

The Groups of Order p^6 (p an Odd Prime)

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Abstract. A complete list of the groups of order p^6 , where p denotes an odd prime number, is given using P. Hall's concept of isoclinism.

1. Introduction. In 1940, P. Hall [7] provided a method for classifying groups (especially p -groups, where p is a prime) into families and in 1964, M. Hall and J. Senior [6] used this method to produce a complete list of all the groups of order 2^n ($n \leq 6$). This paper extends their work to odd primes by providing a complete list of all groups of order p^6 ($p > 2$). The number of non-abelian groups of order 3^6 is found to be 491, and the number of non-abelian groups of order p^6 ($p > 3$) is found to be

$$\frac{1}{4}\{13p^2 + 145p + 1338 + 80(p-1, 3) + 45(p-1, 4) + 8(p-1, 5) + 8(p-1, 6)\},$$

where $(p-1, n)$ denotes the greatest common divisor of $p-1$ and the integer n .

Blackburn [3] includes a list of the groups of order p^6 and class 5, and Leong [11] and Miech [12] have lists of certain subclasses of the class of p -groups with cyclic derived groups, all of which agree with the present list where they overlap it. A list of the groups of order p^5 ($p > 3$) appears in [1], [2], [4] and [13], the first three of which also include the groups of order 3^5 . The present list for $p > 3$ agrees with those of Bender and Schreier, who in turn claim to agree with Bagnera and de Séguier. For $p = 3$, Bender has corrected two errors by Bagnera but has omitted the group $\Delta_{10}(2111)a_2$, which is included by Blackburn and de Séguier. The present list of families (as defined by P. Hall) agrees with that of Easterfield [5], whose ordering I have followed.

This work is a summary of my Ph.D. Thesis [9] which may be consulted for most of the detailed calculations, although errors in that thesis have been corrected and an uncompleted family in the thesis has been completed. As well as the acknowledgements in my thesis, I would also like to thank Richard Keane, who pointed out errors in the original list of 3-groups, and Dr. M. F. Newman, who has hounded me into publishing the present paper and given me a great deal of helpful advice and encouragement.

2. A Guide to the Table and List of Groups. Information concerning the groups of order p^m ($m \leq 6$), with p an odd prime, is given in Table 4.1, and the list of groups in Sections 4.2 to 4.6. Each group of order p^m in the list is presented in terms of

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certain generators and relations, and given a designation of the form $\Phi_s(m_1, m_2, \dots, m_r)x_t$, where $r, s, t, m_1, m_2, \dots, m_r$, are positive integers with $m_1 + m_2 + \dots + m_r = m$ ($m_1 \geq m_2 \geq \dots \geq m_r$), and x is a letter.

The presentation $\langle \alpha_1, \alpha_2, \dots, \alpha_n \mid w_1 = w_2 = \dots = w_k = 1 \rangle$ for the group G means that G is the largest group generated by the symbols $\alpha_1, \alpha_2, \dots, \alpha_n$ subject to the conditions $w_1(\alpha_1, \alpha_2, \dots, \alpha_n) = w_2(\alpha_1, \alpha_2, \dots, \alpha_n) = \dots = w_k(\alpha_1, \alpha_2, \dots, \alpha_n) = 1$, where the w 's represent words in at most n variables. In particular, $[\alpha, \beta]$ is the word $\alpha^{-1}\beta^{-1}\alpha\beta$ and $\alpha_{i+1}^{(p)}$ will denote the word

$\alpha_{i+1}^{(p)}\alpha_{i+2}^{(p)} \dots \alpha_{i+k}^{(p)} \dots \alpha_{i+p}$ where i is a positive integer and $\alpha_{i+2}, \dots, \alpha_{i+p}$ are suitably defined. (Note that, for large enough p , this will often be just $\alpha_{i+1}^{(p)}$.) For economy of space, *all relations of the form $[\alpha, \beta] = 1$ (with α, β generators) have been omitted from the list* and should be assumed when reading the list. No attempt has been made to find minimal presentations, and those chosen are designed to accentuate the groups' structures.

The groups in the list are collected together in *isoclinism* families. Two groups G, H with centers $Z(G), Z(H)$ and derived groups G_2, H_2 are said to be *isoclinic* (written $G \sim H$) if there exist isomorphisms

$$\theta : G/Z(G) \rightarrow H/Z(H),$$

$$\phi : G_2 \rightarrow H_2,$$

such that $\phi([\alpha, \beta]) = [\alpha', \beta']$ for all $\alpha, \beta \in G$, where $\alpha'Z(H) = \theta(\alpha Z(G))$ and $\beta'Z(H) = \theta(\beta Z(G))$. It is easy to show that this relation is well defined and is in fact an equivalence relation. The pair (θ, ϕ) of isomorphisms is called an *isoclinism* (or *autoclinism* if $G = H$), and the equivalence classes are called (isoclinism) *families*. A family of p -groups will be denoted by Φ_s if p is an arbitrary prime and Δ_s if $p = 3$, where s is some positive integer. P. Hall [7] has shown that every family Φ has groups of minimal order $p^{m(0)}$, called the *stem* groups of Φ , and the set $p^{m(0)+k}\Phi$ of groups in Φ of order $p^{m(0)+k}$ is called the *kth branch* of Φ . Thus, the number $m(0)$ is an invariant for the family, called the *rank* of Φ . Some of the family invariants are tabulated in Table 4.1, and these include: nilpotency class, lower central series,* $p^{-k}q_i(G)$ and $p^{-k}r_i(G)$, where G is in the k th branch of a family and $q_i(G), r_i(G)$ denote, respectively, the number of conjugacy classes of G with precisely p^i members and the number of irreducible complex representations of G of degree p^i .

In the designation $\Phi_s(m_1, m_2, \dots, m_r)x_t$ for a group, the symbol Φ_s denotes the isoclinism family containing the group (in Easterfield's ordering). The numbers m_1, m_2, \dots, m_r are the type invariants of the group G when p is large enough for G to be regular (and used analogously when p is small) and are defined as follows:

If G is a regular group, write

$$G^{p^i} = \{x^{p^i} \mid x \in G\}, \quad i = 0, 1, 2, \dots \quad (\text{a group!}),$$

*In this paper, two groups will be regarded as the same if they are isomorphic.

and

$$p^{w(i)} = \text{the order of the factor group } G^{p^{i-1}}/G^{p^i} \quad (i = 1, 2, \dots);$$

then

$$m_j = \text{number of } w(i)\text{'s with } w(i) \geq j \quad (j = 1, 2, \dots, r)$$

(these numbers may be thought of as describing the power structure of G). For simplicity, we will write \mathbf{m} for the partition (m_1, m_2, \dots, m_r) and say that G has type \mathbf{m} . Finally, the symbol x_n means that the group is the n th group of genus x , where two groups G and H of the same family and having the same type are said to have the same genus if there is a bijection $b: M(G) \rightarrow M(H)$ such that M and $b(M)$ are in the same family and have the same type and genus for all $M \in M(G)$, with $M(G) =$ the set of all maximal subgroups of G . In particular, two abelian groups have the same genus if and only if they have the same type (when this happens the symbol x will be omitted).

The list $p^m\Phi_s$ of groups in Φ_s of order p^m is ordered as follows:**

(1) $\Phi_s(m_1, m_2, \dots, m_r)$ occurs before $\Phi_s(m'_1, m'_2, \dots, m'_r)$ if $m_1 = m'_1, \dots, m_i = m'_i, m_{i+1} > m'_{i+1}$ for some $i < \min(r, r')$. When this occurs, we will write $\mathbf{m} < \mathbf{m}'$ where \mathbf{m} represents the partition (m_1, m_2, \dots, m_r) and \mathbf{m}' represents the partition $(m'_1, m'_2, \dots, m'_r)$.

(2) To describe the ordering of the groups in Φ_s with type \mathbf{m} , we introduce for any group G the set $M_i(G)$ of maximal subgroups of G in Φ_i , the set $M_{i,\mathbf{n}}(G)$ of maximal subgroups of G in Φ_i having type \mathbf{n} and the set $M_{i,\mathbf{n},x}(G)$ of maximal subgroups of G in $M_{i,\mathbf{n}}(G)$ having genus x . If G is a group of the form $\Phi_s(m_1, m_2, \dots, m_r)x$ and H is a group of the form $\Phi_s(m_1, m_2, \dots, m_r)x'$ with $x \neq x'$, then we may assume that for some Φ_i there is a bijection $b_j: M_j(G) \rightarrow M_j(H)$ which preserves type and genus for all $j > i$ but no such bijection for $j = i$. G occurs before H (and we write $x < x'$) if

(a) $M_i(G)$ has more elements than $M_i(H)$; or

(b) for some partition \mathbf{n} of $m - 1$, $M_{i,\mathbf{n}'}(G)$ and $M_{i,\mathbf{n}'}(H)$ have the same number of elements for all partitions $\mathbf{n}' < \mathbf{n}$, and $M_{i,\mathbf{n}}(G)$ has more elements than $M_{i,\mathbf{n}}(H)$; or

(c) for some partition \mathbf{n} of $m - 1$ and genus y , $M_{i,\mathbf{n}',z}(G)$ and $M_{i,\mathbf{n}',z}(H)$ have the same number of elements for all partitions $\mathbf{n}' < \mathbf{n}$ and all genera z , $M_{i,\mathbf{n},y'}(G)$ and $M_{i,\mathbf{n},y'}(H)$ have the same number of elements for all genera $y' < y$, and $M_{i,\mathbf{n},y}(G)$ has more elements than $M_{i,\mathbf{n},y}(H)$.

(3) The groups in Φ_s with type \mathbf{m} and genus x are ordered in the way suggested by parameters in the defining relations.

Other notation used is as follows:

**Despite this ordering, the groups which are direct products are listed together for convenience. Also, if the above ordering varies depending on the value of p , the most favorable value is taken in each case.

(1) If G is a group, the *center* of G is

$$Z(G) = \{ \alpha \in G : \alpha\beta = \beta\alpha \text{ for all } \beta \in G \}$$

and the *lower central series* is $G_2 = [G, G]$, $G_{i+1} = [G_i, G]$ for $i > 2$, where

$$[H, K] = \text{the largest subgroup of } G \text{ containing } \{ [\alpha, \beta] : \alpha \in H, \beta \in K \}.$$

The *class* of G is the number c such that $G_c \neq 1$, $G_{c+1} = 1$.

(2) If G, H are groups, their *direct product* is

$$G \times H = \{ (\alpha, \beta) : \alpha \in G, \beta \in H \},$$

where multiplication of ordered pairs is component-wise.

(3) In keeping with usual practice, the symbol Φ_1 will be omitted from all abelian groups (the class Φ_1), which will be designated by their types.

(4) Throughout, ν denotes the smallest positive integer which is a non-quadratic residue (mod p) and g denotes the smallest positive integer which is a primitive root (mod p).

As an illustration of the above, the last five groups in $p^5\Phi_3$ are $\Phi_3(2111)b_\nu = \Phi_3(211)b_\nu \times (1)$ (the direct product of the (cyclic) group (1) of order p and the group $\Phi_3(211)b_\nu$), $\Phi_3(2111)c$, $\Phi_3(2111)d$, $\Phi_3(2111)e$ and $\Phi_3(1^6)$. The first of these has p^2 maximal subgroups in Φ_3 of type (211), the second has $p^2 - 1$ maximal subgroups in Φ_3 of that type, and the others (except the last) have no maximal subgroups in Φ_3 . The third of these groups has p maximal subgroups in Φ_2 of type (211), whereas the fourth group only has $p - 1$ such maximal subgroups.

3. Families of p -Groups of Rank at Most 6. In this section, we outline the method used in [9] to obtain a complete list of the groups of order p^m ($1 \leq m \leq 6$) given in the next section.

In finding these groups, we make use of the well-known list of abelian groups of order p^m , each one corresponding uniquely to a partition of the number m . In particular, since the groups of order p and p^2 are abelian, the only group of order p is the cyclic group (1) and the only two groups of order p^2 are (2) and (11).

