) \to Y$ such that $g \circ p_1 = q_1 \circ f$ on Ω. Let $x_0 \in p_1(\Omega)$ be a closed point. We have to show that the set $q_1 \circ f(\Omega \cap (\{x_0\} \times_k W))$ is reduced to one point $y_0 \in Y$ (which will necessarily be $g(x_0)$). Let C be an irreducible Weil divisor on X containing x_0. We have a morphism

$$\phi_C \ : \Omega \cap (C \times_k W) \subseteq \Omega \xrightarrow{f} Y \times_k Z \xrightarrow{q_1} Y,$$

which can be viewed as a rational map $C \times_k W \dashrightarrow Y$ because $C \times_k W$ is integral and $\Omega \cap (C \times_k W)$ is dense in $C \times_k W$. We claim that ϕ_C is not dominant. Indeed under Condition (1), as $\dim C < \dim Y$, ϕ_C is not dominant by Lemma 5. Under the second condition, if ϕ_C were dominant, then Y would have negative Kodaira dimension (Lemma 7). Let E be the Zariski closure of the image of ϕ_C. It is an irreducible subvariety of Y, of codimension ≥ 1. By construction, $f(\Omega \cap (C \times_k W)) \subseteq E \times_k Z$. As $\dim W = \dim Z$, and f is an isomorphism on Ω, we get $\dim C = \dim E$ and

$$f(\Omega \cap (C \times_k W)) = f(\Omega) \cap (E \times_k Z).$$

Let $C_1, \ldots, C_n$ be irreducible Weil divisors on X containing x_0 such that, for some open neighborhood U_0 of x_0 in $p_1(\Omega)$,

$$U_0 \cap (C_1 \cap C_2 \cap \ldots C_n) = \{x_0\}.$$

For every $1 \leq i \leq n$, the above construction provides an irreducible Weil divisor E_i on Y, such that $f(\Omega \cap (C_i \times_k W)) = f(\Omega) \cap (E_i \times_k Z)$. Taking the intersection for all $i \leq n$, we get

$$f(\Omega \cap (\{x_0\} \times_k W)) = f(\Omega \cap p_1^{-1}(U_0)) \cap ((\cap_{1 \leq i \leq n} E_i) \times_k Z).$$

We can decompose $\cap_{1 \leq i \leq n} E_i$ as union of its irreducible components. As the left-hand side is irreducible, we see that the right-hand side must be $f(\Omega \cap p_1^{-1}(U_0)) \cap (Y_0 \times_k Z)$ for some irreducible component Y_0 with $\dim Y_0 = \dim W - \dim Z = 0$. So Y_0 consists just in a closed point y_0, and we have $q_1 \circ f(\Omega \cap (\{x_0\} \times_k W)) = \{y_0\}$. This implies the existence of the map g as desired, sending x_0 to y_0.

It remains to show that g is a birational morphism. Let $w_0 \in W(k)$ be such that $(x_0, w_0) \in \Omega$ (here we use the assumption k separably closed and W geometrically reduced). Then $x_0 \in V_0 := p_1(\Omega \cap (X \times_k \{w_0\}))$ and the rational point w_0 defines a section $s_0 : X \to X \times_k Y$ which induces a morphism $q_1 \circ f \circ s_0 : V_0 \to Y$. Set-theoretically, $g|_{V_0} = q_1 \circ f \circ s_0|_{V_0}$ and g is a morphism on V_0. This is enough to defined the rational map $g : X \dashrightarrow Y$. However, since X is reduced and Y is separated, the morphisms on various V_0 glue together, and g is defined on $p_1(\Omega)$. Since g is clearly dominant, X is non-uniruled in case (1) and has $\kappa(X) \geq 0$ in case (2). By the uniqueness of g (as rational map) and by the symmetry of the statement, we conclude that g is birational.

Remark 5 Under the hypothesis of the theorem, W is birational to Z over a finite extension of k. Actually, let $x_0 \in X$ be a general closed point, let $k_0 = k(x_0)$. Then

$$h(x_0, .) : W_{k_0} \dashrightarrow Z_{k_0}$$

is birational.

Lemma 5 *Let X be an integral k-variety. Suppose that there exists a rational dominant map*

$$f : S \times_k W \dashrightarrow X$$

with integral k-variety S of dimension $\dim S < \dim X$, *and geometrically integral k-variety W which is rationally chain connected. Then X is uniruled.*

Proof Extending the base field to $\overline{k}$, the condition is satisfied for all irreducible components of $X_{\overline{k}}$. Therefore we are reduced to the case when k is algebraically closed. Moreover, now we can suppose that k is uncountable [6, Sect. 4.1, Remark 4.2 (5)] and W is proper. Let $(s, w) \in S \times_k W$ be a general closed point. Then f induces a rational map $f_s : \{s\} \times_k W \dashrightarrow X$. If f_s were constant, then we would have a dominant rational map $S \dashrightarrow X$, which is impossible because $\dim S < \dim X$. Let $(s, w') \in \{y\} \times_k W$ be a point with image different from $f(s, w)$, let L be a connected curve in W with rational irreducible components and containing w and w' ([22], Corollary IV.3.5.1). Then $f(s, L)$ is a connected curve (because $f(s, L)$ is not a point) passing through $f(s, w)$ with rational irreducible components. In particular, $f(s, w)$ belongs to a rational curve in X. As f is dominant, this implies that X is uniruled (op. cit. Sect. 4.1, Remark 4.2 (4)).

Lemma 6 *Let k be a field of characteristic zero. Let X, Y be connected smooth projective k-varieties of dimension n and r respectively. Let $m \in \mathbb{N}$. Then*

$$P_{(0,\ldots,0,m,0\ldots,0)}(X \times_k Y) = P_{(0,\ldots,0,m)}(X) + P_{(1,0,\ldots,0)}(Y) \cdot P_{(0,\ldots,0,1,m-1)}(X) + \cdots,$$

where m is the nth entry of $(0, \ldots, 0, m, 0, \ldots, 0) \in \mathbb{N}^{n+r}$.

