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## CONTENTS

Pare
Abstract ..... 1
I. Introduction ..... 1
II. Solution of the Equation of Motion ..... 6
A. Initial Growth of the Bubble ..... 8
B. Growth Behavior for $\boldsymbol{R} \gg \boldsymbol{>}$ ( ..... 10
III. Comparison with Experiment and Conclusion ..... 14
Appendix
Evaluation of this Temperature Inisizal ..... 18
(Eq. (33j)
Table ..... 10
Figures ..... 19-32
10


#### Abstract

The growth of a vapor bubblu in a suparheated liquid ia controlled by three factors: the inertia.of the liquid, the aurface tension, and the vapor presaure. As tay habble grown, evaporation takes place at the bubbic boundary, and the temperatire and vapor prassure tn the bubble are thereby decrased. Tha heat laflow :iaqu.rament of evaporstion, however, dapends on the rata of bubbia growth, so that the dynarnic problem is linked with a heat diffuaion problam. Since the hat diffusion problam has been solved, a quantitative formulation of the drnamic problem can be given. A solution for the radius of the vapor bubble 15 a sunction of time is obtaimad which is valld for mutficiently large radius. This asymptotic solution covars the range of physical laterest since the radiue at which it becomes valid is autar the lower limit of experimental observation. It shows the strong effect of hat diffusion on the ratc of bubble growth. Comparison of tha predicted radius-time behavior is mace rith oxerimantal observations in auperheated water, and very good ajramment is found.


## 1. ENTRODUCTIOA

Whan the vapor presuare in a Hquid emceadu the ambient presount, it bscomes posjible for a vafor buoble to groal from a small "auclaus" in hes liquid. This תucleu; is a region of noaliquid phasa and presuinaioly conjists

 1in



 temperamiaz and hance the preasure within the bubble drop and the rate of groviti is decronjed, Onz mizat, thereinere, capect amarimum in the valocity of the bubble wall. The reduction of the temparature within the wisble is a conjequance of the latent heat requirement of the evaporation winc. aken
place at the vapor-liquid Intarface as the bubble grows.
For the quantitativa soiution of the problem, some simplifying phystcal asaumptions may be made. The example considered in detail here is that of a vapor bubble growing in moderately superheated water, and the argumenta for the validity of the goneral assumptions are justified for this casc.

It will te assumad that tha bubbla is spherical throughout its growth. This assumption is rassomabla providad tha radial accaleration and velocity are mall, for with a sphericsily aymmatric extarnal pressure fiald tha spharical shape will then be staibla under the action ofacuriace tension. ${ }^{1}$

Excluded from consideration ts the anymetric, buoyath force of gravity which becomes important if the bubble growth is followed for so long a time that a alguificant translational veloeity is acquired. A transiational velocity of the bubbie ais a whole not only causes $z$ daformation in shape but also increases the ratia of hent inflow to the bubble over that used in the analysis presented zelov. In triter sugerianatad by about $10^{\circ} \mathrm{C}$, no 3 geat error is introduced by the buoyant force provided the bubble growth is not folloysd wayond a. radius of approzarastely 1 mm .

For a superinat of $10^{\circ} \mathrm{K}$ in zater, a vafor bubsle grovis from its initial microscopie sise to a radius of 1 mm in arime of the order of $10 \mathrm{milli}-$ sec. Taz correspondinj uvarage radial relocity of $10 \mathrm{~cm} / \mathrm{sec}$ is vary 3 mall






 that sha y

 siace the acoujtic veloedty in the vapor is 20 lazge, the peasure within ha vapor follow gractically instandasounly itg valuz at tis bubble mall. Whar the

[^0]velocity of tho bubble wall is sufficiontly slow, the prossure of the vapor is given by the equilibrium vapor pressure of the liquid. That this is the casu for the present problem may be soen as follows. The mean valocity appropriate for the rate of evaporation from a liquid aurface is ( $\left.\mathrm{B} / \mathrm{T}_{\mathrm{\pi} M}\right)^{1}$ where $B$ is the gat constant, $T$ the absolute temperature, and $M$ is the molecular weight. This velocity must be reduced by the coefficient for avaporation which has the value 0,04 for water. ${ }^{3}$ The cifitical volocity for a water surface at approximataly $100^{\circ} \mathrm{C}$ is therafore about $3 \mathrm{~m} / \mathrm{sec}$, which is appreciably graater than the radial velociiias ancountared hare, so that tha prosigure deficiancy from the equilibrium vapor pressure may be neglectad.