If G is a group, we will write $|G|$ for the order of G , $Z(G)$ for the center of G and G_2, G_3, \dots, G_c for the lower central series of G (i.e. $G_2 = [G, G]$ and $G_{i+1} = [G_i, G]$ for $i > 2$). If $|G| = p^m$ and the groups of order p, p^2, \dots, p^{m-1} are already listed then, since $|G/Z(G)| < p^m$, the factor group $G/Z(G)$ is listed. Thus, the process consists of determining all those p -groups H , with $|H| < p^m$, capable of being central quotients $G/Z(G)$ for some G and then constructing families of groups G with $G/Z(G)$ isomorphic to H . An example of this may be found in Hall and Senior [6] where the groups of order 2^n ($n \leq 6$) are determined. Similarly, P. Hall [7] used the groups of order p^2, p^3, p^4 to determine the families of rank 5 and, hence, the groups of order p^5 . This was then used by Easterfield [5], and later James, to find all the families of rank ≤ 6 in the Table 4.1. Details of this may be found in [9].

The method of finding the isomorphism classes of groups for an individual family is essentially the same as that used by Blackburn [3] in determining p -groups of

maximal class and is described in [10]. By definition of isoclinism we may suppose that all commutator relations of any group of the family are known, and all relations modulo the center of the group are known. Thus, we only need consider the values of the remaining relations, namely

- (a) the structure of the center of the group;
- (b) the relationship between the center and commutator subgroup of the group;
- (c) the precise value of all words forced to be in the center of the group by the relations modulo the center.

As described in [9] and [10], the isomorphism problem in the family for groups with the relations (a) and (b) specified is equivalent to the determination of equivalence classes of certain matrices (over the field Z_p with p elements) for a certain equivalence relation. The actual calculations were carried out in [9] for all families except Φ_{21} , although there are errors in some of the calculations.

To illustrate the method used, we shall find the groups of order p^6 in this family Φ_{21} for $p > 3$. If $G \in \Phi_{21}$, then Table 4.1 shows that we may suppose $G = \langle (\alpha_1, \alpha_2, \alpha, Z(G)) \rangle$, where $\beta = [\alpha_1, \alpha_2]$, $\beta_i = [\beta, \alpha_i]$, $[\alpha, \alpha_1] = \beta_2$, $[\alpha, \alpha_2] = \beta_1^p$, $\beta^p = \beta_i^p = 1$ and $\alpha^p, \alpha_i^p \in Z(G)$ for $i = 1, 2$. Thus $G_2 = \langle \beta, G_3 \rangle$ and $G_3 = \langle \beta_1, \beta_2 \rangle$ which is in $Z(G)$. If $|G| = p^6$, then $G_3 = Z(G)$ and so

$$(1) \quad \alpha^p = \beta_1^{a(1)}\beta_2^{a(2)}, \quad \alpha_i^p = \beta_1^{a(1,i)}\beta_2^{a(2,i)} \quad (i = 1, 2)$$

for some $a(1), a(2) \in Z_p$ and some matrix

$$A = \begin{pmatrix} a(1, 1) & a(1, 2) \\ a(2, 1) & a(2, 2) \end{pmatrix}$$

over Z_p . Since the centralizer of G_2 contains α but not α_1 or α_2 , an autoclinism of G will map α_i to α_i^* , α to α^* , where

$$(2) \quad \alpha^* \equiv \alpha^z, \quad \alpha_i^* \equiv \alpha_1^{x(1,i)}\alpha_2^{x(2,i)}\alpha^{x(i)} \pmod{G_2} \quad (i = 1, 2)$$

for some $x(1), x(2), z \in Z_p$, and some matrix

$$X = \begin{pmatrix} x(1, 1) & x(1, 2) \\ x(2, 1) & x(2, 2) \end{pmatrix}$$

over Z_p . Since $[\alpha^*, \alpha_1^*] = \beta_2^*$ and $[\alpha^*, \alpha_2^*] = \beta_1^{*p}$, we have

$$(3) \quad x(1, 1) = \epsilon x(2, 2), \quad x(1, 2) = \epsilon \nu x(2, 1), \quad z = x(2, 2)^2 - \nu x(1, 1)^2 \neq 0,$$

where $\epsilon = \pm 1$. This autoclinism yields the relations

$$(1^*) \quad \alpha^{*p} = \beta_1^{*a(1)}\beta_2^{*a(2)*}, \quad \alpha_i^{*p} = \beta_1^{*a(1,i)}\beta_2^{*a(2,i)*} \quad (i = 1, 2);$$

and, substituting (2) into (1*), we obtain

$$(4) \quad \begin{cases} \mathbf{a} = \epsilon X \mathbf{a}^*, \\ AX + (x(1)\mathbf{a}, x(2)\mathbf{a}) = \epsilon z X A^*, \end{cases}$$

where $\mathbf{a} = \begin{pmatrix} a(1) \\ a(2) \end{pmatrix}$ and A^* , \mathbf{a}^* are defined analogously to A , \mathbf{a} , respectively. If $\mathbf{a} \neq \mathbf{0}$, it can easily be seen that (by choice of X) we may suppose $\alpha^p = \beta_1$, $\alpha_1^p = \beta_2^a$, $\alpha_2^p = \beta_2^b$, where the only restriction on a, b is that $b = 0, 1, 2, \dots, \frac{1}{2}(p-1)$. If $\mathbf{a} = \mathbf{0}$, Eq. (4) becomes

$$ezA^* = X^{-1}AX.$$

Writing $x = x(2, 1)$, $y = x(2, 2)$, $a = \frac{1}{2}(a(1, 1) + a(2, 2))$, $b = \frac{1}{2}(a(1, 2) + va(2, 1))$, $c = \frac{1}{2}(a(1, 1) - a(2, 2))$ and $d = \frac{1}{2}(a(1, 2) - va(2, 1))$, this becomes

$$eza^* = a,$$

$$zb^* = b,$$

$$z^2c^* = \epsilon(y^2 + vx^2)c + 2xyd,$$

$$z^2d^* = 2\epsilon vxy c + (y^2 + vx^2)d,$$

forcing $z^2(d^{*2} - vc^{*2}) = d^2 - vc^2$. Thus, by considering the cases $d^2 - vc^2 = 0$, a quadratic residue and a non-quadratic residue, we may determine canonical values for a, b, c, d (and hence A) as described in the list of groups $p^6\Phi_{21}$ in Section 4.6 (21).

4. List of Results.

4.1. Summary of families of rank ≤ 6 .

Family	Rank	Class	G/Z(G)	G ₂	G ₃	G ₄	G ₅	q ₀ *	q ₁ *	q ₂ *	q ₃ *	q ₄ *	r ₀ *	r ₁ *	r ₂ *
Φ_2	3	2	(11)	(1)				p	p^2-1	0	0	0	p^2	p-1	0
Φ_3	4	3	$\Phi_2(1^3)$	(11)	(1)			p	p^2-1	p^2-p	0	0	p^2	p^2-1	0
Φ_4	5	2	(111)	(11)				p^2	p^3-p	p^3-p^2	0	0	p^3	p^3-p	0
Φ_5	5	2	(1 ⁴)	(1)				p	p^4-1	0	0	0	p^4	0	p-1
Φ_6	5	3	$\Phi_2(1^3)$	(111)	(11)			p^2	0	p^3-1	0	0	p^2	p^3-1	0
Φ_7	5	3	$\Phi_2(1^4)$	(11)	(1)			p	p^2-1	p^3-p	0	0	p^3	p^2-p	p-1
Φ_8	5	3	$\Phi_2(22)$	(2)	(1)			p	p^2-1	p^3-p	0	0	p^3	p^2-p	p-1
Φ_9	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)		p	p^3-1	0	p^2-p	0	p^2	p^3-1	0
Φ_{10}	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)		p	p-1	p^2-1	p^2-p	0	p^2	p^2-1	p-1
Φ_{11}	6	2	(111)	(111)				p^3	0	p^4-p	0	0	p^3	p^4-p	0
Φ_{12}	6	2	(1 ⁴)	(11)				p^2	$2p^3-2p$	p^4-2p^2+1	0	0	p^4	$2p^3-2p^2$	p^2-2p+1
Φ_{13}	6	2	(1 ⁴)	(11)				p^2	p^3-p	p^4-p^2	0	0	p^4	p^3-2	p^2-p
Φ_{14}	6	2	(22)	(2)				p^2	p^3-p	p^4-p^2	0	0	p^4	p^3-2	p^2-p
Φ_{15}	6	2	(1 ⁴)	(11)				p^2	0	p^4-1	0	0	p^4	0	p^2-1
Φ_{16}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	p^4-p	0	p^3-p^2	0	p^3	p^4-p	0
Φ_{17}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	$2p^2-2p$	$2p^3-p^2-2p+1$	p^3-2p^2+p	0	p^3	$2p^3-p^2-p$	p^2-2p+1
Φ_{18}	6	3	$\Phi_2(1^4)$	(111)	(1)			p^2	p^2-p	p^3-p	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{19}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	$2p^2-2p$	$2p^3-p^2-2p+1$	p^3-2p^2+p	0	p^3	$2p^3-p^2-p$	p^2-2p+1
Φ_{20}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	p^2-p	p^3-p	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{21}	6	3	$\Phi_2(1^4)$	(111)	(11)			p^2	0	p^2-1	p^3-p	0	p^3	p^2-p	p^2-1
Φ_{22}	6	3	$\Phi_2(1^5)$	(11)	(1)			p	p^3+p^2-p-1	p^4-p^2-p+1	0	0	p^4	p^3-p^2	p^2-p
Φ_{23}	6	4	$\Phi_3(1^4)$	(1 ⁴)	(111)	(1)		p^2	p^2-p	p^3-p	p^3-p^2	0	p^2	$2p^3-p^2-1$	p^2-2p+1
Φ_{24}	6	4	$\Phi_3(1^5)$	(111)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{25}	6	4	$\Phi_3(221)b_1$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{26}	6	4	$\Phi_3(221)b_2$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p

Family	Rank	Class	$G/Z(G)$	G_2	G_3	G_4	G_5	q_0^*	q_1^*	q_2^*	q_3^*	q_4^*	r_0^*	r_1^*	r_2^*
Φ_{27}	6	4	$\Phi_3(1^5)$	(111)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{28}	6	4	$\Phi_3(21)b_1$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{29}	6	4	$\Phi_3(21)b_2$	(21)	(11)	(1)		p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{30}	6	4	$\Phi_7(1^5)$	(111)	(11)	(1)		p	$p-1$	$2p^2-p-1$	p^3-2p+1	0	p^3	p^2-p	p^2-1
Φ_{31}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	p^2-1	p^3+p^2-2p	p^3-p^2-p+1	0	p^3	p^3-p	p^2-p
Φ_{32}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	p^2-1	p^3+p^2-2p	p^3-p^2-p+1	0	p^3	p^3-p	p^2-p
Φ_{33}	6	3	$\Phi_4(1^5)$	(111)	(1)			p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{34}	6	3	$\Phi_4(21)b$	(21)	(1)			p	$2p^2-p-1$	p^3-2p+1	p^3-p^2	0	p^3	p^3-p	p^2-p
Φ_{35}	6	5	$\Phi_9(1^5)$	(1 ⁴)	(11)	(1)	(1)	p	p^4-1	0	0	p^2-p	p^2	p^4-1	0
Φ_{36}	6	5	$\Phi_9(1^5)$	(1 ⁴)	(11)	(1)	(1)	p	p^2-1	p^3-p	0	p^2-p	p^2	p^3-1	p^2-p
Φ_{37}	6	5	$\Phi_9(1^5)$	$\Phi_2(1^4)$	(11)	(1)	(1)	p	$p-1$	p^3-1	p^2-p	p^2-2p+1	p^2	p^3-1	p^2-p
Φ_{38}	6	5	$\Phi_{10}(1^5)$	(1 ⁴)	(11)	(1)	(1)	p	$p-1$	p^2-1	p^2-p	p^2-p	p^2	p^2-1	p^2-1
Φ_{39}	6	5	$\Phi_{10}(1^5)$	$\Phi_2(1^4)$	(11)	(1)	(1)	p	$p-1$	$p-1$	$2p^2-p-1$	p^2-2p+1	p^2	p^2-1	p^2-1
Φ_{40}	6	4	$\Phi_6(1^5)$	(1 ⁴)	(11)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{41}	6	4	$\Phi_6(1^5)$	(1 ⁴)	(11)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{42}	6	4	$\Phi_6(21)b_{\frac{p-1}{2}}$	(21)	(11)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p
Φ_{43}	6	4	$\Phi_5(21)a_3$	(21)	(11)	(1)		p	p^2-1	p^2-p	p^3-p	0	p^2	p^3-1	p^2-p

4.2. The groups of order p, p^2 . The only group of order p is (1), and the groups of order p^2 are (2) and (11).

4.3. The groups of order p^3

(1) Abelian: (3), (21) and (111).

