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The h^*-Polynomials of Locally Anti-Blocking Lattice Polytopes and Their γ-Positivity

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Abstract
A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is called a locally anti-blocking polytope if for any closed orthant $\mathbb{R}^d_\varepsilon$ in $\mathbb{R}^d$, $\mathscr{P} \cap \mathbb{R}^d_\varepsilon$ is unimodularly equivalent to an anti-blocking polytope by reflections of coordinate hyperplanes. We give a formula for the h^*-polynomials of locally anti-blocking lattice polytopes. In particular, we discuss the γ-positivity of h^*-polynomials of locally anti-blocking reflexive polytopes.

Keywords Lattice polytope · Unconditional polytope · Anti-blocking polytope · Locally anti-blocking polytope · Reflexive polytope · h^*-polynomial · γ-positive

Mathematics Subject Classification 05A15 · 05C31 · 13P10 · 52B12 · 52B20

1 Introduction

A *lattice polytope* is a convex polytope all of whose vertices have integer coordinates. A lattice polytope $\mathscr{P} \subset \mathbb{R}^d_{\geq 0}$ of dimension d is called *anti-blocking* if for any $\mathbf{y} = (y_1, \ldots, y_d) \in \mathscr{P}$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $0 \leq x_i \leq y_i$ for all i, it holds that $\mathbf{x} \in \mathscr{P}$. Anti-blocking polytopes were introduced and studied by Fulkerson [11,12] in the context of combinatorial optimization. See, e.g., [35]. For $\varepsilon \in \{-1, 1\}^d$ and $\mathbf{x} \in \mathbb{R}^d$, set $\varepsilon\mathbf{x} := (\varepsilon_1 x_1, \ldots, \varepsilon_d x_d) \in \mathbb{R}^d$. Given an anti-blocking lattice polytope

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$\mathscr{P} \subset \mathbb{R}^d_{\geq 0}$ of dimension d, we define

$$\mathscr{P}^{\pm} := \{\varepsilon \mathbf{x} \in \mathbb{R}^d : \varepsilon \in \{-1, 1\}^d,\ \mathbf{x} \in \mathscr{P}\}.$$

Since $\mathscr{P}$ is an anti-blocking lattice polytope, $\mathscr{P}^{\pm}$ is convex (and a lattice polytope). Moreover, for any $\varepsilon \in \{-1, 1\}^d$ and $\mathbf{x} \in \mathscr{P}^{\pm}$, we have $\varepsilon\mathbf{x} \in \mathscr{P}^{\pm}$. The polytope $\mathscr{P}^{\pm}$ is called an *unconditional lattice polytope* [23]. In general, $\mathscr{P}^{\pm}$ is symmetric with respect to all coordinate hyperplanes. In particular, the origin $\mathbf{0}$ of $\mathbb{R}^d$ is in the interior int $\mathscr{P}^{\pm}$. Given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, let $\mathbb{R}^d_{\varepsilon}$ denote the closed orthant $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i\varepsilon_i \geq 0 \text{ for all } 1 \leq i \leq d\}$. A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is called *locally anti-blocking* [23] if, for each $\varepsilon \in \{-1, 1\}^d$, there exists an anti-blocking lattice polytope $\mathscr{P}_{\varepsilon} \subset \mathbb{R}^d_{\geq 0}$ of dimension d such that $\mathscr{P} \cap \mathbb{R}^d_{\varepsilon} = \mathscr{P}^{\pm}_{\varepsilon} \cap \mathbb{R}^d_{\varepsilon}$. Unconditional polytopes are locally anti-blocking.

In the present paper, we investigate the h^*-polynomials of locally anti-blocking lattice polytopes. First, we give a formula for the h^*-polynomials of locally anti-blocking lattice polytopes in terms of that of unconditional lattice polytopes.

Theorem 1.1 *Let $\mathscr{P} \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope of dimension d and for each $\varepsilon \in \{-1, 1\}^d$, let $\mathscr{P}_{\varepsilon}$ be an anti-blocking lattice polytope of dimension d such that $\mathscr{P} \cap \mathbb{R}^d_{\varepsilon} = \mathscr{P}^{\pm}_{\varepsilon} \cap \mathbb{R}^d_{\varepsilon}$. Then the h^*-polynomial of $\mathscr{P}$ satisfies*

$$h^*(\mathscr{P}, x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} h^*(\mathscr{P}^{\pm}_{\varepsilon}, x).$$

In particular, $h^*(\mathscr{P}, x)$ is γ-positive if $h^*(\mathscr{P}^{\pm}_{\varepsilon}, x)$ is γ-positive for all $\varepsilon \in \{-1, 1\}^d$.

Second, we discuss the γ-positivity of the h^*-polynomials of locally anti-blocking reflexive polytopes. A lattice polytope is called *reflexive* if the dual polytope is also a lattice polytope. Many authors have studied reflexive polytopes from viewpoints of combinatorics, commutative algebra, and algebraic geometry. In [15], Hibi characterized reflexive polytopes in terms of their h^*-polynomials. To be more precise, a lattice polytope of dimension d is (unimodularly equivalent to) a reflexive polytope if and only if the h^*-polynomial is a palindromic polynomial of degree d. On the other hand, in [23], locally anti-blocking reflexive polytopes were characterized. In fact, a locally anti-blocking lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is reflexive if and only if for each $\varepsilon \in \{-1, 1\}^d$, there exists a perfect graph G_{ε} on $[d] := \{1, \ldots, d\}$ such that $\mathscr{P} \cap \mathbb{R}^d_{\varepsilon} = \mathscr{Q}^{\pm}_{G_{\varepsilon}} \cap \mathbb{R}^d_{\varepsilon}$, where $\mathscr{Q}_{G_{\varepsilon}}$ is the stable set polytope of G_{ε}. Moreover, every locally anti-blocking reflexive polytope possesses a regular unimodular triangulation. This fact and the result of Bruns–Römer [5] imply that its h^*-polynomial is unimodal.

In the present paper, we discuss whether the h^*-polynomial of a locally anti-blocking reflexive polytope has a stronger property, which is called *γ-positivity*. In [31], a class of lattice polytopes $\mathscr{B}_G$ arising from finite simple graphs G on $[d]$, which are called *symmetric edge polytopes of type B*, was introduced. Symmetric edge polytopes of type B are unconditional, and they are reflexive if and only if the underlying graphs are bipartite. Moreover, when they are reflexive, the h^*-polynomials are always γ-positive. On the other hand, in [30], another family of lattice polytopes $\mathscr{C}^{(e)}_P$ arising from finite partially ordered sets P on $[d]$, which are called *enriched chain polytopes*, was given. Enriched chain polytopes are unconditional and reflexive, and their h^*-polynomials are always γ-positive. Combining these facts and Theorem 1.1, we know that, for a locally anti-blocking reflexive polytope $\mathscr{P}$, if every $\mathscr{P} \cap \mathbb{R}^d_\varepsilon$ is the intersection of $\mathbb{R}^d_\varepsilon$ and either an enriched chain polytope or a symmetric edge reflexive polytope of type B, then the h^*-polynomial of $\mathscr{P}$ is γ-positive (Corollary 4.2). By using this result, we show that the h^*-polynomials of several classes of reflexive polytopes are γ-positive.

In Sect. 5, we will discuss γ-positivity of the h^*-polynomials of *symmetric edge polytopes of type A*, which are reflexive polytopes arising from finite simple graphs. In [21], it was shown that the h^*-polynomials of the symmetric edge polytopes of type A of complete bipartite graphs are γ-positive. We will show that for a large class of finite simple graphs, which includes complete bipartite graphs, the h^*-polynomials of the symmetric edge polytopes of type A are γ-positive (Sect. 5.1). Moreover, by giving explicit h^*-polynomials of del Pezzo polytopes and pseudo-del Pezzo polytopes, we will show that the h^*-polynomial of every pseudo-symmetric simplicial reflexive polytope is γ-positive (Theorem 5.8).

In Sect. 6, we will discuss γ-positivity of h^*-polynomials of *twinned chain polytopes* $\mathscr{C}_{P,Q} \subset \mathbb{R}^d$, which are reflexive polytopes arising from two finite partially ordered sets P and Q on $[d]$. In [39], it was shown that twinned chain polytopes $\mathscr{C}_{P,Q}$ are locally anti-blocking and each $\mathscr{C}_{P,Q} \cap \mathbb{R}^d_\varepsilon$ is the intersection of $\mathbb{R}^d_\varepsilon$ and an enriched chain polytope. Hence the h^*-polynomials of $\mathscr{C}_{P,Q}$ are γ-positive. We will give a formula for the h^*-polynomials of twinned chain polytopes in terms of the left peak polynomials of finite partially ordered sets (Theorem 6.3). Moreover, we will define *enriched* (P, Q)*-partitions* of P and Q, and show that the Ehrhart polynomial of the twined chain polytope $\mathscr{C}_{P,Q}$ of P and Q coincides with a counting polynomial of enriched (P, Q)-partitions (Theorem 6.8).

