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THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF S^2-BUNDLES OVER S^4, I

Dedicated to Professor Nobuo Shimada on his 60th birthday

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0. Introduction.

The set $Eq(X)$ of homotopy classes of self-homotopy equivalences of a based space X forms a group under the composition of maps, and it is called the group of self-homotopy equivalences of X. The group $Eq(X)$ has been studied by several authors since the paper of W. D. Barcus and M. G. Barratt [1] appeared.

However, we have not yet obtained an effective method for calculating it except classical ones, and its structure also has not been clarified sufficiently. Furthermore, very little is known about this group even when X is a simply connected CW complex with three cells which is not a H-space. In particular, when X is a total space of S^m-bundles over S^n, the group $Eq(X)$ was already considred for $X=S^m\times S^n$ in [7], [8], [17], for a principal S^3-bundle over S^n in [9], [13], [16], and for the real and complex Stiefel manifolds $W_{n,2}$ and $V_{n,2}$ in [10]. Recently, S. Sasao studied the group $Eq(X)$ in [15] for the total space of S^m-bundles over S^n under the stable range, $3<m+1<n<2m-2$.

On the othe hand, it seems to be very difficult to investigate it under the unstable range. However, we would like to consider it when X is simply connected and the total space of S^m-bundles over S^n for a small pair of integers (m, n). Since X is simply connected, $n, m\geqq 2$ and the cases $(m, n)=(2, 2)$ or $(2, 3)$ were already considered by P. J. Kahn [7] and N. Sawashita [8].

Then the purpose of this paper is to study the group $Eq(X)$ for the case $(m, n)=(2, 4)$ and we will treat its application in the subsequent paper in [22].

1. Notations and Results.

All spaces have base points, and all maps and homotopies preserve base points throughout this note. We denote by $[X, Y]$ the set of based homotopy classes of maps from X to Y, and we will not distinguish between a map and its homotopy class. Let $Z\{x\}$ (resp. $Z_m\{x\}$) be the infinite cyclic group (resp. the cyclic group of order m) generated by the element x. Let RP^n (resp. CP^n)

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and SO_n be the n-dimensional real (resp. complex) projective space and the n-th rotation group, respectively. We denote by the map $\rho: S^3 \to RP^3 = SO_3$ the double covering projection, and it is trivial to see $\pi_3(SO_3) = Z\{\rho\}$. In the exact sequence $A \to B \to C \to 1$, we write the group composition in the group A as addition, and the compositions in the groups B and C as multiplication. Then our main results are stated as follows:

THEOREM 1.1. *For each integer m, let X_m be the total space of S^2-bundle over S^4 with its characteristic class $\chi(X_m) = m\rho \in \pi_3(SO_3)$.*

If m is a non-zero integer, the sequence

$$\pi_6(X_m) \xrightarrow{\ \lambda\ } Eq(X_m) \xrightarrow{\ \phi\times\psi\ } G_m \longrightarrow 1$$

is exact, where

$$G_m = \begin{cases} Z_2 & \text{if } (m, 2)=1 \\ Z_2 \times Z_2 & \text{if } (m, 2)=2 \end{cases}$$

and

$$\pi_6(X_m) \cong \begin{cases} 0 & \text{if } (m, 6)=1 \\ Z_3 & \text{if } (m, 6)=3 \\ Z_2 \oplus Z_{(12, m')} & \text{if } m=2m' \text{ for some integer } m' \end{cases}$$

Here we denote by (m, n) the greatest common measure of integers m and n.

COROLLARY 1.2. *If $(m, 6)=1$, then $Eq(X_m) \cong Z_2$.*

PROPOSITION 1.3. *If $(m, 6)=3$, then the group $Eq(X_m)$ is isomorphic to Z_2 or Z_6 and $\mathrm{Im}[\Sigma: Eq(X_m) \to Eq(\Sigma X_m)] \cong Z_2$, where Σ denotes the suspension homomorphism.*

Remark 1.4. (1) If $m=0$, $X_0 = S^2 \times S^4$ and the group $Eq(X_0)$ was already well-known. In fact, the following sequence is split exact [17]:

$$0 \longrightarrow Z_2 \oplus Z_2 \longrightarrow Eq(X_0) \longrightarrow Z_2 \times Z_2 \times Z_2 \longrightarrow 1$$

(2) If $m = 2m' \neq 0$, the homomorphism $\lambda: \pi_6(X_m) \to Eq(X_m)$ is not trivial. In fact, $\mathrm{Im}\,\lambda$ contains the subgroup isomorphic to Z_2.

This paper is organized as follows:

In section 2, we will determine the homotopy groups $\pi_*(L_m)$ and $\pi_*(\Sigma L_m)$, and in section 3, we will calculate $Eq(L_m)$ and $Eq(\Sigma L_m)$. In section 4, we will study the image of the homomorphism $\phi\times\psi$, and in section 5, we will give the proof of our main results.

2. Homotopy Groups $\pi_*(L_m)$ and $\pi_*(\Sigma L_m)$.

Let ι_n be the oriented generator of $\pi_n(S^n)$ and $\eta_2 \in \pi_3(S^2)$ be the Hopf map.

We put $\eta_n=E^{n-2}\eta_2$, $\eta_n^2=\eta_n\circ\eta_{n+1}$ and $\eta_n^3=\eta_n\circ\eta_{n+1}\circ\eta_{n+2}$ for $n>1$, where E^{n-2} denotes the iterated suspension homomorphism. Let $\omega\in\pi_6(S^3)$ be the Blackey-Massey element, and $\rho: S^3\to RP^3=SO_3$ be the double covering projection. Then the following is well-known:

LEMMA 2.1. (H. Toda, [19])

(1) $\pi_n(S^n)=Z\{\iota_n\}$ *and* $\pi_i(S^n)=0$ *for* $i<n$.

(2) $\pi_{n+1}(S^n)=Z_2\{\eta_n\}$ *for* $n>2$ *and* $\pi_3(S^2)=Z\{\eta_2\}$.

(3) $\pi_{n+2}(S^n)=Z_2\{\eta_n^2\}$ *for* $n>1$.

(4) $\pi_5(S^2)=Z_2\{\eta_2^3\}$ *and* $\pi_6(S^3)=Z_{12}\{\omega\}$.

(5) $J(\rho)=\pm\omega$,

where $J:\pi_3(SO_3)=Z\{\rho\}\to\pi_6(S^3)$ *denotes the J-homomorphism.*

(6) $\eta_3^2=\{2\iota_3, \eta_3, 2\iota_4\}$ *modulo zero,*

where $\{\ ,\ ,\ \}$ *denotes the Toda bracket.*

Let L_m be the CW complex formed by attaching the 4-cell e^4 to S^2 with the map $m\eta_2\in\pi_3(S^2)$, and $a_m:(D^4, S^3)\to(L_m, S^2)$ be the characteristic map of the 4-cell in L_m and X_m be the total space of S^2-bundle over S^4 with its characteristic element $\chi(X_m)=m\rho\in\pi_3(SO_3=Z\{\rho\}$ for an integer m. We denote by the map $p: SO_3\to SO_3/SO_2=S^2$ the natural projection map, and by the map $i: S^2\to L_m$ (resp. $i_1: L_m\to(L_m, S^2)$) the inclusion map. Then we have

LEMMA 2.2. (1) $p_*(\rho)=\eta_2$.

