# The H/Q-correspondence and a generalization of the supergravity c-map 

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#### Abstract

Given a hypercomplex manifold with a rotating vector field (and additional data), we construct a conical hypercomplex manifold. As a consequence, we associate a quaternionic manifold to a hypercomplex manifold of the same dimension with a rotating vector field. This is a generalization of the HK/QKcorrespondence. As an application, we show that a quaternionic manifold can be associated to a conical special complex manifold of half its dimension. Furthermore, a projective special complex manifold (with a canonical c-projective structure) associates with a quaternionic manifold. The latter is a generalization of the supergravity c-map. We do also show that the tangent bundle of any special complex manifold carries a canonical Ricci-flat hypercomplex structure, thereby generalizing the rigid c-map.


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## 1 Introduction

The HK/QK-correspondence is a construction of a (pseudo-)quaternionic Kähler manifold from a (pseudo-)hyper-Kähler manifold of the same dimension with a rotating vector field (see Definition 3.1 and [15, 2, 16, 4]). This correspondence gives also the supergravity c-map, which associates a quaternionic Kähler manifold with a projective special Kähler manifold. The supergravity c-map was introduced in theoretical physics (13.

The inverse construction of the HK/QK-correspondence is called the QK/HK-correspondence. It has been generalized to a $\mathrm{Q} / \mathrm{H}$-correspondence, a construction of hypercomplex manifolds from quaternionic manifolds [10]. The purpose of this paper is to construct a quaternionic manifold from a hypercomplex manifold endowed with a rotating vector field and some extra data. We shall call this construction the hypercomplex/quaternionic-correspondence (H/Q-correspondence for short). We briefly explain how we obtain this correspondence. First we define the notion of a conical hypercomplex manifold (Definition 2.11). Next we construct a conical hypercomplex manifold $\hat{M}$ for every hypercomplex manifold $M$ with a rotating vector field $Z$ (Theorem 3.9) and additional data: a two-form $\Theta$ on $M$, a $\mathrm{U}(1)$-bundle over $M$ whose curvature satisfies (3.1) and a function $f$ on $M$ such that $d f=-\iota_{Z} \Theta$. The manifold $\hat{M}$ is endowed with a free action of the Lie algebra Lie $\mathbb{H}^{*} \cong \mathbb{R} \oplus \mathfrak{s u}(2)$ and its quotient space $\bar{M}$ carries a quaternionic structure, provided that the quotient map $\hat{M} \rightarrow \bar{M}$ is a submersion. The H/Q-correspondence is then defined as $M \mapsto \bar{M}$ (Theorems 4.1 and 4.8). In addition, we show that $\bar{M}$ carries not only a quaternionic connection but also an (induced) affine quaternionic vector field (Proposition 4.7). Note that we give an example of our H/Q-correspondence from a hypercomplex Hopf manifold, which does not admit any hyper-Kähler structure (Example 5.3). Therefore the H/Q-correspondence is a proper generalization of the HK/QK-correspondence. Examples like hypercomplex or quaternionic Hopf manifolds show that hypercomplex and quaternionic manifolds arise naturally beyond the context of hyper-Kähler and quaternionic Kähler geometry. We refer to [25, 18, 19] for the theory of quaternionic manifolds and constructions of such manifolds.

The rigid c-map [9] allows to associate with a conical special Kähler manifold its cotangent bundle endowed with a hyper-Kähler structure with a rotating vector field [2]. In the absence of a metric, we show that the tangent bundle of a special complex manifold carries a canonical hypercomplex structure and that its Obata connection is Ricci flat (Theorem 6.5). In this way we establish a generalization of the rigid c-map which assigns a Ricci flat hypercomplex manifold to each special complex manifold. When the special complex manifold is conical, the resulting hypercomplex manifold is shown to admit a canonical rotating vector field (Lemma 8.1). The notion of a (conical) special complex manifold was introduced in [3]. It is a generalization of a (conical)
special Kähler manifold. We give a local example which does not arise as a special Kähler manifold (Example 8.9). In addition, we find many (different) quaternionic structures on the tangent bundle of a conical special complex manifold in this example (Example 8.9), using a generalization of the supergravity c-map.

