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Research Article

The Group Inverse of the Combinations of Two Idempotent Operators

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We present some inverses and group inverses results for linear combinations of two idempotents and their products.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space. Denote by $\mathscr{B}(\mathscr{H})$ the Banach algebra of all bounded linear operators on $\mathscr{H}$. $\mathscr{R}(T)$ and $\mathscr{N}(T)$ represent the range and the null space of T, respectively. The identity onto a Hilbert space $\mathscr{H}$ is denoted by $I_{\mathscr{H}}$ or I if there does not exist confusion. For $T \in \mathscr{B}(\mathscr{H})$, the group inverse [1] of T is the unique element $T^{\#} \in \mathscr{B}(\mathscr{H})$ such that

$$TT^{\#} = T^{\#}T, \qquad T^{\#}TT^{\#} = T^{\#}, \qquad T = TT^{\#}T. \tag{1}$$

$T^{\#}$ exists if and only if T has finite ascent and descent such that $i(T) = \text{asc}(T) = \text{desc}(T) \leq 1$ [2]. When $\text{ind}(T) = 0$, the group inverse reduces to the standard inverse; that is, $T^{\#} = T^{-1}$. In particular, $a^{\#} = a^{-1}$ if $a \neq 0$ and $a^{\#} = 0$ if $a = 0$ for a scalar a. One of the most important applications of group inverses is to derive some closed-form formulas for general solutions to operator equations. An operator $P \in \mathscr{B}(\mathscr{H})$ is said to be idempotent if $P^2 = P$. If T is group invertible, then $\mathscr{R}(T)$ is closed and the spectral idempotent T^{π} is given by $T^{\pi} = I - TT^{\#}$. The operator matrix form of T with respect to the space decomposition $\mathscr{H} = \mathscr{N}(T^{\pi}) \oplus \mathscr{R}(T^{\pi})$ is given by $T = T_1 \oplus 0$, where T_1 is invertible on $\mathscr{N}(T^{\pi})$ [1].

Idempotents are a type of simplest operators. Various expressions or equalities consisting of idempotents occur in operator theory and its applications. Some previous work on linear combinations of idempotents in statistics can be found in [3]. There have been several papers devoted to the invertibility of a linear combination of two idempotent operators in a Hilbert space or in a C^*-algebra. In [4], Buckholtz studied the idempotency of the difference of two operators in a Hilbert space. Du and Li in [5] had established the spectral characterization of generalized projections. In [6], the invertibility of the difference of two orthogonal projectors in a C^*-algebra was studied. Li in [7] had investigated how to express Moore-Penrose inverses of products and differences. In [8], J. K. Baksalary and O. M. Baksalary discussed the invertibility of a linear combination of idempotent matrices. This paper was improved by Koliha and Rakočević [9] by showing that the rank of a linear combination of two idempotents is constant. Du et al. [10] extended the conclusion on an infinite-dimensional Hilbert space.

The purpose of this note is to characterize the invertibility and the group invertibility of the linear combinations of idempotents P, Q and their products $PQ, QP, PQP, QPQ, (PQ)^2$, and $(QP)^2$. These linear combinations were studied by some authors in recent years [2, 4, 6–9, 11–18]. Some formulas for the group inverse of a sum of two bounded operators under some conditions were given (see [7, 11, 14, 19, 20]). Here, we will find group invertibility for a linear combination of two idempotents under the condition $(PQ)^2 = P(QP)^2$. A previous study of the group invertibility of two idempotents was made under the conditions $(PQ)^2 = (QP)^2$ or $PQ = QP$ or $PQP = QPQ$

or $(PQ)^2 = 0$. It is clear that these conditions are special cases of our results.

2. Main Results and Proofs

We start by discussing some lemmas. Let P and Q be two idempotents. Now, we consider the invertibility of $P-Q$. This problem is the subject of Buckholtz's papers [4] and Koliha and Rakočević's paper [12].