(2) $\pi_4(L_m, S^2)=Z\{a_m\}$.

(3) $\pi_5(L_m, S^2)=Z\{[a_m, \iota_2]_r\}\oplus a_{m*}\pi_5(D^4, S^3)$,

where $[\ ,\ ]_r$ *denotes the relative Whitehead product.*

Proof. The statements (1) and (2) are obvious, and the statement (3) follows from (3.1) in [3]. Q. E. D.

Since $p_*(\chi(X_m))=m\eta_2$, we also have

LEMMA 2.3. (I. M. James and J. H. C. Whitehead, [5], [6])

(1) *The space* X_m *has the CW-decomposition* $X_m=L_m\cup_{b_m}e^6$ *for some element* $b_m\in\pi_5(L_m)$, *and*

(2) $i_{1*}(b_m)=[a_m, \iota_2]_r$

where i_{1*} *denotes the induced homomorphism*

$$i_{1*}:\pi_5(L_m)\longrightarrow\pi_5(L_m,\ S^2).$$

Proof. The above statements follow from (3.3) in [5] and (5.1) in [6]. Q. E. D.

LEMMA 2.4. (1) $\pi_1(L_m)=0$ *and* $\pi_2(L_m)=Z\{i\}$.

(2) $\pi_3(L_m)=Z_m\{i\circ\eta_2\}$.

$$(3)\quad \pi_4(L_m)=\begin{cases}0 & \text{if } m\equiv 1 \pmod 2\\ Z_2\{i\circ\eta_2^2\} & \text{if } m\equiv 0 \pmod 2 \text{ and } m\neq 0\\ Z\{i_4\}\oplus Z_2\{i\circ\eta_2^2\} & \text{if } m=0,\end{cases}$$

where the map $i_4:S^3\rightarrow S^2\vee S^4=L_0$ *denotes the inclusion map to the second factor.*

Proof. Without loss of generalities, we may suppose $m\neq 0$ and it suffices only to show the statement (3). Consider the homotopy exact sequence of the pair $(L_m,\ S^2)$:

$$\pi_5(L_m,\ S^2)\xrightarrow{\partial_5}\pi_4(S^2)\xrightarrow{i_*}\pi_4(L_m)\xrightarrow{i_{1*}}\pi_4(L_m,\ S^2)\xrightarrow{\partial_4}\pi_3(S^2)\longrightarrow\pi_3(L_m)\longrightarrow 0.$$

Since $\partial_4(a_m)=m\eta_2$, it follows from (2.2) that the homomorphism i_{1*} is epimorphic. By using the equation $[\eta_2,\ \iota_2]=0$, $\partial_5([a_m,\ \iota_2]_r)=-[\partial_4(a_m),\ \iota_2]=0$. Similarly,

$$\begin{aligned}\partial_5(a_{m*}\pi_5(D^4,\ S^3))&=(m\eta_2)_*\pi_4(S^3)\\ &=\begin{cases}\pi_4(S^2) & \text{if } m\equiv 1 \pmod 2\\ 0 & \text{if } m\equiv 0 \pmod 2).\end{cases}\end{aligned}$$

Hence the assertion (3) easily follows from (2.2). Q. E. D.

COROLLARY 2.5. *If* $m\neq 0$, *then*

$$\operatorname{Ker}[\partial_5:\pi_5(L_m,\ S^2)\longrightarrow\pi_4(S^2)]=\begin{cases}\pi_5(L_m,\ S^2) & \text{if } m\equiv 0 \pmod 2\\ Z\{[a_m,\ \iota_2]_r\} & \text{if } m\equiv 1 \pmod 2).\end{cases}$$

LEMMA 2.6. *If* $m\neq 0$, *then*

$$\operatorname{Im}[\partial_6:\pi_6(L_m,\ S^2)\longrightarrow\pi_5(S^2)]=\begin{cases}\pi_5(S^2) & \text{if } m\equiv 1 \pmod 2\\ 0 & \text{if } m\equiv 0 \pmod 2).\end{cases}$$

Proof. First, we suppose that m is an odd integer. Since $\partial_6(a_{m*}\pi_6(D^4,\ S^3))$ $=(m\eta_2)_*\pi_5(S^3)=\eta_{2*}\pi_5(S^3)=\pi_5(S^2)$, the boundary homomorphism ∂_6 is epimorphic.

Next, we assume that m is a non-zero even integer. Consider the commutative diagram

$$\begin{array}{ccc} \pi_6(L_m, S^2) & \xrightarrow{\partial_6} & \pi_5(S^2)=Z_2\{\eta_2^3\} \\ \downarrow & & \uparrow \Delta_6 \\ \pi_6(X_m, S^2) & \underset{\cong}{\xrightarrow{p_{m*}}} & \pi_6(S^4)=Z_2\{\eta_4^2\} \end{array}$$

where Δ_n denotes the boundary homomorphism induced from the fibration

$$S^2 \longrightarrow X_m \xrightarrow{p_m} S^4, \ \Delta_n : \pi_n(S^4) \longrightarrow \pi_{n-1}(S^2).$$

Since $\chi(X_m)=m\rho$, we have

(2.7) $$\Delta_4(\iota_4)=m\eta_2.$$

Because $\eta_4^2=E(\eta_3^2)$, $\Delta_6(\eta_4^2)=\Delta_4(\iota_4)\circ\eta_3^2=(m\eta_2)\circ\eta_3^2=m(\eta_2^3)=0$. Hence Δ_6 is a trivial homomorphism. Thus the assertion follows from the above diagram. Q. E. D.

Let $q_m : L_m \to L_m/S^2=S^4$ be the pinching map which pinches S^2 in L_m to its base point. Then we have

LEMMA 2.8. (S. Oka)
Let m be an even integer. Then there is an element $\gamma_m \in \pi_5(L_m)$ satisfying the following two conditions:

(1) $q_m \circ \gamma_m = \eta_4$,

and

(2) *the order of γ_m is 2 if $m\equiv 0$ (mod 4) and 4 if $m\equiv 2$ (mod 4).*

Proof. We put $m=2m'$. For each positive integer n, let M_n be the Moore space of type $(3, Z_n)$, $M_n=S^3\cup_{n\iota_3}e^4$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} S^3 & \xrightarrow{2\iota_3} & S^3 & \xrightarrow{i''} & M_2 & \xrightarrow{q''} & S^4 \\ \| & & \downarrow m'\iota_3 & & \downarrow & & \| \\ S^3 & \xrightarrow{m\iota_2} & S^3 & \xrightarrow{i'} & M_m & \xrightarrow{q'} & S^4 \\ \| & & \downarrow \eta_2 & & \downarrow & & \| \\ S^3 & \xrightarrow{m\eta_2} & S^2 & \xrightarrow{i} & L_m & \xrightarrow{q_m} & S^4 \end{array}$$

where three horizontal sequences are cofiber sequences. It follows from the above diagram that there are two maps

$$f : M_2 \longrightarrow M_m \quad \text{and} \quad g : M_m \longrightarrow L_m$$

satisfying the following conditions:

(2.9) $q'\circ f=q''$, $f\circ i''=i'\circ(m'\iota_3)$, $q_m\circ g=q'$, and $g\circ i'=i\circ\eta_2$.