This paper is organized as follows: In Sect. 2, we will review the theory of Ehrhart polynomials, h^*-polynomials, and reflexive polytopes. In Sect. 3, we will introduce several classes of anti-blocking polytopes and unconditional polytopes. In Sect. 4, we will investigate the h^*-polynomials of locally anti-blocking lattice polytopes. In particular, we will prove Theorem 1.1. We will discuss symmetric edge polytopes of type A in Sect. 5, and twinned chain polytopes in Sect. 6.

2 Ehrhart Theory and Reflexive Polytopes

In this section, we review the theory of Ehrhart polynomials, h^*-polynomials, and reflexive polytopes. Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d. Given a positive integer m, we define

$$L_{\mathscr{P}}(m) = |m\mathscr{P} \cap \mathbb{Z}^d|.$$

Ehrhart [10] proved that $L_{\mathscr{P}}(m)$ is a polynomial in m of degree d with the constant term 1. We say that $L_{\mathscr{P}}(m)$ is the *Ehrhart polynomial* of $\mathscr{P}$. The generating function of the lattice point enumerator, i.e., the formal power series

$$\mathrm{Ehr}_{\mathscr{P}}(x) = 1 + \sum_{k=1}^{\infty} L_{\mathscr{P}}(k) x^k$$

is called the *Ehrhart series* of $\mathscr{P}$. It is well known that it can be expressed as a rational function of the form

$$\mathrm{Ehr}_{\mathscr{P}}(x) = \frac{h^*(\mathscr{P}, x)}{(1-x)^{d+1}}.$$

Then $h^*(\mathscr{P}, x)$ is a polynomial in x of degree at most d with nonnegative integer coefficients [36] and it is called the h^*-*polynomial* (or the δ-*polynomial*) of $\mathscr{P}$. Moreover, one has $\mathrm{Vol}(\mathscr{P}) = h^*(\mathscr{P}, 1)$, where $\mathrm{Vol}(\mathscr{P})$ is the normalized volume of $\mathscr{P}$.

A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is called *reflexive* if the origin of $\mathbb{R}^d$ is a unique lattice point belonging to the interior of $\mathscr{P}$ and its dual polytope

$$\mathscr{P}^\vee := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{x} \in \mathscr{P}\}$$

is also a lattice polytope, where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the usual inner product of $\mathbb{R}^d$. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related to mirror symmetry (see, e.g., [3,7]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence [25] and all of them are known up to dimension 4 [24]. In [15], Hibi characterized reflexive polytopes in terms of their h^*-polynomials. We recall that a polynomial $f \in \mathbb{R}[x]$ of degree d is said to be *palindromic* if $f(x) = x^d f(x^{-1})$. Note that if a lattice polytope of dimension d has interior lattice points, then the degree of its h^*-polynomial is equal to d.

Proposition 2.1 [15] *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d with $\mathbf{0} \in \mathrm{int}\, \mathscr{P}$. Then $\mathscr{P}$ is reflexive if and only if $h^*(\mathscr{P}, x)$ is a palindromic polynomial of degree d.*

Next, we review some properties of polynomials. Let $f = \sum_{i=0}^{d} a_i x^i$ be a polynomial with real coefficients and $a_d \neq 0$. We now focus on the following properties.

(RR) We say that f is *real-rooted* if all its roots are real.
(LC) We say that f is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i.
(UN) We say that f is *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d$ for some k.

If all its coefficients are nonnegative, then these properties satisfy the implications

$$\text{(RR)} \Rightarrow \text{(LC)} \Rightarrow \text{(UN)}.$$

On the other hand, the polynomial f is γ-*positive* if f is palindromic and there are $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor} \geq 0$ such that $f(x) = \sum_{i \geq 0} \gamma_i x^i (1+x)^{d-2i}$. The polynomial $\sum_{i \geq 0} \gamma_i x^i$ is called the γ-*polynomial of* f. We can see that a γ-positive polynomial is real-rooted if and only if its γ-polynomial is real-rooted. If f is palindromic and real-rooted, then it is γ-positive. Moreover, if f is γ-positive, then it is unimodal. See, e.g., [2,34] for details.

For a given lattice polytope, a fundamental problem within the field of Ehrhart theory is to determine if its h^*-polynomial is unimodal. One famous instance is given by reflexive polytopes that possess a regular unimodular triangulation.

Proposition 2.2 [5] *Let $\mathscr{P} \subset \mathbb{R}^d$ be a reflexive polytope of dimension d. If P possesses a regular unimodular triangulation, then $h^*(\mathscr{P}, x)$ is unimodal.*

It is known that if a reflexive polytope possesses a flag regular unimodular triangulation all of whose maximal simplices contain the origin, then the h^*-polynomial coincides with the h-polynomial of a flag triangulation of a sphere [5]. For the h-polynomial of a flag triangulation of a sphere, Gal [13] conjectured the following:

Conjecture 2.3 *The h-polynomial of any flag triangulation of a sphere is γ-positive.*

3 Classes of Anti-Blocking Polytopes and Unconditional Polytopes

In this section, we introduce several classes of anti-blocking polytopes and unconditional polytopes. Throughout this section, we associate each subset $F \subset [d]$ with a $(0, 1)$-vector $\mathbf{e}_F = \sum_{i \in F} \mathbf{e}_i \in \mathbb{R}^d$, where each $\mathbf{e}_i$ is the ith unit coordinate vector in $\mathbb{R}^d$.

3.1 (0, 1)-Polytopes Arising from Simplicial Complexes

Let Δ be a simplicial complex on the vertex set $[d]$. Then Δ is a collection of subsets of $[d]$ with $\{i\} \in \Delta$ for all $i \in [d]$ such that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. In particular $\emptyset \in \Delta$ and $\mathbf{e}_\emptyset = \mathbf{0}$. Let $\mathscr{P}_\Delta$ denote the convex hull of $\{\mathbf{e}_F \in \mathbb{R}^d : F \in \Delta\}$. The following is an important observation.

Proposition 3.1 *Let $\mathscr{P} \subset \mathbb{R}^d_{\geq 0}$ be a $(0, 1)$-polytope of dimension d. Then $\mathscr{P}$ is anti-blocking if and only if there exists a simplicial complex Δ on $[d]$ such that $\mathscr{P} = \mathscr{P}_\Delta$.*

3.2 Stable Set Polytopes

Let G be a finite simple graph on the vertex set $[d]$ and $E(G)$ the set of edges of G. (A finite graph G is called simple if G possesses no loop and no multiple edge.) A subset $W \subset [d]$ is called *stable* if, for all i and j belonging to W with $i \neq j$, one has $\{i, j\} \notin E(G)$. We remark that a stable set is often called an *independent set*. Let $S(G)$ denote the set of all stable sets of G. One has $\emptyset \in S(G)$ and $\{i\} \in S(G)$ for each $i \in [d]$. The *stable set polytope* $\mathscr{Q}_G \subset \mathbb{R}^d$ of G is the $(0, 1)$-polytope defined by

$$\mathscr{Q}_G := \operatorname{conv}\{\mathbf{e}_W \in \mathbb{R}^d : W \in S(G)\}.$$

Then one has $\dim \mathscr{Q}_G = d$. Since we can regard $S(G)$ as a simplicial complex on $[d]$, $\mathscr{Q}_G$ is an anti-blocking polytope.

Locally anti-blocking reflexive polytopes are characterized by stable set polytopes. A *clique* of G is a subset $W \subset [d]$ that is a stable set of the complement graph $\overline{G}$ of G.

The *chromatic number* of G is the smallest integer $t \geq 1$ for which there exist stable sets $W_1, \ldots, W_t$ of G with $[d] = W_1 \cup \cdots \cup W_t$. A finite simple graph G is said to be *perfect* if, for any induced subgraph H of G including G itself, the chromatic number of H is equal to the maximal cardinality of cliques of H. See, e.g., [9] for details on graph theoretical terminology.

