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THE H_∞-OPTIMIZATION IN LOCALLY CONVEX SPACES[1]

CHUAN-GAN HU
Nankai University
Department of Mathematics
Tianjin 300071, P.R. China
E-mail: chuganhu@public.tpt.tj.cn, chuangan_hu@hotmail.com

LI-XIN MA
Dezhou University
Department of Mathematics
Dezhou 253023, Shandong, P.R. China

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In this paper, the ordinary H_∞-control theory is extended to locally convex spaces through the form of a parameter. The algorithms of computing the infimal model-matching error and the infimal controller are presented in a locally convex space. Two examples with the form of a parameter are enumerated for computing the infimal model-matching error and the infimal controller.

Key words: Inner-Outer Function, Minimal Realization, Infimal Model-Matching, Locally Convex Space.

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1. Introduction

The ordinary H_∞-control theory was summarized by B.A. Francis, G. Zames and J.C. Doyle *et al.* in [1-3]. In 1993, the H_∞-control theory was extended from finite dimensional spaces to infinite dimensional Hilbert spaces (see [4]). In this paper, we extend the H_∞-control theory to a complete Hausdorff locally convex space containing Hilbert spaces in order to enlarge the scope of solutions of H_∞-control theory.

Assume that $\mathbb{R}$ is the real field and $\mathbb{R}^n$ is the Cartesian product of n copies of $\mathbb{R}$, where n is any positive integer, and that $\boldsymbol{C}$ is a complex plane.

To solve the problem for simplicity, we apply the $G(s)$ in the model matching problem to $G(s,\xi)$, where s in $\boldsymbol{C}$ and ξ in $\mathbb{R}^n$, and $G(s,\xi)$ is in $C^\infty(\mathbb{R}^n)$ (locally convex space) for each fixed s in $\boldsymbol{C}$ and in H_∞ for each fixed ξ in $\mathbb{R}^n$, respectively.

First, we extend several concepts.

Definition 1.1: The locally convex space VH_∞ consists of all complex-valued parameter functions $F(s,\xi)$, of a complex variable s and a parameter ξ, which are analytic and bounded about s in $\mathcal{R}(s) > 0$ (for any fixed ξ in $\mathbb{R}^n$). The VH_∞-norm of $F(s,\xi)$ is defined by

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$$|||F|||_\infty = \sum_{k=1}^{\infty} \frac{g_k}{2^k(1+g_k)},\ g_k = \sup_{-k<\xi<k} \parallel F(\,\cdot\,,\xi) \parallel_\infty.$$

The subset of VH_∞ consisting of all real-rational functions of s and ξ is denoted by VRH_∞.

Let $\parallel F_j(\,\cdot\,,\xi) \parallel_\infty = \alpha_j(\xi),\ \xi \in \mathbb{R}^n,\ j = 1, 2.$ In VH_∞, the order may be defined as follows:

Definition 1.2: For any $F_1, F_2 \in VH_\infty$, we call $\alpha_1 \le \alpha_2$ if $\alpha_1(\xi) \le \alpha_2(\xi)$, $\xi \in \mathbb{R}^n$.

Definition 1.3: We call $F(s,\xi)$ to be strong proper if $F(s,\xi) \in VRH_\infty$ and $\sup_{\xi\in\mathbb{R}^n} \mid F(\infty,\xi) \mid\ < \infty$, strictly strong proper if $F(\infty,\xi) \equiv 0$. We call $F(s,\xi)$ to be stable if $F(s,\xi) \in VRH_\infty$ and $F(s,\xi)$ has no poles in the closed right half-plane $\mathcal{R}(s) \ge 0$ (for any fixed ξ in $\mathbb{R}^n$).

If $F(s,\xi)$ is real-rational about s in $\mathcal{R}(s) > 0$, then $F(s,\xi) \in VRH_\infty$ if and only if F is strong proper and stable (for any fixed $x \in \mathbb{R}^n$).

Similarly, we define the transfer function matrix

$$G(x,\xi) = \begin{bmatrix} T_1(s,\xi) & T_2(s,\xi) \\ T_3(s,\xi) & 0 \end{bmatrix},\ \ K(s,\xi) = -Q(s,\xi),$$

Then the model-matching problem is $\parallel T_1(\,\cdot\,,\xi) - T_2(\,\cdot\,,\xi)Q(\,\cdot\,,\xi)T_3(\,\cdot\,,\xi) \parallel_\infty =$ minimum in the sense of Definition 1.2, where $T_j(j = 1, 2, 3) \in VRH_\infty$, and K represents a controller in VRH_∞.

We shall give the algorithms of computing the model-matching error α and the optimal controller Q in the form of the parameter case.

2. The Minimal Realization

Definition 2.1: A scalar-valued parameter function $T(s,\xi)$ in VRH_∞ is inner if

$$T(-s,\xi)T(s,\xi) = 1 \tag{2.1}$$

and outer if it has no zeros in $\mathcal{R}(s) > 0$. The zeros of an inner-function all lie in $\mathcal{R}(s) > 0$, the number of its zeros is called degree.

