*Mon. Not. R. astr. Soc.* (1977) **179**, 351–358

# The growth of density perturbations in zero pressure Friedmann–Lemaître universes

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Communicated by Dr D. Edwards

Received 1976 September 29; in original form 1975 December 12

**Summary**. Conditions are derived for the formation of bound density-perturbations in low-density universes. A quantitative analysis has then been carried out in the $(q_0, \sigma_0)$ plane into the resulting amplification of density perturbations existing at the decoupling epoch. These perturbations are then enhanced so that they become bound, and subsequently collapse into protogalaxies.

## 1 Introduction

It is well known that gravitational instability alone cannot explain the formation of the galaxies from the big bang models admitted by the General Theory of Relativity. The work of Lifshitz (1946) and Bonnor (1957) showed that, although there is growth, it does not amplify statistically probable fluctuations sufficiently in the time available. In this paper we calculate the resulting amplification of density perturbations existing at the epoch of decoupling, which become bound within the linear regime of the gravitational instability theory and subsequently collapse into protogalaxies. The growth of small density fluctuations in zero-pressure Friedmann–Lemaître universes has also been investigated by Tomita (1969).

Neglecting the pressure of matter and radiation, Bonnor (1957) derived the following differential equation governing the growth of the contrast density, $h(t) = \delta\rho/\rho$, in an expanding universe

$$R^2\ddot{h} + 2R\dot{R}\dot{h} - 4\pi G\rho R^2 h = 0 \qquad (1)$$

where the cosmic scale factor, $R(t)$, satisfies the Friedmann differential equation

$$R^2\dot{R}^2 = \frac{\Lambda}{3}R^4 - kR^2 + AR \qquad (2)$$

and $\Lambda$ is the cosmological constant, $k$ is the curvature index and $A = 8\pi G\rho_0 R_0^3/3$. As usual the subscript zero denotes the current epoch. The various cosmological models obtained as solutions of equation (2) can be classified in terms of the deceleration parameter $q_0$ and the density parameter $\sigma_0$ (see Stabell & Refsdal 1966), where

$$\sigma_0 = 4\pi G\rho_0/3H_0^2, \quad q_0 = -\ddot{R}_0/R_0H_0^2$$

and $H_0 = \dot{R}_0/R_0$ is the Hubble constant. It can be established from the field equations of General Relativity that the cosmological constant and the curvature index can be expressed in terms of these parameters, such that

$$3H_0^2(\sigma_0 - q_0) = \Lambda \tag{3}$$

and

$$H_0^2R_0^2(3\sigma_0 - q_0 - 1) = k. \tag{4}$$

## 2 Solution of the density perturbation equation

In order to solve the differential equation (1) the independent variable is changed from $t$ to the redshift $z$. This change in independent variable is given by

$$\frac{dz}{dt} = -H_0(z+1)P^{1/2}(z), \tag{5}$$

where $P(z)$ is the cubic polynomial

$$P(z) = 2\sigma_0 z^3 + (3\sigma_0 + q_0 + 1)z^2 + 2(1+q_0)z + 1. \tag{6}$$

The differential equation (1) now becomes

$$(z+1)P(z)\frac{d^2h}{dz^2} + Q(z)\frac{dh}{dz} - 3\sigma_0(z+1)^2h(z) = 0 \tag{7}$$

where $Q(z)$ is the cubic polynomial

$$Q(z) = \sigma_0 z^3 + 3\sigma_0 z^2 + 3\sigma_0 z + q_0. \tag{8}$$

It can be verified that a particular solution of equation (7) is $h^-(z) = P^{1/2}(z)$. Using this particular solution the other solution is found to be

$$h^+(z) = CP^{1/2}(z)\int_z^\infty \frac{(u+1)\,du}{P^{3/2}(u)}, \tag{9}$$

where $C$ is a constant. The solution (9) is the growing solution, and in this paper it has been evaluated by means of numerical integration.

It is of interest to investigate the nature of the roots of the cubic equation $P(z) = 0$. Using Cardan's solution the discriminant is

$$\Delta = \frac{(\sigma_0 - q_0)}{432\sigma_0^4}\,[27\sigma_0^2(\sigma_0 - q_0) - (3\sigma_0 - 1 - q)^3].$$

It is known that the nature of the roots is determined by this discriminant such that if

(a) $\Delta = o$ there are two identical roots,
(b) $\Delta > 0$ there is only one real root,
(c) $\Delta < 0$ there are three real roots.