Proof We have

$$\Omega^s_{X \times_k Y/k} = \oplus_{i=0}^s p^* \Omega^i_{X/k} \otimes q^* \Omega^{s-i}_{Y/k},$$

where $p, q : X \times_k Y \to X, Y$ are the canonical projections. Then

$$\left(\Omega^s_{X \times_k Y/k}\right)^{\otimes m} = \left(p^* \mathcal{O}_X \otimes q^* \left(\Omega^s_{Y/k}\right)^{\otimes m}\right) \oplus \left(p^* \Omega^1_{X/k} \otimes q^* \left(\Omega^s_{Y/k}\right)^{\otimes (m-1)} \otimes q^* \Omega^{s-1}_{Y/k}\right) \oplus \cdots.$$

Using the projection formula, we can prove that $H^0(X \times_k Y, p^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_k Y}} q^*\mathcal{G}) \simeq H^0(X, \mathcal{F}) \otimes_k H^0(Y, \mathcal{G})$ for every locally free $\mathcal{O}_X$-module $\mathcal{F}$, and every locally free $\mathcal{O}_Y$-module $\mathcal{G}$. We deduce the result.

Lemma 7 *Let k be a field of characteristic zero. Let X, S, W be projective smooth k-varieties with X, S integral and W geometrically integral. Suppose that there exists a rational dominant map*

$$f : S \times_k W \dashrightarrow X$$

and $\dim S < \dim X$, $\kappa(W) = -\infty$. *Then $\kappa(X) = -\infty$. In particular, uniruled smooth projective k-varieties have Kodaira dimension $\kappa = -\infty$.*

Proof Indeed, for any $m \geq 1$, as f is a rational map between smooth projective k-varieties, the formulas of plurigenera (see Lemma 6, and [20, §5.4, Proposition 5.5 and Theorem 5.3]) imply that

$$0 = P_{(0,\ldots,m,0\ldots,0)}(S \times_k W) \geq P_m(X).$$

Remark 6 In Theorem 2, if Condition (1) is satisfied and if $\mathrm{char}(k) = 0$, the result can be proved using maximal rationally connected (MRC) fibrations. As the question is birational, we can suppose that all varieties are projective and smooth. Let

$$\pi : X \dashrightarrow R(X), \quad \theta : Y \dashrightarrow R(Y)$$

be respectively the maximal rationally connected (MRC) fibrations of X and Y ([22], IV.5). Then the MRC fibrations of $X \times_k W$ and $Y \times_k Z$ are respectively $X \times_k W \to X \dashrightarrow R(X)$ and $Y \times_k Z \to Y \dashrightarrow R(Y)$. As the MRC fibration is functorial with respect to dominant rational maps ([22], IV.5.5), f induces a birational map $g' : R(X) \dashrightarrow R(Y)$. Now as X, Y are not uniruled, $\pi : X \dashrightarrow R(X)$ and $\theta : Y \dashrightarrow R(Y)$ are birational. We can take $g = \theta^{-1} \circ g' \circ \pi$.

Corollary 1 *Let X, Y be integral k-varieties of same dimension. Suppose that they are stably birational (i.e. $X \times_k \mathbb{P}^n_k$ is birational to $Y \times_k \mathbb{P}^n_k$ for some integer $n \geq 1$). Then the following properties are true.*

1. *X is uniruled if and only if Y is uniruled. Moreover, if X is non-uniruled, then X is birational to Y.*
2. *If* $\mathrm{char}(k) = 0$ *and X, Y are projective smooth. Then $\kappa(X) = \kappa(Y)$.*

Proof We have a birational map $X \times_k \mathbb{P}^n_k \dashrightarrow Y \times_k \mathbb{P}^n_k$. If X is uniruled, then X is rationally dominated by some product $S \times_k \mathbb{P}^1_k$ with $\dim S = \dim X - 1$. Then Y is rationally dominated by $S \times_k (\mathbb{P}^1_k \times_k \mathbb{P}^n_k)$. As $W := \mathbb{P}^1_k \times_k \mathbb{P}^n_k$ is rationally chain connected, Y is uniruled by Lemma 5. If X is non-uniruled, then X is birational to Y by Theorem 2. This proves (1).

For (2), one can suppose for instance that $\kappa(Y) \geq 0$. Applying Lemma 6 or using Theorem 2 we get $\kappa(Y) = \kappa(X)$.

Remark 7 After this paper is submitted, we found [18], Lemma 13.5 and the final Remark that state similar results with slightly different methods.

The following lemma is well known.

Lemma 8 *Let k be a perfect field, and let X be a projective regular connected k-variety.*

1. *If* $\dim X = 1$, *then X is uniruled if and only if X isomorphic to $\mathbb{P}^1_{k'}$ or a conic over k', where $k' = H^0(X, \mathcal{O}_X)$.*

2. *If* $\dim X = 2$*, and* k *is algebraically closed and of characteristic zero, then* X *is uniruled if and only if* X *is ruled, i.e. birational to* $C \times_k \mathbb{P}^1_k$*, for some smooth connected projective* k*-curve* C*.*

Proof (1) Suppose X uniruled. Then there exists a dominant morphism $\mathbb{P}^1_{k''} \to X$ for some finite extension k''/k. Thus $H^1(X, \mathcal{O}_X) = 0$. By Lüroth's Theorem (see for instance [26], Corollary 7.4.23), X is isomorphic to $\mathbb{P}^1_{k'}$ or a conic over k'. (2) We can suppose that X is projective and smooth. If X is uniruled, then $\kappa(X) = -\infty$ by Lemma 7. Now apply the classification of surfaces ([15], Theorem V.6.1).

4 Additive invariants and the equality of classes in $K_0(\mathrm{Var}_k)$

4.1 Additive invariants

For details, see, for example, [7].

Definition 4 Let k be a field, and let A be a commutative ring. An *A-additive invariant*

$$\lambda \ : \mathrm{Var}_k \to A$$

assigns to $X \in \mathrm{Var}_k$ an element $\lambda(X) \in A$ such that

1. $\lambda(X) = \lambda(Y)$, if X is k-isomorphic to Y,
2. $\lambda(X) = \lambda(Y) + \lambda(X \backslash Y)$, if Y is a closed subvariety of X,
3. $\lambda(X \times_k Y) = \lambda(X)\lambda(Y)$, for every pair of k-varieties X, Y.