Theorem 2.1: *If scalar-valued parameter function $T(s,\xi) \in VRH_\infty$, then*

(i) *there is a factorization $T(s,\xi) = T_i(s,\xi)T_o(s,\xi)$, where T_i is an inner function, T_o is an outer function;*

(ii) *$\mid T_i(j\omega,\xi) \mid\ = 1$ for any fixed $\xi \in \mathbb{R}^n$;*

(iii) *if $T(j\omega,\xi) \ne 0$ for all ω in $[0,\infty]$ and any fixed ξ in $\mathbb{R}^n$, then $T_o^{-1}(s,\xi)$ exists and $T_o^{-1}(s,\xi) \in VRH_\infty$.*

Proof: (i) Let $T_i(s,\xi)$ be the product of all factors of the form $(a(\xi) - s)/(\overline{a(\xi)} + s)$, here $a(\xi)$ ranges over all zeros of $T(s,\xi)$ in $\mathcal{R}(s) > 0$, counting multiplicities, being a polynomial of ξ, and define $T_o = T(s,\xi)/T_i(s,\xi)$. Then $T_i(s,\xi)$ and $T_o(s,\xi)$ are inner and outer respectively, and $T(s,\xi) = T_i(s,\xi)T_o(s,\xi)$.

(ii) From (i) we derive $T_i(-s,\xi)T_i(s,\xi) = 1$. Particularly, if $s = j\omega$, then $T_i(-j\omega,\xi)T_i(j\omega,\xi) = 1$. Thus $\overline{T_i(j\omega,\xi)}T_i(j\omega,\xi) = 1$, i.e. $\mid T_i(j\omega,\xi) \mid\ = 1$.

(iii) If $T(\infty,\xi) \ne 0$ and $T(j\omega,\xi)$ has no zeros on the imaginary axis for any fixed $\xi \in \mathbb{R}^n$, then so is $T_o(s,\xi)$. Consequently, $T_o^{-1}(s,\xi) \in VRH_\infty$. Q.E.D.

Suppose that

$$K(\lambda) = \lambda^n + K_1\lambda^{n-1} + \cdots + K_n = 0 \tag{2.2}$$

is the characteristic equation of a matrix $A(\xi)$, and that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic roots of $A(\xi)$, where λ is a polynomial of ξ.

The matrix

$$C(\xi) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -K_n & -K_{n-1} & -K_{n-2} & \cdots & -K_1 \end{bmatrix} \tag{2.3}$$

is the companion form matrix associated with $K(\lambda)$ in (2.2).

Definition 2.2: Let $A(\xi)(a_{ij}(\xi))(m \times n)$ and $B(\xi)(p \times q)$ be two matrices. The Kronecker product of $A(\xi)$ and $B(\xi)$ is defined by

$$A(\xi) \otimes B(\xi) = \begin{bmatrix} a_{11}(\xi)B(\xi) & \cdots & a_{1n}(\xi)B(\xi) \\ \vdots & \ddots & \vdots \\ a_{n1}(\xi)B(\xi) & \cdots & a_{nn}(\xi)B(\xi) \end{bmatrix}.$$

Definition 2.3: The linear time invariant system S_1 is defined by

$$\dot{X}(t,\xi) = A(\xi)X(t,\xi) + B(\xi)u(t,\xi) \tag{2.4}$$

$$y(t,\xi) = C(\xi)X(t,\xi) \tag{2.5}$$

where A is an $n \times n$ matrix, B is an $n \times m$ matrix and C is a constant $r \times n$ matrix. S_1 is said to be completely controllable (c.c.) if the $n \times nm$ controllability matrix

$$U(\xi) = [B(\xi) \quad A(\xi)B(\xi) \ \cdots \ A^{n-1}(\xi)B(\xi)] \tag{2.6}$$

has rank n, denoted by $(A(\xi), B(\xi))$. The system S_1 is completely observable (c.o) if the observability matrix

$$V^T(\xi) = [C(\xi) \quad C(\xi)A(\xi) \cdots C(\xi)A^{n-1}(\xi)]^T \tag{2.7}$$

has rank n, denoted by $(A(\xi), C(\xi))$.

Definition 2.4: Given an $r \times m$ matrix $G(s,\xi)$ whose elements are rational functions of s (for any fixed ξ in $\mathbb{R}^n$), we wish to find matrices $A(\xi)$, $B(\xi)$ and $C(\xi)$ depending on ξ, having dimensions $n \times n$, $n \times m$ and $r \times n$, respectively, such that

$$G(s,\xi) = C(\xi)(sI_n - A(\xi))^{-1}B(\xi)$$

where I_n is the unit matrix of order n. $[A(\xi), B(\xi), C(\xi), 0]$ is termed a realization of $G(s,\xi)$ of order n. All the above realizations will include matrices $G(s,\xi)$ having the least dimensions be called the minimal realizations.

Theorem 2.2: *If* $[A(\xi), B(\xi), C(\xi), 0]$ *is c.c. or c.o., then so is* $[\widetilde{A}(\xi), \widetilde{B}(\xi), \widetilde{C}(\xi), 0)]$, *where* $\widetilde{A}(\xi)$, $\widetilde{B}(\xi)$ *and* $\widetilde{C}(\xi)$ *are of algebraical equivalence via a square matrix, respectively.*

Proof: Using the algebraic relation of equivalence among matrices, there is a square matrix $P(\xi)$ such that

$$\widetilde{A}(\xi) = P(\xi)A(\xi)P^{-1}(\xi)\widetilde{B}(\xi) = P(\xi)B(\xi)\widetilde{C}(\xi) = C(\xi)P^{-1}(\xi).$$

It follows that

$$\begin{aligned}&\text{rank}[\widetilde{B}(\xi)\ \ \widetilde{A}(\xi)\widetilde{B}(\xi)\cdots\widetilde{A}^{n-1}(\xi)\widetilde{B}(\xi)]\\ &= \text{rank}(P(\xi)[B(\xi)\quad A(\xi)B(\xi)\cdots A^{n-1}(\xi)B(\xi)]) = n.\end{aligned}$$

So is $[\widetilde{A}(\xi), \widetilde{B}(\xi), \widetilde{C}(\xi), 0]$ since $[A(\xi), B(\xi), C(\xi), 0]$ is c.c. by Definition 2.3. The observability condition follows similarly. Q.E.D.

