

# The group inverse of a product\*

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## Abstract

In this paper, we characterize the existence and give an expression of the group inverse of a product of two regular elements by means of a ring unit.

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## 1 Introduction

In this paper, we consider elements on a general (associative) ring  $R$  with unity 1. We will follow the standard notation regarding generalized inverses. We say  $a$  is regular if  $a \in aRa$ . In this case, a particular solution to  $axa = a$ , called von Neumann inverse of  $a$ , is denoted by  $a^-$ . A reflexive inverse of  $a$ , denoted by  $a^+$ , is a common solution to  $axa = a, x = xax$ . A regular element  $a$  has a reflexive  $a^+$ , namely  $a^-aa^-$ , for any choice of von Neumann inverses  $a^-, a^-$ .

We say  $a$  is group invertible if there is a common solution to  $axa = a, xax = x, ax = xa$ . It is well known that such a solution is unique in case it exists. It is denoted by  $a^\#$ .

Our main goal is to characterize the group inverse of a product of regular elements, as well as to derive an expression of such a group inverse that does not rely on the knowledge of von Neumann regularity of the product.

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## 2 Main result

Let  $a, b$  be regular elements in  $R$ , with reflexive inverses  $a^+, b^+$ , respectively. Let also

$$w = (1 - bb^+)(1 - a^+a)$$

which we will assume to be regular in  $R$ .

Note that the regularity of  $w$  does *not* depend on the choices of  $a^+$  and  $b^+$ . That is to say, if  $w$  is regular for a particular choice of  $a^+$  and of  $b^+$ , then it must be regular for *all* choices of  $a^+$  and  $b^+$ . This can be easily proved by noting that  $w$  being regular is equivalent to the regularity of the matrix  $\begin{bmatrix} a & 0 \\ 1 & b \end{bmatrix}$ , using [6], which it turn is equivalent to  $(1 - bb^-)(1 - a^-a)$  being regular, for any other choices of von Neumann inverses  $a^-$  and  $b^-$  of  $a$  and  $b$ .

Consider the matrix  $M = \begin{bmatrix} ab & a \\ 0 & 1 \end{bmatrix} = AQ$  with  $A = \begin{bmatrix} a & 0 \\ 1 & -b \end{bmatrix}$ ,  $Q = \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix}$ .

It is well known that  $M^\#$  exists if and only if  $(ab)^\#$  exists, using [1]. Furthermore, the (1,1) entry of  $M^\#$  equals  $(ab)^\#$ . Also,  $M^\#$  exists if and only if  $U = AQ + I - AA^-$  is invertible, see [7], [5], in which case  $(AQ)^\# = U^{-2}(AQ)$ .

As  $AQ + I - AA^- = A(Q - A^-) + I$  then  $AQ + I - AA^-$  is invertible if and only if  $(Q - A^-)A + I = QA + I - A^-A$  is invertible, using Jacobson's Lemma, which in turn means  $(QA)^\#$  exists. Therefore, by considering the matrix  $W = QA = \begin{bmatrix} ba + 1 & -b \\ a & 0 \end{bmatrix}$ , then  $(ab)^\#$  exists if and only if  $W$  is group invertible.

Using [4], the matrix  $W$  is group invertible if and only if

$$\begin{aligned} z &= (1 + ba)(1 - a^+a) + ba + (1 - ww^-)(1 - bb^+)(1 + ba) \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+) \end{aligned}$$

is a unit.

We have, hence, the equivalence

$$(ab)^\# \text{ exists if and only if } 1 - a^+a + ba + (1 - ww^-)(1 - bb^+) \text{ is a unit.}$$

Using the expression presented in [4] does not give a tractable algorithm to actually compute  $(ab)^\#$ . We will, therefore, pursue a different strategy and compute the (1,1) entry of  $M^\#$ .

Recall that for  $M = AQ$  and  $Q$  invertible, the group inverse of  $M$  exists if and only if  $U = AQ + I - AA^-$  is invertible. For  $A = \begin{bmatrix} a & 0 \\ 1 & -b \end{bmatrix}$ , there exists  $A^-$  for which

$$AA^- = \begin{bmatrix} aa^+ & 0 \\ -(1 - ww^-)(1 - bb^+)a^+ & bb^+ + ww^-(1 - bb^+) \end{bmatrix},$$

using [6].