Lemma 1 (see [12]). *Let P and Q be two idempotents.*

(i) *$P-Q$ is invertible if and only if $I-PQ$ and $P+Q-PQ$ are invertible.*

(ii) *If $P-Q$ is group invertible, then $I-PQ$ and $P+Q-PQ$ are group invertible.*

Proof. Since the properties in the lemma are similarly invariant, without loss of generality, we can assume that P is an orthogonal projection. In this case, P and Q have the operator matrix representations as follows

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \tag{2}$$

with respect to the space decomposition $\mathscr{H} = \mathscr{R}(P) \oplus \mathscr{N}(P)$, respectively. Since $Q^2 = Q$,

$$\begin{pmatrix} Q_1^2 + Q_2Q_3 & Q_1Q_2 + Q_2Q_4 \\ Q_3Q_1 + Q_4Q_3 & Q_3Q_2 + Q_4^2 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}. \tag{3}$$

So

$$I - PQ = \begin{pmatrix} I - Q_1 & -Q_2 \\ 0 & I \end{pmatrix},$$
$$P + Q - PQ = \begin{pmatrix} I & 0 \\ Q_3 & Q_4 \end{pmatrix}, \qquad (P-Q)^2 = \begin{pmatrix} I - Q_1 & 0 \\ 0 & Q_4 \end{pmatrix}. \tag{4}$$

It is clear that item (i) holds, and if $P-Q$ is group invertible, then $(P-Q)^2$ is group invertible, which implies that $I-Q_1$ and Q_4 are group invertible. □

As for n-idempotents, we have the following decomposition.

Lemma 2 (see [21, Theorem 2.3]). *Let $A \in \mathscr{B}(\mathscr{H})$. Then $A^n = A$ if and only if*

(i) $\sigma(A) \subseteq \{0, e^{i(2k\pi/(n-1))} : 0 \le k \le n-2\}$;

(ii) *there exists a resolution set $\{E_\lambda : \lambda \in \sigma(A)\}$ of the identity I and an invertible operator S such that*

$$SAS^{-1} = \sum_{\lambda \in \sigma(A)} \oplus \lambda E_\lambda, \tag{5}$$

where $\oplus$ denotes the orthogonal direct sum, E_λ is an orthogonal projection with $\sum_{\lambda\in\sigma(A)} E_\lambda = I$, and $E_\lambda E_\mu = E_\mu E_\lambda = 0$ if λ, $\mu \in \sigma(A)$, $\lambda \ne \mu$.

Lemma 3. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathscr{B}(\mathscr{H} \oplus \mathscr{K})$ be such that $A \in \mathscr{B}(\mathscr{H})$ is invertible. Then M is invertible if and only if the Schur complement $S = D - CA^{-1}B$ is invertible.*

Lemma 4 (see [1, Theorem 7.7.3]). *Let $M = \left(\begin{smallmatrix} A & C \\ 0 & D \end{smallmatrix}\right) \in \mathscr{B}(\mathscr{H} \oplus \mathscr{K})$ be such that $A^\#$ exists. Then $M^\#$ exists if and only if $D^\#$ exists and $A^\pi C D^\pi = 0$. In this case,*

$$\begin{pmatrix} A & C \\ 0 & D \end{pmatrix}^\# = \begin{pmatrix} A^\# & Y \\ 0 & D^\# \end{pmatrix}, \tag{6}$$

where $Y = (A^\#)^2 C D^\pi + A^\pi C (D^\#)^2 - A^\# C D^\#$.

Lemma 5 (see [19, Theorem 3.1]). *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathscr{B}(\mathscr{H} \oplus \mathscr{K})$ be such that $A \in \mathscr{B}(\mathscr{H})$ is invertible and the Schur complement $S = D - CA^{-1}B$ is group invertible. Then M is group invertible if and only if $R = A^2 + BS^\pi C$ is invertible.*

As we know, an operator $A \in \mathscr{B}(\mathscr{H})$ is said to be involutory if $A^2 = I$, to be anti-idempotent if $A^2 = -A$, and to be tripotent if $A^3 = A$. Obviously, involutory, idempotent, and anti-idempotent are special cases of tripotent. For linear combinations of two commutative tripotents and their products, we have the following result.

Theorem 6. *Let P, Q satisfy $P^3 = P$, $Q^3 = Q$, and $PQ = QP$. For any scalar a, b, c, d, e, f, g, h, and i, let*

$$\begin{aligned} \Phi &= aI + bP + cQ + dP^2 + eQ^2 \\ &\quad + fPQ + gP^2Q + hPQ^2 + i(PQ)^2. \end{aligned} \tag{7}$$

(i) *If $\lambda_1 \cdots \lambda_9 \ne 0$, then Φ is invertible. In this case, $\Phi^{-1} = \sum_{i=1}^{9} \lambda_i^{-1} E_i$.*