Suppose $R(s,\xi) = [r_{ij}(s,\xi)]$ is a $p \times m$ strictly proper rational-fraction matrix of s (for any fixed ξ in $\mathbb{R}^n$).

Theorem 2.3: *Let*

$$r(s,\xi) = s^q + g_1(\xi)s^{q-1} + \cdots + g_q(\xi)$$

be the monic least common denominator of all element $r_{ij}(s,\xi)$, and let

$$r(s,\xi)R(s,\xi) = s^{q-1}R_0(\xi) + s^{q-2}R_1(\xi) + \cdots + R_{q-1}(\xi), \tag{2.8}$$

where $R_i(\xi)$ is a constant $r \times m$ matrix depending on a parameter ξ. Then a realization of $R(s,\xi)$ is

$$A(\xi) = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ & & & \ddots & \\ -g_q(\xi) & -g_{q-1}(\xi) & & \cdots & -g_1(\xi) \end{bmatrix}, \ B(\xi) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_m \end{bmatrix} \begin{matrix} q \\ \textit{blocks} \end{matrix}$$

$$C(\xi) = [R_{q-1}(\xi)\quad R_{q-2}(\xi)\cdots R_0(\xi)]. \tag{2.9}$$

Proof: Let $\Upsilon(\xi)$ denote the companion matrix of (2.3). Clearly $A(\xi) = \Upsilon(\xi) \otimes I_m$, so that

$$(sI_m - A(\xi))^{-1} = (sI_q \otimes I_m - \Upsilon(\xi) \otimes I_m)^{-1} = [(sI_q - \Upsilon(\xi))^{-1}] \otimes I_m. \tag{2.10}$$

It follows that using $B(\xi) = e_q \otimes I_m$ where e_q denotes the least column of I_q, relation (2.10) yields

$$\begin{aligned}&(sI_m - A(\xi))^{-1}B(\xi)\\ &= [(sI_q - \Upsilon(\xi))^{-1} \otimes I_m][e_q \otimes I_m]\\ &= [(sI_q - \Upsilon(\xi))^{-1}e_q] \otimes I_m\\ &= [1\ \ s\ \ s^2\cdots s^{q-1}]^T \otimes I_m/g(s,\xi).\end{aligned} \tag{2.11}$$

Finally, combining (2.11) and the expression for $C(\xi)$ in (2.9), we have

$$\begin{aligned}&C(\xi)(sI_m - A(\xi))^{-1}B(\xi)\\ &= [R_{q-1}(\xi)\ \ R_{q-2}(\xi)\cdots R_0(\xi)][I_m\ \ sI_m\cdots s^{q-1}I_m]/r(s,\xi)\\ &= R(s,\xi).\end{aligned}$$

By virtue of (2.8), we have that $[A(\xi), B(\xi), C(\xi), 0]$ is a realization of $R(s,\xi)$. Q.E.D.

We now give the central conclusion of this section.

Theorem 2.4: *A realization* $[A(\xi), B(\xi), C(\xi), 0]$ *of a given transfer matrix* $G(s,\xi)$ *is minimal if* $(A(\xi), B(\xi))$ *is c.c. and* $(A(\xi), C(\xi))$ *is c.o.*

Proof: Let $U(\xi)$ and $V(\xi)$ be the controllability and observability matrices in (2.6) and (2.7), respectively. We wish to show that if these both have rank n, then $R(s,\xi)$ has the least order n.

Suppose that there exists a realization $[\widetilde{A}(\xi), \widetilde{B}(\xi), \widetilde{C}(\xi), 0]$ of $R(s,\xi)$, with $\widetilde{A}(\xi)$ having order n_1. Since

$$C(\xi)(sI_m - A(\xi))^{-1}B(\xi) = \widetilde{C}(\xi)(sI_m - \widetilde{A}(\xi))^{-1}\widetilde{B}(\xi),$$

$$C(\xi)e^{A(\xi)t}B(\xi) = \widetilde{C}(\xi)e^{\widetilde{A}(\xi)t}\widetilde{B}(\xi).$$

It follows that

$$C(\xi)A^i(\xi)B(\xi) = \widetilde{C}(\xi)\widetilde{A}^i(\xi)\widetilde{B}(\xi), \text{ for } i = 0, 1, 2, \cdots$$

using the series of $(e^{A(\xi)t})$.