On the other hand, since $(2\iota_3)\circ\eta_3=\eta_3\circ(2\iota_4)=0$, there is a coextension of η_3,

$$\tilde{\eta}: S^5 \longrightarrow M_2$$

satisfying the condition,

(2.10) $$q''\circ\tilde{\eta}=\eta_4.$$

From Prop. 1.8 in [19],

$$\begin{aligned}2\tilde{\eta}&=\tilde{\eta}\circ(2\iota_5)\\&=i''\circ\{2\iota_3,\ \eta_3,\ 2\iota_4\}\\&=i''\circ\eta_3^2 \quad \text{(by (6) in Lemma 2.1)}\end{aligned}$$

Hence we have

(2.11) $$2\tilde{\eta}=i''\circ\eta_3^2.$$

Now we put $\gamma_m=g\circ f\circ\tilde{\eta}$. Then,

$$\begin{aligned}q_m\circ\gamma_m&=q_m\circ g\circ f\circ\tilde{\eta}\\&=q''\circ\tilde{\eta} \quad \text{(by (2.9))}\\&=\eta_4 \quad \text{(by (2.10))}\end{aligned}$$

Similarly,

$$\begin{aligned}2\gamma_m&=g\circ f\circ(2\tilde{\eta})\\&=g\circ f\circ(i''\circ\eta_3^2) \quad \text{(by (2.11))}\\&=i\circ\eta_2\circ(m'\iota_3)\circ\eta_3^2 \quad \text{(by (2.9))}\\&=m'(i\circ\eta_2^3)\\&=\begin{cases} i_*(\eta_2^3) & \text{if } m'\equiv 1 \pmod 2\\ 0 & \text{if } m'\equiv 0 \pmod 2\end{cases}\end{aligned}$$

Since the order of $i_*(\eta_2^3)$ is 2, the order of γ_m is 4 if $m\equiv 2 \pmod 4$ and 2 if $m\equiv 0 \pmod 4$. This completes the proof. Q. E. D.

Remark. 2.12. The order of the element γ_m is essentially determined by the suspension order of $S^2\cup_{m\iota_2}e^3$.

PROPOSITION 2.13. (1) *If* $m\equiv 1 \pmod 2$, *then* $\pi_5(L_m)=Z\{b_m\}$.
(2) *If* $m\equiv 0 \pmod 2$, *then*

$$\pi_5(L_m)=\begin{cases} Z\{b_m\}\oplus Z_4\{\gamma_m\} & \textit{if } m\equiv 2 \pmod 4\\ Z\{b_m\}\oplus Z_2\{\gamma_m\}\oplus Z_2\{i\circ\eta_2^3\} & \textit{if } m\equiv 0 \pmod 4\end{cases}$$

where we can put $b_m=[i, i_4]$ *and* $\gamma_m=i_4\circ\eta_4$ *if* $m=0$, *and* $2\gamma_m=i\circ\eta_2^3$ *if* $m\equiv 2$ (mod 4).

Proof. Consider the homotopy exact sequence of the pair (L_m, S^2),

$$\pi_6(L_m, S^2)\xrightarrow{\partial_6}\pi_5(S^2)\xrightarrow{i_*}\pi_5(L_m)\xrightarrow{i_{1*}}\pi_5(L_m, S^2)\xrightarrow{\partial_5}\pi_4(S^2).$$

First, we suppose that $m\equiv 1$ (mod 2). Then it follows from (2.3), (2.5) and (2.6) that we have $\pi_5(L_m)=Z\{b_m\}$. If $m=0$, then $L_0=S^2\vee S^4$ and the assertion clearly holds. Hence, we assume $m\equiv 0$ (mod 2) and $m\neq 0$. Then, from (2.3), (2.5) and (2.6) we have the following results:

(2.14) (1) $\pi_5(L_m)=Z\{b_m\}\oplus\mathrm{Tor}(\pi_5(L_m))$.

(2) The sequence

$$0\longrightarrow\pi_5(S^2)\xrightarrow{i_*}\mathrm{Tor}(\pi_5(L_m))\xrightarrow{i_{1*}}a_{m*}\pi_5(D^4, S^3)\longrightarrow 0$$

is exact. On the other hand, it follows from Theorem 2.1 in [2] that the sequence

$$0\longrightarrow\pi_2(S^2)\xrightarrow{Q}\pi_5(L_m, S^2)\xrightarrow{q_{m*}}\pi_5(S^4)\longrightarrow 0$$

is exact, where the homomorphism Q is defined by the relative Whitehead product, $Q(\iota_2)=[a_m, \iota_2]_r$. Hence the map q_m induces the isomorphism q_{m*}: $a_{m*}\pi_5(D^4, S^3)\xrightarrow{\cong}\pi_5(S^4)$, and we have the following exact sequence,

$$0\longrightarrow\pi_5(S^2)\xrightarrow{i_*}\mathrm{Tor}(\pi_5(L_m))\xrightarrow{q_{m*}}\pi_5(S^4)\longrightarrow 0. \tag{2.15}$$

Hence it follows from (2.8) and (2.15) that we have

$$\mathrm{Tor}(\pi_5(L_m))=\begin{cases} Z_4\{\gamma_m\} & \text{if } m\equiv 2 \pmod 4\\ Z_2\{\gamma_m\}\oplus Z_2\{i\circ\eta_2^3\} & \text{if } m\equiv 0 \pmod 4).\end{cases} \tag{2.16}$$

Therefore, we obtain the desired results. Q. E. D.

In the rest of this section, we will consider the homotopy group $\pi_*(\Sigma L_m)$. First, we remark that

$$\Sigma L_m=\begin{cases} S^3\vee S^5 & \text{if } m\equiv 0 \pmod 2\\ \Sigma CP^2 & \text{if } m\equiv 1 \pmod 2)\end{cases} \tag{2.17}$$

For each even integer m, let $i_5: S^5\to\Sigma L_m=S^3\vee S^5$ denote the inclusion map to the second factor.

LEMMA 2.18. *Let m be an even integer. Then thesusp ension homomorphism*

$$\Sigma : \mathrm{Tor}(\pi_5(L_m)) \longrightarrow \pi_6(\Sigma L_m) = \pi_6(S^3 \vee S^5)$$

is injective, where $\pi_6(\Sigma L_m) = \pi_6(S^3 \vee S^5) = Z_{12}\{(\Sigma i)\circ\omega\} \oplus Z_2\{i_5 \circ \eta_5\}$ *and we can choose the map* $\gamma_m \in \mathrm{Tor}(\pi_5(L_m))$ *to satisfy the condition*

$$(2.19) \qquad \Sigma\gamma_m = \begin{cases} i_5 \circ \eta_5 & \text{if } m \equiv 0 \pmod 4 \\ i_5 \circ \eta_5 + 3(\Sigma i)\circ\omega & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

Remark 2.20. It is easy to see that there are two possibilities of the choice of γ_m, γ_m and $\gamma_m + i_*(\eta_2^3)$. However, if γ_m satisfies the condition (2.19), then it is uniquely determined.