The tempefiture of the vapor weuld, in general, vary with position in the bubble as well as tith time. The approximation will be made, bowever, that the thermal diffusivity $D^{\prime}$ of the vapor is so large that temparature gradients are negilgible within the bubble. In water vapor the charactariatic ductusion length $(2 \mathrm{DI})^{1 / 2}$ is 0.24 cm for $t=10^{-2} \mathrm{sec}$. For the greateat sursyhzat convidsred here ( $m 10^{\circ} \mathrm{C}$ ) taz'bubbla radius at this tims is approximately 0.1 cm , phici is significaatly omallar than the charactarisic difinsion length. The approximation of uaform tamperature within the vafor improvas with cecrease in juyeriseat.

In aummaryi tiez joynical model uson orhich the calculationu are baved
 pressurs; the temeerature of the vagor is that oif the liquid at cin bis ble mil,

 vazos and in the liguili;

Tha equation of motion for she rasius $R$ of the bubble ian 3 nasiriseous. dacompresoible liguid as a fonction of time is ${ }^{4}$

$$
\begin{equation*}
R \ddot{x}+\frac{3}{2} \dot{n}^{2}=\left[0(\lambda)-x_{0}\right] / p \tag{1}
\end{equation*}
$$

 the presaure at infinity, and $p(2)$ is ins presjure in the liguld at tha bubble

[^1]boundary. A dot denotes differentiation with respect to time. The preseuro $E(R)$ le given In terms of the vapor presicure $P_{v}$ within the bubble by
\[

$$
\begin{equation*}
p(R)=P_{V}(T)-\tau_{\sigma} / R \tag{2}
\end{equation*}
$$

\]

Where $f$ is the surface tengion constant and $P_{V}(T)$ is the equilibrium vapor pressure for the tamperature $T$ at ths bubble boundary. In the following, the emall variations of $\sigma$ and $p$ with temperature are neglacted. It is sonvenient to introduce a radius $R_{0}$ definad by the reiation

$$
\begin{equation*}
2 \sigma / R_{0}=P_{v}\left(T_{0}\right)-P_{0} \tag{3}
\end{equation*}
$$

In which $)_{0}$ is the temperature of superheat of the liquid at a distance from the bubble. Fhysieally, $R_{0}$ is the effective initial radius of the bubble; it regresents an axtrapolation of the free apherical bubble down to the equilibrium radius for the given initial conditions. It should le noted that a bubbls at rest with radiua $R_{0}$ is in unstable equilibrium. The actusl nucleus from which. the bubble groms is not necesoarily opherical and its aurface energy may br: appraciably lasj than $4 \pi \mathbb{F} R_{0}^{2}$; homavir, the nucleus from which tha bubbie g7073 and the free opderical bubble of radius $R_{0}$ are both in unstablz equilibrium at the temporature $T_{0}$ ad.d external p-asjure $P_{0}$. Table 1 gives a set of valuas of $R_{0}$ at various supgrineat temperaturea in miter for a presjure of $P_{0}$ of one atrm.


$$
\begin{equation*}
R \ddot{x}+\frac{3}{2} \dot{z}^{2}=\frac{p_{v}(T)-p_{v}\left(T_{0}\right)+\left(2 \pi / g_{0}\right)\left(1-R_{0} / R\right)}{p} . \tag{1}
\end{equation*}
$$

03

$$
\begin{equation*}
\frac{1}{2 \alpha^{3} \dot{Z}} \frac{d}{d t}\left(R^{3} \dot{R}^{3}\right)=\frac{p_{v}(T)-B_{v}\left(T_{0}\right)+\left(3_{v} / R_{0}\right)\left(1-R_{0} / R\right)}{p} . \tag{3}
\end{equation*}
$$




$$
\begin{equation*}
\dot{R}^{2}=\left(R_{0} / R\right) \dot{R}_{0}^{2}+\left(1 \sigma / 3 \rho R_{0}\right)\left[1-R_{0}^{3} / R^{3}\right]-(2 J / \rho R)\left[1-R_{0}^{2} / R^{2}\right] \tag{6}
\end{equation*}
$$

This solution will bo referred to as the Rayleigh solution, ${ }^{5}$ for $R \gg R_{0}$, Eq. (6) becomes.