Consider the product

$$V(\xi)U(\xi) = [C(\xi)\ \ C(\xi)A(\xi)\cdots C(\xi)A^{n-1}(\xi)]^T[B(\xi)\ \ A(\xi)B(\xi)\cdots A^{n-1}(\xi)B(\xi)]$$

$$= \begin{bmatrix} C(\xi)B(\xi) & C(\xi)A(\xi)B(\xi) & \cdots & C(\xi)A^{n-1}(\xi)B(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ C(\xi)A^{n-1}(\xi) & C(\xi)A^n(\xi)B(\xi) & \cdots & C(\xi)A^{2n-2}(\xi)B(\xi) \end{bmatrix}$$

$$= [\widetilde{C}(\xi)\ \ \widetilde{C}(\xi)\widetilde{A}(\xi)\cdots\widetilde{C}(\xi)\widetilde{A}^{n-1}(\xi)]^T[\widetilde{B}(\xi)\ \ \widetilde{A}(\xi)\widetilde{B}(\xi)\cdots\widetilde{A}^{n-1}(\xi)\widetilde{B}(\xi)]$$

$$= V_1(\xi)U_1(\xi).$$

By assumption, $V(\xi)$ and $U(\xi)$ both have rank n, so the matrix $V_1(\xi)U_1(\xi)$ also has rank n. However, the dimensions of $V_1(\xi)$ and $U_1(\xi)$ are $r_1 n \times n_1$ and $n_1 \times m_1 n$, respectively, where r_1 and m_1 are positive integers, so that the rank of $V_1(\xi)U_1(\xi)$ can not be greater than n_1. This is $n < n_1$, so there can be no realization of $G(s,\xi)$ having order less than n. Q.E.D.

3. Lyapunov Equations

The Lyapunov equations are

$$A(\xi)L_c(\xi) + L_c(\xi)A^T(\xi) = B(\xi)B^T(\xi) \tag{3.1}$$

$$A^T(\xi)L_o(\xi) + L_o(\xi)A(\xi) = C^T(\xi)C(\xi). \tag{3.2}$$

Define the two controllability and observability gramians:

$$L_c(\xi) = \int_0^\infty e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}dt$$

$$L_o(\xi) = \int_0^\infty e^{-A^T(\xi)t} C^T(\xi) C(\xi) e^{-A(\xi)t} dt.$$

Definition 3.1: A matrix $A(\xi)$ is said to be antistable if all the eigenvalues of $A(\xi)$ are in $\mathcal{R}(s) > 0$.

Theorem 3.1: *If $A(\xi)$ is antistable, then $L_c(\xi)$ and $L_o(\xi)$ are the unique solutions of* (3.1) *and* (3.2)*, respectively.*

Proof: From the definition of $L_c(\xi)$ we derive

$$A(\xi)L_c(\xi) + L_c(\xi)A^T(\xi)$$

$$= \int_0^\infty [A(\xi)e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t} + e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}A^T(\xi)]dt$$

$$= \int_0^\infty d[e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}]$$

$$= B(\xi)B^T(\xi) - \lim_{t\to\infty}[e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}].$$

Since $A(\xi)$ is antistable,

$$\lim_{t\to\infty}[e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}] = 0.$$

Then $L_c(\xi)$ is the solution of (3.1).

Proof the uniqueness of $L_c(\xi)$ is as follows.

If $A(\xi)$ and $B(\xi)$ are $n \times n$ and $m \times m$ matrices, having characteristic roots $\lambda_i(\xi)$, $u_i(\xi)$ and vectors $w_i(\xi)$, $y_i(\xi)$, respectively, then

$$(A(\xi) \times B(\xi))(w_i(\xi) \otimes y_i(\xi))$$

$$= A(\xi)w_i(\xi) \otimes B(\xi)y_i(\xi)$$

$$= \lambda_i(\xi)w_i(\xi) \otimes u_j(\xi)y_j(\xi)$$

$$= \lambda_i(\xi)u_j(\xi)w_i(\xi) \otimes y_j(\xi).$$

So, the characteristic roots of $A(\xi) \otimes B(\xi)$ are

$$\lambda_i(\xi)u_j(\xi),\ i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m.$$

Let

$$A(\xi)X(\xi) + X(\xi)B(\xi) = C(\xi), \tag{3.3}$$

be a matrix equation, where $X(\xi)$ and $C(\xi)$ are $n \times m$ matrices.

If $A(\xi)$ is an $n \times n$ matrix and $X(\xi)$ is an $n \times m$ matrix, then the matrix equation $A(\xi)X(\xi) = C(\xi)$ can be written as the form

$$(A(\xi) \otimes I_m)X(\xi) = C(\xi), \tag{3.4}$$

where

$$X(\xi) = [X_{11}, \cdots, X_{1m}, \cdots, X_{n1}, \cdots, X_{nm}] \tag{3.5}$$

is the column mn-vector formed from the roots of $X(\xi)$ taken in order.

Similarly $X(\xi)B(\xi) = C(\xi)$ can be written as

$$(I_n \otimes B^T(\xi))X(\xi) = C(\xi). \tag{3.6}$$

Using (3.4) and (3.6), equation (3.3) can be written as the form

$$(A(\xi) \otimes I_m + I_n \otimes B^T(\xi))X(\xi) = C(\xi). \tag{3.7}$$

Let $D(\xi) = A(\xi) \otimes I_m + I_n \otimes B^T(\xi)$. Thus

$$D(\xi)X(\xi) = C(\xi). \tag{3.8}$$

The solution of (3.8) is unique if and only if the $mn \times mn$ matrix $D(\xi)$ is nonsingular. To find the condition for this to hold, consider

$$(I_n + \epsilon A(\xi)) \otimes (I_m + \epsilon B^T(\xi)) = I_n \otimes I_m + \epsilon D(\xi) + \epsilon^2 A(\xi) \otimes B(\xi)$$

which has characteristic roots

$$(I + \epsilon\lambda_i(\xi))(1 + \epsilon u_j(\xi)) = 1 + \epsilon(\lambda_i(\xi) + u_j(\xi)) + \epsilon^2\lambda_i(\xi)u_j(\xi).$$