Proof. Since $\Sigma L_m = S^3 \vee S^5$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_5(S^2) & \xrightarrow{i_*} & \mathrm{Tor}(\pi_5(L_m)) & \xrightarrow{q_{m*}} & \pi_5(S^4) & \longrightarrow & 0 \\ & & \downarrow E & & \downarrow \Sigma & & \cong \downarrow E' & & \\ 0 & \longrightarrow & \pi_6(S^3) & \xrightarrow{(\Sigma i)_*} & \pi_6(\Sigma L_m) & \underset{i_{5*}}{\rightleftarrows} & \pi_6(S^5) & \longrightarrow & 0 \end{array}$$

where E, E' and Σ denote the suspension homomorphisms. Since E is monic and E' is isomorphic, it follows from the five Lemma that Σ is also monomorphic. Hence the order of $\Sigma\gamma_m$ is 2 if $m \equiv 0 \pmod 4$ and 4 if $m \equiv 2 \pmod 4$. Then it follows from $q_{m*}(\gamma_m) = \eta_4$ that we have $\Sigma\gamma_m \equiv i_5 \circ \eta_5$ modulo $\mathrm{Im}(\Sigma i)_*$. Therefore, there exists some integer n satisfying the condition

$$\Sigma\gamma_m = \begin{cases} i_5 \circ \eta_5 + n(\Sigma i)\circ\eta_3^3 & \text{if } m \equiv 0 \pmod 4 \\ i_5 \circ \eta_5 + 3(\Sigma i)\circ\omega + n(\Sigma i)\circ\eta_3^3 & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

Then by using the base change $\gamma_m \to \gamma_m + n(i \circ \eta_2^3)$, it is easy to see that γ_m satisfies the condition (2.19). Q. E. D.

An easy calculation shows the following results, and we will omit the proof.

LEMMA 2.21. *Let m be an odd integer. Then we have the following results·*

(1) $\pi_i(\Sigma L_m) = 0$ *for* $i = 1, 2$ *or* 4,

(2) $\pi_3(\Sigma L_m) = Z\{\Sigma i\}$,

(3) $\pi_5(\Sigma L_m) = Z\{\widetilde{2\iota_4}\}$, *and*

(4) $\pi_6(\Sigma L_m) = Z_6\{(\Sigma i)\circ\omega\}$,

where we denote by $\widetilde{2\iota_4}$ *the coextension of* $2\iota_4$ *which satisfies the condition* $(\Sigma q_m)\circ\widetilde{2\iota_4} = 2\iota_5$.

3. The Groups $Eq(L_m)$ and $Eq(\Sigma L_m)$.

In this section, we will determine the group structure of $Eq(L_m)$ and $Eq(\Sigma L_m)$.

DEFINITION 3.1. Let K be a CW complex with dim $K<n$. Let X be a CW complex formed by attaching the n-cell e^n to K with the map $f\in\pi_{n-1}(K)$, $X=K\cup_f e^n$. We denote by i, μ and ∇ the inclusion map, a co-action map and a folding map, respectively. Then we define the homomorphism $\lambda:\mathrm{Im}[i_*:\pi_n(K)\to\pi_n(X)]\to Eq(X)$ by the following:

$$\lambda(i\circ g)=\nabla\circ(id_X\vee i\circ g)\circ\mu: X\xrightarrow{\mu} X\vee S^n\xrightarrow{id_X\vee i\circ g} X\vee X\xrightarrow{\nabla} X\quad\text{for } g\in\pi_n(K).$$

Similarly, we define two homomorphisms

$$\phi: Eq(X)\longrightarrow Eq(K)\quad\text{and}\quad \psi: Eq(X)\longrightarrow Eq(S^n)=Z_2$$

by the restriction of maps and the degree of the top cell e^n. (See in detail, [12])

Secondly, we construct the elements of $Eq(L_m)$ and $Eq(\Sigma L_m)$.

DEFINITION 3.2. (1) For $m=0$, we define the map h_0 by the equation,

$$h_0=\iota_2\vee(-\iota_4):\ L_0=S^2\vee S^4\longrightarrow L_0=S^2\vee S^4.$$

For each even integer m, we define the map h'_0 by the equation

$$h'_0=\iota_3\vee(-\iota_5):\ \Sigma L_m=S^3\vee S^5\longrightarrow \Sigma L_m=S^3\vee S^5.$$

Clearly, if $m=0$, $h'_0=\Sigma h_0$. For each odd integer m, let $h'_0:\Sigma L_m\to\Sigma L_m$ be one of the maps which has a degree $+1$ on S^3 and -1 on the cell e^5 in ΣL_m. Since $\eta_3=-\eta_3$, the map h'_0 always exists.

(2) For each integer m, let $h_1: L_m\to L_m$ be one of the maps which has a degree -1 on S^2 and a degree $+1$ on the cell e^4 in L_m. Since

$$H(\eta_2)=\iota_3\quad\text{and}\quad [\iota_2,\iota_2]=2\eta_2,$$

$$\begin{aligned}(-\iota_2)\circ(m\eta_2)&=m((-\iota_2)\circ\eta_2)\\&=m(-\eta_2+[\iota_2,\iota_2]\circ H(\eta_2))=m\eta_2.\end{aligned}$$

Hence the map h_1 always exists. We define the map $h'_1:\Sigma L_m\to\Sigma L_m$ by the equation

$$h'_1=\begin{cases}(-\iota_3)\vee\iota_5:\Sigma L_m=S^3\vee S^5\to\Sigma L_m=S^3\vee S^5 & \text{if } m\equiv 0 \pmod 2\\ \Sigma h_1:\Sigma L_m\to\Sigma L_m & \text{if } m\equiv 1 \pmod 2.\end{cases}$$

(3) For each integer m, we define the map h_2 by the equation $h_2=\lambda(\eta_2^2)$.

In particular, if $m=0$, then $h_2=id_{L_0}+i\circ\eta_2^2\circ pr$, where pr denotes the projection map to the second factor, $pr: L_0=S^2\vee S^4\to S^4$. For each even integer m, we define the map $h_2': \Sigma L_m\to\Sigma L_m$ by the equation $h_2'=\Sigma h_2$. Then it is easy to see that $h_2'=id_{\Sigma L_m}+(\Sigma i)\circ\eta_3^2\circ(\Sigma pr)$.