The matrix  $U$  then becomes

$$U = \begin{bmatrix} ab + 1 - aa^+ & a \\ (1 - ww^-)(1 - bb^+)a^+ & 2 - bb^+ - ww^-(1 - bb^+) \end{bmatrix}.$$

Multiplication on the right by  $K = \begin{bmatrix} 1 & 0 \\ a^+ - b & 1 \end{bmatrix}$  gives

$$G = \begin{bmatrix} 1 & a \\ \alpha & 2 - bb^+ - ww^-(1 - bb^+) \end{bmatrix},$$

where

$$\begin{aligned} \alpha &= (1 - ww^-)(1 - bb^+)a^+ + (2 - bb^+ - ww^-(1 - bb^+))(a^+ - b) \\ &= a^+ - b + 2(1 - ww^-)(1 - bb^+)a^+, \end{aligned}$$

as  $(1 - bb^+)b = 0$ .

We are left with showing when is  $G$  invertible. We do so using the Schur complement on the (1,1) entry. This Schur complement equals

$$\begin{aligned} G/I &= (2 - bb^+ - ww^-(1 - bb^+)) - ((1 - ww^-)(1 - bb^+)a^+ + \\ &\quad + (2 - bb^+ - ww^-(1 - bb^+))(a^+ - b))a \\ &= (2 - bb^+ - ww^-(1 - bb^+))(1 - a^+a + ba) - (1 - ww^-)(1 - bb^+)a^+a \\ &= (1 + (1 - ww^-)(1 - bb^+))(1 - a^+a + ba) - (1 - ww^-)(1 - bb^+)a^+a \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)(1 - 2a^+a) \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)(1 - a^+a) + \\ &\quad + (1 - ww^-)(1 - bb^+)a^+a \\ &= 1 - a^+a + ba + (1 - ww^-)(1 - bb^+) \end{aligned}$$

This gives, and as previously shown,

$$(ab)^\# \text{ exists if and only if } z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+) \text{ is a unit.}$$

As a side note, we construct another unit associated with  $z$ , namely we may show that  $z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$  is a unit if and only if  $z' = 1 - aa^+ + ab - a(1 - ww^-)(1 - bb^+)a^+$  is a unit. This follows by the sequence of identities  $(1 - ww^-)(1 - bb^+) = (1 - ww^-)(1 - bb^+)(1 - a^+a + a^+a) = (1 - ww^-)(1 - bb^+)a^+a$  together with Jacobson's Lemma.

We remark that given a reflexive inverse  $w^+$  of  $w$ , the element  $\tilde{w} = (1 - a^+a)w^+(1 - bb^+)$  is an idempotent reflexive inverse of  $w$ . As such  $z$  and  $z'$  simplify to  $1 - a^+a + ba + 1 - bb^+ - w\tilde{w}$  and  $1 + ab - abb^+a^+ - aw\tilde{w}a^+$ , respectively.

We know, using [5, Corollary 3.3(4)], that  $(AQ)^\#$  exists if and only if  $U$  is invertible, in which case  $(AQ)^\# = U^{-2}(AQ)$ . The matrices  $U$  and  $G$  are equivalent, and we are able to relate their inverses by means of the matrix  $K$ . Indeed, since  $G = UK$ , then  $U^{-1} = KG^{-1}$ . Firstly, we need to compute the inverse of  $G$ , for which we will use the following known result:

**Lemma 2.1.**

$$\begin{bmatrix} 1 & y \\ x & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 + y\zeta^{-1}x & -y\zeta^{-1} \\ -\zeta^{-1}x & \zeta^{-1} \end{bmatrix},$$

where  $\zeta = z - yx$  is the Schur complement.

Our purpose is to derive an expression for  $(ab)^\#$ , which equals the (1,1) entry of  $M^\#$ .