As an application of our H/Q-correspondence, we indeed generalize the supergravity c-map by associating a quaternionic manifold with every conical special complex manifold and therefore with every projective special complex manifold (using the extra data involved in the H/Q-correspondence), see Theorem 8.3. It is shown in Proposition 7.3 that any projective special complex manifold possesses a canonical c-projective structure and in Theorem 7.10 that its c-projective Weyl curvature is of type $(1,1)$. So our generalized supergravity c-map can be formulated as associating a quaternionic manifold to a projective special complex manifold endowed with its canonical c-projective structure with c-projective Weyl curvature of type $(1,1)$. This addresses one of the questions raised in [6], where a different construction of quaternionic manifolds from c-projective structures was obtained, compare Remark 8.5,

In the special case of the HK/QK-correspondence, the two-form $\Theta$, which is part of the data entering the $\mathrm{H} / \mathrm{Q}$-correspondence, is the $Z$-invariant Kähler form $\omega_{1}$ in the hyper-Kähler-triple $\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$. However, in general, we have a freedom in the choice of $\Theta$ in the $\mathrm{H} / \mathrm{Q}$-correspondence (see Section 5). In particular we find two choices of $\Theta$ in Example 5.4 which yield different quaternionic structures on the resulting space. This shows that our H/Q-correspondence is not an inverse construction of the Q/Hcorrespondence without a further specification of $\Theta$. It is left for future studies to find a suitable choice of $\Theta$ which gives an inverse construction.

We summarize our constructions in this paper as the following commutative diagram.


## 2 Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and without boundary and maps are assumed to be smooth unless otherwise mentioned. The space of sections of a vector bundle $E \rightarrow M$ is denoted by $\Gamma(E)$.

In this section we introduce hypercomplex and quaternionic structures and derive some properties of conical hypercomplex manifolds.

We say that $M$ is a quaternionic manifold with the quaternionic structure $Q$ if $Q$ is a subbundle of $\operatorname{End}(T M)$ of rank 3 which at every point $x \in M$ is spanned by endomorphisms $I_{1}, I_{2}, I_{3} \in \operatorname{End}\left(T_{x} M\right)$ satisfying

$$
\begin{equation*}
I_{1}^{2}=I_{2}^{2}=I_{3}^{2}=-\mathrm{id}, I_{1} I_{2}=-I_{2} I_{1}=I_{3}, \tag{2.1}
\end{equation*}
$$

and there exists a torsion-free connection $\nabla$ on $M$ such that $\nabla$ preserves $Q$, that is, $\nabla_{X} \Gamma(Q) \subset \Gamma(Q)$ for all $X \in \Gamma(T M)$. Such a torsion-free connection $\nabla$ is called a quaternionic connection and the triplet $\left(I_{1}, I_{2}, I_{3}\right)$ is called an admissible frame of $Q$ at $x$. Note that we use the same letter $\nabla$ for the connection on $\operatorname{End}(T M)$ induced by $\nabla$. The dimension of the quaternionic manifold $M$ is denoted by $4 n$.

An almost hypercomplex manifold is defined to be a manifold $M$ endowed with 3 almost complex structures $I_{1}, I_{2}, I_{3}$ satisfying the quaternionic relations (2.1). If $I_{1}, I_{2}, I_{3}$ are integrable, then $M$ is called a hypercomplex manifold. There exists a unique torsionfree connection on a hypercomplex manifold for which the hypercomplex structures are parallel. It is called the Obata connection [22]. Obviously, hypercomplex manifolds are quaternionic manifolds with $Q=\left\langle I_{1}, I_{2}, I_{3}\right\rangle$.

Definition 2.1. We say that a hypercomplex manifold ( $M,\left(I_{1}, I_{2}, I_{3}\right)$ ) with a vector field $V$ is conical if $\nabla^{0} V=\mathrm{id}$ holds, where $\nabla^{0}$ is the Obata connection. The vector field $V$ is called the Euler vector field.

We state some lemmas for conical hypercomplex manifolds, which will be used later.
Lemma 2.2. Let $\left(M,\left(I_{1}, I_{2}, I_{3}\right), V\right)$ be a conical hypercomplex manifold. Then we have $L_{V} I_{\alpha}=0, L_{I_{\alpha} V} I_{\alpha}=0$ for $\alpha \in\{1,2,3\}$ and $L_{I_{\alpha} V} I_{\beta}=-2 I_{\gamma}$ for any cyclic permutation $(\alpha, \beta, \gamma)$.

Proof. The formulas follow immediately from $L_{V}=\nabla_{V}^{0}-\nabla^{0} V=\nabla_{V}^{0}-\mathrm{id}$ and $L_{I_{\alpha} V}=$ $\nabla_{I_{\alpha} V}^{0}-I_{\alpha}$.