THEOREM 3.3.

$$\text{(1)}\quad Eq(L_m)=\begin{cases} Z_2\{h_1\} & \text{if } m\equiv 1 \pmod 2 \\ Z_2\{h_1\}\times Z_2\{h_2\} & \text{if } m\equiv 0 \pmod 2 \text{ and } m\neq 0 \\ Z_2\{h_0\}\times Z_2\{h_1\}\times Z_2\{h_2\} & \text{if } m=0. \end{cases}$$

$$\text{(2)}\quad Eq(\Sigma L_m)=\begin{cases} Z_2\{h_0'\}\times Z_2\{h_1'\} & \text{if } m\equiv 1 \pmod 2 \\ Z_2\{h_0'\}\times Z_2\{h_1'\}\times Z_2\{h_2'\} & \text{if } m\equiv 0 \pmod 2). \end{cases}$$

(3) *The suspension homomorphism $\Sigma: Eq(L_m)\to Eq(\Sigma L_m)$ is monomorphic and isomorphic if $m=0$.*

Proof. First, we consider the case $m=0$. Since $L_0=S^2\vee S^4$, the homotopy set $[L_0, L_0]$ has a natural ring structure which is induced from the track addition and the composition of maps. It is easy to see that the ring $[L_0, L_0]$ is isomorphic to the matrix ring

$$\begin{bmatrix} \pi_2(S^2) & \pi_4(S^2) \\ \pi_2(S^4) & \pi_4(S^4) \end{bmatrix}=\begin{bmatrix} Z\{\iota_2\} & Z_2\{\eta_2^2\} \\ 0 & Z\{\iota_4\} \end{bmatrix}$$

Hence $Eq(L_0)=\{\pm h_0, \pm h_1, \pm h_2\}$, where $h_0=\iota_2\vee(-\iota_4)=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$h_1=(-\iota_2)\vee\iota_4=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad\text{and}\quad h_2=id_{L_0}+i\circ\eta_2^2\circ pr=\begin{bmatrix} 1 & \eta_2^2 \\ 0 & 1 \end{bmatrix}.$$

Since $h_n\circ h_n=id_{L_0}$ and $h_n\circ h_k=h_k\circ h_n$ for $0\leqq n,\ k\leqq 2$, we have

$$Eq(L_0)=Z_2\{h_0\}\times Z_2\{h_1\}\times Z_2\{h_2\}.$$

A similar argument shows $Eq(\Sigma L_0)=Z_2\{h_0'\}\times Z_2\{h_1'\}\times Z_2\{h_2'\}$, and it follows from $h_n'=\Sigma h_n$ that the suspension homomorphism $\Sigma: Eq(L_0)\to Eq(\Sigma L_0)$ is isomorphic.

Next, we consider the case $m\neq 0$. It follows from the Barcus-Barratt theorem [2] and Theorem 3.13 in [12] that we obtain the following commutative diagram:

$$(3.4)\quad \begin{array}{ccccccc} \mathrm{Im}\,[i_*: \pi_4(S^2)\to\pi_4(L_m)] & \xrightarrow{\lambda} & Eq(L_m) & \xrightarrow{\phi} & Eq(S^2) & \longrightarrow & 1 \\ \downarrow{\scriptstyle \Sigma_1} & & \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma_2} & & \\ 0\to\mathrm{Im}\,[(\Sigma i)_*: \pi_5(S^3)\to\pi_5(\Sigma L_m)] & \xrightarrow{\lambda} & Eq(\Sigma L_m) & \xrightarrow{\phi\times\phi} & Eq(S^3)\times Eq(S^5) & \longrightarrow & 1 \end{array}$$

where Σ, Σ_1 and Σ_2 denote the suspension homomorphisms, and two horizontal sequences are exact.

Now we suppose that m is an odd integer. Since $\pi_4(L_m)=\mathrm{Im}\,(\Sigma i)_*=0$ and Σ_2 is monomorphic, $Eq(L_m)=Z_2\{h_1\}$, $Eq(\Sigma L_m)=Z_2\{h_0'\}\times Z_2\{h_1'\}$ and Σ is a monomorphism.

Finally we assume that m is a non-zero even integer. Since $\Sigma L_m=S^3\vee S^5$, we obtain

(3.5) $$Eq(\Sigma L_m)=Z_2\{h_0'\}\times Z_2\{h_1'\}\times Z_2\{h_2'\}, \quad \text{and} \quad \mathrm{Im}\,(\Sigma i)_*=Z_2\{(\Sigma i)\circ\eta_3^2\}.$$

Hence Σ_1 is an isomorphism and the homomorphism $\lambda:\mathrm{Im}\,i_*=Z_2\{i\circ\eta_2^2\}\to Eq(L_m)$ is a monomorphism. Because Σ_2 is monic, according to the Five Lemma, Σ is a monomorphism. Therefore, it follows from (3.5) that $Eq(L_m)=Z_2\{h_1\}\times Z_2\{h_2\}$.

Q. E. D.

4. The Image of the Homomorphism $\phi\times\psi$.

The purpose of this section is to determine the image of $\phi\times\psi$,

(4.1) $$G_m=\mathrm{Im}\,[\phi\times\psi: Eq(X_m)\longrightarrow Eq(L_m)\times Eq(S^6)].$$

According to Lemma 2.2 in [12], if we identify $Eq(S^6)=Z_2=\{\pm 1\}$,

LEMMA 4.2. $G_m=\{(h,\varepsilon)\in Eq(L_m)\times\{\pm 1\} : h\circ b_m=\varepsilon b_m\}$.

Thus it suffices only to determine the action of $Eq(L_m)$ to the homotopy group $\pi_5(L_m)$ which is induced from the composition of maps,

(4.3) $$Eq(L_m)\times\pi_5(L_m)\longrightarrow\pi_5(L_m).$$

Let $j': L_m\to L_m\vee S^4$ and $j'': S^4\to L_m\vee S^4$ denote the inclusion maps to the first factor and second factor, respectively. Let $\mu: L_m\to L_m\vee S^4$ be a co-action map. First, we note the following

LEMMA 4.4. (I. M. James, [3])

(1) $\mu_*(b_m)=j'\circ b_m+[i, j'']$.

(2) $\mu_*(\gamma_m)=j'\circ\gamma_m+j'\circ\eta_4$.

(3) $\mu_*(i_*(\eta_2^3))=j'\circ i_*(\eta_2^3)$.

Proof. Let η' be the generator of $\pi_5(D^4, S^3)\cong Z_2$. According to (2.3), (2.8) and (2.13), $i_{1*}(b_m)=[a_m, \iota_2]_r$, $i_{1*}(\gamma_m)=a_{m*}(\eta')$ and $i_{1*}(i_*(\eta_2^3))=0$. Then the above results follow from Lemma 5.4 in [3].

Q. E. D.

LEMMA 4.5. (I. M. James, [4]) $\Sigma b_m=m((\Sigma i)\circ\omega)$.

Proof. Since $J(\chi(X_m))=J(m\rho)=\pm m\omega$, the assertion follows from (3.1) in [4].

Q. E. D.

Then the action (4.3) is described as follows:

THEOREM 4.6. (1) *If* $m\equiv 1 \pmod 2$, *then* $h_1\circ b_m=-b_m$.

(2) *If* $m\equiv 0 \pmod 2$, *then*

(a) $h_1\circ b_m=-b_m$,

(b) $h_2\circ b_m=b_m$,

(c) $h_1\circ\gamma_m=\begin{cases}\gamma_m+i_*(\eta_2^3) & \text{if } m\equiv 2 \pmod 4,\\ \gamma_m & \text{if } m\equiv 0 \pmod 4,\end{cases}$

(d) $h_2\circ\gamma_m=\gamma_m+i_*(\eta_2^3)$,

and

(e) $h_1\circ i_*(\eta_2^3)=h_2\circ i_*(\eta_2^3)=i_*(\eta_2^3)$ *if* $m\equiv 0 \pmod 4$.

(3) *In particular, if* $m=0$, *then*

(a) $h_0\circ b_0=-b_0$,

(b) $h_0\circ\gamma_0=\gamma_0$,

and

(c) $h_0\circ i_*(\eta_2^3)=i_*(\eta_2^3)$.