Proposition 3.2 [23] *Let $\mathscr{P} \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope of dimension d. Then $\mathscr{P} \subset \mathbb{R}^d$ is reflexive if and only if, for each $\varepsilon \in \{-1, 1\}^d$, there exists a perfect graph G_ε on $[d]$ such that $\mathscr{P} \cap \mathbb{R}^d_\varepsilon = \mathscr{Q}^{\pm}_{G_\varepsilon} \cap \mathbb{R}^d_\varepsilon$.*

3.3 Chain Polytopes and Enriched Chain Polytopes

Let $(P, <_P)$ be a partially ordered set (poset, for short) on $[d]$. A subset A of $[d]$ is called an *antichain* of P if all i and j belonging to A with $i \neq j$ are incomparable in P. In particular, the empty set $\emptyset$ and each 1-element subset $\{i\}$ are antichains of P. Let $\mathscr{A}(P)$ denote the set of antichains of P. In [37], Stanley introduced the *chain polytope* $\mathscr{C}_P$ of P defined by

$$\mathscr{C}_P := \mathrm{conv}\,\{\mathbf{e}_A \in \mathbb{R}^d : A \in \mathscr{A}(P)\}.$$

It is known that chain polytopes are stable set polytopes. Indeed, let G_P be the finite simple graph on $[d]$ such that $\{i, j\} \in E(G_P)$ if and only if $i <_P j$ or $j <_P i$. We call G_P the *comparability graph* of P. It then follows that $\mathscr{A}(P) = S(G_P)$. Hence the chain polytope $\mathscr{C}_P$ is the stable set polytope $\mathscr{Q}_{G_P}$. Therefore, chain polytopes are anti-blocking polytopes. We remark that any comparability graph is perfect.

On the other hand, the *enriched chain polytope* $\mathscr{C}^{(e)}_P$ of P is the unconditional lattice polytope defined by $\mathscr{C}^{(e)}_P := \mathscr{C}^{\pm}_P$. In [30], it was shown that the Ehrhart polynomial of $\mathscr{C}^{(e)}_P$ coincides with a counting polynomial of left enriched P-partitions. We assume that P is naturally labeled. A map $f\colon P \to \mathbb{Z} \setminus \{0\}$ is called an *enriched P-partition* [38] if, for all $x, y \in P$ with $x <_P y$, f satisfies

$$|f(x)| \leq |f(y)| \quad \text{and} \quad |f(x)| = |f(y)| \;\Rightarrow\; f(y) > 0.$$

A map $f\colon P \to \mathbb{Z}$ is called a *left enriched P-partition* [33] if, for all $x, y \in P$ with $x <_P y$, f satisfies

$$|f(x)| \leq |f(y)| \quad \text{and} \quad |f(x)| = |f(y)| \;\Rightarrow\; f(y) \geq 0.$$

The symbol $\Omega^{(\ell)}_P(m)$ will denote the number of left enriched P-partitions $f\colon P \to \mathbb{Z}$ with $|f(x)| \leq m$ for any $x \in P$, which is called the *left enriched order polynomial* of P.

Proposition 3.3 [30] *Let P be a naturally labeled finite poset on $[d]$. Then one has*

$$L_{\mathscr{C}^{(e)}_P}(m) = \Omega^{(\ell)}_P(m).$$

Given a linear extension $\pi = (\pi_1, \ldots, \pi_d)$ of a finite poset P on $[d]$, a *left peak* of π is an index $1 \le i \le d-1$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$. Let $\mathrm{pk}^{(\ell)}(\pi)$ denote the number of left peaks of π. Then the *left peak polynomial* $W_P^{(\ell)}(x)$ of P is defined by

$$W_P^{(\ell)}(x) = \sum_{\pi \in \mathscr{L}(P)} x^{\mathrm{pk}^{(\ell)}(\pi)},$$

where $\mathscr{L}(P)$ is the set of linear extensions of P.

Proposition 3.4 [30] *Let P be a naturally labeled finite poset on $[d]$. Then the h^*-polynomial of $\mathscr{C}_P^{(e)}$ is*

$$h^*(\mathscr{C}_P^{(e)}, x) = (x+1)^d \, W_P^{(\ell)}\left(\frac{4x}{(x+1)^2}\right).$$

In particular, $h^*(\mathscr{C}_P^{(e)}, x)$ is γ-positive.

Note that if Q is a finite poset that is obtained from P by reordering the label, then $\mathscr{C}_P^{(e)}$ and $\mathscr{C}_Q^{(e)}$ are unimodularly equivalent. Hence the h^*-polynomials of enriched chain polytopes are always γ-positive.

3.4 Symmetric Edge Polytopes of Type B

Let G be a finite simple graph on $[d]$. We set

$$B_G := \mathrm{conv}\left(\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\} \cup \{\mathbf{e}_i + \mathbf{e}_j : \{i, j\} \in E(G)\}\right).$$

Then $B_G = \mathscr{P}_\Delta$ where Δ is a simplicial complex on $[d]$ obtained by regarding G as a 1-dimensional simplicial complex. The *symmetric edge polytope of type B* of G is the unconditional lattice polytope defined by $\mathscr{B}_G := B_G^{\pm}$.

Proposition 3.5 [31] *Let G be a finite simple graph on $[d]$. Then $\mathscr{B}_G$ is reflexive if and only if G is bipartite.*

A *hypergraph* is a pair $\mathscr{H} = (V, E)$, where $E = \{e_1, \ldots, e_n\}$ is a finite multiset of non-empty subsets of $V = \{v_1, \ldots, v_m\}$. Elements of V are called vertices and the elements of E are the hyperedges. Then we can associate $\mathscr{H}$ to a bipartite graph $\mathrm{Bip}\,\mathscr{H}$ with a bipartition $V \cup E$, such that $\{v_i, e_j\}$ is an edge of $\mathrm{Bip}\,\mathscr{H}$ if $v_i \in e_j$. Assume that $\mathrm{Bip}\,\mathscr{H}$ is connected. A *hypertree* in $\mathscr{H}$ is a function $\mathbf{f}\colon E \to \{0, 1, \ldots\}$ such that there exists a spanning tree Γ of $\mathrm{Bip}\,\mathscr{H}$ whose vertices have degree $\mathbf{f}(e) + 1$ at each $e \in E$. Then we say that Γ induces $\mathbf{f}$. Let $B_{\mathscr{H}}$ denote the set of all hypertrees in $\mathscr{H}$. A hyperedge $e_j \in E$ is said to be *internally active* with respect to the hypertree $\mathbf{f}$ if it is not possible to decrease $\mathbf{f}(e_j)$ by 1 and increase $\mathbf{f}(e_{j'})$, $j' < j$, by 1 so that another hypertree results. We call a hyperedge *internally inactive* with respect to a hypertree if it is not internally active and denote the number of such hyperedges of $\mathbf{f}$ by $\bar{\iota}(\mathbf{f})$. Then the *interior polynomial* of $\mathscr{H}$ is the generating function $I_{\mathscr{H}}(x) = \sum_{\mathbf{f} \in B_{\mathscr{H}}} x^{\bar{\iota}(\mathbf{f})}$. It is known [22, Prop. 6.1] that $\deg I_{\mathscr{H}}(x) \le \min\{|V|, |E|\} - 1$. If $G = \mathrm{Bip}\,\mathscr{H}$, then we set $I_G(x) = I_{\mathscr{H}}(x)$.

Assume that G is a bipartite graph with a bipartition $V_1 \cup V_2 = [d]$. Then let $\widetilde{G}$ be a connected bipartite graph on $[d+2]$ whose edge set is

$$E(\widetilde{G}) = E(G) \cup \{\{i, d+1\} : i \in V_1\} \cup \{\{j, d+2\} : j \in V_2 \cup \{d+1\}\}.$$

Proposition 3.6 [31] *Let G be a bipartite graph on $[d]$. Then the h^*-polynomial of the reflexive polytope $\mathscr{B}_G$ is*

$$h^*(\mathscr{B}_G, x) = (x+1)^d\, I_{\widetilde{G}}\biggl(\frac{4x}{(x+1)^2}\biggr).$$

In particular, $h^*(\mathscr{B}_G, x)$ is γ-positive.

4 *h**-Polynomials of Locally Anti-Blocking Lattice Polytopes

In the present section, we prove Theorem 1.1, that is, a formula for the h^*-polynomials of locally anti-blocking lattice polytopes in terms of that of unconditional lattice polytopes. Given a subset $J = \{j_1, \ldots, j_r\}$ of $[d]$, let

$$\pi_J : \mathbb{R}^d \to \mathbb{R}^r, \qquad \pi_J((x_1, \ldots, x_d)) = (x_{j_1}, \ldots, x_{j_r})$$

denote the projection map. (Here $\pi_\emptyset$ is the zero map.)