(2) Non-abelian: $\Phi_2(21) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$.

$\Phi_2(111) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$.

4.4. The groups of order p^4 .

(1) Abelian: (4), (31), (22), (211), (1⁴).

(2) $\Phi_2(211)a = \Phi_2(21) \times (1)$, $\Phi_2(1^4) = \Phi_2(111) \times (1)$,

$\Phi_2(31) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$

$\Phi_2(22) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$

$\Phi_2(211)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^p = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$

$\Phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$.

(3) $\Phi_3(211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle$

$\Phi_3(211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \alpha_2^p = \alpha_3^p = 1 \rangle$

for $r = 1$ or v

$\Phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_3^p = 1 (i = 1, 2) \rangle$.

4.5. The groups of order p^5 .

(1) Abelian: (5), (41), (32), (311), (221), (2111), (1⁵)

(2) $\Phi_2(311)a = \Phi_2(31) \times (1)$, $\Phi_2(221)a = \Phi_2(22) \times (1)$, $\Phi_2(221)b = \Phi_2(21) \times (2)$,

$\Phi_2(2111)a = \Phi_2(211)a \times (1)$, $\Phi_2(2111)b = \Phi_2(211)b \times (1)$, $\Phi_2(2111)c = \Phi_2(211)c \times (1)$,

$\Phi_2(2111)d = \Phi_2(111) \times (2)$, $\Phi_2(1^5) = \Phi_2(1^4) \times (1)$.

- $\phi_2(41) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^3} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$
 $\phi_2(32)a_1 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^2} = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1 \rangle$
 $\phi_2(32)a_2 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_1^p = \alpha_2, \alpha^{p^3} = \alpha_2^p = 1 \rangle$
 $\phi_2(311)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$
 $\phi_2(311)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^3} = \alpha_1^p = \alpha_2^p = 1 \rangle$
 $\phi_2(221)c = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^p = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle$
 $\phi_2(221)d = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^{p^2} = \alpha_2^p = 1 \rangle$
- (3) $\phi_3(2111)a = \phi_3(211)a \times (1)$, $\phi_3(2111)b_r = \phi_3(211)b_r \times (1)$ for $r = 1$ or v ,
 $\phi_3(1^5) = \phi_3(1^4) \times (1)$,
 $\phi_3(311)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^{p^2} = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle$
 $\phi_3(311)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{p^2} = \alpha_3, \alpha^p = \alpha_2^p = \alpha_3^p = 1 \rangle$
for $r = 1$ or v
 $\phi_3(221)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{p^2} = \alpha_2^p = \alpha_3^p = 1 \rangle$
 $\phi_3(221)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{(p)} = \alpha_3^r, \alpha^{p^2} = \alpha_2^p = \alpha_3^p = 1 \rangle$
for $r = 1$ or v
 $\phi_3(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_1^{(p)} = 1$ ($i=1,2,3$)
 $\phi_3(2111)d = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^{p^2} = \alpha_1^{(p)} = \alpha_3^p = 1$ ($i=1,2$)
 $\phi_3(2111)e = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{p^2} = \alpha_{i+1}^p = 1$ ($i=1,2$)
- (4) $\phi_4(221)a = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^p = \beta_2, \alpha_1^p = \beta_1, \alpha_2^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(221)b = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^p = \beta_2, \alpha_2^p = \beta_1, \alpha_1^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(221)c = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1 = \alpha_1^p, \alpha^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(221)d_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha_1^p = \beta_1^k, \alpha_2^p = \beta_2, \alpha^p = \beta_1^p = 1$ ($i=1,2$)
where $k = g^r$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$.
 $\phi_4(221)e = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2^{-k}, \alpha_2^p = \beta_1\beta_2, \alpha^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(221)f_0 = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1^v, \alpha^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(221)f_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2^k, \alpha_2^p = \beta_1\beta_2, \alpha^p = \beta_1^p = 1$ ($i=1,2$)
where $4k = g^{2r+1} - 1$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$.
 $\phi_4(2111)a = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^p = \beta_2, \alpha_1^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(2111)b = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_1, \alpha^p = \alpha_2^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(2111)c = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_2^p = \beta_1, \alpha^p = \alpha_1^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_4(1^5) = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^p = \alpha_1^p = \beta_1^p = 1$ ($i=1,2$)
- (5) $\phi_5(2111) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$
 $\phi_5(1^5) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$
- (6) $\phi_6(221)a = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1 = \alpha_1^p, \beta^p = \beta_1^p = 1$ ($i=1,2$)
 $\phi_6(221)b_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^p = \beta_1^k, \alpha_2^p = \beta_2, \beta^p = \beta_1^p = 1$ ($i=1,2$)
where $k = g^r$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$.
 $\phi_6(221)c_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^p = \beta_2^{-kr}, \alpha_2^p = \beta_1^r\beta_2^r, \beta^p = \beta_1^p = 1$ ($i=1,2$)
for $r = 1$ or v .

$$\phi_6(221)d_0 = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1^v, \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

$$\phi_6(221)d_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta_2^k, \alpha_2^p = \beta_1 \beta_2, \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

where $4k = g^{2r+1} - 1$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$.

$$\phi_6(2111)a = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta_j, \alpha_2^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $p > 3$

$$\phi_6(2111)b_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_2^p = \beta_1^r, \alpha_1^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 1$ or v , and $p > 3$.

$$\phi_6(1^5) = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

$$(7) \ \phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \alpha^p, \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 \ (i=1,2) \rangle$$

$$\phi_7(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^r = \alpha_3^r = \alpha_1^{(p)}, \alpha^p = \alpha_{i+1}^p = \beta^p = 1 \ (i=1,2) \rangle$$

for $r = 1$ or v .

$$\phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \beta^p, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = 1 \ (i=1,2) \rangle$$

$$\phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 \cdot (i=1,2) \rangle$$

$$(8) \ \phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p, \beta^p = \alpha_2^p = 1 \rangle$$

$$(9) \ \phi_9(2111)a = \langle \alpha, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_4, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$$

$$\phi_9(2111)b_r = \langle \alpha, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha_1^{(p)} = \alpha_4^k = \alpha^p = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$

$$\phi_9(1^5) = \langle \alpha, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$$

$$(10) \ \phi_{10}(2111)a_r = \langle \alpha, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha^p, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 4)$.

$$\phi_{10}(2111)b_r = \langle \alpha, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha_1^{(p)}, \alpha^p = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$, and where $p > 3$.

$$\phi_{10}(1^5) = \langle \alpha, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1,2,3) \rangle$$

4.6 The groups of order p^6 .

$$(1) \ \text{Abelian: } (6), (51), (42), (411), (33), (321), (3111), (222), (2211), (21^4), (1^6).$$

$$(2) \ \phi_2(411)a = \phi_2(41) \times (1), \ \phi_2(321)a_1 = \phi_2(32)a_1 \times (1) \ (i=1,2), \ \phi_2(321)b = \phi_2(31) \times (2),$$

$$\phi_2(321)e = \phi_2(21) \times (3), \ \phi_2(3111)a = \phi_2(311)a \times (1), \ \phi_2(3111)b = \phi_2(311)b \times (1),$$

$$\phi_2(3111)c = \phi_2(311)c \times (1), \ \phi_2(3111)d = \phi_2(111) \times (3), \ \phi_2(222)a = \phi_2(22) \times (2),$$

$$\phi_2(2211)a = \phi_2(221)a \times (1), \ \phi_2(2211)b = \phi_2(221)b \times (1), \ \phi_2(2211)c = \phi_2(221)c \times (1),$$

$$\phi_2(2211)d = \phi_2(221)d \times (1), \ \phi_2(2211)e = \phi_2(211)b \times (2), \ \phi_2(2211)f = \phi_2(211)c \times (2),$$

$$\phi_2(21^4)a = \phi_2(2111)a \times (1), \ \phi_2(21^4)b = \phi_2(2111)b \times (1), \ \phi_2(21^4)c = \phi_2(2111)c \times (1),$$

$$\phi_2(21^4)d = \phi_2(2111)d \times (1), \ \phi_2(1^6) = \phi_2(1^6) \times (1).$$

$$\phi_2(51) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha^{p^4}, \alpha_1^p = \alpha_2^p = 1 \rangle$$

$$\phi_2(42)a_1 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha^{p^3}, \alpha_1^p = \alpha_2^p = 1 \rangle$$

$$\phi_2(42)a_2 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha_1^p, \alpha^{p^4} = \alpha_2^p = 1 \rangle$$

$$\phi_2(411)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2 = \gamma^{p^3}, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$$

- $$\begin{aligned} \phi_2(411)c &= \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle \\ \phi_2(33) &= \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha^{p^2}, \alpha_1^{p^3} = \alpha_2^p = 1 \rangle \\ \phi_2(321)c &= \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2 = \gamma^p, \alpha^{p^3} = \alpha_1^p = \alpha_2^p = 1 \rangle \\ \phi_2(321)d &= \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2 = \gamma^{p^2}, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle \\ \phi_2(321)f &= \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^3} = \alpha_1^{p^2} = \alpha_2^p = 1 \rangle \\ \phi_2(222)b &= \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2 = \gamma^p, \alpha^{p^2} = \alpha_1^{p^2} = \alpha_2^p = 1 \rangle \end{aligned}$$
- (3) $\phi_3(3111)a = \phi_3(311)a \times (1)$, $\phi_3(3111)b_r = \phi_3(311)b_r \times (1)$ ($r = 1$ or v),
 $\phi_3(2211)a = \phi_3(221)a \times (1)$, $\phi_3(2211)b_r = \phi_3(221)b_r \times (1)$ ($r = 1$ or v),
 $\phi_3(2211)c = \phi_3(211)a \times (2)$, $\phi_3(2211)e_r = \phi_3(211)b_r \times (2)$ ($r = 1$ or v),
 $\phi_3(21^4)a = \phi_3(2111)a \times (1)$, $\phi_3(21^4)b_r = \phi_3(2111)b_r \times (1)$ ($r = 1$ or v),
 $\phi_3(21^4)c = \phi_3(2111)c \times (1)$, $\phi_3(21^4)d = \phi_3(2111)d \times (1)$, $\phi_3(21^4)e = \phi_3(2111)e \times (1)$,
 $\phi_3(21^4)f = \phi_3(1^4) \times (2)$, $\phi_3(1^6) = \phi_3(1^5) \times (1)$,
- $$\begin{aligned} \phi_3(411)a &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(411)b_r &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{p^3} = \alpha_3^r, \alpha^p = \alpha_2^p = \alpha_3^p = 1 \rangle \quad r = 1 \text{ or } v \\ \phi_3(321)a &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha_1^{p^2} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(321)b_r &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{(p)} = \alpha_3^{p^3}, \alpha^{p^3} = \alpha_2^p = \alpha_3^p = 1 \rangle \quad r = 1 \text{ or } v \\ \phi_3(321)c_r &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{p^2} = \alpha_3^r, \alpha^{p^2} = \alpha_2^p = \alpha_3^p = 1 \rangle \quad r = 1 \text{ or } v \\ \phi_3(321)d &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha_1^{p^3} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(3111)c &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^{p^2} = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(3111)d &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^{p^3} = \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(3111)e &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_1^{p^3} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(2211)d &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_1^{p^2} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(2211)f &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^{p^2} = \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle \\ \phi_3(2211)g &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^{p^2} = \alpha_1^{p^2} = \alpha_2^p = \alpha_3^p = 1 \rangle \end{aligned}$$
- (4) $\phi_4(2211)a = \phi_4(221)a \times (1)$, $\phi_4(2211)b = \phi_4(221)b \times (1)$, $\phi_4(2211)c = \phi_4(221)c \times (1)$,
 $\phi_4(2211)d_r = \phi_4(221)d_r \times (1)$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$, $\phi_4(2211)e = \phi_4(221)e \times (1)$,
 $\phi_4(2211)f_r = \phi_4(221)f_r \times (1)$ for $r = 0, 1, \dots, \frac{1}{2}(p-1)$, $\phi_4(21^4)a = \phi_4(2111)a \times (1)$,
 $\phi_4(21^4)b = \phi_4(2111)b \times (1)$, $\phi_4(21^4)c = \phi_4(2111)c \times (1)$, $\phi_4(1^6) = \phi_4(1^5) \times (1)$
- $$\begin{aligned} \phi_4(321)a &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^{p^2} = \beta_1, \alpha^p = \beta_2, \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(321)b &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^{p^2} = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(321)c &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^p = \beta_2, \alpha_2^p = \beta_1, \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(321)d &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^p = \beta_2, \alpha_1^{p^2} = \beta_1, \alpha_2^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(321)e_r &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^{p^2} = \beta_1, \alpha_2^p = \beta_2^r, \alpha^p = \beta_1^p = 1 (i=1, 2) \rangle \text{ for } r=1, 2, \dots, p-1 \\ \phi_4(321)f_r &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2^r, \alpha_2^p = \beta_1, \alpha^p = \beta_1^p = 1 (i=1, 2) \rangle \text{ for } r=1 \text{ or } v \\ \phi_4(3111)a &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha^{p^2} = \beta_1, \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(3111)b &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^{p^2} = \beta_1, \alpha^p = \alpha_2^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(3111)c &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_2^p = \beta_1, \alpha^p = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(222)a &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \alpha^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \\ \phi_4(222)b_r &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_1^k, \alpha_2^p = \beta_2, \alpha^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \end{aligned}$$
- where $k = g^r$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$.