The (1,1) entry of  $M^\#$  is obtained by multiplying the first row of  $U^{-2}$  by the first column of  $AQ$ , which is  $\begin{bmatrix} ab \\ 0 \end{bmatrix}$ . So, in fact we just need the (1,1) entry of  $U^{-2}$ , which is then multiplied on the right by  $ab$  to give  $(ab)^\#$ .

We recall that  $G = UK$ , where  $K = \begin{bmatrix} 1 & 0 \\ a^+ - b & 1 \end{bmatrix}$ , which gives  $U^{-1} = KG^{-1}$  and  $U^{-2} = KG^{-1}KG^{-1}$ . Pre-multiplication with  $K$  does not affect the first row, and so we just need the (1,1) element of  $G^{-1}KG^{-1}$ . Calculations show that

$$U^{-2} = (KG^{-1})^{-2} = \begin{bmatrix} (1 + az^{-1}\alpha)^2 - az^{-1}(a^+ - b)(1 + az^{-1}\alpha) + az^{-2}\alpha & ? \\ ? & ? \end{bmatrix}$$

We will need the simplification

$$b - zb = a^+ab - bab, \quad (1)$$

from where we obtain

$$\begin{aligned} \alpha ab &= (a^+ - b)ab = b - zb \\ (1 + az^{-1}\alpha)ab &= az^{-1}b. \end{aligned}$$

Indeed,  $\alpha ab = aa^+ab - bab + 2(1 - ww^-)(1 - bb^+)a^+ab$  whose last summand can be expressed as  $2(1 - ww^-)(1 - bb^+)a^+ab = -2(1 - ww^-)(1 - bb^+)(1 - a^+a - 1)b = -2(1 - ww^-)w + 2(1 - ww^-)(1 - bb^+)b = 0$ , and therefore  $\alpha ab = a^+ab - bab$ .

Therefore,

$$(AQ)^\# = \begin{bmatrix} (ab)^\# & ? \\ 0 & ? \end{bmatrix} = \begin{bmatrix} ((1 + az^{-1}\alpha)^2 - az^{-1}(a^+ - b)(1 + az^{-1}\alpha) + az^{-2}\alpha) ab & ? \\ 0 & ? \end{bmatrix}$$

from which we obtain the general formula

$$\begin{aligned} (ab)^\# &= ((1 + az^{-1}\alpha)^2 - az^{-1}(a^+ - b)(1 + az^{-1}\alpha) + az^{-2}\alpha) ab \\ &= ab + 2(az^{-1}b - ab) + az^{-1}\alpha(az^{-1}b - ab) - \\ &\quad az^{-1}(a^+ - b)az^{-1}b + az^{-1}(z^{-1}b - b) \\ &= (az^{-1}\alpha az^{-1}b - az^{-1}(a^+ - b)az^{-1}b) + az^{-2}b \\ &= 2az^{-1}(1 - ww^-)(1 - bb^+)z^{-1}b + az^{-2}b \\ &= 2az^{-1}b - 2(az^{-1}b)^2 + az^{-2}b. \end{aligned}$$

From  $1 - a^+a = z^{-1} - z^{-1}a^+a$  we obtain, by post-multiplying by  $b$ ,

$$b - a^+ab = z^{-1}b - z^{-1}a^+ab \quad (2)$$

which implies

$$az^{-1}b = az^{-1}a^+ab. \quad (3)$$

Now, from (1) we have  $z^{-1}b = b + z^{-1}a^+ab - z^{-1}bab$  which implies, using (2), that

$$z^{-1}bab = a^+ab \quad (4)$$

which in turns delivers

$$ab = az^{-1}bab. \quad (5)$$

Using (4) and (5), together with  $(ab)^\# = 2az^{-1}b - 2(az^{-1}b)^2 + az^{-2}b$ , we write the idempotent  $(ab)^\#(ab)$  as

$$\begin{aligned} (ab)^\#ab &= 2az^{-1}b - 2(az^{-1}b)^2 + az^{-2}b \\ &= 2az^{-1}bab - 2az^{-1}bab + az^{-1}z^{-1}bab \\ &= az^{-1}a^+ab \end{aligned}$$

Using (3), this equals  $az^{-1}b$  and therefore  $az^{-1}b$  is an idempotent, the unit of the group generated by  $ab$ . This simplifies the expression of  $(ab)^\#$  to

$$(ab)^\# = az^{-2}b.$$

It comes with no surprise that the expression of  $(ab)^\#$  is of the form  $aXb$ , for a suitable  $X$ .