It is clear that such an additive invariant defines, in a unique way, a ring homomorphism

$$\lambda \ : K_0(\mathrm{Var}_k) \to A.$$

There are many examples of such additive invariants. For instance, recall that :

Euler characteristics are $\mathbb{Z}$-additive invariants. Let k be a field. The assignment of (*topological*) *Euler characteristics*

$$X \mapsto \chi_{\mathrm{top}}(X) := \sum_{i=0}^{2\dim X} (-1)^i b_i(X)$$

defines a $\mathbb{Z}$-additive invariant.

Hodge polynomials are $\mathbb{Z}[u, v]$-additive invariants. Let k be a field of characteristic zero. By the theory of Deligne's mixed Hodge theory, the assignment of *Hodge polynomials* $X \mapsto H_X(u, v)$ defines a unique $\mathbb{Z}[u, v]$-additive invariant (see [35], Theorem 1.1). Recall that, if X is a projective and smooth k-variety, its Hodge polynomial is defined as

$$H_X(u, v) := \sum_{p,q \geq 0} h^{p,q}(X) u^p v^q.$$

Poincaré polynomials are $\mathbb{Z}[u]$-additive invariants. Let k be a field of characteristic zero. We call *Poincaré polynomial* of X the polynomial $P_X \in \mathbb{Z}[u]$, defined by $P_X(u) := H_X(u, u)$. This assignment defines a $\mathbb{Z}[u]$-additive invariant and $P_X(u) = \sum_{n=0}^{2\dim X} \dim_k H^n_{\mathrm{dR}}(X) u^n$

if X is smooth and projective. In this case, the topological Euler characteristic of X is also given by the integer $\chi_{\text{top}}(X) := P_X(-1)$.

Counting points. Let k be a finite field. Then the map N_k, which associates to a k-variety X its number of rational points, gives rise to a $\mathbb{Z}$-additive invariant.

4.2 The work of Larsen–Lunts and of Bittner

In [25], Larsen and Lunts have obtained the following very strong result (we give here a slightly different presentation). They work over $\mathbb{C}$. The statement for any field of characteristic 0 is an immediate consequence of Bittner [3], Theorem 3.1. However, as Kollár already observed in [21], top of page 28, $\mathbb{Z}[\mathrm{SB}]$ is non longer a monoid ring because the product of two irreducible varieties need not be irreducible. So the statement must be adjusted.

Theorem 3 (Larsen and Lunts [25], Bittner [3]) *Let k be a field of characteristic zero. Let M be the monoid of isomorphism classes of smooth (non-necessarily connected) projective k-varieties, and let $\psi : M \to A$ be a multiplicative map to a commutative ring such that:*

1. $\psi(\{X \coprod Y\}) = \psi(\{X\}) + \psi(\{Y\})$, *where $\{Z\}$ denotes the isomorphism class of Z;*
2. *if X and Y are birationally equivalent smooth connected projective varieties over k, then* $\psi(\{X\}) = \psi(\{Y\})$;
3. $\psi(\{\mathbb{P}^1_k\}) = 1$.

Then there exists a unique ring homomorphism

$$\Psi : K_0(\mathrm{Var}_k) \to A$$

such that $\Psi([X]) = \psi(\{X\})$ whenever X is a smooth projective k-variety.

Now let us show some known corollaries.

- Let $\mathbb{Z}[\mathrm{SB}]$ be the free abelian group generated by the equivalence classes of smooth connected projective k-varieties under stably birational equivalence. If X is a smooth connected projective k-variety, we denote by $\mathrm{SB}(X)$ the stably birational class of X. Then $\mathbb{Z}[\mathrm{SB}]$ is a commutative ring with the multiplication $\mathrm{SB}(X) \cdot \mathrm{SB}(Y) = \sum_{1 \leq i \leq d} \mathrm{SB}(Z_i)$ if $Z_1, \ldots, Z_d$ are the connected components of $X \times_k Y$.
- Suppose further that k is algebraically closed. Let AV be the monoid of isomorphism classes of abelian varieties over k. Let Alb be the functor from the category of smooth connected projective k-varieties to the category of abelian varieties, which associates to a smooth connected projective k-variety X its Albanese $\mathrm{Alb}(X)$.

Corollary 2 *Let k be a field of characteristic zero.*

1. *The assignment $X \mapsto \mathrm{SB}(X)$ for smooth connected projective varieties induces a $\mathbb{Z}[\mathrm{SB}]$-additive invariant.*
2. *If k is algebraically closed, the assignment $X \mapsto \mathrm{Alb}(X)$ for smooth connected projective varieties defines a $\mathbb{Z}[\mathrm{AV}]$-additive invariant.*

Proof See [33, Sect. 4] for (2). This property is a key point in the proof of [33], Theorem 1.

Corollary 3 *Let k be a field of characteristic zero. Let $A_1, \ldots, A_n$ be a family of abelian varieties over k such that A_i, A_j have no common isogeny factor for all $i \neq j$. Then the classes $[A_1], \ldots, [A_n]$ in $K_0(\mathrm{Var}_k)$ are algebraically independent over $\mathbb{Z}$.*

Proof Otherwise, there exist $(r_1, \dots, r_n) \neq (s_1, \dots, s_n)$ with $r_i, s_i \geq 0$ such that $\mathrm{SB}(A_1^{r_1} \times_k \cdots \times_k A_n^{r_n}) = \mathrm{SB}(A_1^{s_1} \times_k \cdots \times_k A_n^{s_n})$, thus $A_1^{r_1} \times_k \cdots \times_k A_n^{r_n}$ is birational (Theorem 2) hence isomorphic to $A_1^{s_1} \times_k \cdots \times_k A_n^{s_n}$ (if k is algebraically closed, we can just apply Alb(.)). Therefore at least two A_i's have a common isogeny factor. Contradiction.