$$\begin{aligned} \Phi_4(222)c &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_1, \alpha_2^p = \beta_2, \alpha_1^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(222)d_1 &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2^{-k}, \alpha_2^p = \beta_1 \beta_2, \alpha_1^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(222)d_2 &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1, \alpha_1^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(222)e_0 &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1^v, \alpha_1^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(222)e_r &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2^k, \alpha_2^p = \beta_1 \beta_2, \alpha_1^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \end{aligned}$$

where $4k = g^{2r+1} - 1$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$.

$$\begin{aligned} \Phi_4(2211)g &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha] = \beta_1, \gamma^p = \beta_2, \alpha_1^p = \beta_1, \alpha_1^{p^2} = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)h &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha] = \beta_1, \gamma^p = \beta_2, \alpha_1^p = \beta_1, \alpha_1^p = \alpha_2^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)i &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha] = \beta_1, \gamma^p = \beta_2, \alpha_2^p = \beta_1, \alpha_1^p = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)j_1 &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_1, \alpha_1^{p^2} = \alpha_2^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)j_2 &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_1, \alpha_1^{p^2} = \alpha_2^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)k &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^p = \beta_2, \alpha_1^{p^2} = \alpha_2^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)l &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_2^p = \beta_1, \alpha_1^{p^2} = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)m &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_2^p = \beta_2, \alpha_1^{p^2} = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(2211)n &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_2^p = \beta_1, \alpha_1^{p^2} = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(21^4)d &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha] = \beta_1, \gamma^p = \beta_1, \alpha_1^p = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(21^4)e &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^{p^2} = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_4(21^4)f &= \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \beta_1, \alpha_1^{p^2} = \alpha_2^p = \alpha_1^p = \beta_1^p = 1 (i=1, 2) \rangle \end{aligned}$$

(5) $\Phi_5(21^4)a = \Phi_5(2111) \times (1)$, $\Phi_5(1^6) = \Phi_5(1^5) \times (1)$,

$$\begin{aligned} \Phi_5(311) &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^{p^2} = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle \\ \Phi_5(2211)a &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^{p^2} = \beta, \alpha_1^{p^2} = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle \\ \Phi_5(2211)b &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^p = \beta, \alpha_1^{p^2} = \alpha_2^p = \alpha_4^p = \beta^p = 1 \rangle \\ \Phi_5(21^4)b &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta, \gamma \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \gamma^p = \beta, \alpha_1^p = \beta^p = 1 (i=1, 2, 3, 4) \rangle \\ \Phi_5(21^4)c &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^{p^2} = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle \end{aligned}$$

(6) **Note:** In this family, $\alpha_1^{(p)} = \alpha_1^p \beta_2^{\binom{p}{2}}$

$$\begin{aligned} \Phi_6(2211)a &= \Phi_6(2211)a \times (1), \quad \Phi_6(2211)b_r = \Phi_6(2211)b_r \times (1) \quad r=1, 2, \dots, \frac{1}{2}(p-1), \\ \Phi_6(2211)c_r &= \Phi_6(2211)c_r \times (1) \quad r=1 \text{ or } v, \quad \Phi_6(2211)d_r = \Phi_6(2211)d_r \times (1) \quad r=0, 1, \dots, \frac{1}{2}(p-1), \\ \Phi_6(21^4)a &= \Phi_6(2111)a \times (1), \quad \Phi_6(21^4)b_r = \Phi_6(2111)b_r \times (1) \quad r=1 \text{ or } v, \quad \Phi_6(1^6) = \Phi_6(1^5) \times (1), \\ \Phi_6(321)a_r &= \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^{p^2} = \beta_1, \alpha_2^p = \beta_2^r, \beta^p = \beta_1^p = 1 (i=1, 2) \rangle \end{aligned}$$

for $r = 1, 2, \dots, p-1$,

$$\Phi_6(321)b_{r,s} = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^{(p)} = \beta_2^r, \alpha_2^{p^2} = \beta_1^s, \beta^p = \beta_1^p = 1 (i=1, 2) \rangle$$

for $r, s = 1$ or v ,

$$\begin{aligned} \Phi_6(3111)a &= \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^{p^2} = \beta_1, \alpha_2^p = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_6(3111)b_r &= \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_2^{p^2} = \beta_1^r, \alpha_1^{(p)} = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle \end{aligned}$$

for $r = 1$ or v ,

$$\begin{aligned} \Phi_6(2211)e &= \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \gamma^p = \beta_2, \alpha_1^{(p)} = \beta_1, \alpha_2^p = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_6(2211)f_r &= \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \gamma^p = \beta_2, \alpha_2^p = \beta_1^r, \alpha_1^{(p)} = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle \end{aligned}$$

for $r = 1$ or v ,

$$\begin{aligned} \Phi_6(2211)g &= \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_2^p = \beta_2, \alpha_1^{p^2} = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle \\ \Phi_6(2211)h_r &= \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_2^p = \beta_1^r, \alpha_1^{p^2} = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle \end{aligned}$$

for $r=1$ or v

$$\phi_6(21^4)c = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \gamma^p = \beta_1, \alpha_1^{(p)} = \alpha_2^p = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle$$

$$\phi_6(21^4)d = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^{p^2} = \alpha_2^p = \beta^p = \beta_1^p = 1 (i=1, 2) \rangle$$

$$(7) \quad \phi_7(21^4)a = \phi_7(2111)a \times (1), \quad \phi_7(21^4)b_r = \phi_7(2111)b_r \times (1) \quad r=1 \text{ or } v, \quad \phi_7(21^4)c = \phi_7(2111)c \times (1),$$

$$\phi_7(1^6) = \phi_7(1^5) \times (1),$$

$$\phi_7(3111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

$$\phi_7(3111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^r = \alpha_3^r, \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

for $r=1$ or v

$$\phi_7(3111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2) \rangle$$

$$\phi_7(2211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha^p = \alpha_3, \alpha_1^{p^2} = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

$$\phi_7(2211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^r = \alpha_3^r, \alpha_1^{(p)} = \alpha_3^r, \alpha_1^{p^2} = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

for $r=1$ or v

$$\phi_7(2211)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha^p = \alpha_3, \beta^{p^2} = \alpha_1^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2) \rangle$$

$$\phi_7(2211)d_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^r = \alpha_3^r, \beta^{p^2} = \alpha_1^{(p)} = \alpha_3^r, \beta^{p^2} = \alpha_{i+1}^p = 1 (i=1, 2) \rangle$$

for $r=1$ or v

$$\phi_7(2211)e = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \beta^p = \alpha_3, \alpha_1^{p^2} = \alpha_{i+1}^p = \alpha_{i+1}^p = 1 (i=1, 2) \rangle$$

$$\phi_7(2211)f = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \beta^p = \alpha_3, \alpha_1^{p^2} = \alpha_1^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2) \rangle$$

$$\phi_7(21^4)d = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \gamma^p = \alpha_3, \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

$$\phi_7(21^4)e = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^{p^2} = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

$$\phi_7(21^4)f = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^{p^2} = \alpha^p = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

$$\phi_7(21^4)g = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \beta^{p^2} = \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2) \rangle$$

$$(8) \quad \phi_8(42) = \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \alpha_1^p = \beta, [\beta, \alpha_2] = \beta^p, \alpha_2^3 = \beta^{p^2} = 1 \rangle$$

$$\phi_8(33) = \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta = \alpha_1^p, [\beta, \alpha_2] = \beta^p, \alpha_2^3 = \beta^{p^2} = 1 \rangle$$

$$\phi_8(321)a = \phi_8(32) \times (1)$$

$$\phi_8(321)b = \langle \alpha_1, \alpha_2, \beta, \gamma | [\alpha_1, \alpha_2] = \beta = \alpha_1^p, [\beta, \alpha_2] = \beta^p = \gamma^p, \alpha_2^2 = \beta^{p^2} = 1 \rangle$$

$$\phi_8(321)c_r = \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_2]^{r+1} = \beta^{p(r+1)} = \alpha_1^{p^2}, \alpha_2^2 = \beta^{p^2} = 1 \rangle \quad \text{for } r=0, 1, \dots, p-2$$

$$\phi_8(321)c_{p-1} = \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_2] = \beta^p = \alpha_2^{p^2}, \alpha_1^{p^2} = \beta^{p^2} = 1 \rangle$$

$$\phi_8(222) = \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_2] = \beta^p, \alpha_1^{p^2} = \alpha_2^2 = \beta^{p^2} = 1 \rangle$$

$$(9) \quad \phi_9(21^4)a = \phi_9(2111)a \times (1), \quad \phi_9(21^4)b_r = \phi_9(2111)b_r \times (1) \quad \text{for } r+1=1, 2, \dots, (p-1, 3),$$

$$\phi_9(1^6) = \phi_9(1^5) \times (1),$$

$$\phi_9(3111)a = \langle \alpha, \alpha_1, \dots, \alpha_k | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1, 2, 3) \rangle$$

$$\phi_9(3111)b_r = \langle \alpha, \alpha_1, \dots, \alpha_k | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^{p^2} = \alpha_k^k, \alpha^p = \alpha_{i+1}^{(p)} = 1 (i=1, 2, 3) \rangle$$

where $k = g^r$ for $r+1=1, 2, \dots, (p-1, 3)$

$$\phi_9(2211)a = \langle \alpha, \alpha_1, \dots, \alpha_k | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_k, \alpha_1^{p^2} = \alpha_{i+1}^{(p)} = 1 (i=1, 2, 3) \rangle$$

$$\phi_9(2211)b_r = \langle \alpha, \alpha_1, \dots, \alpha_k | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^{(p)} = \alpha_k^k, \alpha^{p^2} = \alpha_{i+1}^{(p)} = 1 (i=1, 2, 3) \rangle$$

where $k = g^r$ for $r+1=1, 2, \dots, (p-1, 3)$

$$\phi_9(21^4)c = \langle \alpha, \alpha_1, \dots, \alpha_k, \gamma | [\alpha_i, \alpha] = \alpha_{i+1}, \gamma^p = \alpha_k, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1, 2, 3) \rangle$$

$$\phi_9(21^4)d = \langle \alpha, \alpha_1, \dots, \alpha_k | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1, 2, 3) \rangle$$

$$\phi_9(21^4)e = \langle \alpha, \alpha_1, \dots, \alpha_k | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^{p^2} = \alpha^p = \alpha_{i+1}^{(p)} = 1 (i=1, 2, 3) \rangle$$