$$
\begin{equation*}
\dot{R}^{2} \sim \frac{4 \dot{\sigma}^{\prime}}{3 p R_{0}}=\frac{2}{3 p}\left[P_{V}\left(T_{0}\right)-P_{0}\right] \tag{7}
\end{equation*}
$$

which is a constant.
The actual motion deviates markedly from that pradictad by the Rayleigh solution because of the cooling effect. The heat $\dot{\phi}$ which must be supplied to the bubble per unit time is

$$
\begin{equation*}
\dot{Q}=(4 \pi / 3) L \frac{d}{d t}\left(R^{3} p^{\prime}\right) \tag{8}
\end{equation*}
$$

where L is the latent heat of evaporation per unit mass and $p^{\prime}$ is the vapor density. This heat is supplied by conduction from the liquid into the bubble so that

$$
\dot{Q}=4 \pi R^{2} k(a T / \partial r)_{R}
$$

Where $k$ is the thermal conductivity of the lIquid and ( $\partial T / \partial r)_{\text {R }}$ is tie temperature gradient in the liquid at the bubble boundary. Hence,

$$
\begin{equation*}
(\partial T / \partial x)_{R}=\frac{L}{3 x} \frac{1}{R^{2}} \frac{d}{d t}\left(R^{3} \rho^{\prime}\right) \tag{10}
\end{equation*}
$$

 sic:
 lated, therefore, by

$$
\begin{equation*}
(0 T / 3)_{a} \approx(1, p / x) \dot{R} \tag{11}
\end{equation*}
$$


 the small tamparature decrease during the bubble growth.

Equation (11) tojatiar with the specification of the temperature at
${ }^{5}$ The solution for the motion of a ruble weller suit constant prasiay.
 to the can? of a roll :psia; is?
infinity $T_{0}$ detorminos the ternperaiure field in the Hquid, The alution for this tomperature problem with the moving boundary $R(t)$ has been found ${ }^{6}$ under the assumption that the drop in temperature from $T_{0}$ to the value $T$ at the bubble boundary takes place in a layer of IIquid adjacent to the bubble which has small thickness compared with $R(t)$. This approximation of the "thin tharmal boundary linyer" is juatified physically because the thermai diffusivity of tha liquid is amall. The approximate expression for the tampersture at the bubble wall is ${ }^{7}$

$$
\begin{equation*}
T=T_{0}-\left(\frac{D}{x}\right)^{1 / 2} \int_{0}^{t} \frac{R^{2}(x)(\partial T / \partial x)_{z=R(x)}}{\left[\int_{x}^{t} R^{4}(y) d y\right]^{1 / 2}} d x \tag{12}
\end{equation*}
$$

The temperature $T$ is thus given in terms of $R$ and $\dot{R}$ by Eq. (12) so that $P_{v}(T)$ is also specified in terme of thase variables. Equations (5) and (12) tharzfore datarmina the dynanie proslem of the bubble groyth,

## II. SOLUTION OF THTE ECU.ATICN OF MOTION

 sure is specifizd aja function of temperaburz, For superheats not too tur

 turs:

$$
\begin{equation*}
\frac{P_{v}(T)-P_{0}}{P}=A\left(T-T_{b}\right) \tag{i3}
\end{equation*}
$$

For $T=T_{0}$, Eq. (13) sives

$$
\begin{equation*}
2 \sigma / \rho R_{0}=A\left(T_{0}-T_{b}\right) \tag{11}
\end{equation*}
$$

bacause of Eq. (3).

[^2]Equation (5) may now ba'writton as

$$
\begin{equation*}
\frac{1}{2 R^{2} \dot{f}} \frac{d}{d t}\left(R^{3} \dot{R}^{2}\right)=A\left(T-A_{0}\right)+\frac{2 \sigma}{p R_{0}}\left(1-\frac{R}{R_{0}}\right) \tag{15}
\end{equation*}
$$

It is convenient to uso in place of $R$ the dimensionless volume ratio

$$
\begin{equation*}
p=R^{3} / R_{0}^{3} \tag{16a}
\end{equation*}
$$

and in place of the dirysnionleas variable

$$
\begin{equation*}
u=\left(a / R_{0}^{4}\right) \int_{0}^{t} R^{4}(y) d y \tag{16b}
\end{equation*}
$$

where the constant a is defined at

$$
\begin{equation*}
a=\left[\frac{2 \sigma}{p R_{0}^{3}}\right]^{1 / 2} \tag{life}
\end{equation*}
$$

and has the dimensions of reciprocal time. Equation (15) then is transformed to

$$
\begin{equation*}
\frac{1}{6 y^{\prime}} \cdot \frac{d}{d u}\left(p^{7 / 3} p^{2}\right)=1-\frac{1}{p^{1 / 3}}-p \int_{0}^{a} \frac{p^{\prime}(y) d y}{(u-y)^{1 / 2}}, \tag{17}
\end{equation*}
$$

in which $p^{2}=d z / d u$ and the dimensionless parameter $\mu$ is given by

$$
j \quad \mu=\frac{A^{1} \frac{o^{\prime}}{36} \int_{0}^{0}}{j}\left[\frac{0}{\pi a}\right]^{1 / 2}
$$



$$
\begin{align*}
t & =\frac{1}{a} \int_{0}^{u} \frac{d v}{p^{7 / 3}(v)}:  \tag{19a}\\
R & =R_{0} p^{1 / 3}:  \tag{19b}\\
\dot{R} & =\left(a R_{0} / 3\right) p^{2 / 3} p^{\prime}:  \tag{19c}\\
T-T_{0} & =-\frac{\sigma^{2} R_{0}^{2} \mu}{A} \int_{0}^{u} \frac{p^{\prime}(v) d v}{(u-v)^{1 / 2}} \tag{19d}
\end{align*}
$$