It follows by comparing terms in ϵ that $D(\xi)$ has characteristic roots $\lambda_i(\xi) + u_j(\xi)$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. Hence, $D(\xi)$ is nonsingular if and only if there are no characteristic roots of $A(\xi)$ and $B(\xi)$ such that $\lambda_i(\xi) + u_j(\xi) = 0$, and this is the condition for the solution $X(\xi)$ of matrix equation (3.3) to be unique. Because $A(\xi)$ is antistable, the characteristic roots $\lambda_i(\xi) + \overline{\lambda_i(\xi)}$ of $A(\xi) + A^T(\xi)$ are not zero. Consequently, the solution of the Lyapunov equation (3.1) is unique.

$L_o(\xi)$ is the unique solution of (3.2) with similar proof above. Q.E.D.

4. Infimal Model-Matching Error

Define

$$\begin{aligned} f(s,\xi) &= [A(\xi), w(\xi), C(\xi), 0], \\ g(s,\xi) &= [-A^T(\xi), \lambda^{-1}(\xi)L_o(\xi)w(\xi), B^T(\xi), 0] \end{aligned} \tag{4.1}$$

and

$$X(s,\xi) = R(s,\xi) - \lambda(\xi)f(s,\xi)/g(s,\xi). \tag{4.2}$$

So

$$f(s,\xi) = C(\xi)(sI - A(\xi))^{-1}w(\xi) \in VRH_2^{\perp}$$

and

$$g(s,\xi) = B^T(\xi)(sI + A^T(\xi))^{-1}\lambda^{-1}(\xi)L_o(\xi)w(\xi) \in VRH_2.$$

Definition 4.1: Let $\alpha(\xi)$ denote the infimal model-matching error:

$$\alpha(\xi) = \inf\{\, \| T_1(\,\cdot\,,\xi) - T_2(\,\cdot\,,\xi)Q(\,\cdot\,,\xi)T_3(\,\cdot\,,\xi) \|_\infty : Q \in VRH_\infty\}. \tag{4.3}$$

A matrix Q in VRH_∞ satisfying $\alpha(\xi) = \| T_1(\cdot,\xi) - T_2(\cdot,\xi)Q(\cdot,\xi)T_3(\cdot,\xi) \|_\infty$ is called optimal.

If $T_i(s,\xi)$ are scalar-valued, then there is no need for both $T_2(s,\xi)$ and $T_3(s,\xi)$. So we may as well suppose $T_3(s,\xi) = 1$. It is also assumed that $T_2^{-1}(s,\xi) \in VRH_\infty$ to avoid the trivial instance of the problem.

Returning to the model-matching problem, bringing in an inner-outer factorization of $T_2(s,\xi)$: $T_2(s,\xi) = T_{2i}(s,\xi)T_{2o}(s,\xi)$, we have

$$\begin{aligned} &\| T_1(\cdot,\xi) - T_2(\cdot,\xi)Q(\cdot,\xi) \|_\infty \\ &= \| T_{2i}(\cdot,\xi)(T_{2i}^{-1}(\cdot,\xi)T_1(\cdot,\xi) - T_{2o}(\cdot,\xi)Q(\cdot,\xi)) \|_\infty \\ &= \| T_{2i}^{-1}(\cdot,\xi)T_1(\cdot,\xi) - T_{2o}(\cdot,\xi)Q(\cdot,\xi) \|_\infty \\ &= \| R(\cdot,\xi) - X(\cdot,\xi) \|_\infty . \end{aligned} \tag{4.4}$$

Theorem 4.1: *The infimal in* (4.3) *is achieved if* $T_2(s,\xi)$ *has no zeros on the extended imaginary axis. In this case, the optimal* $Q(s,\xi)$ *is determined by the following property*: $T_1(s,\xi) - T_2(s,\xi)Q(s,\xi)$ *is a scalar multiple of an inner function of degree less than the number of zeros of* $T_2(s,\xi)$ *in* $\mathcal{R}(s) > 0$ (*for any fixed* ξ *in* $\mathbb{R}^n$).

Proof: Suppose $B_1 = VHR_\infty$ and $B = VRL_\infty$, then from Proposition A 2.2 in [7], the infimum in (4.4) is achieved.

Assume $\widetilde{X}(s,\xi) = T_1(s,\xi) - T_2(s,\xi)Q(s,\xi)$, then using (4.4) we have

$$\| \widetilde{X}(\cdot,\xi) \|_\infty = \| X(\cdot,\xi)T_{2i}^{-1}(\cdot,\xi) \|_\infty .$$

Consequently, the minimization of $\widetilde{X}(s,\xi)$ can be accomplished by minimizing $\widetilde{X}(s,\xi)T_{2i}^{-1}(s,\xi)$ and multiplying the result by $T_{2i}^{-1}(s,\xi)$. Now as $\widetilde{X}(s,\xi)T_{2i}^{-1}(s,\xi)$ is analytic in $\mathcal{R}(s) > 0$, except for the poles of $T_{2i}^{-1}(s,\xi)$, which are b_j, $j = 1,\cdots,r$ (depending on ξ). So $\widetilde{X}(s,\xi)T_{2i}^{-1}(s,\xi)$ must have a continuation to the entire plane, with poles at b_j and $-b_j$. Therefore, $\widetilde{X}(s,\xi)T_{2i}^{-1}(s,\xi)$ is rational, and has the form

$$\widetilde{X}(s,\xi)T_{2i}^{-1}(s,\xi) = C(\xi)\frac{\prod_{i=1}^{n}(s-c_i(\xi))(s+c_i(\xi))}{\prod_{j=1}^{m}(s-b_j(\xi))(s+b_j(\xi))} \tag{4.5}$$

where $m < r$, $\mathcal{R}(c_i(\xi)) > 0$, $C(\xi) > 0$.