- (10) $\phi_{10}(21^4)_r = \phi_{10}(2111)_r \times (1)$ for $r+1 = 1, 2, \dots, (p-1, 4)$,
 $\phi_{10}(21^4)_r = \phi_{10}(2111)_r \times (1)$ for $r+1 = 1, 2, \dots, (p-1, 3)$ and where $p > 3$,
 $\phi_{10}(1^6) = \phi_{10}(1^5) \times (1)$,
 $\phi_{10}(3111)_r = \langle \alpha_1, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1, 2, 3) \rangle$
 where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 4)$
 $\phi_{10}(3111)_r = \langle \alpha_1, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1, 2, 3) \rangle$
 where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$
 $\phi_{10}(2211)_r = \langle \alpha_1, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1, 2, 3) \rangle$
 where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 4)$
 $\phi_{10}(2211)_r = \langle \alpha_1, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1, 2, 3) \rangle$
 where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$
 $\phi_{10}(21^4)_c = \langle \alpha_1, \alpha_1, \dots, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma^p = \alpha_4, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1, 2, 3) \rangle$
 $\phi_{10}(21^4)_d = \langle \alpha_1, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1, 2, 3) \rangle$
 $\phi_{10}(21^4)_e = \langle \alpha_1, \alpha_1, \dots, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i=1, 2, 3) \rangle$
- (11) $\phi_{11}(222)_r = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^r \beta_2^{-1}, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 for $r = 1$ or v
 $\phi_{11}(222)_r = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^r \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 for $r = 1, 2, \dots, p-2$, where at least two of $-r, -(r+1), -(r+1)/r$ are quadratic residues (mod p)
 $\phi_{11}(222)_c = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_3^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_2^{-1} \beta_3, \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(222)_d = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \alpha_1^{-p} = \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(222)_r = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^r \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 for $r = 1, 2, \dots, p-2$, where at most one of $-r, -(r+1), -(r+1)/r$ is a quadratic residue (mod p)
 $\phi_{11}(222)_d = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \alpha_1^p = \beta_1, [\alpha_3, \alpha_1] = \alpha_2^p = \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(2211)_a = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_2^{-1} \beta_3, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(2211)_b = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \alpha_3^p = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \alpha_2^p = \beta_1 \beta_2, \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(2211)_c = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \alpha_1^{-p} = \beta_2, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(2211)_d = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_1^p = \beta_1, [\alpha_3, \alpha_1]^r = \alpha_2^p = \beta_2^r, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 for $r = 0, 1$
 $\phi_{11}(2211)_d = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^{-r} \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 for $r = 2, 3, \dots, \frac{1}{2}(p-1)$
 $\phi_{11}(2211)_e = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_2^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1 \beta_2^{-1}, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(2211)_f = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^k \beta_2^{-1}, \alpha_2^p = \beta_1 \beta_2, \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 where $k = -vr^2$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$
 $\phi_{11}(21^4)_a = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, \alpha_3, \alpha_1] = \alpha_1^p = \beta_2, \alpha_2^p = \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(21^4)_b = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \alpha_1^p = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_2^p = \alpha_3^p = \beta_1^p = \beta_2^p = \beta_3^p = 1 \rangle$
 $\phi_{11}(1^6) = \langle \alpha_1, \beta_1, \dots, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_1^p = \beta_1^p = 1 \ (i=1, 2, 3) \rangle$
- (12) $\phi_{12}(2211)_a = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \mid [\alpha_1, \beta_1] = \gamma_1, \alpha_1^p = \gamma_1, \beta_1^p = \gamma_2, \alpha_2^p = \beta_2^p = \gamma_1^p = 1 \ (i=1, 2) \rangle$
 $\phi_{12}(2211)_b = \phi_2(21) \times \phi_2(21)$
 $\phi_{12}(2211)_c = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \mid [\alpha_1, \beta_1] = \gamma_1, \alpha_1^p = \gamma_1 \gamma_2, \alpha_2^p = \gamma_2, \beta_1^p = \gamma_1^p = 1 \ (i=1, 2) \rangle$
 $\phi_{12}(2211)_d = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \mid [\alpha_1, \beta_1] = \gamma_1, \alpha_1^p = \gamma_2, \alpha_2^p = \gamma_1, \beta_1^p = \gamma_1^p = 1 \ (i=1, 2) \rangle$

$$\begin{aligned} \phi_{12}(2211)e &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_i^p = \gamma_1 \gamma_2, \alpha_2^p = \gamma_1, \beta_i^p = \gamma_i^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(2211)f &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \alpha_2^p = \gamma_1, \beta_1^p = \gamma_2, \beta_2^p = \gamma_1^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(2211)g &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_i^p = \gamma_1, \alpha_2^p = \beta_1^p = \gamma_2, \beta_2^p = \gamma_1^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(2211)h &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \beta_2^p = \gamma_1, \alpha_2^p = \beta_1^p = \gamma_2, \gamma_i^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(2211)i &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \gamma_1, \alpha_2^p = \gamma_1 \gamma_2, \beta_1^p = \gamma_2, \beta_2^p = \gamma_1^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(21^4)a &= \phi_2(21) \times \phi_2(111) \\ \phi_{12}(21^4)b &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \gamma_1 \gamma_2, \alpha_2^p = \beta_1^p = \gamma_1^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(21^4)c &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \gamma_2, \alpha_2^p = \beta_1^p = \gamma_1^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(21^4)d &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \alpha_2^p = \gamma_1, \beta_1^p = \gamma_1^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(21^4)e &= \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 | [\alpha_i, \beta_i] = \gamma_i, \alpha_1^p = \alpha_2^p = \gamma_1 \gamma_2, \beta_1^p = \gamma_1^p = 1 \ (i=1,2) \rangle \\ \phi_{12}(1^6) &= \phi_2(111) \times \phi_2(111) \end{aligned}$$

$$\begin{aligned} (13) \ \phi_{13}(2211)a &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \alpha_2^p = \beta_2, \alpha_1^p = \beta_1, \alpha_3^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(2211)b &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \alpha_3^p = \beta_2, \alpha_1^p = \beta_1, \alpha_2^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(2211)c_r &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u]^r = \alpha_2^p = \beta_2^r, \alpha_3^p = \beta_1, \alpha_1^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \end{aligned}$$

for $r = 1$ or v

$$\begin{aligned} \phi_{13}(2211)d &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \alpha_1^p = \beta_2, \alpha_3^p = \beta_1, \alpha_2^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(2211)e_r &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u]^r = \alpha_4^p = \beta_2^r, \alpha_1^p = \beta_1, \alpha_2^p = \alpha_3^p = \beta_1^p = 1 \ (i=1,2) \rangle \end{aligned}$$

for $r = 1, 2, \dots, p-1$

$$\begin{aligned} \phi_{13}(2211)f &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \alpha_4^p = \beta_2, \alpha_3^p = \beta_1, \alpha_1^p = \alpha_2^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(21^4)a &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \beta_2, \alpha_1^p = \beta_1, \alpha_{i+1}^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(21^4)b &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \alpha_1^p = \beta_2, \alpha_{i+1}^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(21^4)c &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \alpha_3^p = \beta_2, \alpha_1^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(21^4)d &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \beta_2, \alpha_3^p = \beta_1, \alpha_1^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{13}(1^6) &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_u] = \beta_2, \alpha_1^p = \alpha_3^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \end{aligned}$$

$$\begin{aligned} (14) \ \phi_{14}(42) &= \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta, \alpha_1^p = \beta, \alpha_2^p = \beta^p = 1 \rangle \\ \phi_{14}(321) &= \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta, \alpha_1^p = \beta^p, \alpha_2^p = \beta^p = 1 \rangle \\ \phi_{14}(222) &= \langle \alpha_1, \alpha_2, \beta | [\alpha_1, \alpha_2] = \beta, \alpha_1^p = \alpha_2^p = \beta^p = 1 \rangle \end{aligned}$$

$$\begin{aligned} (15) \ \phi_{15}(2211)a &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_u] = \alpha_1^p = \beta_1, [\alpha_2, \alpha_u] = \alpha_2^p = \beta_2^p, \alpha_3^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{15}(2211)b_{r,s} &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_u] = \beta_1, [\alpha_2, \alpha_u]^k = \alpha_2^p = \beta_2^p, \alpha_1^p = \beta_1 \beta_2^r, \alpha_3^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \end{aligned}$$

where $k = g^{\lfloor \frac{1}{2}n \rfloor + s}$ and $g^n = g^2(g-r^2)$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$ and $s = 0, 1, \dots, m$, with

$m = \frac{1}{2}(p-3) + n - 2\lfloor \frac{1}{2}n \rfloor$ and $\lfloor \frac{1}{2}n \rfloor =$ integral part of $\frac{1}{2}n$;

$$\begin{aligned} \phi_{15}(2211)c &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_u] = \alpha_1^p = \beta_1, [\alpha_2, \alpha_u] = \alpha_4^p = \beta_2^p, \alpha_{i+1}^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{15}(2211)d_r &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_u] = \alpha_1^p = \beta_1, [\alpha_2, \alpha_u] = \beta_2^p, \alpha_4^p = \beta_2^k, \alpha_{i+1}^p = \beta_1^p = 1 \ (i=1,2) \rangle \end{aligned}$$

where $k = g^r$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$;

$$\begin{aligned} \phi_{15}(21^4) &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_u] = \alpha_1^p = \beta_1, [\alpha_2, \alpha_u] = \beta_2^p, \alpha_{i+1}^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \\ \phi_{15}(1^6) &= \langle \alpha_1, \dots, \alpha_u, \beta_1, \beta_2 | [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_u] = \beta_1, [\alpha_2, \alpha_u] = \beta_2^p, \alpha_1^p = \alpha_3^p = \alpha_4^p = \beta_1^p = 1 \ (i=1,2) \rangle \end{aligned}$$

$$(16) \quad \begin{aligned} \Phi_{16}(2211)a &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \beta^p = \gamma, \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle \\ \Phi_{16}(2211)b &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \alpha^p = \gamma, \beta^p = \alpha_3, \alpha_1^{(p)} = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle \\ \Phi_{16}(2211)c_r &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \beta^p = \gamma, \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle \end{aligned}$$

for $r = 1, 2, \dots, p-1$

$$\Phi_{16}(2211)d_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha]^k = \alpha_1^{(p)} = \gamma^k, \beta^p = \alpha_3, \alpha^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$

$$\Phi_{16}(2211)e = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \alpha_1^{(p)} = \gamma, \alpha^p = \alpha_3, \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{16}(2211)f_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \alpha^p = \gamma, \alpha_1^{(p)} = \alpha_3^r, \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v

$$\Phi_{16}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \gamma, \alpha^p = \alpha_3, \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{16}(21^4)b = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \alpha^p = \gamma, \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{16}(21^4)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \beta^p = \gamma, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{16}(21^4)d = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \gamma, \beta^p = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{16}(21^4)e_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \gamma, \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v

$$\Phi_{16}(21^4)f = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \alpha_1^{(p)} = \gamma, \alpha^p = \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{16}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \gamma, \alpha^p = \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$(17) \quad \Phi_{17}(2211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \beta^p = \gamma, \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{17}(2211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \beta^p = \gamma, \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v

$$\Phi_{17}(2211)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha^p = \alpha_3, \beta^p = \alpha_3 \gamma, \alpha_1^{(p)} = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{17}(2211)d = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \alpha^p = \gamma, \beta^p = \alpha_3, \alpha^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{17}(2211)e_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha_1^{(p)} = \alpha_3^r, \beta^p = \alpha_3 \gamma, \alpha^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v

$$\Phi_{17}(2211)f = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \alpha^p = \gamma, \beta^p = \alpha_3, \alpha_1^{(p)} = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{17}(2211)g_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \beta^p = \gamma, \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_3^r, \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1, 2, \dots, p-1$

$$\Phi_{17}(2211)h_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \alpha_1^p = \gamma, \alpha^p = \gamma^k, \beta^p = \alpha_3, \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1, 2) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$

$$\Phi_{17}(2211)i_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_3^r, \beta^p = \alpha_3 \gamma, \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1, 2, \dots, p-1$

$$\Phi_{17}(2211)j = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \alpha_1^p = \gamma, \alpha^p = \alpha_3, \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{17}(2211)k_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha^p = \alpha_3^{1-s+r+s} \gamma^s, \alpha_1^{(p)} = \alpha_3^r, \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1, 2) \rangle$$