The values of the radius $R$ for which experimental observations aro raadily obtainable are much greater than $R_{0}$ so that the esymptotic solution of Eg. ( 17 ) ( $p \gg 1$ ) is of the grentest physical interest. It is of importance bowever, to examiace the initial stages of the bubble growth since a quilitative dascripition oif this portion of the problom is nacessery for an undarotanding of in apyraptoke solution. It will bs shom that the solution in tis asymptotic ragze is not aifectad by the datailo of the matisumatical modal uned to deserike tha kethevior of tha', bbla for $R$ mase $R_{0}$ so that the uneartaiaty in the sur-
 aydications tr be ments herz.







 par tait volums infom $t=0$ to $t=t$ is

$$
\begin{equation*}
\eta=a t \text {. } \tag{20}
\end{equation*}
$$

Than, if the temperature of the bule liquidis $T_{0}$ at $t=0$, at tims $t$ it will be

$$
\begin{equation*}
T_{\infty}=T_{0}+\eta / \rho c=T_{0}+\eta D / k \tag{2!}
\end{equation*}
$$

The heat flow problem may be solved ${ }^{6}$ for the boundary conditions of Eng. (21) and (10), and the temperature at the bubble wall is given by replacing $T_{0}$ in Eq. (12) by $T_{\infty}$. In place of Eq. (19d) one has

$$
\begin{equation*}
T=T_{0}+\eta D / 2-\frac{a^{2} R_{0}^{2} \mu}{A} \int_{0}^{u} \frac{p^{\prime}(v) d v}{(n-v)^{1 / 2}} . \tag{22}
\end{equation*}
$$

The extension of Eg. (17) to include the heat source term is thus

$$
\begin{equation*}
\frac{1}{6 p^{\prime}} \frac{d}{d x}\left(p^{7 / 3} p^{2}\right)=1-\frac{1}{p^{173}}-\mu \int_{0}^{u} \frac{p^{\prime}(v) d v}{(u-v)^{1 / 2}}+\frac{A D}{a^{2} R_{0}^{2} k} \eta . \tag{23}
\end{equation*}
$$

For values of $p$ near the initial value of unity, one may write

$$
\eta=a t=\frac{a}{a} \int_{0}^{a} \frac{d v}{p^{4 / 3}(v)} \approx \frac{a}{a} u
$$

where the substitution for follo:r3 from Eq. (191). Equation (23) so: becomes

$$
\begin{equation*}
\frac{1}{6 p} \frac{d}{d \eta}\left(p^{7 / 3} p^{2}\right)=1-\frac{1}{p^{1 / 3}}-\mu \int_{0}^{u} \frac{p^{\prime}(v) d v}{(u-v)^{1 / 2}}+r u \tag{23'}
\end{equation*}
$$

whirs

$$
\begin{equation*}
Y=\frac{A D_{3}}{a^{3} R_{0}^{2} k} \tag{IA}
\end{equation*}
$$

This constant $Y$ is extracmaly mall; for example, ia water with a temperaturd zips of $1^{\circ} \mathrm{C} /$ aec and $R_{0}=10^{-3} \mathrm{~cm}, Y$ is approximately $10^{-6}$. The smallazes of the constant if fmplisu that the forced gro:yts away from the unstable equilibrium point $p=1, p^{\prime}=0$ is very slow until the bubble: radius has increased sufficiently for the surface tension to be partially "relaxed". This initial slow growth is a delay prod in the bubble: grover, for th:

Lutbile radiun ohangun very littlo untll thu remalhitig fetma on the right side of Eqa (23!) bocome approciablei The buiblle growith Ia than ato rapld that the ehnnge th the bulk tomperatura of the liquild in insignificant, and the term in $Y$ may be neglected. The delay period effectively gives an initial value of $R$ alightly greater than $R_{0}$ from which the imporatant growth begins. An approximate solution may be found for this initial period of forced growth from the equilibrium point by linearizing Eq. (23') ; that is, by neglecting terms of the second order in $p-1$ and its derivativas. The details of this calculation will not be presented here ; but, as would be expectad, the forced growth away from the equillibrium has an exponential behavior:

$$
\begin{equation*}
R \approx R_{0}\left[1+\frac{2 \gamma}{\beta\left(3 \beta^{2}+1\right)} e^{a \beta t}\right] \text {. } \tag{25}
\end{equation*}
$$