(ii) *Φ is always group invertible and $\Phi^\# = \sum_{i=1}^{9} \lambda_i^\# E_i$.*

The E_i and $\lambda_i, i = 1, 2, \ldots, 9$, in items (i) and (ii) are defined as

$$E_1 = \frac{1}{4}\left(P + P^2\right)\left(Q + Q^2\right),$$
$$E_2 = -\frac{1}{4}\left(P + P^2\right)\left(Q - Q^2\right),$$
$$E_3 = \frac{1}{2}\left(P + P^2\right)\left(I - Q^2\right),$$
$$E_4 = -\frac{1}{4}\left(P - P^2\right)\left(Q + Q^2\right),$$
$$E_5 = \frac{1}{4}\left(P - P^2\right)\left(Q - Q^2\right),$$
$$E_6 = -\frac{1}{2}\left(P - P^2\right)\left(I - Q^2\right),$$
$$E_7 = \frac{1}{2}\left(I - P^2\right)\left(Q + Q^2\right),$$
$$E_8 = -\frac{1}{2}\left(I - P^2\right)\left(Q - Q^2\right),$$

$$E_9 = \left(I - P^2\right)\left(I - Q^2\right),$$
$$\lambda_1 = a + b + c + d + e + f + g + h + i,$$
$$\lambda_2 = a + b - c + d + e - f - g + h + i,$$
$$\lambda_4 = a - b + c + d + e - f + g - h + i,$$
$$\lambda_5 = a - b - c + d + e + f - g - h + i,$$
$$\lambda_3 = a + b + d, \qquad \lambda_6 = a - b + d,$$
$$\lambda_7 = a + c + e, \qquad \lambda_8 = a - c + e, \qquad \lambda_9 = a. \tag{8}$$

Proof. Since $P^3 = P$, by Lemma 2, $\sigma(P) \subseteq \{0, 1, -1\}$ and there exists an invertible operator S_0 such that $P = S_0^{-1}[I \oplus -I \oplus 0]S_0$. We consider a partition Q conforming with P. Since $PQ = QP$, Q can be written as $Q = S_0^{-1}[Q_1 \oplus Q_2 \oplus Q_3]S_0$, where $Q_i^3 = Q_i$, $i = 1, 2, 3$. In a similar way, Q_i, $i = 1, 2, 3$, can be written as $Q_i = S_i^{-1}[I \oplus -I \oplus 0]S_i$, $i = 1, 2, 3$. Let $S = (S_1 \oplus S_2 \oplus S_3)S_0$. Now, we get

$$P = S^{-1}\left[I \oplus I \oplus I \oplus -I \oplus -I \oplus -I \oplus 0 \oplus 0 \oplus 0\right] S,$$
$$Q = S^{-1}\left[I \oplus -I \oplus 0 \oplus I \oplus -I \oplus 0 \oplus I \oplus -I \oplus 0\right] S. \tag{9}$$

Let E_i and λ_i, $i = 1, 2, \ldots, 9$, be defined as in (8). By (9), SE_iS^{-1} is a diagonal block matrix such that the ith diagonal element is the identity I and the remaining diagonal elements are 0, $i = 1, 2, \ldots, 9$. Moreover, $\sum_{i=1}^{9} E_i = I$, $E_i^2 = E_i$, and $E_iE_j = E_jE_i = 0$, $i \neq j$, $i, j = 1, 2, \ldots, 9$. We get $\Phi = aI + bP + cQ + dP^2 + eQ^2 + fPQ + gP^2Q + hPQ^2 + i(PQ)^2 = \sum_{i=1}^{9} \lambda_i E_i$. Hence, if $\lambda_1 \cdots \lambda_9 \neq 0$, Φ is invertible and $\Phi^{-1} = \sum_{i=1}^{9} \lambda_i^{-1} E_i$. If $\lambda_1 \cdots \lambda_9 = 0$, Φ is group invertible and $\Phi^{\#} = \sum_{i=1}^{9} \lambda_i^{\#} E_i$. □

The matrix case of Theorem 6 was first investigated by Tian [17]. The commutative relations ensure that idempotents (or tripotents) have simple block matrix forms. It is natural to ask whether this kind of combinational properties still hold when a pair of idempotents $P, Q \in \mathscr{B}(\mathscr{H})$ is noncommutative. Next, let $P^2 = P$, $Q^2 = Q$, and $(PQ)^2 = P(QP)^2$. It is easy to verify that this condition includes some specific cases:

(i) $PQP = 0$,

(ii) $(PQ)^2 = 0$ (see [14]),

(iii) $PQP = Q$,

(iv) $PQP = PQ$,

(v) $PQP = QPQ$,

(vi) $PQ = QP$ (see [17]),

(vii) $(PQ)^2 = (QP)^2$ (see [14]).