As $T_{2i}(s,\xi)$ is an inner function, obviously rational, so $X(s,\xi)$ is also rational.
Since $|\widetilde{X}(j\omega,\xi)| = \alpha$ a.e., $\widetilde{X}(s,\xi)$ is inner and the zeros of $\widetilde{X}(s,\xi)$ must be among c_i in (4.5). Q.E.D.

Definition 4.2: The VL_p space, $1 \leq p < \infty$, will be viewed as pth power integrable functions about s and ξ. When $p = \infty$, VL_∞ is the space of essentially bounded functions (for any fixed ξ in $\mathbb{R}^n$).

Definition 4.3: The VRL_p space, $1 \leq p \leq \infty$, will be viewed as a subset of VL_p, which consists of all real-variational functions of s and ξ.

Definition 4.4: Let $F(s,\xi) \in VL_\infty$ and $g(s,\xi) \in VL_2$. Then the operator

$$\Lambda_{F(s,\xi)}: \Lambda_{F(x,\xi)}g(s,\xi) = F(s,\xi)g(s,\xi)$$

is called the Laurent operator.

A related operator is $\Lambda_{F(s,\xi)} \mid VH_2$, the restriction of $\Lambda_{F(s,\xi)}$ to VH_2, which maps VH_2 to VL_2 where $F(s,\xi) \in VL_\infty$.

For $F(s,\xi)$ in VL_∞, the Hankel operator with symbol $F(s,\xi)$, denoted by $\Gamma_{F(s,\xi)}$, maps VH_2 to $VH_2^\perp$ and is defined as

$$\Gamma_{F(s,\xi)} \colon = \prod_1 \Lambda_{F(s,\xi)} \mid VH_2,$$

where $VL_2 = VH_2 \oplus VH_2^\perp$, and Π_1 is the projection from VL_2 onto $VH_2^\perp$.

Using a similar method to the classical methods we have the following conclusion:

Theorem 4.2: *There exists a closest VRH_∞-function $X(s,\xi)$ to a given VRL_∞-function $R(s,\xi)$, and $\| R(\,\cdot\,,\xi) - X(\,\cdot\,,\xi) \|_\infty = \| \Gamma_{R(\cdot,\xi)} \|$.*

From Section 3, a factor $R(s,\xi)$ can be written as $R_1(s,\xi) + R_2(s,\xi)$ with $R_1(s,\xi)$ strictly proper and analytic in $\mathcal{R}(s) < 0$ and $R_2(s,\xi)$ in VRH_∞. Then $R_1(s,\xi)$ has the minimal state-space realization

$$R_1(s,\xi) = [A(\xi), B(\xi), C(\xi), 0].$$

And from Section 3, with $v(\xi) = \lambda^{-1}(\xi)L_o(\xi)w(\xi)$ we derive

$$L_c(\xi)v(\xi) = \lambda(\xi)w(\xi) \tag{4.6}$$

and

$$L_o(\xi)w(\xi) = \lambda(\xi)v(\xi). \tag{4.7}$$

Theorem 4.3: *The infimal model-matching error $\alpha(\xi)$ equals $\| \Gamma_R(\,\cdot\,,\xi) \|$ and the unique optimal X equals $R(s,\xi) - \lambda(\xi)f(s,\xi)/g(s,\xi)$.*

Proof: From Theorem 4.2, we derive that there is a function $X(s,\xi)$ in VH_∞ such that

$$\| R(\,\cdot\,,\xi) - X(\,\cdot\,,\xi) \|_\infty = \| \Gamma_{R(\cdot,\xi)} \|. \tag{4.8}$$

It is claimed that every $X(s,\xi)$ in VH_∞ satisfying (4.8) also satisfies

$$R(s,\xi) - X(s,\xi)g(s,\xi) = \Gamma_R(s,\xi)g(s,\xi). \tag{4.9}$$

But (4.9) has a unique solution $X(s,\xi) = R(s,\xi) - \lambda(\xi)f(s,\xi)/g(s,\xi)$. We know that

$$\Gamma_{R(s,\xi)}g(s,\xi) = \lambda(\xi)f(s,\xi) \tag{4.10}$$

holds. In fact, add and subtract $sL_c(\xi)$ on the left-hand side in (3.1) to get

$$-(sI - A(\xi))L_c(\xi) + L_c(\xi)(sI + A^T(\xi)) = B(\xi)B^T(\xi).$$

Now pre-multiply by $C(\xi)(sI - A(\xi))^{-1}$ and post-multiply by $(sI + A^T(\xi))^{-1}v(\xi)$ to get

$$\begin{aligned} &-C(\xi)L_c(\xi)(sI + A^T(\xi))v(\xi) + C(\xi)(sI - A(\xi))^{-1}L_c(\xi)v(\xi) \\ &= C(\xi)(sI - A(\xi))^{-1}B(\xi)B^T(\xi)(sI + A^T(\xi))^{-1}v(\xi). \end{aligned} \tag{4.11}$$