Remark 8 This statement is proved in [29], Theorem 13 for elliptic curves over number fields. See also [23], Corollary 5.8. Over finite fields [23], Theorem 5.7 shows similar results for elliptic curves; [29], Theorem 12 exhibits curves whose classes in $K_0(\mathrm{Var}_k)$ are algebraically independent over $\mathbb{Z}$.

Corollary 4 *Let k be a field of characteristic zero. Then the ring* $K_0(\mathrm{Var}_k)$ *is not noetherian.*

Proof Let $(C_n)_{n\geq 1}$ be a sequence of projective smooth and geometrically connected curves over k such that the genus $g(C_n) \geq n$. If the ideal generated by the $[C_n]$'s is generated by $\{[C_i]\}_{i\leq m}$, then the same is true in $\mathbb{Z}[\mathrm{SB}]$, and for any $n \gg m$, $\mathrm{SB}(C_n) = \mathrm{SB}(C_i \times X_n)$ for some $i \leq m$ and for some projective smooth connected variety X_n. Using Theorem 2.(2), we see that $\kappa(X_n) < 0$ and C_n is birational, hence isomorphic, to C_i. Contradiction.

4.3 First consequences of the equality of classes in $K_0(\mathrm{Var}_k)$

Corollary 5 *Let k be a field of characteristic zero. Let X and Y be k-varieties such that* $[X] = [Y]$.

1. *Suppose X and Y are smooth, connected and projective. Then they are stably birational; they have the same Hodge polynomial, the same Hodge numbers and the same Betti numbers. If* $\kappa(X) \geq 0$*, then X is birational to Y. Over* $\bar{k}$*, they have isomorphic Albanese varieties and isomorphic fundamental groups.*
2. *We have* $\dim X = \dim Y$;
3. *The varieties* $X_{\bar{k}}$ *and* $Y_{\bar{k}}$ *have the same number of irreducible components of maximal dimension*;

Proof (1) If $\kappa(X) \geq 0$, the assertion is a direct application of Theorem 2/(2). The middle part comes directly from the additivity of the Hodge polynomials and Corollary 2. See [14], Exposé X/Corollaire 3.4, and Exposé XI/Proposition 1.1 for assertion on the fundamental groups.

For any smooth connected projective k-variety X, the Poincaré polynomial H_X has degree $2\dim X$, and the leading term is the number of geometric irreducible components of dimension $\dim X$ of X. This implies (2) and (3) by using Lemma 1.

Remark 9 In [2], Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer have exhibited projective smooth varieties stably rational, but not rational (in dimension two over non algebraically closed field and in dimension three over $\mathbb{C}$). These examples are counterexamples to the generalization of Theorem 2/(2) to the case of varieties with negative Kodaira dimension, but, *a priori*, they do not constitute counterexamples to Larsen and Lunts' question, in so far as it is not clear that the class of such examples in the Grothendieck ring of varieties is equal (or not) to the class of a rational variety.

Recall that the *index* $\delta(X/k)$ of an algebraic variety X over k is the gcd of the degrees over k of the closed points of X. The integer $\nu(X/k)$ is the minimum of the degrees over k of the closed points of X. We denote by $\mathrm{CH}_0(X)$ the Chow group of 0-cycles on X modulo rational equivalence relation.

Corollary 6 *Let X, Y be smooth projective varieties over a field k of characteristic* 0*. If* $[X] = [Y]$*, then*

$$\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y), \quad \delta(X/k) = \delta(Y/k), \quad \nu(X/k) = \nu(Y/k).$$

In particular, X has a rational point (*or, equivalently,* $\nu(X/k) = 1$) *if and only if Y has a rational point.*

Proof First notice that CH_0, δ and ν are inchanged if we replace X with $X \times \mathbb{P}^n_k$. Thus we can suppose that X and Y are birational. A birational map $f : X \dashrightarrow Y$ induces an isomorphism $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y)$ (cf. [4], Proposition 6.3 or [8], Example 16.1.11 where the hypothesis k algebraically closed is not necessary) compatible with the degree maps (hence $\delta(X/k) = \delta(Y/k)$). Let $x_0 \in X$ be a closed point, let $\widetilde{X} \to X$ be the blowing-up of X along $\{x_0\}$ with exceptional divisor $E \simeq \mathbb{P}^d_{k(x_0)}$. Then f extends to a rational map $f' : \widetilde{X} \dashrightarrow Y$, which is defined in a neighborhood of the generic point of E by the valuative criterion of properness. As k is infinite, the neighborhood contains a point rational over $k(x_0)$, and its image in Y is a point of degree $\leq \deg x_0$. Hence $\nu(X/k) \geq \nu(Y/k)$, and the equality holds by symmetry.

Remark 10 Let X and Y be smooth and projective varieties over an algebraically closed field k of characteristic zero. As remarked by Göttsche in [10, Conjecture 2.5], under the validity of a conjecture of Beilinson and Murre, if the class of the effective motives of X and Y are equal in the Grothendieck ring of effective Chow motives $K_0(M_k)$, then they have the same Chow groups (with rational coefficients). Assume that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Thanks to the existence of a ring morphism $K_0(\mathrm{Var}_k) \to K_0(M_k)$ which send the class of a smooth projective variety to its motive (see [10, Theorem 2.1] for example), we conclude that, under the validity of this conjecture of Beilinson and Murre, the Chow groups $CH_*(X)$ of X are isomorphic to the Chow groups $CH_*(Y)$ of Y.

Over a field k of characteristic $p > 0$, we are much less armed to work with $K_0(\mathrm{Var}_k)$. However, part of Corollary 5 still hold. The idea is, as Antoine Chambert-Loir suggested, to use the method of reduction *à la* Miyaoka-Mori [28] to reduce to finite base fields. The following proposition is also obtained by Johannes Nicaise using Poincaré polynomials over arbitrary fields ([30, Sect. 8 Appendix, Proposition 8.7]).