The tomperature at the bubble wall is given approximately by

$$
\begin{equation*}
T \approx T_{\infty}-\left[\frac{R_{0}^{2} a_{0}^{2}}{A}\right] \frac{2 \gamma\left(1-\beta^{2}\right)}{\beta\left(3 \beta^{2}+1\right)} e^{\alpha \beta t} \tag{25}
\end{equation*}
$$

The eonstant $\beta$ is that root of the equation

$$
\beta^{2}+3 \mu(\pi \beta)^{3 / 2}-1=0
$$

for phaici $\beta^{1 / 2}$ is posisive.
B. Grorgh 5inarios for $R=1 \mathrm{R}_{0}$

Aftar has busole growin has bean initiated, thare is a rapid rina in the vilocity $\dot{k}$ witil the cooling , fect escomes innoortant, The bubble wall valo-
 tinlo rejion are as yet availople and the detaila ois this analysis will be omitted.

 ston from the lifuid to the vapor. As $R$ incresaes, the temperiturs at the bubjle wall da jasass atzadily, but it camot tall betion $T_{b}$; for, if tha vapor camperabure 1 below $T_{b}$, the pressure diffarence $p_{v}-P_{0}$ would become negative and t Jubble growth would be arrested and eventually reversed. Such
behavior is excluded on physical grounds, It therefore follows that the intogral on the right hand sido of Eq. (17), which ie proportional to the temperature drop at the bubble wall, must approach a limit as $t$ or $u \rightarrow \infty$. A further physical argument determines more precisely the asymptotic behavior of this integral: The left hand side of Eq. (17) represents eazentially the acceleration effects of the bubble growth in the liquid. As the bubble grows, this acceleration tends toward zero because of the cooling effact. It therefore follows that

$$
\begin{equation*}
\mu \int_{0}^{u} \frac{p^{\prime}(v) d v}{(u-v)^{7 / 2}} \sim 1 \quad \text { as } u \rightarrow \infty \tag{27}
\end{equation*}
$$

This asymptotic relation may be inverted to yield ${ }^{8}$

$$
\begin{equation*}
p(u) \sim \frac{2}{\pi \mu} u^{1 / 2} \quad \text { at } u \infty \tag{28}
\end{equation*}
$$

If this result is substituted in Eq. (i7), it may be rerified that Eq3. (17), (27) and (23) are, in fact, consiatent.

Equation (28) is not yet useful since it provides no means of matching the indicated asymptotic solution of Eq. (17) with a solution valid for small valuss of p. The possibility of matching solutions depends on :3z yossibility of sbifiting the asymptotic solution in $t$ (or in u) $s 0$ as to accouat for the delay geriod in bubble grovith. It is necasjary that one be fres to shift the asymptotic solution since the duration of the dele y period depends congletely on the choice of the heat source t32m wille the subsequent benavior of tha bubbla is independent of this term. A manns for making an azsitrary tima shift is furnisised by notiag that, in addition to the asymptotic solution of Eq, (23), Ty. (17) al30 posa33コ33 the solution $p(u)=1$. It will sharaiore be asounasd tias s:s asymptotic solusion is describsd by

$$
\begin{gather*}
p(u)=1, \quad 0<u \leqslant u_{1},  \tag{29}\\
\mu \int_{u_{1}}^{u} \frac{p^{\prime}(v) d v}{(u-v)^{172}}=1-\frac{1}{p^{1 / 3}}-\frac{1}{6 p^{\prime}} \frac{d}{d u}\left(p^{7 / 3} p^{2}\right), u>u_{1} . \tag{30}
\end{gather*}
$$

[^3]From Eq. (19a), onu has corresponding to Eqs. (29) and (30)

$$
t=\frac{1}{a} \int_{0}^{u} \frac{d \cdot}{p^{4 / 3}(v)}=\left[\begin{array}{ll}
\frac{u}{a}, & 0<u \leqslant u_{1},  \tag{31}\\
\frac{u_{1}}{a}+\frac{1}{a} & \int_{u_{1}}^{u} \frac{d v}{p^{4 / 3}}, u>u_{1} .
\end{array}\right.
$$

wo that $u_{1} / a$ repreaents the duration of the delay period in the growth. The time delay may ba introduced explicitly in the amymptotic solution for $u>\mathrm{u}_{1}$ by use of the fact that, if $p(u)$ is a solution of Eq. (30), $P\left(u+u_{0}\right)$ is also a solution where $u_{0}$ is a constant. A consistent acheme for continuing the asymptotic solution may then be found by taking the solution to be of the form