Applying Lemmas 3 and 4, we get the following main result.

Theorem 7. *Let P and Q be two idempotents such that $(PQ)^2 = P(QP)^2$. For any scalar a, b, c, d, e, f, g and h with $a \neq 0, b \neq 0$, and $a + b + c + d + e + f + g + h \neq 0$, let*

$$\Gamma = aP + bQ + cPQ + dQP + ePQP + fQPQ + g(PQ)^2 + h(QP)^2. \tag{10}$$

(i) *Γ is invertible if and only if $P + Q - PQ$ is invertible.*

(ii) *Γ is always group invertible.*

Proof. Let P and Q have the forms as in (2). Then

$$(PQ)^2 = \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}\right]^2 = \begin{pmatrix} Q_1^2 & Q_1Q_2 \\ 0 & 0 \end{pmatrix},$$
$$(QP)^2 = \left[\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right]^2 = \begin{pmatrix} Q_1^2 & 0 \\ Q_3Q_1 & 0 \end{pmatrix}. \tag{11}$$

If $(PQ)^2 = P(QP)^2$, then $Q_1Q_2 = 0$. Moreover, by (3), $Q_2Q_4 = Q_2$ and $Q_3Q_2 + Q_4^2 = Q_4$. These imply that $\mathscr{R}(Q_2) \subset \mathscr{N}(Q_1)$, $\mathscr{R}(I - Q_4) \subset \mathscr{N}(Q_2)$, and $Q_3\mathscr{R}(Q_2) \subset \mathscr{R}(I - Q_4)$. So P and Q can be rewritten as 4×4 block matrix forms as

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & Q_{11} & 0 & Q_{21} \\ 0 & Q_{12} & 0 & 0 \\ Q_{31} & Q_{32} & Q_{41} & Q_{42} \\ 0 & Q_{33} & 0 & I \end{pmatrix} \tag{12}$$

with respect to the space decomposition $\mathscr{H} = \overline{\mathscr{R}(Q_2)} \oplus [\mathscr{R}(P) \ominus \overline{\mathscr{R}(Q_2)}] \oplus \overline{\mathscr{R}(I - Q_4)} \oplus [\mathscr{N}(P) \ominus \overline{\mathscr{R}(I - Q_4)}]$, respectively. From $Q^2 = Q$, by (12), we deduce that

$$Q_{12}^2 = Q_{12}, \qquad Q_{41}^2 = Q_{41}, \qquad Q_{41}Q_{31} = Q_{31},$$
$$Q_{11}Q_{12} + Q_{21}Q_{33} = Q_{11}, \qquad Q_{33}Q_{12} = 0,$$
$$Q_{31}Q_{11} + Q_{32}Q_{12} + Q_{41}Q_{32} + Q_{42}Q_{33} = Q_{32},$$
$$Q_{41}Q_{42} + Q_{31}Q_{21} = 0. \tag{13}$$

By (2) and (12) we get Γ as an operator on $\overline{\mathscr{R}(Q_2)} \oplus (\mathscr{R}(P) \ominus \overline{\mathscr{R}(Q_2)}) \oplus \overline{\mathscr{R}(I - Q_4)} \oplus (\mathscr{N}(P) \ominus \overline{\mathscr{R}(I - Q_4)})$ which can be represented as 4×4 operator matrix form:

$$\begin{aligned}\Gamma &= aP + bQ + cPQ + dQP + ePQP + fQPQ + g(PQ)^2 + h(QP)^2 \\ &= \begin{pmatrix} aI + (b+c+d+e)Q_1 + (f+g+h)Q_1^2 & (b+c)Q_2 \\ (b+d)Q_3 + (f+h)Q_3Q_1 & bQ_4 + fQ_3Q_2 \end{pmatrix} \\ &= \begin{pmatrix} aI & (b+c+d+e)Q_{11} + (f+g+h)Q_{11}Q_{12} & 0 & (b+c)Q_{21} \\ 0 & aI + (b+c+d+e+f+g+h)Q_{12} & 0 & 0 \\ (b+d)Q_{31} & (b+d)Q_{32} + (f+h)(Q_{31}Q_{11} + Q_{32}Q_{12}) & bQ_{41} & bQ_{42} + fQ_{31}Q_{21} \\ 0 & (b+d)Q_{33} & 0 & bI \end{pmatrix}. \end{aligned} \tag{14}$$