Proposition 4 *Let k be a field of characteristic $p > 0$. Let X, Y be k-varieties such that* $[X] = [Y]$ *in* $K_0(\mathrm{Var}_k)$*. Then* $\dim X = \dim Y$*, and $X_{\bar{k}}$ and $Y_{\bar{k}}$ have the same number of irreducible components of maximal dimension.*

Proof First remark that, by [13], Proposition IV.8.9.1, there exist a sub-$\mathbb{F}_p$-algebra A of k of finite type, and reduced A-schemes $\mathcal{X}, \mathcal{Y}$ separated and of finite type, such that $\mathcal{X} \otimes_A k \simeq X$, $\mathcal{Y} \otimes_A k \simeq Y$. Moreover, $[\mathcal{X}] = [\mathcal{Y}]$ in $K_0(\mathrm{Var}_A)$ if A is big enough. This implies that $[\mathcal{X}_s] = [\mathcal{Y}_s]$ in $K_0(\mathrm{Var}_{k(s)})$ for all $s \in \operatorname{Spec} A$. In particular, if s is a closed point, then $\mathcal{X}_s$ and $\mathcal{Y}_s$ have the same number of rational points in any finite extension F of $k(s)$.

Let η be the generic point of $\operatorname{Spec} A$. Shrinking $\operatorname{Spec} A$ if necessary, $\mathcal{X}_s$ has the same dimension and the same number of geometric irreducible components of maximal dimension than $\mathcal{X}_\eta$ for all $s \in \operatorname{Spec} A$ ([13], combine Corollaire IV.9.5.6 and Proposition IV.9.7.8). Denote this number by m. Let $s \in \operatorname{Spec} A$ is a closed point. According to Lang-Weil estimate ([24], Corollary 2), for finite extensions F of $k(s)$,

$$\mathrm{Card}(\mathcal{X}_s(F))\mathrm{Card}(F)^{-\dim \mathcal{X}_s} - m \to 0$$

as $[F : k(s)]$ tends to infinity. As $\mathrm{Card}(\mathcal{X}_s(F)) = \mathrm{Card}(\mathcal{Y}_s(F))$, we conclude that $\mathcal{X}_\eta$ and $\mathcal{Y}_\eta$ have the same dimension and the same number of geometric irreducible components of maximal dimension. The same is true for $X \simeq (\mathcal{X}_\eta)_k$ and $Y \simeq (\mathcal{Y}_\eta)_k$.

Remark 11 As $\dim X$ depends only on the class $[X] \in K_0(\mathrm{Var}_k)$, we have a filtration (actually a graduation) on $K_0(\mathrm{Var}_k)$ by dimension. This fact is stated in [5], letter of 16 Aug. 1964.

We have the following funny consequences:

Corollary 7 *Let k be a field of characteristic zero and let X be a smooth projective connected k-variety. Assume that there exists $a \in K_0(\mathrm{Var}_k)$ and a smooth projective connected k-variety Y, with $\dim Y < \dim X$, such that $[X] = [Y] + a\mathbb{L}$. Then $\kappa(X) = -\infty$.*

Proof Assume that $\kappa(X) \geq 0$. Then $\mathrm{SB}(X) = \mathrm{SB}(Y \times_k \mathbb{P}_k^{\dim X - \dim Y})$ and, by Theorem 2, X is birational to $Y \times_k \mathbb{P}_k^{\dim X - \dim Y}$. This is a contradiction.

Corollary 8 *Let $f : X \to Y$ be a proper birational morphism of normal irreducible schemes of finite type over a field k. If $[X] = [Y]$ (e.g. $X = Y$), then f is an isomorphism.*

Proof There exists a closed subset F of Y such that $f^{-1}(Y \backslash F) \to Y \backslash F$ is an isomorphism and $f^{-1}(F) \to F$ has connected fibers of positive dimensions. In particular $\dim f^{-1}(F) > \dim F$. As $[X] = [Y]$, we have

$$[X \backslash f^{-1}(F)] + [f^{-1}(F)] = [Y \backslash F] + [F].$$

Hence $[f^{-1}(F)] = [F]$ and $\dim f^{-1}(F) = \dim F$ by Proposition 4 and Corollary 5, therefore $F = \emptyset$ and f is an isomorphism.

5 Applications

We prove Assertion 1 in some cases. The base field k has characteristic zero except in Proposition 5.

5.1 Zero-dimensional varieties

Proposition 5 *Let X, Y be zero-dimensional k-varieties such that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. If k has characteristic zero, or is a finite field, or is algebraically closed, then X is isomorphic to Y.*

Proof (1) Suppose that $\mathrm{char}(k) = 0$. Let x be point of X. By Corollary 2, $\mathrm{Spec}\, k(x)$ is stably birational to $\mathrm{Spec}\, k(y)$ for some point $y \in Y$. This implies that $k(x) \simeq k(y)$. So $X \simeq Y$ by induction on the cardinality of X.

(2) When k is finite, our assertion is an immediate consequence of [29], Theorem 25 (see especially the first line of the proof).

(3) When k is algebraically closed, the isomorphism class of X is determined by its cardinality $\mathrm{Card}(X)$. As $\mathrm{Card}(X)$ can be computed using Euler characteristic of X, the assertion follows.

5.2 The case of curves

Proposition 6 *Let k be an algebraically closed field of characteristic zero. Let X and Y be k-varieties such that* $\dim X \leq 1$ *and* $[X] = [Y]$ *in* $K_0(\mathrm{Var}_k)$. *Then X is piecewise isomorphic to Y.*

Proof We can suppose that X (and hence Y) have dimension 1. Lemma 1 implies

$$[X] = \sum_{i=0}^{n}[X_i] + a \quad \text{and} \quad [Y] = \sum_{j=0}^{m}[Y_j] + b,$$

where X_i, Y_j are smooth connected projective k-curves, birational to the irreducible components of dimension 1 of X and Y respectively, and $a, b \in \mathbb{Z}$. By Corollary 5.3, $n = m$. By Corollary 2, $\mathrm{SB}(X_1)$ is equal to a $\mathrm{SB}(Y_j)$ or $\mathrm{SB}(\mathrm{Spec}\, k)$. If X_1 is not rational, $\mathrm{SB}(X_1) \neq \mathrm{SB}(\mathrm{Spec}\, k)$, so $\mathrm{SB}(X_1) = \mathrm{SB}(Y_j)$ for some $j \leq n$, and then $X_1 \simeq Y_j$ by Corollary 1.1. After removing all non-rational X_i's, we can suppose that the X_i's (hence all Y_j's by symmetry) are rational. Therefore there exist open subsets $U \subseteq X$, $V \subseteq Y$ such that $U \simeq V$ and $X \backslash U$, $Y \backslash V$ are zero-dimensional. By Proposition 5, $X \backslash U \simeq Y \backslash V$, and X is piecewise isomorphic to Y.