Proof. According to the cellular aproximation theorem, we may assume $h_1(S^2)\subset S^2$. Hence $h_1\circ i_1=i_1\circ h_1$. Therefore,

$$\begin{aligned} i_{1*}(h_1\circ b_m) &= h_1\circ i_{1*}(b_m)\\ &= h_{1*}([a_m, \iota_2]_r) \qquad \text{(by (2.3))}\\ &= [h_1\circ a_m, (h_1|S^2)\circ\iota_2]_r\\ &= [a_m, -\iota_2]_r\\ &= -[a_m, \iota_2]_r\\ &= -i_{1*}(b_m)\,. \end{aligned}$$

Thus, we have

$$h_1\circ b_m=-b_m \quad \text{modulo } i_*\pi_5(S^2). \tag{4.7}$$

(1) First, we suppose $m\equiv 1 \pmod 2$,
According to (2.13), $i_*\pi_5(S^2)=0$ and we have $h_1\circ b_m=-b_m$.

(2) Next, we assume $m\equiv 0 \pmod 2$.

(a) Since $i_*\pi_5(S^2)$ is contained in $\mathrm{Tor}(\pi_5(L_m))$, there exists some element $\gamma\in\mathrm{Tor}(\pi_5(L_m))$ such that $h_1\circ b_m=-b_m+\gamma$. Then

$$\begin{aligned}\Sigma\gamma &= \Sigma(h_1\circ b_m + b_m)\\ &= (\Sigma h_1)\circ\Sigma b_m + \Sigma b_m\\ &= h_1'\circ(m(\Sigma i\circ\omega)) + m(\Sigma\imath\circ\omega) \qquad \text{(by (4.5))}\\ &= -m(\Sigma i\circ\omega) + m(\Sigma i\circ\omega)\\ &= 0.\end{aligned}$$

Therefore, according to (2.18), $\gamma=0$. Hence $h_1\circ b_m=-b_m$.

(b) Since $[\iota_2, \eta_2^2]=0$,

$$\begin{aligned}h_2\circ b_m &= \lambda(\eta_2^2)\circ b_m\\ &= \nabla\circ(id_{L_m}\vee i\circ\eta_2^2)\circ\mu_*(b_m)\\ &= \nabla\circ(id_{L_m}\vee i\circ\eta_2^2)\circ(j'\circ b_m + [i, j'']) \qquad \text{(by (4.4))}\\ &= b_m + i\circ[\iota_2, \eta_2^2]\\ &= b_m.\end{aligned}$$

(c) If $m\equiv 2 \pmod 4$, then

$$\begin{aligned}\Sigma(h_1\circ\gamma_m) &= h_1'\circ\Sigma\gamma_m\\ &= ((-\iota_3)\vee\iota_5)\circ(i_5\circ\eta_5 + 3(\Sigma\imath\circ\omega)) \qquad \text{(by (2.19)}\\ &= -(i_5\circ\eta_5 + 3(\Sigma\imath\circ\omega))\\ &= -\Sigma\gamma_m\\ &= \Sigma(\gamma_m + i_*(\eta_2^3)).\end{aligned}$$

Hence, according to (2.18), we have $h_1\circ\gamma_m=\gamma_m+i_*(\eta_2^3)$. If $m=0 \pmod 4$, then it follows from (2.19) that $\Sigma\gamma_m=i_5\circ\eta_5$. Hence, a similar calculation shows $h_1\circ\gamma_m=\gamma_m$.

(d) $$\begin{aligned}h_2\circ\gamma_m &= \lambda(\eta_2^2)\circ\gamma_m\\ &= \nabla\circ(id_{L_m}\vee\imath\circ\eta_2^2)\circ\mu_*(\gamma_m)\\ &= \nabla\circ(id_{L_m}\vee\imath\circ\eta_2^2)\circ(j'\circ\gamma_m + j''\circ\eta_4) \qquad \text{(by (4.4))}\\ &= \gamma_m + \imath_*(\eta_2^3).\end{aligned}$$

(e) $$\begin{aligned}\Sigma(h_1\circ\imath_*(\eta_2^3)) &= h_1'\circ(\Sigma i)\circ\eta_3^3\\ &= ((-\iota_3)\vee\iota_5)\circ(\Sigma i)\circ\eta_3^3\\ &= -\Sigma\imath\circ\eta_3^3\\ &= \Sigma(i\circ\eta_2^3).\end{aligned}$$

Hence, according to (2.18), we have $h_1\circ i_*(\eta_2^3)=i_*(\eta_2^3)$.

$$\begin{aligned} h_2\circ i_*(\eta_2^3) &= \nabla\circ(id_{L_m}\vee i\circ\eta_2^2)\circ\mu_*(i_*(\mu_2^3)) \\ &= \nabla\circ(id_{L_m}\vee i\circ\eta_2^2)\circ j'\circ i_*(\eta_2^3) \\ &= i_*(\eta_2^3). \end{aligned}$$

Thus,

$$h_1\circ i_*(\eta_2^3)=h_2\circ i_*(\eta_2^3)=i_*(\eta_2^3).$$

(3) If $m=0$, then it follows from (2.13) and (3.2) that $h_0=\iota_2\vee(-\iota_4)$, $b_0=[i, i_4]$ and $\gamma_0=i_4\circ\eta_4$. Therefore, it is easy to see the assertion (3) and we omit the proof. Q. E. D.

In particular, it follows from (4.2) that we obtain the following

COROLLARY 4.8.

$$G_m\cong\begin{cases} Z_2 & \textit{if } m\equiv 1 \pmod 2 \\ Z_2\times Z_2 & \textit{if } m\equiv 0 \pmod 2 \textit{ and } m\neq 0 \\ Z_2\times Z_2\times Z_2 & \textit{if } m=0. \end{cases}$$

5. The Proof of the Main Results.

In this section, we will prove (1.1) and (1.3).

Let $j: L_m\to X_m$ be an inclusion map and $b'_m\in\pi_6(X_m, L_m)$ denote the characteristic map of the top cell e^6 in X_m. Consider the homotopy exact sequence

$$\pi_6(L_m)\xrightarrow{j_*}\pi_6(X_m)\longrightarrow\pi_6(X_m, L_m)=Z\{b'_m\}\xrightarrow{\partial}\pi_5(L_m)=Z\{b_m\}\oplus\mathrm{Tor}(\pi_5(L_m))$$

Since $\pi_6(X_m, L_m)=Z\{b'_m\}$ and $\partial(b'_m)=b_m$, we have

LEMMA 5.1. $\mathrm{Im}[j_*:\pi_6(L_m)\longrightarrow\pi_6(X_m)]=\pi_6(X_m)$.

Hence, according to the Barcus-Barratt Theorem and (4.8), we obtain

LEMMA 5.2. *The sequence*

$$\pi_6(X_m)\xrightarrow{\lambda} Eq(X_m)\xrightarrow{\phi\times\phi} G_m\longrightarrow 1$$

is exact.