$$
\begin{gather*}
p(u)=1, \quad 0<u \leqslant u_{1}  \tag{32a}\\
p(u) \sim \frac{2}{\pi \mu}\left(u-u_{0}\right)^{3 / 2}\left[1+\frac{b_{1}}{\left(u-u_{0}\right)^{1 / 6}}+\cdots+\frac{b_{5}}{\left(u-u_{0}\right)^{\prime / 6}}+\frac{b_{6} \ln \left(u-u_{0}\right)}{\left(u-u_{0}\right)}\right]
\end{gather*}
$$

$$
\begin{equation*}
u>u_{1} \tag{32k}
\end{equation*}
$$

 dijis rence $\left(u_{1}-u_{0}\right)$ is fited by the requiramant that $p\left(u_{1}\right)=1$. Tias dalay pariodis than dizerminad by ine choice of $u_{0}$.

IVhen (32b) is substituted in the integral on the lait side of Eg. (30) the rasultis 10
and $n$ are integurs. ${ }^{9}$ Higher terms are of the form $\left[\ln \left(u-u_{0}\right)\right]^{m} N\left(u-u_{0}\right)^{n / \delta}$, where m ${ }^{10}$ Sue the appendiz for the evaluation of the integral.

By aq. (30), this oxprecsion te also asymptotic to

$$
\begin{equation*}
1-\frac{1}{p^{1 / 3}} \cdot \frac{1}{6 p^{\prime}} \frac{d}{d u}\left(p^{7 / 3} p^{\prime 2}\right) \tag{34}
\end{equation*}
$$

If (34) is expanded by (32b), the coefficient a of corresponding powers of ( $u \cdot u_{0}$ ) may be equated to give a set of successive equations for the para meters $b_{1}, b_{2}, \ldots b_{6}$. At each ate, one hal a linear uquation for the unknown parameter. A tabulation of these paramatars for various superheat conditions in water for an external pressure of 1 atm. is given in Table il. The leading terms in the asymptotic solution are

$$
\begin{align*}
& P=\left(R / R_{0}\right)^{3} \sim \frac{2}{\pi \mu} u^{1 / 2}\left[1+0\left(u^{-1 / 6}\right)\right]: \\
& \left.t \sim \frac{3}{\left(\frac{\pi}{2}\right.}\right)^{4 / 3} u^{1 / 3}\left[1+0\left(u^{-1 / 6}\right)\right]: \\
& T-T_{0} \sim-\frac{e^{2} R_{0}^{2}}{A}\left[1+0\left(u^{-1 / 6}\right)\right] .
\end{align*}
$$

Thus,

$$
\begin{align*}
& \text { RN } R_{0}\left(\frac{2}{\pi N^{2}\left(\frac{a t}{3}\right)^{1 / 2}}\left[1+0\left(t^{-1 / 2}\right)\right] ;\right.  \tag{35}\\
& T-T_{0} \sim \frac{a^{2} R_{0}^{2}}{A}\left[1+0\left(t^{-1 / 2}\right)\right] . \tag{3i}
\end{align*}
$$




$$
\rho^{\prime} \rightarrow \cos 3 t,\left[1+\infty\left(t^{-1 / 3}\right)\right] \text {. an } t \rightarrow \infty \text {. }
$$



$$
R^{3} \sim \text { con } 3 t_{0} \cdot t^{3 / 2}\left[1+c\left(t^{-j / 3}\right)\right] \quad \text { as } t \rightarrow \infty
$$

It is evident that the vapor density varies alowly in the asymptotic range compared whit h the bubble $y$ jame so that the neglect of $d$ pods compared


## IH, COMPARISON VYRE EXPERIMENT AND CONCLUSION

With the values of the consiants given in Table II, radius-time curves have been computed for water at 1 atm for auperheata in the range $0: 102^{\circ} \mathrm{C}$ to $106^{\circ} \mathrm{C}$. These curvis are prosented in Fig. 1. Inasmuch as the delay pariod may be chosen arbitrarily so far as these asymptotic solutions are concernet, the time scals is determined only within an arbitrary constant which varies from one curve to anothor. The actual spacing of the curves as presented was chosen to that the time intercepts at $R=0,04 \mathrm{~cm}$ were equally spaced.

Observations have been made by Dergarabedian ${ }^{11}$ on the growth of vapor bubbles in euperheated water. The comparison between the theoretieal curves and the observed vaiues is ghown in Figs. 2, 3, and 4. The theoretical curves were obtained by interpolation from the values graphed in Fig. 1. The tims origins for both the theoretical curves and the experimental points are both arbitrary so that a time translation of the theoretical curve has been made in each case to give the best fit. The agreement is, however, seen to be very good. The importance of the cooling effect is evident from Fig. 2 whare the Rayleigh solution (Eg. (7)) is aloo shown.