We have, from the above, our main result:

**Theorem 2.2.** *Let  $a, b$  be regular elements in  $R$  with reflexive inverses  $a^+$  and  $b^+$ , respectively. Assume, also, that  $w = (1 - bb^+)(1 - a^+a)$  is regular. Then  $(ab)^\#$  exists if and only if  $z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$  is a unit. In this case,*

$$(ab)^\# = az^{-2}b.$$

### 3 Special cases

#### On $ag$ where $g$ is a unit

We consider a special instance of a product, where one element is a unit. Precisely, let  $a, g \in R$  with  $g$  a unit. In [5, Corollary 3.2], the existence of the group inverse of  $ag$  was related to the existence of a unit, and an expression of  $(ag)^\#$  was given. By Theorem 2.2, we know that

$(ag)^\#$  exists if and only if  $z = 1 - a^+a + ga + (1 - ww^-)(1 - gg^+)$  is a unit. But  $g^+ = g^{-1}$  so that  $(ag)^\#$  exists if and only if  $z = 1 - a^+a + ga$  is a unit, which is equivalent to the criterion of [5, Corollary 3.2] by Jacobson's Lemma. However, Theorem 2.2 gives us

$$(ag)^\# = a(1 - a^+a + ga)^{-2}g$$

which is different from [5, Corollary 3.2]. On the other hand, we can also consider the case of a group inverse of  $ga$ . In this case the  $w$  in the previous theorems is 0.  $(ga)^\#$  exists if and only if  $ag - 1 + aa^+$  is a unit, and

$$(ga)^\# = g(ag - 1 + aa^+)^{-2}a.$$

As the existence of  $(ag)^\#$  is equivalent with the existence of  $(ga)^\#$  (and using  $-g$  instead of  $g$  in the second case), we get:

**Corollary 3.1.** *The following conditions are equivalent:*

1.  $ag$  is group invertible;
2.  $ga$  is group invertible;
3.  $z = 1 - a^+a + ga$  is a unit;
4.  $\eta = 1 - aa^+ + ag$  is a unit;

*In which case:*

$$(ag)^\# = az^{-2}g \text{ and } (ga)^\# = g\eta^{-2}a.$$

The units in the previous corollary are strongly related to the existence of the inverse of  $g$  along  $a$ , see [3].

As a special case, when  $g = 1$ , we recover the classical result ([5, Corollary 3.3.]):

**Corollary 3.2.** *The following conditions are equivalent:*

1.  $a$  is group invertible;
2.  $z = 1 - a^+a + a$  is a unit;
3.  $\eta = 1 - aa^+ + a$  is a unit;

*In which case:*

$$(a)^\# = az^{-2} \text{ and } (a)^\# = \eta^{-2}a.$$

## On the sum

We now apply the results of the previous section to the sum of ring elements to obtain a known characterization of the group inverse of a sum.

Let  $a, b$  be ring elements such that  $a + b$  is regular. Consider  $A = \begin{bmatrix} a & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}$ , for which  $AB = \begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix}$ . This is a key factorization that allows us to characterize the group inverse of  $a + b$  by the group inverse of  $AB$ .

Using the results on the previous section,  $(AB)^\#$  exists if and only if  $H = I - A^+A + BA - (I - WW^-)(I - BB^+)$  is invertible. We will now undertake the computation of this matrix, for particular choices of inner and reflexive inverses.

We will take  $A^+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$  and  $B^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , which will deliver  $A^+A = \begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix}$  and  $BB^+ = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}$ .