Therefore, to prove (1.1), it suffices only to show the following

PROPOSITION 5.3.

$$\pi_6(X_m)\cong\begin{cases} 0 & \textit{if } (m, 6)=1 \\ Z_3 & \textit{if } (m, 6)=3 \\ Z_2\oplus Z_{(12, m')} & \textit{if } m=2m'\neq 0. \end{cases}$$

Consider the homotopy exact sequence of the fibration $\xi_m : S^2 \xrightarrow{j\circ i} X_m \xrightarrow{p_m} S^4$,

$$\pi_7(S^4) \xrightarrow{\Delta_7} \pi_6(S^2) \longrightarrow \pi_6(X_m) \xrightarrow{p_{m*}} \pi_6(S^4) \xrightarrow{\Delta_6} \pi_5(S^2). \tag{5.4}$$

Here, according to [19], we note

$$\pi_5(S^2)=Z_2\{\eta_2^3\}, \quad \pi_6(S^2)=Z_{12}\{\eta_2\circ\omega\}, \quad \pi_6(S^4)=Z_2\{\eta_4^2\}, \tag{5.5}$$
$$\textit{and} \quad \pi_7(S^4)=Z\{\nu_4\}\oplus Z_{12}\{E\omega\}.$$

Then we have

LEMMA 5.6.

(1) $\Delta_6(\eta_4^2)=m(\eta_2^3)=\begin{cases} \eta_2^3 & \textit{if } m\equiv 1 \pmod 2 \\ 0 & \textit{if } m\equiv 0 \pmod 2. \end{cases}$

(2) $\Delta_7(E\omega)=m(\eta_2\circ\omega)$.

(3) $\Delta_7(\nu_4)=\pm(m(m-1)/2)(\eta_2\circ\omega)$.

Proof. (1) Since $\eta_4^2=E(\eta_3^2)$, according to (2.7),

$$\Delta_6(\eta_4^2)=\Delta_4(\iota_4)\circ\eta_3^2=(m\eta_2)\circ\eta_3^2=m(\eta_2^3).$$

(2) Similarly, since $[\iota_3, \iota_3]=0$, $\Delta_7(E\omega)=\Delta_4(\iota_4)\circ\omega=(m\eta_2)\circ\omega=m(\eta_2\circ\omega)$.

(3) Consider the induced fibration $\nu_4^*\xi_m$:

$$\begin{array}{cccccc}
\xi_m : & S^2 & \xrightarrow{j\circ i} & X_m & \xrightarrow{p_m} & S^4 \\
 & \| & & \uparrow h & & \uparrow \nu_4 \\
\nu_4^*\xi_m : & S^2 & \longrightarrow & X & \longrightarrow & S^7
\end{array}$$

Then we have the commutative diagram

$$\begin{array}{ccccc}
\pi_7(S^4) & \xrightarrow{\Delta_7} & \pi_6(S^2) & \xrightarrow{(j\circ i)_*} & \pi_6(X_m) \\
\uparrow \nu_{4*} & & \| & & \uparrow h_* \\
\pi_7(S^7) & \xrightarrow{\Delta_7'} & \pi_6(S^2) & \longrightarrow & \pi_6(X) \\
\| & & & & \\
Z\{\iota_7\} & & & &
\end{array}$$

Hemce

(5.7) $$\Delta_7(\nu_4)=\Delta_7(\nu_{4*}(\iota_7))=\Delta'_7(\iota_7).$$

Since $\rho: S^3 \to RP^3=SO_3$ is a double covering projection, the induced map $B\rho: HP^\infty = BS^3 \to BSO_3$ is a fibration with its fiber $BZ_2=K(Z_2, 1)$. Then, if $n>2$, we have the composite of isomorphisms

(5.8) $$\pi_n(HP^\infty)=\pi_n(BS^3) \xrightarrow[\cong]{B\rho_*} \pi_n(BSO_3) \xrightarrow[\cong]{ad} \pi_{n-1}(SO_3),$$

where ad denotes the adjoint isomorphism. Since $\pi_3(SO_3)=Z\{\rho\}$, there exists a map $\rho' \in \pi_4(HP^\infty)$ such that,

(5.9) $$ad \circ B\rho_*(\rho')=\rho \quad \text{and} \quad \pi_4(HP^\infty)=Z\{\rho'\}.$$

Furthermore, according to the celluar approximation theorem, we have

(5.10) $$\rho'=j_2 \circ j_1,$$

where the maps $j_1: S^4 \to HP^2=S^4 \cup_{\nu_4} e^8$ and $j_2: HP^2 \to HP^\infty$ denote the natural inclusion maps. Let $c(\xi_m) \in \pi_4(BSO_3)$ and $c(\nu_4^* \xi_m) \in \pi_7(BSO_3)$ denote the characteristic classes of the S^2-bundles ξ_m and $\nu_4^*\xi_m$. We put $\rho''=B\rho_*(\rho') \in \pi_4(BSO_3)$. Since $\chi(X_m)=m\rho$, according to (5.8) and (5.9), we have

(5.11) $$c(\xi_m)=m(B\rho_*(\rho'))=m\rho''.$$

Since $[\iota_4, \iota_4]=2\nu_4 \pm E\omega$ and $H(\nu_4)=\iota_7$,

$$\begin{aligned} c(\nu_4^*\xi_m) &= \nu_4^* c(\xi_m) \\ &= c(\xi_m) \circ \nu_4 \\ &= (m\rho'') \circ \nu_4 \\ &= m(\rho'' \circ \nu_4)+(m(m-1)/2)[\rho'', \rho''] \circ H(\iota_4) \\ &= m(\rho'' \circ \nu_4)+(m(m-1)/2)\rho'' \circ [\iota_4, \iota_4] \circ \iota_7 \\ &= m(\rho'' \circ \nu_4)+(m(m-1)/2)\rho'' \circ (2\nu_4 \pm E\omega) \\ &= m^2(\rho'' \circ \nu_4) \pm (m(m-1)/2)(\rho'' \circ E\omega). \end{aligned}$$

Because HP^2 is a mapping cone of ν_4, $j_1 \circ \nu_4=0$. Hence $\rho'' \circ \nu_4=(B\rho \circ \rho') \circ \nu_4 = B\rho \circ (j_2 \circ j_1) \circ \nu_4=0$, and

$$c(\nu_4^*\xi_m)=\pm(m(m-1)/2)(\rho'' \circ E\omega)=\pm(m(m-1)/2)B\rho_*(\rho') \circ E\omega.$$

Thus, according to (5.8) and (5.9), we obtain

(5.12) $$\chi(X)=\pm(m(m-1)/2)\rho \circ \omega,$$

where $\chi(X) \in \pi_6(BSO_3)$ denotes the characteristic element of $\nu_4^*\xi_m$. Then, if $p: SO_3 \to SO_3/SO_2=S^2$ is a natural projection map, $p_*(\rho)=\eta_2$ and

$$\begin{aligned}\Delta_7(\nu_4)&=\Delta_7'(\iota_7) && \text{(by (5.7))}\\ &=p_*(\chi(X))\\ &=\pm(m(m-1)/2)p_*(\rho)\circ\omega && \text{(by (5.12))}\\ &=\pm(m(m-1)/2)(\eta_2\circ\omega).\end{aligned}$$

This completes the proof. Q. E. D.