($i=1, 2$)

for $r = 1$ or v and $s = 0$ or 1

$$\Phi_{17}(2211)l_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha^p = \alpha_3^{r+s} \gamma, \alpha_1^{(p)} = \alpha_3^r \gamma, \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v and $s = 1, 2, \dots, p-1$

$$\Phi_{17}(2211)m_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha^p = \alpha_3^s \gamma, \alpha_1^{(p)} = \alpha_3^r, \beta^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v and $s = 0$ or 1

$$\Phi_{17}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha^p = \alpha_3, \beta^p = \alpha_1^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r=1$ or v and $s=2,3, \dots, p-1$

$$\phi_{19}(2211)d_{0,0,0} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^{-v} = \alpha_2^p = \beta_1^{-v}, \alpha_1^p = \beta_2^v, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

$$\phi_{19}(2211)d_{r,0,t} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^t = \alpha_2^p = \beta_1^t, \alpha_1^p = \beta_1^t \beta_2^t, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1$ or v , $t=1,2, \dots, p-1$, and $1+4rt$ is a quadratic residue (mod p)

$$\phi_{19}(2211)d_{r,s,t} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_1^t \beta_2^t, \alpha_2^p = \beta_1^s \beta_2^k, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1, v$ and $s=1,2, \dots, p-1$, where $k=g^t$ ($t=0,1, \dots, \frac{1}{2}(p-1)$), $k \neq rs$ and $(1-k)^2 + 4rs$ is a quadratic residue (mod p), and where $-s$ is a non-quadratic residue (mod p) whenever $r=v$ and $k \neq \pm 1$

$$\phi_{19}(2211)e_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_1^t \beta_2^t, \alpha_2^p = \beta_2^t, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1$ or v

$$\phi_{19}(2211)f_{r,0} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^t = \alpha_2^p = \beta_1^t, \alpha_1^p = \beta_1^t \beta_2^t, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

where $rt = -k$ for $r=1$ or v

$$\phi_{19}(2211)f_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_1^t \beta_2^t, \alpha_2^p = \beta_1^s \beta_2^k, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1, v$ where $rt = -k(1-k)^2$ and $k=g^s$ ($s=1,2, \dots, \frac{1}{2}(p-3)$)

$$\phi_{19}(2211)g_{r,0,0} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^r = \alpha_2^p = \beta_1^r, \alpha_1^p = \beta_2^r, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1$ or v

$$\phi_{19}(2211)g_{r,0,t} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^t = \alpha_2^p = \beta_1^t, \alpha_1^p = \beta_1^t \beta_2^t, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1$ or v , $t=1,2, \dots, p-1$, and $1+4rt$ is a non-quadratic residue (mod p)

$$\phi_{19}(2211)g_{r,s,t} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_1^t \beta_2^t, \alpha_2^p = \beta_1^s \beta_2^k, \alpha^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1, v$ and $s=1,2, \dots, p-1$, where k, r and s satisfy the same conditions as for $d_{r,s,t}$

except that $(1-k)^2 + 4rs$ is a non-quadratic residue (mod p)

$$\phi_{19}(2211)h_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha^p = \beta_1, \alpha_1^p = \beta_2^r, \alpha_2^p = \beta_2^r, \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1,2, \dots, p-1$

$$\phi_{19}(2211)i = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha^p = \beta_1, \alpha_2^p = \beta_2, \alpha_1^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

$$\phi_{19}(2211)j_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1]^r = \alpha_1^p = \beta_1^r, [\alpha, \alpha_1] = \beta_1, \alpha^p = \beta_1 \beta_2, \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1,2, \dots, \frac{1}{2}(p-1)$

$$\phi_{19}(2211)k_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^r = \alpha_1^p = \beta_1^r, \alpha^p = \beta_1 \beta_2, \alpha_2^p = \beta_2^{s-r}, \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1,2, \dots, p-1$ and $s=0,1, \dots, \frac{1}{2}(p-1)$ where $s-r$ and $2r-s$ are not divisible by p

$$\phi_{19}(2211)l_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^r = \alpha_1^p = \beta_1^r, \alpha^p = \beta_1 \beta_2, \alpha_2^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1,2, \dots, p-1$

$$\phi_{19}(2211)m_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha^p = \beta_1, \alpha_1^p = \beta_2^r, \alpha_2^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

for $r=1$ or v

$$\phi_{19}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha_1^p = \beta_1, \alpha^p = \alpha_2^p = \beta^p = \beta_1^p = 1 \rangle \quad (i=1,2)$$

$$\phi_{19}(21^4)b_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_1 \beta_2^r, \alpha^p = \alpha_2^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{19}(21^4)c_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha_1^p = \beta_1, \alpha_2^p = \beta_1^r, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{19}(21^4)d_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_1 \beta_2^r, \alpha_2^p = \beta_1^{k/r} \beta_2^k, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v , where $k = g^s$ ($s = 0, 1, \dots, \frac{1}{2}(p-3)$) and $d_{v,0}$ only exists for $p \equiv 1 \pmod{4}$

$$\phi_{19}(21^4)e_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_2^r, \alpha^p = \alpha_2^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{19}(21^4)f_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha_1^p = \beta_1 \beta_2^r, \alpha_2^p = \beta_1^{-1/r} \beta_2^{-1}, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{19}(21^4)g = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha^p = \beta_1, \alpha_1^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

$$\phi_{19}(21^4)h = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha^p = \beta_1 \beta_2, \alpha_1^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

$$\phi_{19}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_1, \alpha^p = \alpha_1^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

(20) Note: In this family, $\alpha_2^{(p)} = \alpha_2^p \beta_1^{-\binom{p}{3}}$.

$$\phi_{20}(2211)a = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha_2^{(p)} = \beta_2, \alpha_1^p = \beta_1, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

$$\phi_{20}(2211)b_{r-1} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^r = \alpha_2^{(p)} = \beta_2^r, \alpha_1^p = \beta_1, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 2, 3, \dots, p-1$, and $p > 3$

$$\phi_{20}(2211)c_{r,0} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^r = \alpha_1^p = \beta_2^r, \alpha_2^{(p)} = \beta_1^r, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{20}(2211)c_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^s = \alpha_1^p = \beta_2^s, \alpha_2^{(p)} = \beta_1^r \beta_2, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v and $s = 1, 2, \dots, p-1$, where $1 + 4rs$ is a quadratic residue (mod p)

$$\phi_{20}(2211)d_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha_2^{(p)} = \beta_2, \alpha_1^p = \beta_1 \beta_2^r, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{20}(2211)e_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^s = \alpha_1^p = \beta_2^s, \alpha_2^{(p)} = \beta_1^r \beta_2, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v , where $1 + 4rs = 0$

$$\phi_{20}(2211)f_{r,0} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^r = \alpha_1^p = \beta_2^r, \alpha_2^{(p)} = \beta_1^{v/r}, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{20}(2211)f_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^s = \alpha_1^p = \beta_2^s, \alpha_2^{(p)} = \beta_1^r \beta_2, \alpha^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1$ or v and $s = 1, 2, \dots, p-1$, where $1 + 4rs$ is a non-quadratic residue (mod p)

$$\phi_{20}(2211)g = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha^p = \beta_2, \alpha_1^p = \beta_1, \alpha_2^{(p)} = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

$$\phi_{20}(2211)h_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^k = \alpha_2^{(p)} = \beta_2^k, \alpha^p = \beta_1, \alpha_1^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{20}(2211)i_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1]^r = \alpha_1^p = \alpha_2^{(p)} = \beta_2^r, \alpha^p = \beta_1, \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r = 1, 2, \dots, p-1$

$$\phi_{20}(2211)j_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha^p = \beta_2, \alpha_2^{(p)} = \beta_1^r, \alpha_1^p = \beta^p = \beta_1^p = 1 \quad (i=1,2) \rangle$$

for $r=1,2, \dots, p-1$

$$\phi_{20}(2211)k_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]^{\ell}=\alpha_1^p \beta_2^{\ell}, \alpha^p=\beta_1, \alpha_2^{(p)}=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

where $\ell = g^r$ for $r+1=1,2, \dots, (p-1,4)$

$$\phi_{20}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, \alpha_1^p=\beta_1, \alpha^p=\alpha_2^{(p)}=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $p > 3$

$$\phi_{20}(21^4)b = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\alpha_2^{(p)}=\beta_2, \alpha^p=\alpha_1^p \beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $p > 3$

$$\phi_{20}(21^4)c_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, \alpha_2^{(p)}=\beta_1^r \beta_2, \alpha^p=\alpha_1^p \beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=1$ or v

$$\phi_{20}(21^4)d_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]^r=\alpha_1^p \beta_2^r, \alpha^p=\alpha_2^{(p)}=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=1$ or v

$$\phi_{20}(21^4)e_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, \alpha_2^{(p)}=\beta_1^r, \alpha^p=\alpha_1^p \beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=1$ or v

$$\phi_{20}(21^4)f = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\alpha^p \beta_2, \alpha_1^p=\alpha_2^{(p)}=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

$$\phi_{20}(21^4)g = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, \alpha^p=\beta_1, \alpha_1^p=\alpha_2^{(p)}=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

$$\phi_{20}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, \alpha^p=\alpha_1^p \alpha_2^{(p)}=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

(21) Note: In this family, $\alpha_1^{(p)} = \alpha_1^p \beta_2^{-\binom{p}{3}}$, $\alpha_2^{(p)} = \alpha_2^p \beta_1^{\binom{p}{3}}$ and $t = 1 - \frac{1}{2}(p-1,4)$.

$$\phi_{21}(2211)a = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\alpha_2^{(p)}=\beta_2, [\alpha, \alpha_2]=\beta_1^v, \alpha_1^{(p)}=\beta_1, \alpha^p=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

$$\phi_{21}(2211)b_{r,s,1} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, [\alpha, \alpha_2]=\beta_1^v, \alpha_1^{(p)}=\beta_1^r \beta_2^{(s-1)/v}, \alpha_2^{(p)}=\beta_1^{s+1} \beta_2^r, \alpha^p=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=0,1, \dots, \frac{1}{2}(p-1)$ and $s=2,3, \dots, p-2$ or 0 , where $vr^2 \not\equiv s^2 - 1 \pmod{p}$ and $s^2 - 1$ is a non-quadratic residue \pmod{p}

$$\phi_{21}(2211)b_{r,s,2} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, [\alpha, \alpha_2]=\beta_1^v, \alpha_1^{(p)}=\beta_1^{r+1} \beta_2^{(s-t)/v}, \alpha_2^{(p)}=\beta_1^{s+t} \beta_2^{r-1}, \alpha^p=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=0,1, \dots, p-1$ and $s=0,1, \dots, \frac{1}{2}(p-1)$, where $vr^2 \not\equiv s^2 - t^2 + v \pmod{p}$ and $s^2 - t^2 + v$ is a non-quadratic residue \pmod{p}

$$\phi_{21}(2211)c_{r,1} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, [\alpha, \alpha_2]=\beta_1^v, \alpha_1^{(p)}=\beta_1^r, \alpha_2^{(p)}=\beta_1^2 \beta_2^r, \alpha^p=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=1,2, \dots, \frac{1}{2}(p-1)$

$$\phi_{21}(2211)c_{r,2} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]^r=\alpha_2^{(p)}=\beta_2^r, [\alpha, \alpha_2]=\beta_1^v, \alpha_1^{(p)}=\beta_1^r \beta_2^{-2/v}, \alpha^p=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=1,2, \dots, \frac{1}{2}(p-1)$