The first function on the left-hand side belongs to VH_2; from (4.1) and (4.6), the second function equals $\lambda(\xi)f(s,\xi)$; and from (4.2) and (4.6), the function on the right-hand side equals $R_1(s,\xi)g(s,\xi)$. Project both sides of (4.12) onto $VRH_2^\perp$ to get

$$\lambda(\xi)f(s,\xi) = \prod_1 R_1(s,\xi)g(s,\xi) = \Gamma_{R_1(s,\xi)}g(s,\xi).$$

But $\Gamma_{R_1(s,\xi)} = \Gamma_{R(s,\xi)}$; hence (4.10) holds. It follows that (4.10) and Theorem 4.2 imply $\alpha(\xi) = \lambda(\xi)$. There is $X(s,\xi) = R(s,\xi) - \alpha(\xi)f(s,\xi)/g(s,\xi)$. Set $\alpha(\xi) = \lambda(\xi)$ and

$$Q(s,\xi) = T_2^{-1}(s,\xi)X(s,\xi). \tag{4.12}$$

Since $Tw_{2o}(s,\xi)$ and $T_{2o}^{-1}(s,\xi) \in VRH_\infty$, (4.12) sets up the one-to-one correspondence between functions $Q(s,\xi)$ in VRH_∞ and functions $X(s,\xi)$ in VRH_∞. The optimal $X(s,\xi)$ yields the optimal $Q(s,\xi)$ via (4.9).

5. Steps of Computation

From Section 2 through Section 4, we derive that in the form of parameter valued case, the steps in the design procedure on the H_∞-optimization in locally convex spaces are as follows:

Step 1: Do an inner-outer factorization

$$T_2(s,\xi) = T_{2i}(s,\xi)T_{2o}(s,\xi).$$

Step 2: Define

$$R(s,\xi) = T_{2i}^{-1}(s,\xi)T_1(s,\xi)$$

and find a minimal realization

$$R(s,\xi) = [a(\xi), B(\xi), C(\xi), 0] + (\text{a function in } VRH_\infty).$$

Step 3: Solve the equations

$$A(\xi)L_c(\xi) + L_c(\xi)A^T(\xi) = B(\xi)B^T(\xi)$$

and

$$A^T(\xi)L_o(\xi) + L_o(\xi)A(\xi) = C^T(\xi)C(\xi).$$

Step 4: Find the maximum eigenvalue λ^2 of $L_c(\xi)L_o(\xi)$ and a corresponding eigenvector $w(\xi)$.

Step 5: Define

$$f(s,\xi) = [A(\xi), w(\xi), C(\xi), 0]$$

$$g(s,\xi) = [-A^T(\xi), \lambda^{-1}(\xi)L_o(\xi)w(\xi), B^T(\xi), 0]$$

and

$$X(s,\xi) = R(s,\xi) - \lambda(\xi)\tfrac{f(s,\xi)}{g(s,\xi)}.$$

Step 6: Set $\alpha(\xi) = \lambda(\xi)$ and $Q(s,\xi) = T_{2o}^{-1}(s,\xi)X(s,\xi)$.

For a single-input and single-output design in the form of parameter valued case, we have a similar to ordinary computing method.

Example 1:

$$P(s,\xi) = [(s-1)(s-2)]/[(s+1)(s^2+s+1+\xi^2)] \in VRH_\infty,\ \omega_1 = 0.01, \epsilon = 0.1.$$

Step 1: $$-P(s,\xi) = N(s,\xi)/M(s,\xi),$$

$$N(s,\xi) = -P(s,\xi), M(s,\xi) = 1 = X(s,\xi), Y(s,\xi) = 0.$$

Step 2: $$W(s,\xi) = (s+1)/(10s+1).$$

Step 3: $$T_1(s,\xi) = (s+1)^k/(10s+1)^k,$$

$$T_2(s,\xi) = -[(s+1)^k(s-1)(s-2)]/[(10s+1)^k(s+1)(s^2+s+1+\xi^2)],$$

$$V(s) = s+1.$$

Step 4: When $k = 1$

Step (1) $T_{2i}(s,\xi) = [(s-1)(s-2)]/[(s+1)(s+2)],$
$T_{2o}(s,\xi) = -[(s+1)(s+2)[/[(10s+1)(s^2+s+1+\xi^2)].$

Step (2) $R(s,\xi) = [(s+1)^2(s+2)]/[(10s+1)(s^2+s+1+\xi^2)],$

and A, B, C in the minimal realization of $R(s,\xi)$ are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ B = \begin{bmatrix} -12/11 \\ 12/7 \end{bmatrix}, \ C = [1 \quad 1],$$

respectively.

Step (3)

$$L_c = \begin{bmatrix} 72/121 & -48/77 \\ -48/77 & 36/49 \end{bmatrix}, \ L_o = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}.$$

Step (4)

$$L_c L_o = \begin{bmatrix} 0.0898 & 0.0425 \\ -0.0668 & -0.0853 \end{bmatrix},$$

then $\alpha_1 = 0.2299 > 0.1 = \epsilon$. Thus we take $k = 2$.