Denote

$$\begin{gathered} A_1 = \begin{pmatrix} aI & (b+c+d+e)Q_{11} + (f+g+h)Q_{11}Q_{12} \\ 0 & aI + (b+c+d+e+f+g+h)Q_{12} \end{pmatrix}, \\ B_1 = \begin{pmatrix} 0 & (b+c)Q_{21} \\ 0 & 0 \end{pmatrix}, \\ C_1 = \begin{pmatrix} (b+d)Q_{31} & (b+d)Q_{32} + (f+h)(Q_{31}Q_{11} + Q_{32}Q_{12}) \\ 0 & (b+d)Q_{33} \end{pmatrix}, \\ D_1 = \begin{pmatrix} bQ_{41} & bQ_{42} + fQ_{31}Q_{21} \\ 0 & bI \end{pmatrix}. \end{gathered} \tag{15}$$

Since $a+b+c+d+e+f+g+h \neq 0$, then $aI + (b+c+d+e+f+g+h)Q_{12}$ and A_1 are invertible. The Schur complement has the form

$$\begin{aligned} S &= D_1 - C_1 A_1^{-1} B_1 \\ &= \begin{pmatrix} bQ_{41} & bQ_{42} + \left[f - a^{-1}(b+c)(b+d)\right]Q_{31}Q_{21} \\ 0 & bI \end{pmatrix}. \end{aligned} \tag{16}$$

Hence, by Lemma 3, Γ is invertible if and only if Q_{41} (see (12)) is invertible if and only if Q_4 (see (2)) is invertible, which is equivalent to that

$$P + Q - PQ = \begin{pmatrix} I & 0 \\ Q_3 & Q_4 \end{pmatrix} \tag{17}$$

is invertible.

If the idempotent operator Q_{41} is not invertible, by Lemma 4, S is group invertible:

$$\begin{aligned} S^{\#} &= \begin{pmatrix} b^{-1}Q_{41} & * \\ 0 & b^{-1}I \end{pmatrix}, \\ S^{\pi} &= \begin{pmatrix} I - Q_{41} & * \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{18}$$

where the omitted element $*$ can be got by Lemma 4. Note that $B_1 S^{\pi} = 0$ and $A_1^2 + B_1 S^{\pi} C_1$ are invertible. By Lemma 5, Γ is group invertible. □

If $a+b+c+d+e+f+g+h = 0$, we get the following main result.

Theorem 8. *Let P and Q be two idempotents such that $(PQ)^2 = P(QP)^2$. For any scalar a, b, c, d, e, f, g, and h with $a \neq 0, b \neq 0$, and $a+b+c+d+e+f+g+h = 0$, let Γ be defined as in* (10). *Then*

(i) *Γ is invertible if and only if $P - Q$ is invertible;*

(ii) *if any one of $I - PQ$ and $P + Q - PQ$ is invertible, then Γ is always group invertible.*

Proof. We use the notations from the proof of Theorem 7. By (14), if $a+b+c+d+e+f+g+h = 0$, then the upper left submatrix A_1 fails to be invertible. Perturb it a little. Γ as an operator on $\overline{\mathscr{R}(Q_2)} \oplus (\mathscr{N}(P) \ominus \overline{\mathscr{R}(I - Q_4)}) \oplus \overline{\mathscr{R}(I - Q_4)} \oplus (\mathscr{R}(P) \ominus \overline{\mathscr{R}(Q_2)})$ can be written as

$$\Gamma = \begin{pmatrix} aI & (b+c)Q_{21} & 0 & (b+c+d+e)Q_{11} + (f+g+h)Q_{11}Q_{12} \\ 0 & bI & 0 & (b+d)Q_{33} \\ (b+d)Q_{31} & bQ_{42} + fQ_{31}Q_{21} & bQ_{41} & (b+d)Q_{32} + (f+h)(Q_{31}Q_{11} + Q_{32}Q_{12}) \\ 0 & 0 & 0 & a(I - Q_{12}) \end{pmatrix}. \tag{19}$$