Proposition 4.1 *Let $\mathscr{P} \subset \mathbb{R}^d_{\ge 0}$ be an anti-blocking lattice polytope. Then we have*

$$h^*(\mathscr{P}^{\pm}, x) = \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{J \subset [d],\, |J| = j} h^*(\pi_J(\mathscr{P}), x).$$

Proof The proof is similar to the discussion in [31, proof of Prop. 3.1]. The intersection of $\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon}$ and $\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon'}$ is of dimension $d-1$ if and only if $\varepsilon - \varepsilon' \in \{\pm 2\mathbf{e}_1, \ldots, \pm 2\mathbf{e}_d\}$. Moreover, if $\varepsilon - \varepsilon' = 2\mathbf{e}_k$, then

$$\begin{aligned}(\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon}) \cap (\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon'}) = \mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon} \cap \mathbb{R}^d_{\varepsilon'} &\simeq \pi_{[d]\setminus\{k\}}(\mathscr{P}^{\pm}) \cap \mathbb{R}^{d-1}_{\pi_{[d]\setminus\{k\}}(\varepsilon)} \\ &\simeq \pi_{[d]\setminus\{k\}}(\mathscr{P}).\end{aligned}$$

Hence the Ehrhart polynomial $L_{\mathscr{P}^{\pm}}(m)$ satisfies the following:

$$L_{\mathscr{P}^{\pm}}(m) = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{J \subset [d],\, |J| = j} L_{\pi_J(\mathscr{P})}(m).$$

Thus the Ehrhart series satisfies

$$\frac{h^*(\mathscr{P}^{\pm}, x)}{(1-x)^{d+1}} = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{J \subset [d],\, |J|=j} \frac{h^*(\pi_J(\mathscr{P}), x)}{(1-x)^{j+1}},$$

as desired. □

We now prove Theorem 1.1.

Proof of Theorem 1.1 Given $J = \{j_1, \ldots, j_r\} \subset [d]$ and $\varepsilon \in \{-1, 1\}^r$, let

$$\mathbb{R}^d_{J,\varepsilon} = \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : \pi_J(\mathbf{x}) \in \mathbb{R}^r_\varepsilon \text{ and } x_j = 0 \text{ for all } j \notin J\}.$$

It then follows that $\mathscr{P} \cap \mathbb{R}^d_{J,\varepsilon}$ is equal to $\pi_J(\mathscr{P}_{\varepsilon'})^{\pm} \cap \mathbb{R}^r_\varepsilon$, where $\pi_J(\varepsilon') = \varepsilon$. Note that, given $J = \{j_1, \ldots, j_r\} \subset [d]$ and $\varepsilon \in \{-1, 1\}^r$, we have $|\{\varepsilon' \in \{-1, 1\}^d : \pi_J(\varepsilon') = \varepsilon\}| = 2^{d-r}$. Thus

$$\begin{aligned}
h^*(\mathscr{P}, x) &= \sum_{j=0}^{d} (x-1)^{d-j} \sum_{J \subset [d],\, |J|=j} \ \sum_{\varepsilon \in \{-1,1\}^j} h^*(\mathscr{P} \cap \mathbb{R}^d_{J,\varepsilon}, x) \\
&= \sum_{j=0}^{d} (x-1)^{d-j} \sum_{\varepsilon \in \{-1,1\}^d} \ \sum_{J \subset [d],\, |J|=j} \frac{h^*(\pi_J(\mathscr{P}_\varepsilon), x)}{2^{d-j}} \\
&= \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} \ \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{J \subset [d],\, |J|=j} h^*(\pi_J(\mathscr{P}_\varepsilon), x) \\
&= \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} h^*(\mathscr{P}^{\pm}_\varepsilon, x)
\end{aligned}$$

by Proposition 4.1. □

Combining Theorem 1.1 with Propositions 3.4 and 3.6, we have

Corollary 4.2 *Let $\mathscr{P} \subset \mathbb{R}^d$ be a locally anti-blocking reflexive polytope. If every $\mathscr{P} \cap \mathbb{R}^d_\varepsilon$ is the intersection of $\mathbb{R}^d_\varepsilon$ and either an enriched chain polytope or a symmetric edge reflexive polytope of type B, then the h^*-polynomial of $\mathscr{P}$ is γ-positive.*

Finally, we conjecture the following.

Conjecture 4.3 *The h^*-polynomial of any locally anti-blocking reflexive polytope is γ-positive.*

Thanks to Theorem 1.1 and Proposition 3.2, in order to prove Conjecture 4.3, it is enough to study unconditional lattice polytopes $\mathscr{Q}^{\pm}_G$ where $\mathscr{Q}_G$ is the stable set polytope of a perfect graph G.

5 Symmetric Edge Polytopes of Type A

Let G be a finite simple graph on the vertex set $[d]$ and the edge set $E(G)$. The *symmetric edge polytope* $\mathscr{A}_G \subset \mathbb{R}^d$ of type A is the convex hull of the set

$$A(G) = \{\pm(\mathbf{e}_i - \mathbf{e}_j) \in \mathbb{R}^d : \{i, j\} \in E(G)\}.$$

The polytope $\mathscr{A}_G$ is introduced in [26,28] and called a "symmetric edge polytope of G."

Example 5.1 Let G be a tree on $[d]$. Then $\mathscr{A}_G$ is unimodularly equivalent to a $(d-1)$-dimensional cross polytope. Hence we have $h^*(\mathscr{A}_G, x) = (x+1)^{d-1}$.

It is known [26, Prop. 4.1] that the dimension of $\mathscr{A}_G$ is $d-1$ if and only if G is connected. Higashitani [20] proved that $\mathscr{A}_G$ is simple if and only if $\mathscr{A}_G$ is smooth Fano if and only if G contains no even cycles. It is known [26,28] that $\mathscr{A}_G$ is unimodularly equivalent to a reflexive polytope having a regular unimodular triangulation. In particular, the h^*-polynomial of $\mathscr{A}_G$ is palindromic and unimodal. For a complete bipartite graph $K_{\ell,m}$, it is known [21] that the h^*-polynomial of $\mathscr{A}_{K_{\ell,m}}$ is real-rooted and hence γ-positive.

5.1 Recursive Formulas for *h**-Polynomials

In this section, we give several recursive formulas of h^*-polynomials of $\mathscr{A}_G$ when G belongs to certain classes of graphs. By the following fact, we may assume that G is 2-connected if needed.

Proposition 5.2 *Let G be a graph and let $G_1, \ldots, G_s$ be 2-connected components of G. Then the h^*-polynomial of $\mathscr{A}_G$ satisfies*

$$h^*(\mathscr{A}_G, x) = h^*(\mathscr{A}_{G_1}, x) \cdots h^*(\mathscr{A}_{G_s}, x).$$

Proof Since $\mathscr{A}_G$ is the free sum of reflexive polytopes $\mathscr{A}_{G_1}, \ldots, \mathscr{A}_{G_s}$, a desired conclusion follows from [4, Thm. 1]. $\square$

The *suspension* $\widehat{G}$ of a graph G is the graph on the vertex set $[d+1]$ and the edge set

$$E(G) \cup \{\{i, d+1\} : i \in [d]\}.$$

We now study the h^*-polynomial of $\mathscr{A}_{\widehat{G}}$. Given a subset $S \subset [d]$,

$$E_S := \{e \in E(G) : |e \cap S| = 1\}$$

is called a *cut* of G. For example, we have $E_\emptyset = E_{[d]} = \emptyset$. In general, it follows that $E_S = E_{[d]\setminus S}$. We identify E_S with the subgraph of G on the vertex set $[d]$ and the edge set E_S. By definition, E_S is a bipartite graph. Let $\mathrm{Cut}(G)$ be the set of all cuts of

G. Note that $|\mathrm{Cut}(G)| = 2^{d-1}$. From Theorem 1.1 and Proposition 3.6, we have the following.

Theorem 5.3 *Let G be a finite graph on $[d]$. Then $\mathscr{A}_{\widehat{G}}$ is unimodularly equivalent to a locally anti-blocking reflexive polytope whose h^*-polynomial is*

$$h^*(\mathscr{A}_{\widehat{G}}, x) = \frac{1}{2^{d-1}} \sum_{H \in \mathrm{Cut}(G)} h^*(\mathscr{B}_H, x) = (x+1)^d f_G\left(\frac{4x}{(x+1)^2}\right),$$

where

$$f_G(x) = \frac{1}{2^{d-1}} \sum_{H \in \mathrm{Cut}(G)} I_{\widetilde{H}}(x).$$

In particular, $h^*(\mathscr{A}_{\widehat{G}}, x)$ is γ-positive. Moreover, $h^*(\mathscr{A}_{\widehat{G}}, x)$ is real-rooted if and only if $f_G(x)$ is real-rooted.