Remark 12 If k is not necessarily algebraically closed, and if X, Y are k-varieties of dimension 1 with $[X] = [Y]$, then one can show that there exist open subsets U, V of X, Y respectively, such that $U \simeq V$, $X \backslash U$ is the disjoint union

$$\mathbb{P}^1_{k_1} \sqcup \cdots \sqcup \mathbb{P}^1_{k_n} \sqcup \mathrm{Spec}\, k'_1 \sqcup \cdots \sqcup \mathrm{Spec}\, k'_m,$$

and $Y \backslash V$ is the disjoint union

$$\mathbb{P}^1_{k'_1} \sqcup \cdots \sqcup \mathbb{P}^1_{k'_m} \sqcup \mathrm{Spec}\, k_1 \sqcup \cdots \sqcup \mathrm{Spec}\, k_n,$$

where k_i, k'_j are finite extensions of k. Furthermore, we have $\sum_i [k_i : k] = \sum_j [k'_j : k]$. If Assertion 1 is true here, then we must have $n = m$ and, up to re-numbering, $k_i \simeq k'_i$. The proof is similar to the algebraically closed case, and we use [16], Theorem 1.2 to deal with the conics.

Remark 13 Let k be a field of characteristic zero. Let X be a k-variety. Let $f : X \dashrightarrow X$ be a birational map. Is it true that f can be extended in a piecewise isomorphism? Of course, if Larsen and Lunts' question admits a positive answer, this question admits a positive answer. Besides, if k is algebraically closed and if the dimension of X is less or equal to 2, then, by Proposition 6, this question admits a positive answer.

Remark 14 In [5], letter of 16 Aug. 1964, Grothendieck sketched the construction of a homomorphism from $K_0(\mathrm{Var}_k)$ to the Grothendieck ring of Chow motives with rational coefficients $K_0(\mathrm{M}_k)$ (see [9], Theorem 4 when k has characteristic zero, or [3], Corollary 4.3), and he asked whether it is far from being bijective. For k of haracteristic zero, it is known that this homomorphism is not injective (take two isogeneous but not isomorphic abelian k-varieties A, B, then they have the same image in $K_0(\mathrm{M}_k)$, but $[A] \neq [B]$ by Corollary 2.2). Using the above remark, we can construct similar examples with curves. Let X, Y be two non-isomorphic, projective smooth and geometrically connected curves over k, with isomorphic jacobian varieties (see for instance [17]). Then they have the same class in $K_0(\mathrm{M}_k)$. But $[X] \neq [Y]$ in $K_0(\mathrm{Var}_k)$.

Proposition 7 *Let k be an algebraically closed field of characteristic zero. Let X be a k-variety of dimension at most one. Then X is rational* (*i.e. the irreducible components of X are rational*) *if and only if there exist $\alpha, \beta \in \mathbb{Z}$ such that, in $K_0(\mathrm{Var}_k)$*

$$[X] = \alpha\mathbb{L} + \beta.$$

In particular, if X is smooth and projective, then α is the number of connected components of dimension 1.

Proof Applying SB and using Lemma 1, it is clear, by Corollary 1, that, if $[X] = \alpha\mathbb{L} + \beta$, then X is rational. Conversely, as $[\mathbb{P}^1_k] = \mathbb{L} + 1$ in $K_0(\mathrm{Var}_k)$, applying Lemma 1, we obtain the result.

5.3 The case of surfaces

Lemma 9 *Let k be an algebraically closed field of characteristic zero. Let X, Y be smooth connected projective k-surfaces. If X is stably birational to Y, then it is birational to Y.*

Proof If X is non-uniruled, then X is birational to Y by Corollary 1.1. Suppose now that X, hence Y, are uniruled. Then they are birational respectively to $C \times_k \mathbb{P}^1_k$ and $D \times_k \mathbb{P}^1_k$ for some connected smooth projective k-curves C, D (Lemma 8). We have $\mathrm{SB}(C) = \mathrm{SB}(D)$. If C is non-uniruled, then C is birational to D, hence X is birational to Y. Otherwise $C = \mathbb{P}^1_k$ and D is uniruled, hence isomorphic to $\mathbb{P}^1_k$. So in this case X, Y are both rational.

Theorem 4 *Let k be an algebraically closed field of characteristic zero. Let X and Y be smooth projective k-varieties of dimension two. Suppose that their one-dimensional connected components are rational curves. If $[X]=[Y]$, then X is piecewise isomorphic to Y.*

Proof It is sufficient to prove that the irreducible components of dimension 2 of X and Y are pairwise birational because we then are reduced to the case of curves, which is done in Proposition 6. We have

$$[X] = \sum_{1\le i\le n} [X_i] + \sum_{1\le j\le m} [C_j], \quad [Y] = \sum_{1\le i\le n} [Y_i] + \sum_{1\le j\le m'} [D_j],$$

where the X_i's (resp. Y_i's) are the irreducible components of X (resp. Y) of dimension 2 (Corollary 5.3), and the C_j, D_j's are projective lines or $\mathrm{Spec}\, k$. It comes that in the free abelian group $\mathbb{Z}[\mathrm{SB}]$, we have

$$\sum_{1\le i\le n} \mathrm{SB}(X_i) + a\mathrm{SB}(\mathrm{Spec}\, k) = \sum_{1\le i\le n} \mathrm{SB}(Y_i) + b\mathrm{SB}(\mathrm{Spec}\, k)$$

for some integers $a, b \in \mathbb{Z}$. So $a = b$ and $\sum_{i\in I} \mathrm{SB}(X_i) = \sum_{i\in I} \mathrm{SB}(Y_i)$. Up to renumbering, we have $\mathrm{SB}(X_i) = \mathrm{SB}(Y_i)$, hence X_i is birational to Y_i (Lemma 9) for all i.