Denote

$$A_0 = \begin{pmatrix} aI & (b+c)\,Q_{21} \\ 0 & bI \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & (b+c+d+e)\,Q_{11} + (f+g+h)\,Q_{11}Q_{12} \\ 0 & (b+d)\,Q_{33} \end{pmatrix},$$

$$C_0 = \begin{pmatrix} (b+d)\,Q_{31} & bQ_{42} + fQ_{31}Q_{21} \\ 0 & 0 \end{pmatrix},$$

$$D_0 = \begin{pmatrix} bQ_{41} & (b+d)\,Q_{32} + (f+h)\,(Q_{31}Q_{11} + Q_{32}Q_{12}) \\ 0 & a\,(I - Q_{12}) \end{pmatrix}. \tag{20}$$

The Schur complement of A_0 in (19) has the structure

$$S_0 = D_0 - C_0 A_0^{-1} B_0 = \begin{pmatrix} bQ_{41} & * \\ 0 & a\,(I - Q_{12}) \end{pmatrix}. \tag{21}$$

Hence, by Lemma 3, Γ is invertible if and only if idempotents Q_{41} and $I - Q_{12}$ (see (12)) are invertible if and only if idempotents Q_4 and $I - Q_1$ are invertible, which is equivalent to the fact that $P - Q$ (or $(P - Q)^2$; see (4)) is invertible by Lemma 1.

If $I - PQ$ is invertible, then $I - Q_{12}$ is invertible by (12). Since $I - Q_{12}$ is idempotent, then $Q_{12} = 0$. By Lemma 4, S_0 is group invertible:

$$S_0^{\#} = \begin{pmatrix} b^{-1}Q_{41} & * \\ 0 & a^{-1}I \end{pmatrix},$$
$$S_0^{\pi} = I - S_0 S_0^{\#} = \begin{pmatrix} I - Q_{41} & * \\ 0 & 0 \end{pmatrix}. \tag{22}$$

If $P + Q - PQ$ is invertible, then Q_{41} is invertible by (12). Since Q_{41} is idempotent, then $Q_{41} = I$. By Lemma 4, S_0 is group invertible:

$$S_0^{\#} = \begin{pmatrix} b^{-1}I & * \\ 0 & a^{-1}\,(I - Q_{12}) \end{pmatrix}, \tag{23}$$

$$S_0^{\pi} = I - S_0 S_0^{\#} = \begin{pmatrix} 0 & * \\ 0 & Q_{12} \end{pmatrix}. \tag{24}$$

So, if any one of $I - PQ$ and $P + Q - PQ$ is invertible, then $B_0 S_0^{\pi} C_0 = 0$. By Lemma 5, $A_0^2 + B_0 S_0^{\pi} C_0 = A_0^2$ is invertible. Hence Γ is group invertible. □

3. Concluding Remarks

In Theorems 7 and 8, the inverse and the group inverse formulae can be obtained by using the results in Lemma 3 and [19, Theorem 3.1], respectively. This is a trivial and redundant work. If $P - Q$ is group invertible and $a, b \neq 0$, by definition (1), it is also trivial to check

$$\begin{aligned}(aP + bQ + cPQ)^{\#} &= b^{-1}(P-Q)^{\#}(P-Q) - b^{-1}P(P-Q)^{\#} \\ &\quad - \frac{a+c}{ab}(P-Q)^{\#}P + (a+b+c)^{\#}P(P-Q)^{\pi} \\ &\quad + \frac{a+b+c}{ab}(P-Q)^{\#}P(P-Q)^{\#}.\end{aligned} \tag{25}$$

Hence, $aP + bQ + cPQ$ is always group invertible. In particular, if $P - Q$ is invertible (see [18, Theorem 3.1]), then

$$\begin{aligned}(aP + bQ + cPQ)^{-1} &= -\frac{1}{b}Q(P-Q)^{-1} - \frac{a+c}{ab}(P-Q)^{-1}P \\ &\quad + \frac{a+b+c}{ab}(P-Q)^{-1}P(P-Q)^{-1}.\end{aligned} \tag{26}$$

From Theorems 7 and 8 we know that Γ in (10) is always group invertible if $(PQ)^2 = P(QP)^2$. It seems very difficult to find the minimum requirements that guarantee that Γ in (10) is group invertible, which could be the topic of some future research. Hence, we suggest the following question: what are the minimum requirements that guarantee that Γ in (10) is group invertible?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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