$$\phi_{21}(2211)c_{r,3} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, [\alpha, \alpha_2]=\beta_1^v, \alpha_1^{(p)}=\beta_1^{r+1} \beta_2^{-(s+t)}, \alpha_2^{(p)}=\beta_1^{s+t} \beta_2^{r-1}, \alpha^p=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r=1,2, \dots, p-1$, where s is the smallest positive solution to $s^2 \equiv t^2 - v \pmod{p}$

$$\phi_{21}(2211)d_{r,0,0} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2]=\beta, [\beta, \alpha_1]=\beta_1, [\alpha, \alpha_1]=\beta_2, [\alpha, \alpha_2]=\beta_1^v, \alpha_1^{(p)}=\beta_1^r \alpha_2^{1/v}, \alpha_2^{(p)}=\beta_1 \beta_2^r, \alpha^p=\beta^p \beta_1^p=1 \ (i=1,2) \rangle$$

for $r = 0, 1, \dots, \frac{1}{2}(p-1)$

$$\Phi_{21}(2211)d_{r,s,1} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^v, \alpha_1^{(p)} = \beta_1^r \beta_2^{(s-1)/v}, \alpha_2^{(p)} = \beta_1^{s+1} \beta_2^r, \alpha^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 0, 1, \dots, \frac{1}{2}(p-1)$ and $s = 2, 3, \dots, p-2$ or 0 , where $s^2 - 1$ is a quadratic residue (mod p)

$$\Phi_{21}(2211)d_{r,s,2} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^v, \alpha_1^{(p)} = \beta_1^{r+1} \beta_2^{(s-t)/v}, \alpha_2^{(p)} = \beta_1^{s+1} \beta_2^{r-1}, \alpha^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 0, 1, \dots, p-1$ and $s = 0, 1, \dots, \frac{1}{2}(p-1)$, where $s^2 - t^2 + v$ is a quadratic residue (mod p)

$$\Phi_{21}(2211)e_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha_2^r = \beta_2^r, [\alpha, \alpha_2] = \alpha^v \beta_1^v, \alpha_1^{(p)} = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 1, 2, \dots, \frac{1}{2}(p-1)$

$$\Phi_{21}(2211)f_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha_2^r = \beta_2^r, [\alpha, \alpha_2] = \alpha^v \beta_1^v, \alpha_1^{(p)} = \beta_2^s, \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 1, 2, \dots, \frac{1}{2}(p-1)$ and $s = 1, 2, \dots, p-1$

$$\Phi_{21}(2211)g_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \alpha_1^r = \beta_2^r, [\alpha, \alpha_2] = \alpha^v \beta_1^v, \alpha_2^{(p)} = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 1, 2, \dots, p-1$

$$\Phi_{21}(21^4)a_{r,1} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^v, \alpha_1^{(p)} = \beta_1^s \beta_2^{(r-1)/v}, \alpha_2^{(p)} = \beta_1^{r+1} \beta_2^s, \alpha^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 2, 3, \dots, p-2$ or 0 , where $r^2 - 1$ is a non-quadratic residue (mod p) and s is the smallest positive solution to $vs^2 \equiv r^2 - 1 \pmod{p}$

$$\Phi_{21}(21^4)a_{r,2} = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^v, \alpha_1^{(p)} = \beta_1^{r+1} \beta_2^{(s-t)/v}, \alpha_2^{(p)} = \beta_1^{s+t} \beta_2^{r-1}, \alpha^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 0, 1, \dots, p-1$, where $v(r^2-1) + t^2$ is a quadratic residue (mod p), and s is the smallest positive solution to $s^2 - t^2 \equiv v(r^2-1) \pmod{p}$

$$\Phi_{21}(21^4)b_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^v, \alpha_1^{(p)} = \beta_2^{-2r/v}, \alpha_2^{(p)} = \beta_2^{2(1-r)}, \alpha^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 0$ or 1

$$\Phi_{21}(21^4)b_r = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^v, \alpha_1^{(p)} = \beta_1 \beta_2^{(s-t)/v}, \alpha_2^{(p)} = \beta_1^{s+t} \beta_2^{-1}, \alpha^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

for $r = 2$ or 3 , corresponding respectively to the two smallest positive solutions to $s^2 - t^2 \equiv -v \pmod{p}$

$$\Phi_{21}(21^4)c = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \alpha^v \beta_1^v, \alpha_1^{(p)} = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

$$\Phi_{21}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^v, \alpha^p = \beta^p = \beta_1^p = 1 \ (i=1,2) \rangle$$

$$(22) \ \Phi_{22}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 | [\alpha_1, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \beta_1^p = \alpha_{i+1}^p = 1 \ (i=1,2) \rangle$$

$$\Phi_{22}(21^4)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 | [\alpha_1, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \alpha_3, \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \beta_1^p = \alpha_{i+1}^p = 1 \ (i=1,2) \rangle$$

for $r = 1$ or v

$$\Phi_{22}(21^4)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 | [\alpha_1, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \beta_1^p = \alpha_3, \alpha^p = \alpha_1^{(p)} = \beta_2^p = \alpha_{i+1}^p = 1 \ (i=1,2) \rangle$$

$$\Phi_{22}(21^4)d_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 | [\alpha_1, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \beta_1^p = \alpha_3, \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \beta_2^p = \alpha_{i+1}^p = 1 \ (i=1,2) \rangle$$

for $r = 1$ or v

$$\phi_{22}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \beta_1^p = \beta_2^p = 1 \quad (i=1,2) \rangle$$

(23) $\phi_{23}(2211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_1^p = \gamma, \alpha^p = \alpha_4, \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1,2,3) \rangle$

for $p > 3$

$$\phi_{23}(2211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^p = \alpha_4, \alpha_1^p = \alpha_4^r \gamma, \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

for $r = 1, 2, \dots, p-1$ and $p > 3$

$$\Delta_{23}(2211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^3 = \alpha_4 \gamma^r, \alpha_1^3 \alpha_2^3 \alpha_3 = \alpha_4, \alpha_{i+1}^{(3)} = \gamma^3 = 1 \quad (i=1,2,3) \rangle$$

for $r = 1$ or 2

$$\phi_{23}(2211)c_{r,s} = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^p = \gamma^r, \alpha_1^{(p)} = \alpha_4^k, \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

for $r = 1$ or v , where $k = g^s$ for $s + 1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{23}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^p = \alpha_4, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

$$\phi_{23}(21^4)b_{r,0} = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^p = \gamma^r, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

for $r = 1$ or v

$$\phi_{23}(21^4)b_{r,1} = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^p = \alpha_4 \gamma^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 4)$

$$\phi_{23}(21^4)c_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha_1^{(p)} = \alpha_4^k, \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{23}(21^4)d = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_1^p = \gamma, \alpha^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

for $p > 3$

$$\phi_{23}(21^4)e_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha_1^p = \alpha_4^k \gamma, \alpha^p = \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 3)$ and $p > 3$

$$\Delta_{23}(21^4)e = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^3 = \alpha_1^3 \alpha_2^3 \alpha_3 = \alpha_4, \alpha_{i+1}^{(3)} = \gamma^3 = 1 \quad (i=1,2,3) \rangle$$

$$\phi_{23}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = \gamma^p = 1 \quad (i=1,2,3) \rangle$$

(24) $\phi_{24}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha^p = \alpha_4, \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^{(p)} = 1 \quad (i=1,2,3) \rangle$

$$\phi_{24}(21^4)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^k = \alpha_4^k, \alpha^p = \beta^p = \alpha_{i+1}^{(p)} = 1 \quad (i=1,2,3) \rangle$$

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{24}(21^4)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \beta^p = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \quad (i=1,2,3) \rangle$$

$$\phi_{24}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^{(p)} = 1 \quad (i=1,2,3) \rangle$$

(25) and (26) $\phi_{25+x}(321) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_3, \alpha] = \alpha_4, \alpha^2 = \alpha_1^y, \alpha_1^{(p)} = \alpha_{i+2}^y, \alpha_{i+2}^p = 1 \quad (i=1,2) \rangle$

$$\phi_{25+x}(222) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_3, \alpha] = \alpha_4, \alpha_1^{(p)} = \alpha_{i+2}^y, \alpha^2 = \alpha_{i+2}^p = 1 \quad (i=1,2) \rangle$$

where $y = v^x$ and $x = 0$ (for ϕ_{25}) or $x = 1$ (for ϕ_{26})

(27) $\phi_{27}(21^4)a_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^k = [\alpha_1, \alpha_2]^k = \alpha^p = \alpha_4^k, \alpha_1^{(p)} = \beta^p = \alpha_{i+1}^{(p)} = 1 \quad (i=1,2,3) \rangle$

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 4)$

$$\phi_{27}(21^4)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^k = [\alpha_1, \alpha_2]^k = \alpha_1^k, \alpha^p = \beta^p = \alpha_{i+1}^{(p)} = 1 \quad (i=1,2,3) \rangle$$

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{27}(21^4)c_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^r = [\alpha_1, \alpha_2]^r = \beta^p = \alpha_4^r, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \quad (i=1,2,3) \rangle$$

for $r = 1$ or v

$$\phi_{27}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \beta] = [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \beta^p = \alpha_{1+1}^{(p)} = 1 \quad (i=1, 2, 3) \rangle$$

$$(28) \text{ and } (29) \quad \phi_{28+x}(321)a_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_3, \alpha] = [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_4^r, \alpha_1^{(p)} = \alpha_{1+2}^y, \alpha_{1+2}^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1, 2, \dots, p-1$

$$\phi_{28+x}(222) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_3, \alpha] = [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^{(p)} = \alpha_{1+2}^y, \alpha^p = \alpha_{1+2}^p = 1 \quad (i=1, 2) \rangle$$

where $y = v^x$ and $x = 0$ (for ϕ_{28}) or $x = 1$ (for ϕ_{29})

$$(30) \quad \phi_{30}(21^4)a = \langle \alpha, \alpha_1, \dots, \alpha_4, \beta | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \beta] = \alpha_{1+2}, [\alpha_3, \alpha] = \alpha^p = \alpha_4, \alpha_1^p = \beta^p = \alpha_{1+2}^p = 1 \quad (i=1, 2) \rangle$$

for $p > 3$

$$\phi_{30}(21^4)b_r = \langle \alpha, \alpha_1, \dots, \alpha_4, \beta | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \beta] = \alpha_{1+2}, [\alpha_3, \alpha]^k = \alpha_1^{(p)} = \alpha_4^k, \alpha^p = \beta^p = \alpha_2^{(p)} = \alpha_{1+2}^p = 1 \quad (i=1, 2) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{30}(21^4)c = \langle \alpha, \alpha_1, \dots, \alpha_4, \beta | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \beta] = \alpha_{1+2}, [\alpha_3, \alpha] = \beta^p = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_{1+2}^p = 1 \quad (1, 2) \rangle$$

$$\phi_{30}(21^4)d_r = \langle \alpha, \alpha_1, \dots, \alpha_4, \beta | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \beta] = \alpha_{1+2}, [\alpha_3, \alpha] = \beta^p = \alpha_4, \alpha_1^{(p)} = \alpha_4^k, \alpha^p = \alpha_2^{(p)} = \alpha_{1+2}^p = 1 \quad (i=1, 2) \rangle$$

where $k = g^r$ for $r+1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{30}(1^6) = \langle \alpha, \alpha_1, \dots, \alpha_4, \beta | [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \beta] = \alpha_{1+2}, [\alpha_3, \alpha] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \beta^p = \alpha_{1+2}^p = 1 \quad (i=1, 2) \rangle$$

$$(31) \text{ and } (32) \quad \phi_{31+x}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\alpha_1, \beta_1] = \alpha^p = \gamma, [\alpha_2, \beta_2] = \gamma^y, \alpha_1^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\phi_{31+x}(21^4)b = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\alpha_1, \beta_1] = \alpha_2^p = \gamma, [\alpha_2, \beta_2] = \gamma^y \alpha^p = \alpha_1^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\phi_{31+x}(21^4)c = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\alpha_1, \beta_1] = \alpha_1^p = \gamma, [\alpha_2, \beta_2] = \gamma^y, \alpha_2^p = \gamma^j, \alpha^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