Also,  $BA = \begin{bmatrix} a & 1 \\ ba & b \end{bmatrix}$  and  $W = (I - BB^+)(I - A^+A) = \begin{bmatrix} 0 & 0 \\ -a - b & 0 \end{bmatrix}$ . Since  $a+b$  is regular, then  $W$  is regular and we may take  $W^- = \begin{bmatrix} 0 & (-a - b)^- \\ 0 & 0 \end{bmatrix}$ . The associated idempotents take the form  $WW^- = \begin{bmatrix} 0 & 0 \\ 0 & (a + b)(a + b)^- \end{bmatrix}$  and  $I - WW^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 - (a + b)(a + b)^- \end{bmatrix}$ .

We therefore obtain

$$(I - WW^-)(I - BB^+) = \begin{bmatrix} 0 & 0 \\ -(1 - (a + b)(a + b)^-)b & 1 - (a + b)(a + b)^+ \end{bmatrix}.$$

The invertible matrix takes the form

$$\begin{bmatrix} 1 + a & 1 \\ -a + ba + 1(1 - (a + b)(a + b)^+)b & b - 1 + (a + b)(a + b)^+ \end{bmatrix},$$

whose Schur complement equals  $(a + b) + 1 - (a + b)^+(a + b)$  and which has to be a ring unit. This is coherent with the known result.

## The example of a trace product

Let  $a, b$  be two elements of the ring  $R$  such that  $Rab = Rb$  and  $abR = aR$  (one says that  $ab$  is a trace product). Then it is known, by a theorem of Clifford ([2, Proposition 2.3.7]), that this is equivalent with the existence of an idempotent  $e \in R$  such that  $Ra = Re$  and  $bR = eR$ . In particular,  $a$  and  $b$  are von Neumann regular and we can find  $a^+$  and  $b^+$  such that  $a^+a = bb^+ = e$ . For this particular choice,  $w = (1 - bb^+)(1 - a^+a) = (1 - e)$  is idempotent, and the element  $z = 1 - a^+a + ba + (1 - ww^-)(1 - bb^+)$  reduces to  $z = 1 - a^+a + ba$ . As the

invertibility of  $z$  implies the invertibility of all the elements of the form  $1 - a^+a + ba$ , we get by Theorem 2.2:

**Corollary 3.3.** *Let  $ab$  be a trace product. Then  $a$  and  $b$  are regular, and for any choice of  $a^+$ ,  $ab$  is group invertible if and only if  $z = 1 - a^+a + ba$  is a unit. In this case,*

$$(ab)^\# = a(1 - a^+a + ba)^{-2}b.$$

Once again, we recognize the criterion of invertibility of  $b$  along  $a$ .

## 4 Final remarks

1. In this paper, we are primarily interested in the group invertibility (on the group inverse) of the product  $ab$ . Since the group inverse of  $ab$  equals  $az^{-2}b$ , what can be said about  $ba$ ?

If  $z^{-1}$  is an inner inverse, and since  $ab$  is group invertible, then  $1 + ab - (ab)(ab)^\#$  is a unit, that is  $1 + ab - az^{-1}b$  is a unit. By Jacobson lemma,  $1 + ba - baz^{-1}$  is a unit, and if  $z^{-1}$  is an inner inverse of  $ba$ , then  $ba$  is actually group invertible.

Let us compute  $z(baz^{-1}ba - ba)$ .

$$\begin{aligned} z(baz^{-1}ba - ba) &= (1 - a^+a + ba + (1 - ww^-)(1 - bb^+)) (baz^{-1}ba - ba) \\ &= (1 - a^+a + ba) (baz^{-1}ba - ba) \text{ as } (1 - bb^+)b = 0 \\ &= (baz^{-1}ba - ba) + (-a^+ + b) (abaz^{-1}ba - aba) \end{aligned}$$