It can be seen that the discriminant will vanish when either the cosmological constant is zero (i.e. $\sigma_0 = q_0$) or when

$$27\sigma_0^2(\sigma_0 - q_0) = (3\sigma_0 - 1 - q_0)^3. \tag{10}$$

The condition (10) is satisfied in the case of model universes with $\Lambda = \pm\Lambda_c$, where $\Lambda_c$ denotes the critical value of the cosmological constant in the static Einstein universe. In the case when $\Lambda = \Lambda_c$ and $k = +1$, we have the A1 model (Robertson 1933) which expands from the singularity and asymptotically approaches the static Einstein state. When $\Lambda = -\Lambda_c$ and $k = -1$, we have an oscillating model which expands out to a maximum radius and then collapses back to a singular state.

It follows from the above discussion that, if either the cosmological constant is zero or the condition (10) is satisfied, we expect that the equation $P(z) = 0$ has three real roots, two of which are coincident. In the case of three real roots, these are given by

$$z_i = [(6\sigma_0 - 2 - 2q_0)\,x_i - (1 + q_0 + 3\sigma_0)]/6\sigma_0, \quad i = 1, 2, 3$$

where

$$x_1 + x_2 + x_3 = 0$$

and

$$x = \cos(\theta/3)$$

satisfies the equation

$$\cos\theta = 4\cos^3(\theta/3) - 3\cos(\theta/3).$$

When the cosmological constant is zero $\cos\theta = 1$, and when $\Lambda = \pm\Lambda_c$ $\cos\theta = -1$. This gives the following expressions for $P(z)$

*When* $\Lambda = 0$:

$$P(z) = (2\sigma_0 z + 1)(z + 1)^2. \tag{11}$$

*When* $\Lambda = \pm\Lambda_c$:

$$P(z) = \frac{1}{27\sigma_0^2}(3\sigma_0 z + q_0 + 1)^2(6\sigma_0 z + 9\sigma_0 - 1 - q_0). \tag{12}$$

The integral in the growing solution (9) can be determined when the cosmological constant is zero and when $\Lambda = \pm\Lambda_c$. These have been published elsewhere (Edwards & Heath 1976) in terms of the development angle, $\theta$, but these solutions are now presented in terms of $z$, $\sigma_0$ and $q_0$.

*When* $\Lambda = 0$, $k = \pm 1$:

$$h^+(z) = \frac{(6\sigma_0 z + 4\sigma_0 + 1)}{|2\sigma_0 - 1|} - \frac{3\theta\sigma_0 P^{1/2}(z)}{|2\sigma_0 - 1|^{3/2}}, \tag{13}$$

where

$$\cos\theta = \frac{\sigma_0 z - \sigma_0 + 1}{\sigma_0(z + 1)} \quad \text{for} \quad \sigma_0 > \tfrac{1}{2}, k = 1,$$

and

$$\cosh\theta = \frac{\sigma_0 z - \sigma_0 + 1}{\sigma_0(z + 1)} \quad \text{for} \quad \sigma_0 < \tfrac{1}{2}, k = -1,$$

and $P(z)$ is given by the expression (11).

*When* $\Lambda = \pm\Lambda_c$:

$$h^+(z) = \frac{6\sigma_0 z + 3\sigma_0 + 1 + q_0}{|3\sigma_0 - 1 - q_0|} - \frac{3\theta\sigma_0 P^{1/2}(z)}{|3\sigma_0 - 1 - q_0|^{3/2}} + \frac{3(1 + q_0 - 3\sigma_0)}{(3\sigma_0 z + 1 + q_0)}, \tag{14}$$

where $P(z)$ is given by the expression (12), and

$$\cos\theta = \frac{3\sigma_0 z + 9\sigma_0 - 2 - 2q_0}{(3\sigma_0 z + q_0 + 1)} \quad \text{when } \Lambda = -\Lambda_c, k = -1$$

and

$$\cosh\theta = \frac{3\sigma_0 z + 9\sigma_0 - 2 - 2q_0}{(3\sigma_0 z + q_0 + 1)} \quad \text{when} \quad \Lambda = \Lambda_c, k = +1.$$

## 3 The conditions for bound perturbations

If a perturbation is bound it will eventually collapse into a discrete system. In model universes with a density parameter greater than or equal to the critical density, any relative perturbation must necessarily be bound. In the particular case of $\Lambda = 0$ cosmological models, the critical density is that of the Einstein–de Sitter universe. Since the work of Gott *et al.* (1974) suggests that an upper limit for the density parameter is $\sigma_0 = 0.045$ approximately, we now investigate the conditions for the evolution of bound perturbations in low-density universes.