Proof Let $\mathscr{P} \subset \mathbb{R}^d$ be the convex hull of

$$\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\} \cup \{\pm(\mathbf{e}_i - \mathbf{e}_j) : \{i, j\} \in E(G)\}.$$

Then $\mathscr{A}_{\widehat{G}}$ is lattice isomorphic to $\mathscr{P}$. Given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, let $S_\varepsilon = \{i \in [d] : \varepsilon_i = 1\}$. Then $\mathscr{P} \cap \mathbb{R}^d_\varepsilon$ is the convex hull of

$$\{\mathbf{0}\} \cup \{\varepsilon_i \mathbf{e}_i : i \in [d]\} \cup \{\mathbf{e}_i - \mathbf{e}_j : \{i, j\} \in E_{S_\varepsilon}, \ i \in S_\varepsilon\}.$$

Hence $\mathscr{P} \cap \mathbb{R}^d_\varepsilon = \mathscr{B}_{E_{S_\varepsilon}} \cap \mathbb{R}^d_\varepsilon$. Thus $\mathscr{P}$ is a locally anti-blocking polytope and

$$h^*(\mathscr{A}_{\widehat{G}}, x) = \frac{1}{2^{d-1}} \sum_{H \in \mathrm{Cut}(G)} h^*(\mathscr{B}_H, x)$$

by Theorem 1.1. □

Let G be a graph and let $e = \{i, j\}$ be an edge of G. Then the graph G/e obtained by the procedure

(i) Delete e and identify the vertices i and j
(ii) Delete the multiple edges that may be created while (i)

is called the graph obtained from G by *contracting* the edge e. Next, we will show that, for any bipartite graph G and $e \in E(G)$, $h^*(\mathscr{A}_G, x)$ is γ-positive if and only if so is $h^*(\mathscr{A}_{G/e}, x)$. In order to show this fact, we need the theory of Gröbner bases of toric ideals. Given a graph G on the vertex set $[d]$ and the edge set $E(G) = \{e_1, \ldots, e_n\}$, let

$$\mathscr{R} = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s]$$

be the Laurent polynomial ring over a field K and let

$$\mathscr{S} = K[x_1, \dots, x_n, y_1, \dots, y_n, z]$$

be the polynomial ring over K. We define the ring homomorphism $\pi : \mathscr{S} \to \mathscr{R}$ by setting $\pi(z) = s$, $\pi(x_k) = t_i t_j^{-1} s$ and $\pi(y_k) = t_i^{-1} t_j s$ if $e_k = \{i, j\} \in E(G)$ and $i < j$. The *toric ideal* $I_{\mathscr{A}_G}$ of $\mathscr{A}_G$ is the kernel of π. (See, e.g., [14] for details on toric ideals and Gröbner bases.) We now recall the notation given in [21]. For any oriented edge e_i, let p_i denote the corresponding variable, i.e., $p_i = x_i$ or $p_i = y_i$ depending on the orientation, and let $\{p_i, q_i\} = \{x_i, y_i\}$. Let $\mathscr{G}(G)$ be the set of all binomials f satisfying one of the following:

$$f = \prod_{e_i \in I} p_i - \prod_{e_i \in C \setminus I} q_i, \tag{1}$$

where C is an even cycle in G of length $2k$ with a fixed orientation, and I is a k-subset of C such that $e_\ell \notin I$ for $\ell = \min\{i : e_i \in C\}$;

$$f = \prod_{e_i \in I} p_i - z \prod_{e_i \in C \setminus I} q_i, \tag{2}$$

where C is an odd cycle in G of length $2k + 1$ and I is a $(k + 1)$-subset of C;

$$f = x_i y_i - z^2, \tag{3}$$

where $1 \le i \le n$. Then $\mathscr{G}(G)$ is a Gröbner basis of $I_{\mathscr{A}_G}$ with respect to a reverse lexicographic order $<$ induced by the ordering $z < x_1 < y_1 < \cdots < x_n < y_n$ [21, Prop. 3.8]. Here the initial monomial of each binomial is the first monomial. Using this Gröbner basis, we have the following.

Proposition 5.4 *Let G be a bipartite graph on $[d]$ and let $e \in E(G)$. Then we have*

$$h^*(\mathscr{A}_G, x) = (x + 1) h^*(\mathscr{A}_{G/e}, x).$$

Proof Let $E(G) = \{e_1, \dots, e_n\}$ with $e = e_1 = \{i, j\}$. Since G is a bipartite graph, the Gröbner basis $\mathscr{G}(G)$ above consists of the binomials of the form (1) and (3).

Since G has no triangles, the procedure (ii) does not occur when we contract e of G. Hence $E(G/e) = \{e'_2, \dots, e'_n\}$ where e'_k is obtained from e_k by identifying i with j. Let G' be a graph obtained by adding an edge $e'_1 = \{d + 1, d + 2\}$ to the graph G/e. Then $\mathscr{G}(G')$ consists of all binomials f satisfying one of the following:

$$f = \prod_{e_i \in I} p_i - \prod_{e_i \in C \setminus I} q_i,$$

where C is an even cycle in G of length $2k$ with a fixed orientation and $e_1 \notin C$, and I is a k-subset of C such that $e_\ell \notin I$ for $\ell = \min\{i : e_i \in C\}$;

$$f = \prod_{e_i \in I} p_i - z \prod_{e_i \in C \setminus I} q_i,$$

where $C \cup \{e_1\}$ is an even cycle in G of length $2k+2$ and I is a $(k+1)$-subset of C;

$$f = x_i y_i - z^2,$$

where $1 \le i \le n$. Hence $\{\mathrm{in}_<(f) : f \in \mathscr{G}(G)\} = \{\mathrm{in}_<(f) : f \in \mathscr{G}(G')\}$. By a similar argument as in the proof of [19, Thm. 3.1], it follows that

$$h^*(\mathscr{A}_G, x) = h^*(\mathscr{A}_{G'}, x) = h^*(\mathscr{A}_{\{e_1'\}}, x)h^*(\mathscr{A}_{G/e}, x) = (x+1)h^*(\mathscr{A}_{G/e}, x),$$

as desired. □

From Theorem 5.3, Propositions 5.2 and 5.4 we have the following immediately.

Corollary 5.5 *Let G be a bipartite graph on $[d]$. Then we have that:*

(a) *The h^*-polynomial $h^*(\mathscr{A}_{\widetilde{G}}, x) = (x+1)h^*(\mathscr{A}_{\widehat{G}}, x)$ is γ-positive.*
(b) *If G is obtained by gluing bipartite graphs G_1 and G_2 along with an edge e, then*

$$\begin{aligned} h^*(\mathscr{A}_G, x) &= (x+1)h^*(\mathscr{A}_{G/e}, x) \\ &= (x+1)h^*(\mathscr{A}_{G_1/e}, x)h^*(\mathscr{A}_{G_2/e}, x) \\ &= h^*(\mathscr{A}_{G_1}, x)h^*(\mathscr{A}_{G_2}, x)/(x+1). \end{aligned}$$

Remark Corollary 5.5 (b) was recently generalized in [8, Thm. 4.17].

5.2 Pseudo-Symmetric Simplicial Reflexive Polytopes

A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is called *pseudo-symmetric* if there exists a facet $\mathscr{F}$ of $\mathscr{P}$ such that $-\mathscr{F}$ is also a facet of $\mathscr{P}$. Nill [27] proved that any pseudo-symmetric simplicial reflexive polytope $\mathscr{P}$ is a free sum of $\mathscr{P}_1, \ldots, \mathscr{P}_s$, where each $\mathscr{P}_i$ is one of the following:

- cross polytope;
- del Pezzo polytope $V_{2m} = \mathrm{conv}\,(\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{2m}, \pm(\mathbf{e}_1 + \cdots + \mathbf{e}_{2m}))$;
- pseudo-del Pezzo polytope $\widetilde{V}_{2m} = \mathrm{conv}\,(\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{2m}, -\mathbf{e}_1 - \cdots - \mathbf{e}_{2m})$.