Remark 15 (Uniruled normal surfaces) Let k be a field of characteristic zero. Let C be a projective connected smooth k-curve of genus $g(C) > 0$. Let us construct a singular normal k-surface X_C as follows. First blow-up $C \times_k \mathbb{P}^1_k$ along a rational point of $C_0 = C \times \{0\} \subset C \times_k \mathbb{P}^1_k$; then contract in the new surface the strict transform of C_0. Denote by X_C the normal singular projective k-surface obtained in this way. It is uniruled and we have

$$[X_C] = [C \times_k \mathbb{P}^1_k] - [C].$$

Now pick another projective smooth connected k-curve D and consider the surface X_D. Suppose that $[X_C] = [X_D]$. Then $g(C) = g(D)$ by using Poincaré polynomials. On the other hand, Larsen–Lunts's theorem is helpless here because X_C and X_D have the same (trivial) image in $\mathbb{Z}[SB]$.

Remark 16 Let k be as in Theorem 4. Suppose that $\mathbb{L}$ is not zero divisor in $K_0(\mathrm{Var}_k)$, then Assertion 1 is true for varieties of dimension 2. Indeed, we can write

$$[X] = \sum_{1\le i\le n} [X_i] + \sum_{1\le j\le m} \pm[C_j]$$

as in Lemma 1, with moreover X_i non-uniruled or isomorphic to $E_i \times_k \mathbb{P}^1_k$ for some projective smooth connected k-curve E_i. Similarly write

$$[Y] = \sum_{1\le i\le n} [Y_i] + \sum_{1\le j\le m'} \pm[C'_j]$$

with Y_i non-uniruled or isomorphic to $F_i \times_k \mathbb{P}^1_k$. After removing components in X (resp. Y) which are birational to an irreducible component of Y (resp. X) and after adding the necessary counterpart in the C_j, C'_j's, we can suppose that X_i is not birational to $Y_{i'}$ for all $i, i' \le n$. Then

$$0 = \sum_i ([E_i \times_k \mathbb{P}^1_k] - [F_i \times_k \mathbb{P}^1_k]) + \sum_j \pm[D_j] + a\mathbb{L}$$

with $a \in \mathbb{Z}$, D_j smooth, connected and projective of dimension 1 (replace $\mathrm{Spec}\, k$ by $[\mathbb{P}^1_k]-\mathbb{L}$). Applying SB, we see that every E_i and F_i is isomorphic to some D_j. As $\mathrm{SB}(D_j) = \mathrm{SB}(D_{j'})$ implies $[D_j] = [D_{j'}]$, we finally get

$$0 = \left(a + \sum_i ([E_i] - [F_i])\right)\mathbb{L}.$$

So if $\mathbb{L}$ does not divide zero, then $a + \sum_i ([E_i] - [F_i]) = 0$. Proposition 6 implies that $\sqcup_i E_i$ is birational to $\sqcup_i F_i$. Contradiction.

5.4 The non-uniruled case

Theorem 5 *Let k be a field of characteristic zero. Let X be a k-variety of dimension $d \ge 0$. Suppose that*

1. *either k is algebraically closed and X contains only finitely many rational curves, or,*
2. *$X_{\bar{k}}$ does not contain any irreducible rational curve (e.g. if X is a subvariety of an abelian variety).*

Let Y be a k-variety such that $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$. Then X is piecewise isomorphic to Y.

Proof We proceed by induction on d. The case $d \le 1$ is done in Propositions 5, 6 and Remark 12. Suppose $d \ge 2$. Write $[X] = \sum_{1\le i\le n}[X_i] + \sum_j \epsilon_j [C_j]$ and $[Y] = \sum_{1\le q\le n}[Y_q] + \sum_\ell \epsilon'_j [D_\ell]$ as in Lemma 1. Then X_i is non-uniruled, hence not stably birational to any D_ℓ. Applying SB we see that, up to renumbering, each X_i is stably birational, hence birational (Corollary 1), to one Y_i. Consequently, there are open subsets U, V in X, Y respectively such that $U \simeq V$ and $X \setminus U$ has dimension at most $d-1$. As $[X \setminus U] = [Y \setminus V]$, by induction hypothesis, $X \setminus U$ is piecewise isomorphic to $Y \setminus V$, hence X is piecewise isomorphic to Y.

Corollary 9 *Let A be an abelian variety over a field k of characteristic zero. Suppose that there exists a non-trivial torsor X under A. Then $[A]$ is a zero divisor in $K_0(\mathrm{Var}_k)$.*

Proof By definition, there exists an isomorphism $A \times_k A \simeq A \times_k X$. Therefore $[A]([A] - [X]) = 0$. It remains to show that $[A] \neq [X]$. Otherwise A would be birational to X. As A is an abelian variety, we have a birational morphism $f : X \to A$. The inverse rational map $A \dashrightarrow X$ is a morphism over $\bar{k}$ because $X_{\bar{k}}$ has a structure of abelian variety. Therefore $f_{\bar{k}} : X_{\bar{k}} \to A_{\bar{k}}$ is an isomorphism, hence f is an isomorphism. Contradiction.

5.5 Some final computations in the Grothendieck ring

We can also reinterpret, via the Grothendieck ring of varieties, some classical results of the classification of surfaces. Denote by K_X the canonical divisor on X and by $\chi(\mathcal{O}_X)$ the coherent Euler characteristic of the structural sheaf $\mathcal{O}_X$.