For a connection $\nabla$ and $X \in \Gamma(T M)$, we define

$$
\begin{equation*}
\left(L_{X} \nabla\right)_{Y} Z:=L_{X}\left(\nabla_{Y} Z\right)-\nabla_{L_{X} Y} Z-\nabla_{Y}\left(L_{X} Z\right) \tag{2.2}
\end{equation*}
$$

where $Y, Z \in \Gamma(T M)$. Note that $L_{X} \nabla$ is a tensor.
Lemma 2.3. Let $\left(M,\left(I_{1}, I_{2}, I_{3}\right), V\right)$ be a conical hypercomplex manifold. Then we have $L_{V} \nabla^{0}=0$ and $L_{I_{\alpha} V} \nabla^{0}=0$.
Proof. By Lemma2.2, $V$ and $I_{\alpha} V$ are quaternionic vector fields, namely $L_{V} \Gamma(Q) \subset \Gamma(Q)$ and $L_{I_{\alpha} V} \Gamma(Q) \subset \Gamma(Q)$, where $Q=\left\langle I_{1}, I_{2}, I_{3}\right\rangle$. By [10, Proposition 4.2], it is enough to check $\operatorname{Ric}^{\nabla^{0}}(V, \cdot)=0$ and $\operatorname{Ric}^{\nabla^{0}}\left(I_{\alpha} V, \cdot\right)=0$. We have

$$
\operatorname{Ric}^{\nabla^{0}}(V, Y)=-\operatorname{Ric}^{\nabla^{0}}(Y, V)=-\operatorname{Tr} R^{\nabla^{0}}(\cdot, Y) V=0
$$

Here we used the skew-symmetry of the Ricci tensor of the Obata connection. It follows that also $\operatorname{Ric}^{\nabla^{0}}\left(I_{\alpha} V, \cdot\right)=-\operatorname{Ric}^{\nabla^{0}}\left(V, I_{\alpha} \cdot\right)=0$, by the hermitian property of the Ricci tensor of the Obata connection.

Alternatively we could have used Lemma 2.2 and the explicit form of the Obata connection to check $L_{I_{\alpha} V} \nabla^{0}=0$. Note that $L_{V} \nabla^{0}=0$ follows from the uniqueness of the Obata connection, since the vector field $V$ preserves the hypercomplex structure.

Example 2.4 (The Swann bundle). The principal $\mathbb{R}^{>0} \times \mathrm{SO}(3)$ bundle over a quaternionic manifold, whose fibers consist of all volume elements and admissible frames at each point, possesses a hypercomplex structure (see [24, 10]). It is conical and is called the Swann bundle. The fundamental vector field generated by $c(\neq 0) \in T_{1} \mathbb{R}^{>0}=\mathbb{R}$ is the Euler vector field, as can be easily checked from the explicit representation of the Obata connection (see [5] for example). In the notation of [10] with $\varepsilon=-1$ and $c=-4(n+1)$, a basis of fundamental vector fields for the principal action is given by the vector fields $V=Z_{0}$ and $Z_{\alpha}=-I_{\alpha} Z_{0}$ with non-trivial commutators $\left[Z_{\alpha}, Z_{\beta}\right]=-2 Z_{\gamma}$ and Lie derivatives $L_{Z_{\alpha}} I_{\beta}=-2 I_{\gamma}$ for any cyclic permutation of $\{1,2,3\}$, where we have denoted by $\left(I_{1}, I_{2}, I_{3}\right)$ the hypercomplex structure of the Swann bundle. Specializing to the Swann bundle $\mathbb{H}^{*} /\{ \pm 1\}$ of a point, we see that $Z_{0}$ corresponds to 1 and $\left(Z_{1}, Z_{2}, Z_{3}\right)$ to $(i, j, k)$ in $T_{1}\left(\mathbb{H}^{*} /\{ \pm 1\}\right)=T_{1} \mathbb{H}=\mathbb{H}$.

Lemma 2.5. On any conical hypercomplex manifold $\left(M,\left(I_{1}, I_{2}, I_{3}\right), V\right)$, the distribution $\mathcal{D}:=\left\langle V, I_{1} V, I_{2} V, I_{3} V\right\rangle$ on $\left\{x \in M \mid V_{x} \neq 0\right\}$ is integrable.

Proof. This follows from Lemma 2.2.