Note that a del Pezzo polytope is unimodularly equivalent to $\mathscr{A}_{C_{2m+1}}$ where C_{2m+1} is an odd cycle of length $2m+1$ (see [20]). The h^*-polynomial of $\mathscr{A}_{C_d}$ was essentially studied in the following papers (see also the OEIS sequence A204621):

- Conway and Sloane [6, p. 2379] computed $h^*(\mathscr{A}_{C_d}, x)$ for small d by using results of O'Keeffe [32] and gave a conjecture on the γ-polynomial of $h^*(\mathscr{A}_{C_d}, x)$ (coincides with the γ-polynomial in Proposition 5.7 below).

- General formulas for the coefficients of $h^*(\mathscr{A}_{C_d}, x)$ were given in Ohsugi–Shibata [29] and Wang–Yu [40].

In order to give the h^*-polynomial of $\widetilde{V}_{2m}$, we need the following lemma.

Lemma 5.6 *Let G be a connected graph. Suppose that an edge $e = \{i, j\}$ of G is not a bridge. Let $\mathscr{P}_e$ be the convex hull of $A(G) \setminus \{\mathbf{e}_i - \mathbf{e}_j\}$. Then we have*

$$h^*(\mathscr{P}_e, x) = \frac{h^*(\mathscr{A}_G, x) + h^*(\mathscr{A}_{G\setminus e}, x)}{2},$$

where $G \setminus e$ is the graph obtained by deleting e from G.

Proof Note that $\mathscr{A}_{G\setminus e} \subset \mathscr{P}_e \subset \mathscr{A}_G$. Since G is connected and e is not a bridge of G, the dimension of both $\mathscr{A}_G$ and $\mathscr{A}_{G\setminus e}$ is $d-1$. Let $\mathscr{P}'_e$ denote the convex hull of $A(G) \setminus \{-\mathbf{e}_i + \mathbf{e}_j\}$, which is unimodularly equivalent to $\mathscr{P}_e$. Then $\mathscr{A}_G$ and $\mathscr{P}_e$ are decomposed into the following disjoint union:

$$\mathscr{A}_G = \mathscr{A}_{G\setminus e} \cup (\mathscr{P}_e \setminus \mathscr{A}_{G\setminus e}) \cup (\mathscr{P}'_e \setminus \mathscr{A}_{G\setminus e}),$$
$$\mathscr{P}_e = \mathscr{A}_{G\setminus e} \cup (\mathscr{P}_e \setminus \mathscr{A}_{G\setminus e}).$$

Since $\mathscr{P}_e \setminus \mathscr{A}_{G\setminus e}$ is unimodularly equivalent to $\mathscr{P}'_e \setminus \mathscr{A}_{G\setminus e}$, we have a desired conclusion. □

The h^*-polynomials of V_{2m} and $\widetilde{V}_{2m}$ are as follows:

Proposition 5.7 *Let C_d denote a cycle of length $d \geq 3$ and let $1 \leq m \in \mathbb{Z}$. Then we have*

$$h^*(\mathscr{A}_{C_d}, x) = \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{2i}{i} x^i (x+1)^{d-2i-1},$$
$$h^*(V_{2m}, x) = \sum_{i=0}^{m} \binom{2i}{i} x^i (x+1)^{2m-2i},$$
$$h^*(\widetilde{V}_{2m}, x) = (x+1)^{2m} + \sum_{i=1}^{m} \binom{2i-1}{i-1} x^i (x+1)^{2m-2i}.$$

In particular, the h^*-polynomials of $\mathscr{A}_{C_d}$, V_{2m}, and $\widetilde{V}_{2m}$ are γ-positive.

Proof The proof for C_d is by induction on d. First, we have $h^*(\mathscr{A}_{C_3}, x) = x^2+4x+1 = (x+1)^2 + \binom{2}{1}x$. If $d \geq 4$ is even, then

$$\begin{aligned} h^*(\mathscr{A}_{C_d}, x) &= (x+1) h^*(\mathscr{A}_{C_{d-1}}, x) \\ &= \sum_{i=0}^{(d-2)/2} \binom{2i}{i} x^i (x+1)^{d-2i-1} = \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{2i}{i} x^i (x+1)^{d-2i-1}. \end{aligned}$$

Moreover, if $d = 2m+1$, $2 \le m \in \mathbb{Z}$, then the coefficient of x^m in

$$\sum_{i=0}^{(d-1)/2} \binom{2i}{i} x^i (x+1)^{d-2i-1} = (x+1)h^*(\mathscr{A}_{C_{d-1}}, x) + \binom{2m}{m} x^m$$

is

$$\sum_{i=0}^{m} \binom{2i}{i}\binom{2m-2i}{m-i} = 4^m = 2^{d-1},$$

and the other coefficient is arising from $(x+1)h^*(\mathscr{A}_{C_{d-1}}, x)$. By a recursive formula in [29, Thm. 2.3], we have

$$h^*(\mathscr{A}_{C_d}, x) = \sum_{i=0}^{(d-1)/2} \binom{2i}{i} x^i (x+1)^{d-2i-1}.$$

Since V_{2m} is unimodularly equivalent to $\mathscr{A}_{C_{2m+1}}$, we have $h^*(V_{2m}, x) = h^*(\mathscr{A}_{C_{2m+1}}, x)$. By Lemma 5.6, it follows that

$$\begin{aligned} h^*(\widetilde{V}_{2m}, x) &= \frac{h^*(\mathscr{A}_{C_{2m+1}}, x) + h^*(\mathscr{A}_{P_{2m+1}}, x)}{2} \\ &= \frac{1}{2}\sum_{i=0}^{m} \binom{2i}{i} x^i (x+1)^{2m-2i} + \frac{(x+1)^{2m}}{2} \\ &= (x+1)^{2m} + \sum_{i=1}^{m} \binom{2i-1}{i-1} x^i (x+1)^{2m-2i}. \end{aligned}$$

□

Thus it turns out that any pseudo-symmetric simplicial reflexive polytope is a free sum of reflexive polytopes whose h^*-polynomials are γ-positive. By [4, Thm. 1], we have the following.

Theorem 5.8 *The h^*-polynomial of any pseudo-symmetric simplicial reflexive polytope is γ-positive.*

Proof From results by Nill [27], any pseudo-symmetric simplicial reflexive polytope is a free sum of cross polytopes, del Pezzo polytopes, and pseudo-del Pezzo polytopes. On the other hand, by [4, Thm. 1], the h^*-polynomial of a free sum of reflexive polytopes $\mathscr{P}_1, \ldots, \mathscr{P}_s$ is equal to the product of h^*-polynomials of $\mathscr{P}_1, \ldots, \mathscr{P}_s$. Hence, by Example 5.1 and Proposition 5.7, it follows that the h^*-polynomial of any pseudo-symmetric simplicial reflexive polytope is γ-positive. □

5.3 Classes of Graphs with $h^*(\mathscr{A}_G, x)$ Being γ-Positive

With the results of the present section one can show that, for example, $h^*(\mathscr{A}_G, x)$ is γ-positive if one of the following holds:

- $G = \widehat{H}$ for some graph H (e.g., G is a complete graph, a wheel graph);
- $G = \widetilde{H}$ for some bipartite graph H (e.g., G is a complete bipartite graph);
- G is a cycle;
- G is an outerplanar bipartite graph.

Moreover, one can compute $h^*(\mathscr{A}_G, x)$ explicitly in some cases. We give such calculations for some known formulas (for complete [1] and complete bipartite graphs [21]).