Lemma 10 *Let k be a field of characteristic zero. Let X and Y be smooth connected projective k-surfaces such that $[X] = [Y]$. Then $(K_X^2) = (K_Y^2)$ and $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$.*

Proof The result comes from the following formulas for X:

$$\chi(\mathcal{O}_X) = \frac{1}{12}\left(\chi_{\mathrm{top}}(X) + (K_X^2)\right),$$

$$(K_X^2) = 10 - 8h^{0,1}(X) + 12h^{0,2}(X) - b_2(X)$$

and for Y.

Lemma 11 *Let k be an algebraically closed field of characteristic zero. Let X be an irrational smooth and projective k-surface. Assume that X is ruled over a smooth projective k-curve C. Then X is relatively minimal if and only if $[X] = (\mathbb{L} + 1)[C]$ in $K_0(\mathrm{Var}_k)$.*

Proof If X is relatively minimal, then it is a projective bundle $\mathbb{P}(\mathcal{E})$ over C, where $\mathcal{E}$ is a locally free sheaf on C of rank two. So $[X] = (\mathbb{L} + 1)[C]$ (see Example 1).

Conversely, let Y be any smooth connected projective k-surface. Let $Y' \to Y$ be the blowing-up of Y along a point. Then $[Y'] = [Y] + \mathbb{L}$. As X is obtained by successive blowing-ups of points from the relatively minimal model X_0 of X, we have $[X] = [X_0] + m\mathbb{L}$, where m is the number of blowing-ups. As $[X_0] = (\mathbb{L} + 1)[C] = [X]$, we have $m\mathbb{L} = 0$, so $m = 0$ by computing the Poincaré polynomials. Therefore $X \to X_0$ is an isomorphism.

Lemma 12 *Let k be an algebraically closed field of characteristic zero. Let X be a k-surface. If*

$$[X] = \alpha\mathbb{L}^2 + \beta\mathbb{L} + \gamma \in \mathbb{Z}[\mathbb{L}],$$

then X is rational (i.e. all irreducible components of X are rational). The converse holds if X is smooth and projective. Moreover, in this case, $\gamma = \alpha = b_4(X)$ is the number of irreducible components of X, and $\beta = b_2(X)$.

Proof Write a decomposition

$$[X] = \sum_{1 \le i \le n} [X_i] + \sum_{1 \le j \le m} \epsilon_j [C_j]$$

as in Lemma 1. We have

$$\sum_{1\leq i\leq n}[X_i]=\alpha\mathbb{L}^2+\beta\mathbb{L}+\gamma-\sum_j\epsilon_j[C_j].$$

Applying SB, we see that for all $i\leq n$, we have either $\mathrm{SB}(X_i)=\mathrm{SB}(\operatorname{Spec}k)$ and X_k is rational, or, $\mathrm{SB}(X_i)$ is equal to a $\mathrm{SB}(C_{\sigma(i)})$ with $\dim C_{\sigma(i)}=1$. Computing the Poincaré polynomials in the relation above, we see that $b_3(X_i)=0$, and by duality, $b_1(X_i)=0$. It follows that $C_{\sigma(i)}$ has genus 0, thus X_i is again rational.

Now let X be a smooth projective rational surface. Then each connected component X_i of X is birational to $\mathbb{P}^2_k$. The factorization theorem for birational maps of surfaces implies that

$$[X_i]=[\mathbb{P}^2_k]+a\mathbb{L}=\mathbb{L}^2+(a+1)\mathbb{L}+1,\quad a\in\mathbb{Z}.$$

Thus $[X]=\sum_{1\leq i\leq n}[X_i]=\alpha\mathbb{L}^2+\beta\mathbb{L}+\alpha$. The last assertion comes directly from a computation of Poncaré polynomials.

More generally:

Lemma 13 *Let k be a field of characteristic zero. Let X be a smooth projective connected k-variety of dimension $d\geq 2$, ruled over a smooth projective connected variety D of dimension $d-1$. Then there exists a finite number of smooth projective connected k-varieties C_i, $i\in I$, of dimension at most $d-2$ such that*

$$[X]=\mathbb{L}^d+\mathbb{L}([D]+\sum_{i\in I}\varepsilon_i[C_i])+[D],$$

with $\varepsilon_i\in\{-1,1\}$.

Proof By assumption, X is birational to $D\times_k\mathbb{P}^1_k$. The formula comes from a direct application of the weak factorization theorem (see [3, Sect. 2]).

Lemma 14 *Let k be a field of characteristic zero. Let X be a connected smooth k-curve such that $[X]=\mathbb{L}$ in $K_0(\mathrm{Var}_k)$. Then X is isomorphic to $\mathbb{A}^1_k$.*

Proof Consider a smooth completion $\tilde{X}$ of X. Let $Y=\tilde{X}\backslash X=\{y_1,\ldots,y_n\}$. We have

$$\sum_{1\leq i\leq n}[\operatorname{Spec}k(y_i)]+[\mathbb{P}^1_k]=[\tilde{X}]+1.$$

Applying SB, we conclude first that $n=1$, so $Y=\operatorname{Spec}k(y)$. Secondly, we have either $\mathrm{SB}(\operatorname{Spec}k(y))=\mathrm{SB}(\operatorname{Spec}k)$ or $\mathrm{SB}(\operatorname{Spec}k(y))=\mathrm{SB}(\tilde{X})$. In the first case, $k(y)=k$, $\tilde{X}\simeq\mathbb{P}^1_k$ and $X\simeq\mathbb{A}^1_k$. In the second case, $\tilde{X}\simeq\mathbb{P}^1_{k(y)}$ and

$$[\operatorname{Spec}k(y)]+[\mathbb{P}^1_k]=[\mathbb{P}^1_{k(y)}]+1.$$

Over $\bar{k}$, we have

$$[k(y):k]+[\mathbb{P}^1_{\bar{k}}]=[k(y):k][\mathbb{P}^1_{\bar{k}}]+1,$$

which implies that $k(y)=k$.

Acknowledgments We would like to thank Dennis Eriksson for pointing out ideas to treat dimension 1 case, Johannes Nicaise and Hélène Esnault for useful conversations and Antoine Chambert-Loir for indicating us the general arguments for Proposition 4.

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