Step (1) $T_{2i}(s,\xi) = [(s-1)(s-2)]/[(s+1)(s+2)],$

$$T_{2o}(s,\xi) = -[(s+1)(s+2)]/[(10s+1)(10s+1)(s^2+s+1+\xi^2)].$$

Step (2) $R(s,\xi) = [(s+1)^3(s+2)]/[(10s+1)^2(s^2+s+1+\xi^2)],$

and A, B, C in the minimal realization of $R(s,\xi)$ are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ B = \begin{bmatrix} -24/121 \\ 12/49 \end{bmatrix}, \ C = [1 \quad 1],$$

respectively.

Step (3)

$$L_c = \begin{bmatrix} (24\times 12)/(121\times 121) & -(8\times 12)/(121\times 49) \\ -(8\times 12)/(121\times 49) & (12\times 3)/(49\times 49) \end{bmatrix}, \ L_o = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}.$$

Step (4)

$$L_cL_o = \begin{bmatrix} 0.0044 & -0.0025 \\ 0.0031 & -0.0017 \end{bmatrix},\ \lambda = 0.05113,\ w = \begin{bmatrix} 1 \\ -0.7209 \end{bmatrix}.$$

Step (5) $f(s) = (0.2791s - 1.2791)/[(s-1)(s-2)]$,

$$g(s) = \lambda^{-1}(-0.0141s - 0.0657)/[(s+1)(s+2)],$$

$$X(s) = 6.15[(s+1)(s+2)(s+0.4)]/[(10s+1)^2(s+4.66)].$$

Step (6) Set

$$\alpha = \lambda = 0.05113,$$

$$Q(s) = -6.15[(s+0.4)(s^2+s+1+\xi^2)]/[(s+1)(s+4.66)].$$

Step 5: $Q_1(s,\xi) = -6.15[(s+0.4)(s^2+s+1+\xi^2)]/[(s+1)^2(s+4.66)]$.
Step 6:

$$K(s,\xi) = \frac{0.615(s+0.4)(s+1)(s^2+s+1+\xi^2)}{(s^4+6.145s^3+12.54s^2+13.53s+0.0232).}$$

Note $K(s,\xi) \notin RH_\infty$, but $K(s,\xi) \in VRH_\infty$.
Example 2:

$$P(s,\xi) = \frac{(10s+1)(s-2-\xi^2)}{(s+1)(s^2+s+1+\xi^2)} \in VRH_\infty, \omega_1 = 0.01, \epsilon = 0.15.$$

Step 1:

$$-P(s,\xi) = N(s,\xi)/M(s,\xi),$$

$$N(s,\xi) = -P(s,\xi), M(s,\xi) = 1 = X(s,\xi),\ Y(s,\xi) = 0.$$

Step 2:

$$W(s,\xi) = (s+1)/(10s+1).$$

Step 3:

$$T_1(s,\xi) = (s+1)^k/(10s+1)^k,$$

$$T_2(s,\xi) = -\frac{(s+1)^k(10s+1)(s-2-\xi^2)}{(10s+1)^k(s+1)(s^2+s+1+\xi^2)},\ V(s) = s+1.$$

Step 4: When $k = 1$,
Step (1) $T_{2i}(s,\xi) = -\frac{s-2-\xi^2}{s+2+\xi^2},\ T_{2o}(s,\xi) = -\frac{s+2+\xi^2}{s^2+s+1+\xi^2}$.

Step (2) $R(s,\xi) = -[(s+1)(s+2+\xi^2)]/[(10s+1)(s-2-\xi^2)]$,
then $A(\xi), B(\xi)$ and $C(\xi)$ in the minimal realization of $R(s,\xi)$ are $A(\xi) = 2+\xi^2$, $B(\xi) = -2[(2+\xi^2)(3+\xi^2)]/(10\xi^2+21)$ and $C(\xi) = 1$, respectively.
Step (3)

$$L_c(\xi) = \frac{2(2+\xi^2)(3+\xi^2)^2}{(10\xi^2+21)^2},\ L_o(\xi) = \frac{1}{2(2+\xi^2)}.$$

Step (4) $L_c(\xi)L_o(\xi) = (2+\xi^2)^2/(10\xi^2+21)^2$,
then

$$\alpha_1(\xi) = \frac{3+\xi^2}{10\xi^2+21} < 0.15.$$

Step (5)

$$f(s,\xi) = \frac{1}{s-2-\xi^2},\ g(s,\xi) = \frac{\lambda^{-1}(\xi)(\xi^2+3)}{(s+2+\xi^2)(10\xi^2+21)},$$

$$X(s,\xi) = \frac{(s+2+\xi^2)(9s+9\xi^2-18)}{(s-2-\xi^2)(10s+1)(10\xi^2+21)}.$$

Step (6) Set

$$\alpha(\xi) = \frac{3+\xi^2}{10\xi^2+21}, Q(s,\xi) = -\frac{9(s^2+s+1+\xi^2)}{(10s+1)(10\xi^2+21)}.$$

Step 5:

$$Q_1(s,\xi) = -\frac{9(s^2+s+1+\xi^2)}{(10s+1)(10\xi^2+21)(s+1)}.$$

Step 6:

$$K(s,\xi) = \frac{9(s^2+s+1+\xi^2)(s+1)}{(10s+1)[(10\xi^2+21)s^2+(110\xi^2+22)s+(19\xi^2+39)]}.$$

Note $K(s,\xi) \notin RH_\infty$, but $K(s,\xi) \in VRH_\infty$.

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