Example 5.9 [1] By Theorem 5.3, we have

$$h^*(\mathscr{A}_{K_d}, x) = h^*(\mathscr{A}_{\widehat{K}_{d-1}}, x) = \frac{(x+1)^{d-1}}{2^{d-2}} \sum_{H \in \mathrm{Cut}(K_{d-1})} I_{\widetilde{H}}\left(\frac{4x}{(x+1)^2}\right).$$

If the edge set of $H \in \mathrm{Cut}(K_{d-1})$ is E_S with $S \subset [d-1]$, then H is a complete bipartite graph $K_{|S|, d-1-|S|}$ and

$$I_{\widetilde{H}}(x) = \sum_{i \geq 0} \binom{|S|}{i}\binom{d-|S|-1}{i} x^i.$$

(Here $K_{0,d-1}$ denotes an empty graph.) It then follows that

$$\begin{aligned} h^*(\mathscr{A}_{K_d}, x) &= \frac{1}{2^{d-1}} \sum_{k=0}^{d-1} \binom{d-1}{k} \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} 4^i \binom{k}{i}\binom{d-k-1}{i} x^i (x+1)^{d-1-2i} \\ &= \frac{1}{2^{d-1}} \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} 4^i x^i (x+1)^{d-1-2i} \sum_{k=i}^{d-i-1} \binom{d-1}{k}\binom{k}{i}\binom{d-k-1}{i} \\ &= \frac{1}{2^{d-1}} \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} 4^i x^i (x+1)^{d-1-2i} \sum_{k=i}^{d-i-1} \binom{d-1}{2i}\binom{2i}{i}\binom{d-1-2i}{k-i} \\ &= \frac{1}{2^{d-1}} \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} 4^i x^i (x+1)^{d-1-2i} 2^{d-1-2i} \binom{d-1}{2i}\binom{2i}{i} \\ &= \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{d-1}{2i}\binom{2i}{i} x^i (x+1)^{d-1-2i}. \end{aligned}$$

Example 5.10 [21] Let $G = K_{m,n}$. Then $\widetilde{G} = K_{m+1,n+1}$ and

$$\begin{aligned} h^*(\mathscr{A}_{K_{m+1,n+1}}, x) &= (x+1) h^*(\mathscr{A}_{\widehat{K}_{m,n}}, x) \\ &= \frac{(x+1)^{m+n+1}}{2^{m+n-1}} \sum_{H \in \mathrm{Cut}(K_{m,n})} I_{\widetilde{H}}\left(\frac{4x}{(x+1)^2}\right). \end{aligned}$$

Let $V_1 \cup V_2$ be the partition of the vertex set of $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$. If the edge set of $H \in \mathrm{Cut}(K_{m,n})$ is E_S with $S \subset [m+n]$, then H is the disjoint union of two complete bipartite graphs $K_{k,\ell}$ and $K_{m-k,n-\ell}$, and hence

$$I_{\widetilde{H}}(x) = \sum_{i\geq 0} \binom{k}{i}\binom{\ell}{i} x^i \times \sum_{j\geq 0} \binom{m-k}{j}\binom{n-\ell}{j} x^j,$$

where $k = |V_1 \cap S|$ and $\ell = n - |V_2 \cap S|$. It then follows that

$$\begin{aligned} h^*(\mathscr{A}_{K_{m+1,n+1}}, x) &= \frac{x+1}{2^{m+n}} \sum_{k=0}^{m}\sum_{\ell=0}^{n} \binom{m}{k}\binom{n}{\ell} \sum_{i=0}^{\min(k,\ell)} 4^i \binom{k}{i}\binom{\ell}{i} x^i (x+1)^{k+\ell-2i} \\ &\quad \times \sum_{j=0}^{\min(m-k,n-\ell)} 4^j \binom{m-k}{j}\binom{n-\ell}{j} x^j (x+1)^{m+n-k-\ell-2j} \\ &= \frac{1}{2^{m+n}} \sum_{i,j\geq 0} 4^{i+j} x^{i+j} (x+1)^{n+m-2(i+j)+1} \\ &\quad \times \sum_{k=i}^{m-j} \binom{m}{k}\binom{k}{i}\binom{m-k}{j} \sum_{\ell=i}^{n-j} \binom{n}{\ell}\binom{\ell}{i}\binom{n-\ell}{j}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=i}^{m-j} \binom{m}{k}\binom{k}{i}\binom{m-k}{j} &= \sum_{k=i}^{m-j} \binom{m}{i+j}\binom{i+j}{i}\binom{m-(i+j)}{k-i} \\ &= 2^{m-(i+j)} \binom{m}{i+j}\binom{i+j}{i}, \end{aligned}$$

we have

$$\begin{aligned} h^*(\mathscr{A}_{K_{m+1,n+1}}, x) &= \sum_{i\geq 0}\sum_{j\geq 0} \binom{i+j}{i}^2 \binom{m}{i+j}\binom{n}{i+j} x^{i+j} (x+1)^{m+n-2(i+j)+1} \\ &= \sum_{\alpha=0}^{\min(m,n)} \sum_{i=0}^{\alpha} \binom{\alpha}{i}^2 \binom{m}{\alpha}\binom{n}{\alpha} x^\alpha (x+1)^{m+n-2\alpha+1} \\ &= \sum_{\alpha=0}^{\min(m,n)} \binom{2\alpha}{\alpha}\binom{m}{\alpha}\binom{n}{\alpha} x^\alpha (x+1)^{m+n-2\alpha+1}. \end{aligned}$$

Finally, we conjecture the following:

Conjecture 5.11 *The h^*-polynomial of any symmetric edge polytope of type A is γ-positive.*

6 Twinned Chain Polytopes

In this section, we will apply Theorem 1.1 to twinned chain polytopes. For two lattice polytopes $\mathscr{P}, \mathscr{Q} \subset \mathbb{R}^d$, we set

$$\Gamma(\mathscr{P}, \mathscr{Q}) := \mathrm{conv}\,(\mathscr{P} \cup (-\mathscr{Q})) \subset \mathbb{R}^d.$$

Let P and Q be two finite posets on $[d]$. The *twinned chain polytope* of P and Q is the lattice polytope defined by $\mathscr{C}_{P,Q} := \Gamma(\mathscr{C}_P, \mathscr{C}_Q)$. Then $\mathscr{C}_{P,Q}$ is reflexive. Moreover, $\mathscr{C}_{P,Q}$ has a flag, regular unimodular triangulation all of whose maximal simplices contain the origin [16, Prop. 1.2]. Hence we obtain

Corollary 6.1 *Let P and Q be two finite posets on $[d]$. Then the h^*-polynomial of $\mathscr{C}_{P,Q}$ coincides with the h-polynomial of a flag triangulation of a sphere.*

In [39, Prop. 2.2] it was shown that $\mathscr{C}_{P,Q}$ is locally anti-blocking. In general, for two finite posets $(P, <_P)$ and $(Q, <_Q)$ with $P \cap Q = \emptyset$, the *ordinal sum* of P and Q is the poset $(P \oplus Q, <_{P\oplus Q})$ on $P \oplus Q = P \cup Q$ such that $i <_{P\oplus Q} j$ if and only if (a) $i, j \in P$ and $i <_P j$, or (b) $i, j \in Q$ and $i <_Q j$, or (c) $i \in P$ and $j \in Q$. Given a subset I of $[d]$, we define the *induced subposet* of P on I to be the finite poset $(P_I, <_{P_I})$ on I such that $i <_{P_I} j$ if and only if $i <_P j$. For $I \subset [d]$, let $\overline{I} := [d] \setminus I$.

Proposition 6.2 [39, Prop. 2.2] *Let P and Q be two finite posets on $[d]$. Then for each $\varepsilon \in \{-1, 1\}^d$, it follows that*

$$\mathscr{C}_{P,Q} \cap \mathbb{R}^d_\varepsilon = \mathscr{C}^{\pm}_{P_{I_\varepsilon} \oplus Q_{\overline{I_\varepsilon}}} \cap \mathbb{R}^d_\varepsilon,$$

where $I_\varepsilon = \{i \in [d] : \varepsilon_i = 1\}$.

From this result, Theorem 1.1, and Proposition 3.4 we obtain the following:

Theorem 6.3 *Let P and Q be two finite posets on $[d]$. Then one has*

$$h^*(\mathscr{C}_{P,Q}, x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} h^*(\mathscr{C}^{(e)}_{R_\varepsilon}, x) = (x+1)^d f_{P,Q}\left(\frac{4x}{(x+1)^2}\right),$$

where $I_\varepsilon = \{i \in [d] : \varepsilon_i = 1\}$ and R_ε is a naturally labeled poset that is obtained from $P_{I_\varepsilon} \oplus Q_{\overline{I}_\varepsilon}$ by reordering the label and

$$f_{P,Q}(x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} W^{(\ell)}_{R_\varepsilon}(x).$$

In particular, $h^*(\mathscr{C}_{P,Q}, x)$ is γ-positive. Moreover, $h^*(\mathscr{C}_{P,Q}, x)$ is real-rooted if and only if $f_{P,Q}(x)$ is real-rooted.