Let $H_i$ denote the value of the Hubble parameter corresponding to the epoch $z_i$. It is found that the relationship connecting $H_i$ and $H_0$ is

$$H_i^2 = H_0^2[(z_i + 1)^2(2\sigma_0 z_i + q_0 - \sigma_0 + 1) + (\sigma_0 - q_0)]. \tag{15}$$

The density within the perturbation at the epoch $z_i$ will be denoted by $\bar{\rho}_i$ and the corresponding external density by $\rho_{ei}$. In terms of these, the contrast density satisfying the differential equation (1) is defined as

$$h(z_i) = \frac{\bar{\rho}_i - \rho_{ei}}{\rho_{ei}}. \tag{16}$$

The critical density at the epoch $z_i$ is given by

$$\rho_{ci} = \frac{3H_i^2}{8\pi G} - \frac{\Lambda}{8\pi G} \tag{17}$$

and the condition for a bound perturbation is $\bar{\rho}_i \geqslant \rho_{ci}$. Using equation (15) the critical density can be expressed as

$$\rho_{ci} = \frac{3H_0^2}{8\pi G}(z_i + 1)^2(2\sigma_0 z_i + q_0 - \sigma_0 + 1). \tag{18}$$

Using these results the relationship between $\rho_{ei}$ and $\rho_{ci}$ is

$$\frac{\rho_{ei}}{\rho_{ci}} = \frac{2\sigma_0(z_i + 1)}{(2\sigma_0 z_i + q_0 - \sigma_0 + 1)}. \tag{19}$$

on putting $\sigma_0 = q_0$ in equations (15) and (19), we recover the $\Lambda = 0$ relationships given by Gunn & Gott (1972). Using the relationship between $\rho_{ei}$ and $\rho_{ci}$, the condition for the contrast density to be bound is