But equation (5) gives  $ab = az^{-1}bab$  and since we have shown that  $az^{-1}b = ab(ab)^\#$ , then it commutes with  $ab$  and  $ab = abaz^{-1}b$ . This gives that the second term in the above sum is 0, and  $z(baz^{-1}ba - ba) = (baz^{-1}ba - ba)$ , or equivalently,  $(1 - z)(baz^{-1}ba - ba) = 0$ . Multiplying by  $z^{-1}$  on the left gives  $(baz^{-1}ba - ba) = z^{-1}baz^{-1}ba - z^{-1}ba$  or

$$(z^{-1} - 1)(baz^{-1}ba - ba) = 0. \quad (6)$$

Note that  $1 - z^{-1}$  is a unit if and only if  $1 - z$  is a unit. Indeed, setting  $\chi = 1 - z$  then  $1 = z^{-1} - z^{-1}\chi = z^{-1} - \chi z^{-1}$  which imply  $z^{-1} - 1 = z^{-1}\chi = \chi z^{-1}$  and  $\chi = 1 - z$  is a unit. Conversely, if  $\chi = 1 - z$  is a unit then  $z^{-1} - 1 = z^{-1}\chi$  we obtain the desired conclusion.

Suppose  $1 - z$  is a unit. Then  $z^{-1}$  is an inner inverse of  $ba$  (and  $z^{-1}ba$  is idempotent). Of course, if  $z^{-1}ba$  is idempotent then  $z^{-1}$  is always an inner inverse of  $ba$  ( $ba = zz^{-1}ba = zz^{-1}ba z^{-1}ba = baz^{-1}ba$ .)



2. Consider  $(G_z, \cdot)$  the group generated by  $z$ , and  $(G_{ab}, \cdot)$  the group generated by  $ab$ . Then we can prove that  $\phi : (G_z, \cdot) \rightarrow (G_{ab}, \cdot)$  defined by  $\phi(z^k) = az^{k-1}b$  is an isomorphism, or put in an other form:  $az^n b = (ab)^{n+1}$  for all  $n \in \mathbb{Z}$ .

This can be proved by induction on  $\mathbb{Z}$ . First, we have proved in our paper that  $z(b - a^+ab) = b - a^+ab$  and  $z^{-1}bab = a^+ab$ . By induction on  $\mathbb{Z}$ , this imply that  $z^n(b - a^+ab) = b - a^+ab$  for all  $n \in \mathbb{Z}$  hence  $az^n b = az^n a^+ab = az^{n-1}bab$  for all  $n \in \mathbb{Z}$ . Now our equation is true for  $n = 0$ . Assume it is true for  $n \in \mathbb{N}$ . Then  $az^{n+1}b = az^n bab = (ab)^{n+1}$ . Suppose now it is true for  $-n \in \mathbb{N}$ , that is  $az^{-n}b = (ab)^{1-n}$ . Then we prove that  $az^{-n-1}b$  is the group inverse of  $(ab)^n$ . We have  $az^{-n-1}b(ab)^n = az^{-n-1}b(ab)(ab)^{n-1} = az^n b(ab)^{n-1} = (ab)^0$ . As symmetrically,  $(ab)^n az^{-n-1}b = (ab)^0$ , then  $az^{-n-1}b$  is the group inverse of  $(ab)^n$ .

3. We can compute other units by duality, using the opposite ring  $(R, \times)$ ,  $x \times y = yx$ . Precisely,  $ab$  is group invertible in  $(R, \cdot)$  if and only if  $b \times a$  is group invertible in  $(R, \times)$ , and by our theorem this happens if and only if  $1 - b^+ \times b + a \times b + (1 - w \times w^-) \times (1 - a \times a^+)$  (if  $a^+$  and  $b^+$  are inverses in  $(R, \cdot)$  then they are also inverses in  $(R, \times)$ ), with  $w = (1 - a \times a^+) \times (1 - b^+ \times b)$ , or equivalently if and only if  $t = 1 - bb^+ + ba + (1 - a^+a)(1 - w^-w)$  is a unit with  $w = (1 - bb^+)(1 - a^+a)$  (classical one). Note that this unit is *a priori* different from the other ones in the paper. If we continue the duality principle, we end with  $b \times t^{-2} \times a$  is the group inverse of  $b \times a$ , that is  $at^{-2}b$  is the group inverse of  $ab$ , so that this unit works equivalently as our  $z$ .

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