On the other hand, it is known that from $h^*(\mathscr{C}_{P,Q}, x)$ we obtain h^*-polynomials of several non-locally anti-blocking lattice polytopes arising from the posets P and Q. The *order polytope* $\mathscr{O}_P$ [37] of P is the (0, 1)-polytope defined by

$$\mathscr{O}_P := \{\mathbf{x} \in [0, 1]^d : x_i \le x_j \text{ if } i <_P j\}.$$

Given two lattice polytopes $\mathscr{P}, \mathscr{Q} \subset \mathbb{R}^d$, we define

$$\mathscr{P} * \mathscr{Q} := \mathrm{conv}\,((\mathscr{P} \times \{0\}) \cup (\mathscr{Q} \times \{1\})) \subset \mathbb{R}^{d+1},$$

which is called the *Cayley sum* of $\mathscr{P}$ and $\mathscr{Q}$, and define

$$\Omega(\mathscr{P}, \mathscr{Q}) := \mathrm{conv}\,((\mathscr{P} \times \{1\}) \cup (-\mathscr{Q} \times \{-1\})) \subset \mathbb{R}^{d+1}.$$

Proposition 6.4 [16, Thm. 1.1] *Let P and Q be two finite posets on $[d]$. Then*

$$h^*(\mathscr{C}_{P,Q}, x) = h^*(\Gamma(\mathscr{O}_P, \mathscr{C}_Q), x).$$

Furthermore, if P and Q have a common linear extension, then

$$h^*(\mathscr{C}_{P,Q}, x) = h^*(\Gamma(\mathscr{O}_P, \mathscr{O}_Q), x).$$

Proposition 6.5 [18, Thm. 1.4] *Let P and Q be two finite posets on $[d]$. Then*

$$(1 + x)h^*(\mathscr{C}_{P,Q}, x) = h^*(\Omega(\mathscr{O}_P, \mathscr{C}_Q), x).$$

Furthermore, if P and Q have a common linear extension, then

$$(1 + x)h^*(\mathscr{C}_{P,Q}, x) = h^*(\Omega(\mathscr{O}_P, \mathscr{O}_Q), x).$$

Proposition 6.6 [17, Thm. 4.1] *Let P and Q be two finite posets on $[d]$. Then*

$$h^*(\mathscr{C}_{P,Q}, x) = h^*(\mathscr{O}_P * \mathscr{C}_Q, x).$$

From these propositions and Theorem 6.3, we obtain the following:

Corollary 6.7 *Let P and Q be two finite posets on $[d]$. Then the h^*-polynomials of $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$, $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$, $\mathscr{O}_P * \mathscr{C}_Q$, and $\Omega(\mathscr{C}_P, \mathscr{C}_Q)$ are γ-positive. Furthermore, if P and Q have a common linear extension, then the h^*-polynomials of $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ and $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$ are also γ-positive.*

In the rest of this section, we introduce enriched (P, Q)-partitions and we show that the Ehrhart polynomial of $\mathscr{C}_{P,Q}$ coincides with a counting polynomial of enriched (P, Q)-partitions. Assume that P and Q are naturally labeled. We say that a map $f : [d] \to \mathbb{Z}$ is an *enriched* (P, Q)*-partition* if, for all $x, y \in [d]$, it satisfies

- $x <_P y$, $f(x) \geq 0$, and $f(y) \geq 0 \Rightarrow f(x) \leq f(y)$;
- $x <_Q y$, $f(x) \leq 0$, and $f(y) \leq 0 \Rightarrow f(x) \geq f(y)$.

For a map $f\colon [d] \to \mathbb{Z}$, we set

$$m(f) = \min\{\{0\} \cup \{f(x) : x \in [d]\}\} \quad \text{and} \quad M(f) = \max\{\{0\} \cup f(x) : x \in [d]\}\}.$$

For each $0 < m \in \mathbb{Z}$, let $\Omega^{(e)}_{P,Q}(m)$ denote the number of enriched (P, Q)-partitions $f\colon [d] \to \mathbb{Z}$ with $M(f) - m(f) \leq m$.

Theorem 6.8 *Let P and Q be two finite posets on $[d]$. Then one has*

$$L_{\mathscr{C}_{P,Q}}(m) = \Omega^{(e)}_{P,Q}(m).$$

Proof Let $F(m)$ stand for the set of enriched (P, Q)-partitions with $M(f) - m(f) \leq m$. We show that there exists a bijection from $m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$ to $F(m)$. Take $f \in F(m)$ and set $m(f) = a$ and $M(f) = b$. We set

$$I = \{i \in [d] : f(i) \geq 0\}.$$

Let

$$x_i = \begin{cases} f(i) & \text{if } i \in I \text{ is minimal in } P_I, \\ \min\{f(i) - f(j) : i \text{ covers } j \text{ in } P_I\} & \text{if } i \in I \text{ is not minimal in } P_I, \\ -|f(i)| & \text{if } i \in \overline{I} \text{ is minimal in } Q_{\overline{I}}, \\ -\min\{|f(i)| - |f(j)| : i \text{ covers } j \text{ in } Q_{\overline{I}}\} & \text{if } i \in \overline{I} \text{ is not minimal in } Q_{\overline{I}}. \end{cases}$$

Assume that $I = \{1, \ldots, k\}$ and $\overline{I} = \{k+1, \ldots, d\}$. Then we have $(x_1, \ldots, x_k) \in b\mathscr{C}_{P_I}$ and $(x_{k+1}, \ldots, x_d) \in a\mathscr{C}_{Q_{\overline{I}}}$ by a result of Stanley [37, Thm. 3.2]. Hence one obtains $(x_1, \ldots, x_d) \in b\mathscr{C}_{P_I} \oplus a\mathscr{C}_{Q_{\overline{I}}} \subset m\mathscr{C}_{P,Q}$, where $b\mathscr{C}_{P_I} \oplus a\mathscr{C}_{Q_{\overline{I}}}$ is the free sum of $b\mathscr{C}_{P_I}$ and $a\mathscr{C}_{Q_{\overline{I}}}$. Similarly, in general, it follows that $(x_1, \ldots, x_d) \in m\mathscr{C}_{P,Q}$. Therefore, the map $\varphi\colon F(m) \to m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$, $\varphi(f) = (x_1, \ldots, x_d)$ for each $f \in F(m)$, is well defined.

Take $(x_1, \ldots, x_d) \in m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$. We set $I = \{i \in [d] : x_i \geq 0\}$ and define a map $f\colon [d] \to \mathbb{Z}$ by

$$f(i) = \begin{cases} \max\{x_{j_1} + \cdots + x_{j_k} : j_1 <_{P_I} \cdots <_{P_I} j_k = i\} & \text{if } i \in I, \\ -\max\{|x_{j_1}| + \cdots + |x_{j_k}| : j_1 <_{Q_{\overline{I}}} \cdots <_{Q_{\overline{I}}} j_k = i\} & \text{if } i \in \overline{I}. \end{cases}$$

Assume that $I = \{1, \ldots, k\}$ and $\overline{I} = \{k+1, \ldots, d\}$. Then one has $(x_1, \ldots, x_d) \in m(\mathscr{C}_{P_I} \oplus (-\mathscr{C}_{Q_{\overline{I}}})) \cap \mathbb{Z}^d$. Moreover, for some integers a and b with $a \leq 0 \leq b$ and $b - a \leq m$, it follows that $(x_1, \ldots, x_k) \in b\mathscr{C}_{P_I}$ and $(x_{k+1}, \ldots, x_d) \in a\mathscr{C}_{Q_{\overline{I}}}$. We define $f_1\colon I \to \mathbb{Z}$ by $f_1(i) = f(i)$, and $f_2\colon \overline{I} \to \mathbb{Z}$ by $f_2(i) = -f(i)$. From [37, proof of Thm. 3.2], it follows that $0 \leq f_1(i) \leq b$ for any $i \in I$ and $f_1(x) \leq f_1(y)$ if $x_{<_{P_I}} y$, and $0 \geq f_2(i) \geq a$ for any $i \in \overline{I}$ and $f_2(x) \leq f_2(y)$ if $x_{<_{Q_{\overline{I}}}} y$. Therefore,

$f\colon [d] \to \mathbb{Z}$ is an enriched (P, Q)-partition with $M(f) - m(f) \le b - a \le m$, namely, $f \in F(m)$. Similarly, in general, it follows that $f \in F(m)$. Thus, the map $\psi\colon m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d \to F(m)$, $\psi(\mathbf{x})(i) = f(i)$ for each $\mathbf{x} = (x_1, \ldots, x_d) \in m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$, is well defined.

Finally, we show that φ is a bijection. However, this immediately follows by the above and the argument in [37, proof of Thm. 3.2]. ◻

Since $\mathscr{C}_{P,Q}$ is reflexive, we obtain

Corollary 6.9 *Let P and Q be two finite naturally labeled posets on* $[d]$. *Then* $\Omega^{(e)}_{P,Q}(m)$ *is a polynomial in m of degree d and one has*

$$\Omega^{(e)}_{P,Q}(m) = (-1)^d \Omega^{(e)}_{P,Q}(-m-1).$$

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