where j is the smallest positive solution to $j^2 + y \equiv 0 \pmod{p}$

$$\phi_{31+x}(21^4)d = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\alpha_1, \beta_1] = \alpha^p = \alpha_1^p = \gamma, [\alpha_2, \beta_2] = \gamma^y, \alpha_2^p = \gamma^j, \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

where j is as in the previous group

$$\phi_{31+x}(21^4)e_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\alpha_1, \beta_1] = \alpha_2^p = \gamma, [\alpha_2, \beta_2] = \gamma^y, \alpha^p = \gamma^r, \alpha_1^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1, v$

$$\phi_{31+x}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^y, \alpha^p = \alpha_1^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

where $y = v^x$ and $x = 0$ (for ϕ_{31}) or $x = 1$ (for ϕ_{32})

$$(33) \text{ Note: In this family, } \alpha_2^{(p)} = \alpha_2^p \gamma^{\binom{p}{3}}$$

$$\phi_{33}(21^4)a = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\beta_2, \alpha] = [\alpha_1, \beta_1] = \alpha_1^p = \gamma, \alpha^p = \alpha_2^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\phi_{33}(21^4)b_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\beta_2, \alpha]^r = [\alpha_1, \beta_1]^r = \alpha^p = \gamma^r, \alpha_1^p = \alpha_2^{(p)} = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v

$$\phi_{33}(21^4)c_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\beta_2, \alpha]^r = [\alpha_1, \beta_1]^r = \alpha_2^{(p)} = \gamma^r, \alpha^p = \alpha_1^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

for $r = 1$ or v

$$\phi_{33}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma | [\alpha_1, \alpha] = \beta_1, [\beta_2, \alpha] = [\alpha_1, \beta_1] = \gamma, \alpha^p = \alpha_1^p = \alpha_2^{(p)} = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

(34) $\phi_{34}(321)a = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha] = \beta_1 (i=1,2), [\beta_2, \alpha] = [\alpha_1, \beta_1] = \beta_1^p = \gamma, \alpha^p = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_2^p = \gamma^p = 1 \rangle$
 $\phi_{34}(321)b_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha] = \beta_1 (i=1,2), [\beta_2, \alpha] = [\alpha_1, \beta_1] = \beta_1^p = \gamma, \alpha^p = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \gamma^r, \beta_2^p = \gamma^p = 1 \rangle$
 for $r = 1$ or v

(35) $\phi_{35}(21^4)a = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_5, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$
 $\phi_{35}(21^4)b_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha_1^{(p)} = \alpha_5^k, \alpha^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$
 where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 4)$
 $\phi_{35}(1^6) = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$

(36) $\phi_{36}(21^4)a_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha^p = \alpha_5^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$
 where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 6)$
 $\phi_{36}(21^4)b_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_5, \alpha_1^{(p)} = \alpha_5^k, \alpha^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$
 where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 4)$ and $p > 3$
 $\phi_{36}(1^6) = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_5, \alpha^{(p)} = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$

(37) Note this family does not exist for $p = 3$
 $\phi_{37}(21^4)a_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_2, \alpha_3]^r = [\alpha_3, \alpha_1]^r = [\alpha_4, \alpha_1]^r = \alpha^p = \alpha_5^r, \alpha_1^p = \alpha_{i+1}^p = \alpha_5^p = 1 (i=1,2,3) \rangle$
 for $r = 1$ or v

$\phi_{37}(21^4)b_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = [\alpha_4, \alpha_1] = \alpha_1^p = \alpha_5, \alpha^p = \alpha_5^k, \alpha_{i+1}^p = \alpha_5^p = 1 (i=1,2,3) \rangle$
 where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 4)$
 $\phi_{37}(21^4)b_p = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = [\alpha_4, \alpha_1] = \alpha_1^p = \alpha_5, \alpha^p = \alpha_{i+1}^p = \alpha_5^p = 1 (i=1,2,3) \rangle$
 $\phi_{37}(1^6) = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = [\alpha_4, \alpha_1] = \alpha_5, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = \alpha_5^p = 1 (i=1,2,3) \rangle$

(38) Note this family does not exist for $p = 3$
 $\phi_{38}(21^4)a_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3]^k = \alpha^p = \alpha_5^k, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$
 where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 5)$
 $\phi_{38}(21^4)b_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3]^r = \alpha^p = \alpha_1^r = \alpha_5^r, \alpha_{i+1}^p = 1 (i=1,2,3,4) \rangle$
 for $r = 1, 2, \dots, p-1$

$\phi_{38}(21^4)b_{p+r} = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3]^k = \alpha_1^{(p)} = \alpha_5^k, \alpha^p = \alpha_{i+1}^p = 1 (i=1,2,3,4) \rangle$
 where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 4)$
 $\phi_{38}(1^6) = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3] = \alpha_5, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 (i=1,2,3,4) \rangle$

(39) Note this family does not exist for $p = 3$
 $\phi_{39}(21^4)a_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, [\alpha_2, \alpha_3]^k = [\alpha_3, \alpha_1]^k = [\alpha_4, \alpha_1]^k = \alpha^p = \alpha_5^k, \alpha_{i+1}^p = \alpha_5^p = 1 (i=1,2,3) \rangle$
 where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 6)$

$\phi_{39}(21^4)b_r = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, [\alpha_2, \alpha_3]^r = [\alpha_3, \alpha_1]^r = [\alpha_4, \alpha_1]^r = \alpha_1^p = \alpha_5^r, \alpha_{i+1}^p = \alpha_5^p = 1 (i=1,2,3) \rangle$

for $r = 1, 2, \dots, p-1$
 $\phi_{39}(21^4)b_{p+r} = \langle \alpha, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, [\alpha_2, \alpha_3]^k = [\alpha_3, \alpha_1]^k = [\alpha_4, \alpha_1]^k = \alpha_1^p = \alpha_5^k, \alpha_{i+1}^p = \alpha_5^p = 1 (i=1,2,3) \rangle$

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 5)$

$$\phi_{39}(1^6) = \langle \alpha_1, \alpha_1, \dots, \alpha_5 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = [\alpha_4, \alpha_1] = \alpha_5, \alpha^P = \alpha_1^P = \alpha_{i+1}^P = \alpha_5^P = 1 \rangle$$

(i=1, 2, 3)

(40) $\phi_{40}(21^4)a_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta_1, \alpha_2] = [\beta_2, \alpha_1] = \alpha_1^P = \gamma, \alpha_2^P = \gamma^k, \beta^P = \beta_1^P = \gamma^P = 1 \rangle$

(i=1, 2)

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 3)$

$$\phi_{40}(21^4)a_p = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta_1, \alpha_2] = [\beta_2, \alpha_1] = \alpha_1^P = \gamma, \alpha_2^P = \beta^P = \beta_1^P = \gamma^P = 1 \rangle$$

(i=1, 2)

$$\phi_{40}(1^6) = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta_1, \alpha_2] = [\beta_2, \alpha_1] = \gamma, \alpha_1^P = \beta^P = \beta_1^P = \gamma^P = 1 \rangle$$

(i=1, 2)

(41) $\phi_{41}(21^4)a_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_1]^k = \alpha_1^P = \gamma^k, [\alpha_2, \beta_2] = \gamma^{-v}, \alpha_2^P = \beta^P = \beta_1^P = \gamma^P = 1 \rangle$

(i=1, 2)

where $k = g^r$ for $r + 1 = 1, 2, \dots, (p-1, 3)$

$$\Delta_{41}(21^4)a_1 = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_1] = \alpha_1^3 = \gamma, \beta^P = \beta_1^P = \gamma^P = 1 \rangle$$

(i=1, 2)

for $p = 3$

$$\phi_{41}(1^6) = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_1]^{-v} = [\alpha_2, \beta_2] = \gamma^{-v}, \alpha_1^P = \beta^P = \beta_1^P = \gamma^P = 1 \rangle$$

(i=1, 2)

(42) $\phi_{42}(222)a_0 = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_2] = [\alpha_2, \beta_1] = \beta^P = \gamma, \alpha_1^P = \beta_1^{-1} \gamma^{-\frac{1}{2}}, \alpha_2^P = \beta_2 \gamma^{\frac{1}{2}}, \beta_1^P = \gamma^P = 1 \rangle$

(i=1, 2)

$\phi_{42}(222)a_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_2] = [\alpha_2, \beta_1] = \beta^P = \gamma, \alpha_1^P = \beta_1^{-1} \gamma^{\frac{1}{2}}, \alpha_2^P = \beta_2 \gamma^{r+\frac{1}{2}}, \beta_1^P = \gamma^P = 1 \rangle$

(i=1, 2)

for $r = 1, 2, \dots, p$

(43) $\phi_{43}(222)a_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_1]^{-v} = [\alpha_2, \beta_2] = \gamma^{-v}, \alpha_1^P = \beta_2 \gamma^k \alpha_2^P = \beta_1^v \gamma^{\ell}, \beta^P = \gamma^n; \beta_1^P = \gamma^P = 1 \rangle$

(i=1, 2)

where $n = v + \binom{p}{3}$, and k, ℓ are the smallest positive integers satisfying $(k-v)^2 - v(\ell+v)^2 \equiv r \pmod{p}$, for $r = 0, 1, \dots, p-1$.

Added in Proof. In her M.Sc. thesis (Australian National University, 1979), Miss A. M. Küpper has pointed out an error in the second line of the above list 4.6 (25) and (26). This should read:

$$“ \phi_{25+r}(222)a_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_3, \alpha]^y = \alpha_2^{(p)} = \alpha_4^y \alpha_1^{(p)} = \alpha_3^y \alpha_4^{r-y}, \alpha^P = \alpha_{i+2}^P = 1 \rangle$$

(i = 1, 2)

for $r = 0, 1, \dots, \frac{1}{2}(p-1)$ ” yielding another $p-1$ groups of order p^6 (and so another 2 groups of order 3^5). I am indebted to her for this correction.

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1. G. BAGNERA, "Normalformen ohne Verwendung Galoisscher Imaginären geben," *Ann. di Mat.* (3), v. 1, 1898, pp. 137–228.
2. H. BENDER, "A determination of the groups of order p^5 ," *Ann. of Math.* (2), v. 29, 1927, pp. 61–94.
3. N. BLACKBURN, "On a special class of p -groups," *Acta Math.*, v. 100, 1958, pp. 45–92. MR 21 #1349.
4. J. DE SÉGUIER, *Éléments de la Théorie des Groupes Abstracts*, Gauthier-Villars, Paris, 1904.
5. T. EASTERFIELD, *A Classification of Groups of Order p^6* , Ph.D. Dissertation, Cambridge, 1940.
6. M. HALL & J. SENIOR, *The Groups of Order 2^n ($n \leq 6$)*, Macmillan, New York, 1964. MR 29 #5889. See also review in this journal, *Math. Comp.*, v. 19, 1965, pp. 335–337.
7. P. HALL, "The classification of prime-power groups," *J. Reine Angew. Math.*, v. 182, 1940, pp. 130–141. MR 2 #211.
8. B. HUPPERT, *Endliche Gruppen. I, Die Grundlehren der Math. Wissenschaften, Band 134*, Springer-Verlag, Berlin and New York, 1967. MR 37 #302.
9. R. JAMES, *The Groups of Order p^6 ($p \geq 3$)*, Ph.D. Thesis, Univ. of Sydney, 1968.
10. R. JAMES & J. CANNON, "Computation of isomorphism classes of p -groups," *Math. Comp.*, v. 23, 1969, pp. 135–140. MR 39 #313.
11. Y. LEONG, "Odd order nilpotent groups of class two with cyclic centre," *J. Austral. Math. Soc.*, v. 17, 1974, pp. 142–153. MR 50 #470.
12. R. MIECH, "On p -groups with a cyclic commutator subgroup," *J. Austral. Math. Soc.*, v. 20, 1975, pp. 178–198. MR 53 #8243.
13. O. SCHREIER, "Über die Erweiterung von Gruppen II," *Hamburg Abh.*, v. 4, 1926, pp. 321–346.