$$h(z_i) \geqslant \frac{(1 + q_0 - 3\sigma_0)}{2\sigma_0(z_i + 1)}. \tag{20}$$

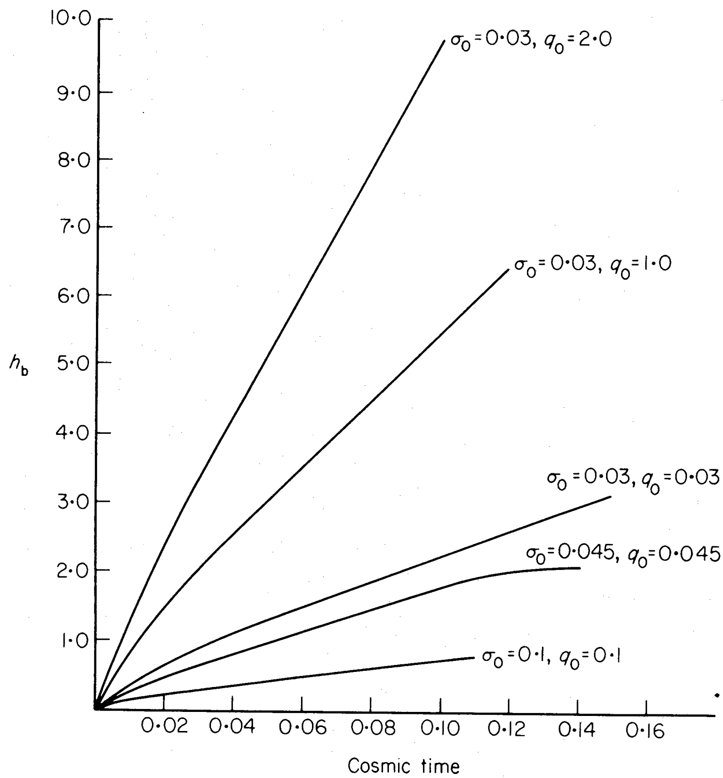


**Figure 1.** Each curve is terminated at the cosmic time corresponding to a redshift of 4.

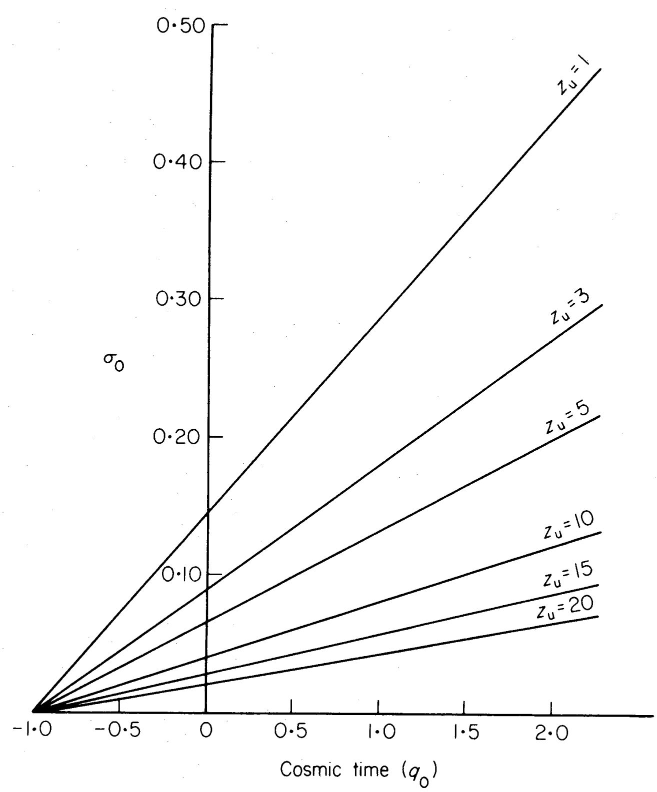


**Figure 2.** A contrast density of unity must be achieved at the latest by the redshift $z_u$. Loci of points in the $(q_0, \sigma_0)$ plane corresponding to $z_u$ = constant are shown above.

The inequality (20) determines the minimum value of the contrast density in order that a bound perturbation is produced at the epoch $z_i$. This minimum value will be denoted by $h_b(z_i)$.

It is of interest to note how the inequality (20) can be related back to the curvature of the universe. It can be seen from equation (4) that if the universe is flat or spherical then $(3\sigma_0 - q_0 - 1) \geq 0$, and the right-hand side of (20) becomes negative for all values of $z$. Any perturbation evolving within a flat or spherical universe must therefore necessarily be bound. Conversely, the curvature of the universe is hyperbolic if $(1 + q_0 - 3\sigma_0) > 0$. In this case the inequality (20) sets a positive value which must be exceeded in order that a bound perturbation is produced.

In Fig. 1, $h_b(z_i)$ has been plotted against cosmic time for different values of the density and deceleration parameters. A significant feature of this diagram is the steep increase in $h_b$ for low-density parameters, and the appreciable variation in its value caused by changes in $q_0$. Fig. 2 has been constructed by putting the contrast density equal to unity in the inequality (20), and calculating the minimum value of the redshift, $z_u$, that satisfies it. This gives rise to a pencil of straight lines in the $(q_0, \sigma_0)$ plane. Each ray has an equation $q_0 = (5\sigma_0 + 2\sigma_0 z_u - 1)$ and the vertex of the pencil is the point $(q_0 = -1,\ \sigma_0 = 0)$, corresponding to the De Sitter universe. The significance of choosing a contrast density of unity is that this is usually take as corresponding to the end of the linear regime for the growth of density perturbations, and $z_u$ is the redshift of the epoch of undecelerated expansion of the model (see Appendix 1). Once the contrast density rises above unity we pass over into the non-linear regime, and the differential equation (1) is then no longer valid.

## 4 The amplification of density perturbations

In the tables presented here we have evaluated the amplification of density perturbations existing at the epoch of decoupling. The choice of $\sigma_0 = 0.03$ corresponds approximately to the lower limit of the density parameter proposed by Gott *et al.* (1974) and $\sigma_0 = 0.045$ to the upper limit. With regard to the deceleration parameter, Gott *et al.* (1974) felt safe in concluding only that $q_0 < 2$. We assume the perturbation starts growing at the decoupling epoch at a redshift denoted in the tables by $z_d$. In the analysis presented here we have taken two values of $z_d$, 1500 and 1000, both of which figures are commonly used for the decoupling redshift.