Equation (35) grives for ths leading term in the asymptotic valocity

While the number of terms given in eq. (33b) is oubiscient foz high accuracy in


 should jiva the esuantial vapiation of 玉g. (37). The argument is as follo:7s: At a time $t$ at which tise bubbla sadius $R$ is much greater tinn $R_{0}$, ths dilizrsace betve an tiae tamparatura in tina liquid at the bubole wall and that in the liquid at a diobance is only alightly lasj than $T_{0}-T_{b}$. This temarature droy tales place principally in a liquid layer ayound the bubble of approximate thicieness given by the diffusion length $(\mathrm{D} t)^{1 / 2}$. The heat flow into tha butbia
${ }^{11}$ P. Dergarabedina, J. Appl. Mech., In preso.
par unit time is thorefore given roughly by

$$
\begin{equation*}
\dot{Q} \approx k \frac{\left(T_{0}-T_{b}\right)}{(D t)^{5 / 2}}+\pi R^{2} \tag{38}
\end{equation*}
$$

The heat requirement per unit Hme of evaporation, on the other hand is

$$
\begin{equation*}
\dot{Q}=L \frac{d}{d}\left(\frac{4}{3} w R^{3} \rho^{\prime}\right) \approx 4 w R^{2} \dot{R} L \rho^{\prime} \tag{39}
\end{equation*}
$$

When (39) is equated to (33), there results

$$
\begin{equation*}
\dot{L} \sim \sim \frac{k\left(T_{0}-T_{b}\right\rangle}{L p^{\prime}(D t)^{7 / 2}} \tag{40}
\end{equation*}
$$

which agrees in order of magnitude with the leading tarm as given by Eq. (37).
Some expariments kave recently been performed by Dergarabedian on vapor bubble growth in pure CCf $_{4}$. At moderate superheats, ito vapor. presaure curve is approsimazaly parallel to that of sater, displaced to iover temperatures. If tha ratas of bubble growth in these two liquids are compared at tha aame valua of the tameseratura difiserence ( $T_{0}-T_{b}$ ), thay should thus be soujaly in the sarge ratio as u(L $p^{1} D^{1 / 2}$ ) for the two Hquid. This conaizat is atout 3, 3 grater in misr than in carbon tsizachlorids.
 this viluz,

## APPENDIX

## Evaluation of thu Tomperature integral (Eq. (33))

By differentiating $p(u)$ (Eq. (32b)) and aubstituting into the temperature integral there rosults, after a change of variabla,

$$
\left.-\left[\begin{array}{c}
u_{s}  \tag{a}\\
u-u_{0}
\end{array}\right]\left[\operatorname{tav} v \operatorname{sn}\left(u-u_{0}\right)-2\right] \frac{1}{v}\right\}
$$

Gonaicar a typical interral appeariag in (7),

$$
\begin{equation*}
I_{s}(x)=\int_{x}^{1} \frac{v^{s-1} d 7}{(1-v)^{72}} \tag{s}
\end{equation*}
$$



By an argumant bajed on the thaory of analytite continuation, however, it may zeediay be shown tat $(c)$ is valid providad only that $<x<1$, s $\mathcal{A} 0,-1,-2, \ldots$, and hance that $(d)$ holds ior $0<x<1, R e(s)>-1 .\left\{\begin{array}{l}s=0, \\ (d) \text { is }\end{array}\right.$

[^4]\[

$$
\begin{align*}
& I_{3}\left(:=\frac{1}{3}\left[5(1 / 2,3 ; 3+1 ; 1)-3 r^{3}(1 / 2,3 ; 3+1 ;: \pi)\right]\right. \text {. }  \tag{c}\\
& \sim \frac{\Gamma(1 / 2) \Gamma(0)}{\Gamma(1+1 / 2)}-\frac{1}{3} x^{3} \quad \text { a3 } x \rightarrow 0^{+} \text {. } \tag{J}
\end{align*}
$$
\]

$$
\begin{aligned}
& \mu \int_{u_{1}}^{u} \frac{p^{\prime}(u) d u}{(u-v)^{1 / 2}} \\
& \sim \frac{1}{\pi} \int_{\left(\frac{u_{1}-u_{0}}{u-u_{0}}\right)}^{1} \frac{d v}{v^{7 / 2}\left(1-v v^{7 / 2}\right.}\left\{1+\left[\frac{2}{3} \frac{b_{1}}{\left(u-u_{0}\right)^{76}}\right] \cdot \frac{1}{1^{7 / 6}}+\left[\frac{1}{3} \frac{b_{2}}{\left(u-u_{0}\right)^{7 / 6}}\right] \frac{1}{v^{2 / 6}}\right. \\
& -\left[\frac{1}{3} \frac{b_{4}}{\left(u-u_{0}\right)^{7 / 6}}\right] \frac{1}{v^{2 / 6}}-\left[\frac{c}{3}-\cdots b_{5}^{\left(u-u_{0}\right)^{76}}\right] \frac{1}{5 / 6}
\end{aligned}
$$