Proof of Proposition 5.3. Consider the homotopy exact sequence (5.4). First, we suppose $m\equiv 1 \pmod 2$. Then according to (5.6), we have

$$\pi_6(X_m)=Z_{(12,m)}\{(j\circ i)_*(\eta_2\circ\omega)\}=Z_{(m,3)}\{(j\circ i)_*(\eta_2\circ\omega)\}\cong\begin{cases}0 & \text{if } (m,3)=1\\ Z_3 & \text{if } (m,3)=3.\end{cases}$$

Now, we assume that $m=2m'\neq 0$ for some integer m'. According to (5.6), the following is exact:

$$\text{(5.13)}\qquad 0\longrightarrow Z_{(12,m')}\{\eta_2\circ\omega\}\xrightarrow{(j\circ i)_*}\pi_6(X_m)\xrightarrow{p_{m*}}\pi_6(S^4)\longrightarrow 0.$$

Since $(m\eta_2)\circ\eta_3^2=m(\eta_2^3)=0$, there exists a coextension of η_3^2, $\tilde{\eta}\in\pi_6(L_m)$ such that,

$$\text{(5.14)}\qquad q_m\circ\tilde{\eta}=\eta_4^2.$$

Furthermore, because $\pi_6(L_m)$ is a finite group, the order of $\tilde{\eta}$ is 2. Hence it follows from $p_m|L_m=q_m$ that the sequence (5.13) is split exact. Hence $\pi_6(X_m)=Z_{(12,m')}\{(j\circ i)_*(\eta_2\circ\omega)\}\oplus Z_2\{\tilde{\eta}\}\cong Z_{(12,m')}\oplus Z_2$. Q. E. D.

Proof of Theorem 1.1.
The assertion easily follows from (4.8), (5.2) and (5.3). Q. E. D.

Proof of Proposition 1.3.
Let m be an integer which satisfies the condition $(m, 6)=3$. Then it follows from the proof of (5.3) that $\pi_6(X_m)=Z_3\{(j\circ i)_*(\eta_2\circ\omega)\}$. According to [19], $\pi_6(S^3)=Z_{12}\{\omega\}=Z_4\{\nu'\}\oplus Z_3\{\alpha_1(3)\}$. Hence $(j\circ i)_*(\eta_2\circ\omega)=(j\circ i)_*(\eta_2\circ\alpha_1(3))$. Thus, it follows from (1.1) that the group $Eq(X_m)$ is generated by two elements,

$$\text{(5.15)}\qquad \theta_1=\lambda((j\circ i)_*(\eta_2\circ\alpha_1(3))) \text{ and } \theta_2,$$

where θ_2 denotes the map in $[X_m, X_m]$ which satisfies the conditions,

$$\text{(5.16)}\qquad \phi(\theta_2)=h_1\in Eq(L_m)=Z_2\{h_1\} \text{ and } \psi(\theta_2)=1\in Eq(S^4)=Z_2=\{\pm 1\}.$$

According to (3.3), the suspension homomorphism $\Sigma: Eq(L_m)\to Eq(\Sigma L_m)$ is monomorphic. Therefore

$$\text{(5.17)}\qquad \Sigma\theta_2\neq id_{\Sigma X_m} \text{ and } (\Sigma\theta_2)\circ(\Sigma\theta_2)=id_{\Sigma X_m}.$$

Since $E(\eta_2\circ\alpha_1(3))=\eta_3\circ E\alpha_1(3)=0$, according to the naturality of the Barcus-Barratt operation λ, we have

$$\begin{aligned}\Sigma\theta_1&=\Sigma\lambda((j\circ i)_*(\eta_2\circ\alpha_1(3)))\\&=\lambda(\Sigma(j\circ i)_*(E(\eta_2\circ\alpha_1(3))))\\&=\lambda(0)\\&=id_{\Sigma X_m}.\end{aligned}$$

Then it follows from (5.15) and (5.17) that we obtain

$$\operatorname{Im}[\Sigma : Eq(X_m) \longrightarrow Eq(\Sigma X_m)]=Z_2\{\Sigma\theta_2\}\cong Z_2.$$

This completes the proof. Q. E. D

REFERENCES

[1] W. D, BARCUS AND M. G. BARRATT, On the homotopy classification of a fixed map, Tran. Amer. Math. Soc. 88 (1958), 57-74.

[2] I. M. JAMES, On the homotopy groups of certain pairs and triads, Quart. J. Math. Oxford, 5 (1954), 260-270.

[3] I. M. JAMES, Note on cup-products, Proc. Amer. Math. Soc., 8 (1957), 374-383.

[4] I. M. JAMES, On sphere bundles over spheres, Comment. Math. Helv., 35 (1961), 126-135.

[5] I. M. JAMES AND J. H. C. WHITEHEAD, The homotopy theory of sphere bundles over spheres (I), Proc. London Math. Soc., 4 (1954), 196-218.

[6] I. M. JAMES AND J. H. C. WHITEHEAD, The homotopy theory of sphere bundles over spheres (II), Proc. London Math. Soc., 5 (1955), 148-166.

[7] P. J. KAHN, Self-equivalences of $(n-1)$-connected $2n$-manifolds, Math. Ann., 180 (1969), 26-47.

[8] J. LEVINE, Self-equivalences of $S^n\times S^k$, Trans. Amer. Math. Soc., 143 (1969), 523-543.

[9] M. MIMURA AND N. SAWASHITA, On the group of self-homotopy equivaleces of principal S^3-bundles over spheres, Hiroshima Math. J., 14 (1984), 415-424.

[10] Y. NOMURA, Self-homotoy equivalences of Stiefel manifolds $W_{n,2}$ and $V_{n,2}$, Osaka J. Math., 20 (1983), 79-93.

[11] S. OKA, Groups of self-equivalences of certain complexes, Hiroshima Math, J., 2 (1972), 285-293.

[12] S. OKA, N. SAWASHITA AND M. SUGAWARA, On the group of self-equivalences of a mapping cone, Hiroshima Math. J., 4 (1974), 9-28.

[13] J. W. RUTTER, The group of self-homotopy equivalences of principal three sphere bundles over the seven sphere, Math. Proc. Camb. Phil. Soc., 84 (1978), 303-311.

[14] J. W. RUTTER, The group of homotopy self-equivalence classes of CW complexes, Math. Proc. Camb. Phil. Soc., 93 (1983), 275-293.

[15] S. SASAO, On self-homotopy equivalences of the total spaces of a sphere bundle over a sphere, Kodai Math. J., 7 (1984), 365-381.

[16] S. SASAO, On self-homotopy equivalences of S^3-principal bundles over S^n, Kodai Math. J., 8 (1985), 285-295.

[17] N. SAWASHITA, On the group of self-equivalences of the product of spheres, Hiroshima Math. J., **5** (1975), 69-86.

[18] A. J. SIERADSKI, Twisted self-homotopy equivalences, Pacific J. Math., **34** (1970), 789-802.

[19] H. TODA, Composition methods in homotopy groups of spheres, Annals of Math. Studies, **49** (1962), Princeton Univ. Press.

[20] K. YAMAGUCHI, On the homotopy type of CW complexes with the form $S^2 \cup e^4 \cup e^6$, Kodai Math. J., **5** (1982), 303-312.

[21] K. YAMAGUCHI, Corrections on the homotopy type of CW complexes with the form $S^2 \cup e^4 \cup e^6$, Kodai Math. J., **6** (1983), 443.

[22] K. YAMAGUCHI, The group of self-homotopy equivalences of S^2-bundles over S^4. II Applications, preprint.

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