In order that a bound perturbation is produced, its growth is terminated at $z_u$. The effect of this is that the phase of undecelerated expansion of the model is taken to coincide with the end of the linear regime, providing the perturbation with the maximum possible time for growth. The cosmological models chosen in Tables 1 and 2 give either $z_u = 10$ or $z_u = 20$ in order that the inequality (20) is satisfied.

**Table 1.**

$\sigma_0 = 0.045$

| $q_0$ | $z_d$ | $z_u$ | Cosmic time | Amplification | Decoupling fluctuation |
|---|---|---|---|---|---|
| 0.13 | 1000 | 10 | 0.049 | 59.6 | 0.017 |
| 0.13 | 1500 | 10 | 0.049 | 89.2 | 0.011 |
| 1.03 | 1000 | 20 | 0.018 | 31.4 | 0.032 |
| 1.03 | 1500 | 20 | 0.018 | 46.9 | 0.021 |

**Table 2.**

| | | | $\sigma_0 = 0.03$ | | |
|---|---|---|---|---|---|
| $q_0$ | $z_d$ | $z_u$ | Cosmic time | Amplification | Decoupling fluctuation |
| −0.25 | 1000 | 10 | 0.060 | 59.6 | 0.017 |
| −0.25 | 1500 | 10 | 0.060 | 89.2 | 0.011 |
| 0.35 | 1000 | 20 | 0.023 | 31.4 | 0.032 |
| 0.35 | 1500 | 20 | 0.023 | 46.9 | 0.021 |

If a perturbation starts to grow just after the end of the radiation era, Peebles (1967) argued that the contribution of the decaying solution must be negligible. Consequently, the amplification of a perturbation which starts growing after decoupling is

$$A = \frac{h^+(z_u)}{h^+(z_d)}.$$

The amplification formulae corresponding to the exact solutions (13) and (14) have been given elsewhere (Edwards & Heath 1976). In more general cases it is necessary to go back to equation (9) and calculate the amplification via numerical integration. The fluctuations that must be postulated at decoupling, in order to achieve a contrast density of unity by the redshift $z_u$, are obtained by taking the reciprocal of the amplification factors given in the tables.

It is seen from the tables that the amplification factors are identical to within the accuracy quoted for points lying on the straight lines $z_u$ = constant. This is due to the fact that each ray of the pencil of straight lines given in Fig. 2 has an equation

$$q_0 = (5\sigma_0 + 2\sigma_0 z_u - 1),$$

and the polynomial $P(z)$ occurring in the growing mode of equation (9) becomes dependent solely on the value assigned to the density parameter. Since $\sigma_0$ has been given the values corresponding to the upper and lower limits of the small range of the density parameter proposed by Gott *et al.* (1974), the amplification factors are effectively identical.

## 5 Comments

It is seen from the tables that the amplification increases with decreasing $q_0$. The range of fluctuations varies from $11 \times 10^{-3}$ to $32 \times 10^{-3}$, depending on the values assigned to $z_d, z_u$ and the density and deceleration parameters. The effect of a non-zero cosmological constant on the epoch $z_u$ is shown in the tables and also in Fig. 2.

## Acknowledgments

I thank Dr D. Edwards for initially suggesting this research, and also an anonymous referee for constructive criticisms. My sincere thanks also go to Dr B. J. T. Jones for useful discussions at the Institute of Astronomy, Cambridge.

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## Appendix 1

### The phase of undecelerated expansion

If we make the substitution $r = R/R_0$ in equation (2) and use relationships (3) and (4) we find that

$$H^2 = H_0^2 \left[(\sigma_0 - q_0) + \frac{(1 + q_0 - 3\sigma_0)}{r^2} + \frac{2\sigma_0}{r^3}\right]. \tag{A1}$$

At sufficiently early epochs $r \ll 1$ and the main contribution to the expansion rate comes from the term involving $r^{-3}$. It is seen from equation (A1) that the contributions of the curvature and the cosmological constant terms to the expansion rate in the early universe are virtually negligible.

A critical stage in the evolution of the universe is reached when the contributions from the terms involving $r^{-2}$ and $r^{-3}$ become equal, at a redshift given by

$$z + 1 = \frac{1 + q_0 - 3\sigma_0}{2\sigma_0} \tag{A2}$$

which marks the transition from behaviour like $k = 0$ to undecelerated expansion. The significance of this epoch is that, if bound density perturbations are to evolve, they must do so at the latest by the redshift given by (A2).