#### Abstract

difforontiating (d) with respoct to at $s=-1 / 2$, one readily finds


$$
\begin{equation*}
\int_{x}^{1} \frac{\ln v d v}{3^{3 / 2}(1-v)^{1 / 2}} \sim-2 \pi+\frac{2}{x^{1 / 2}}(\ln x+2) \text { as } x \rightarrow 0^{+} \tag{c}
\end{equation*}
$$

With the add of (d), (c), Eq. (a) therefore becomes

$$
\begin{align*}
& \mu \int_{u_{1}}^{u} \frac{p(v) d v}{(u-v)^{2 / 2}} \\
& \sim\left\{i+.89266 \frac{b_{1}}{\left(u-u_{0}\right)^{76}}+.77306 \frac{. b_{2}}{\left(u-u_{0} f^{76}\right.}\right. \\
& \left.+.47545 \frac{b_{4}}{\left(u-u_{0}\right)^{\prime}}+.27450 \frac{b_{5}}{\left(u-u_{0}\right)^{5} / 6}+2 \frac{b_{6}}{\left(u-u_{0}\right)}\right\} \\
& -\frac{2}{-} \frac{1}{\left(u-u_{0}\right)^{1 / 2}}\left(u_{1}-u_{0}\right)^{3 / 6}\left\{1+\frac{b_{1}}{\left(u_{1}-u_{0}\right)^{1 / 6}}+\frac{b_{2}}{\left(u_{1}-u_{0}\right)^{7 / 6}}+\frac{b_{4}}{\left(u_{1}-u_{0}\right)^{4 / 6}}\right. \\
& \left.+\frac{b_{5}}{\left(u_{1}-u_{0}\right)^{j b}}+b_{6}-\frac{\ln \left(u_{1}-u_{0}\right)}{u_{1}-u_{0}}\right\} \tag{f}
\end{align*}
$$

 ing in (d) have bean cyaluated numerically in (f). By comparison with the a.yrmptotic form for p(u) of 玉q. (30) the lajt term in (f) may be writtan

$$
\frac{1}{\left(u-u_{0}\right)^{T Z}} \quad \frac{2}{n} b_{3}-\mu p\left(u_{1}\right)
$$

Inasmuch as $p\left(u_{1}\right)=1$ by assumption, Eq. (f) reduces to Eq. (33) of the text.

| $R_{0}\left(\mathrm{~cm} \times 10^{3}\right)$ | 1.56 | 1.02 | 0.751 | 0.590 | 0.183 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| T. ${ }^{\circ} \mathrm{C}$ | 102 | 103 | 104 | 105 | 106 |




Fig. 1 - Caiculated bubble radius versus time curves at


Fig. 3 - Comparison of theoretical radius-time values with three sets of experimental values obtained in water superheated in $104.5^{\circ} \mathrm{C}$ at an external pressure of 1 atm .


Fig. 4 - Comparioon of theoretical radius-time values with tivo sets of experimental values obtained in water superheated to 105.3 C at an external pressure of 1 atm .


[^0]:    ${ }^{1}$ M. S. Plesset, J. Appl. Pinys., in press.
    
    

[^1]:    ${ }^{3}$ G. Wylliz, Pror. Roy. Soc. (Lo,don) A 177, 333 (19.19).
    ${ }^{4}$ H. Lamb, Hydrodynamies ( Cover Fub. Netv Yortc, 1945); M. S. Pleaset, J, Ar.al, M. :ch, 16, 277 (1917).

[^2]:    ${ }^{6}$ M. S. Plesset and S. A. Zwic':, J. Appl. Phy3, 23, 95 (1952).
    ${ }^{7}$ Ref. 6, Eq. (20). The error in Eq. (20) may be estimated from Eq. (31) of this refercenc. For the problem discusized here it was fount to be lesu than $10 \%$ of the difference $T-T_{0}$ at any tiain.

[^3]:    ${ }^{8}$ Equation (27) is multiplied by $(z-u)^{-1 / 2}$ and integrated iron $u=0$ to $u=z$ to give Eq. (28).

[^4]:    
    