## 3 Conification of hypercomplex manifolds

The main result of this section is a construction of conical hypercomplex manifolds $\hat{M}$ of dimension $\operatorname{dim} \hat{M}=\operatorname{dim} M+4$ from hypercomplex manifolds $M$ with a rotating vector field.

Let $M$ be a hypercomplex manifold of dimension $4 n$ with a hypercomplex structure $H=\left(I_{1}, I_{2}, I_{3}\right)$.

Definition 3.1. A vector field $Z$ on a hypercomplex manifold $\left(M,\left(I_{1}, I_{2}, I_{3}\right)\right)$ is called rotating if $L_{Z} I_{1}=0$ and $L_{Z} I_{2}=-2 I_{3}$.

Note that if $Z$ is rotating, then $L_{Z} I_{3}=2 I_{2}$. In this section we will essentially show that by choosing a (local) primitive of the one-form $\iota_{Z} \Theta$ we can construct a conical hypercomplex manifold $(\hat{M}, \hat{H}, V)$ for a hypercomplex manifold $(M, H)$ with a rotating vector field $Z$ and a closed two-form $\Theta$ such that $L_{Z} \Theta=0$.

Let $f$ be a smooth function on $M$ such that $d f=-\iota_{Z} \Theta$ and $f_{1}:=f-(1 / 2) \Theta\left(Z, I_{1} Z\right)$ is nowhere vanishing. Consider a principal $\mathrm{U}(1)$-bundle $\pi: P \rightarrow M$ with a connection form $\eta$ whose curvature form is

$$
\begin{equation*}
d \eta=\pi^{*}\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z} \Theta\right) \circ I_{1}\right)\right) \tag{3.1}
\end{equation*}
$$

Since the curvature $d \eta$ is a basic form, we will usually identify it with its projection $\Theta-\frac{1}{2} d\left(\left(\iota_{Z} \Theta\right) \circ I_{1}\right)$ on $M$. With this understood we have the following lemma, which follows immediately from the definition of $f_{1}$.

Lemma 3.2. $d f_{1}=-\iota_{Z} d \eta$.
Define a vector field $Z_{1}$ on $P$ by $Z_{1}=Z^{h_{\eta}}+\left(\pi^{*} f_{1}\right) X_{P}$, where $Z^{h_{\eta}}$ is the $\eta$-horizontal lift and $X_{P}$ is the fundamental vector field such that $\eta\left(X_{P}\right)=1$. We will write $f_{1}$ for $\pi^{*} f_{1}$.

Remark 3.3. Note that $\left[X_{P}, Z_{1}\right]=0$. Therefore if $Z_{1}$ generates a $\mathrm{U}(1)$-action on $P$, then its action commutes with the principal action of $\pi: P \rightarrow M$.

Set $\tilde{M}=\mathbb{H}^{*} \times P$. Let $\left(e_{0}^{R}, e_{1}^{R}, e_{2}^{R}, e_{3}^{R}\right)\left(\right.$ resp. $\left.\left(e_{0}^{L}, e_{1}^{L}, e_{2}^{L}, e_{3}^{L}\right)\right)$ be the right-invariant (resp. the left-invariant) frame of $\mathbb{H}^{*}$ which coincides with $(1, i, j, k)$ at $1 \in \mathbb{H}^{*}$. Note that $\left[e_{1}^{R}, e_{2}^{R}\right]=-2 e_{3}^{R}$. We will use the same letter for vectors or vector fields canonically lifted to the product $\tilde{M}=\mathbb{H}^{*} \times P$ as for those on the factors $\mathbb{H}^{*}$ and $P$. Set

$$
V_{1}:=e_{1}^{L}-Z_{1}
$$

We denote the space of integral curves of $V_{1}$ by $\hat{M}$. We assume that the quotient map $\tilde{\pi}: \tilde{M} \rightarrow \hat{M}$ is a submersion. Note that "submersion" requires that the quotient space $\hat{M}$ is smooth.

Lemma 3.4. We assume that the equation (3.1) holds. If $L_{Z} I_{1}=0$ and $L_{Z} \Theta=0$, we have

$$
L_{V_{1}} Y^{h_{\eta}}=-[Z, Y]^{h_{\eta}}
$$

for all $Y \in \Gamma(T M)$.
Proof.

$$
\begin{aligned}
-L_{V_{1}} Y^{h_{\eta}} & =-\left[e_{1}^{L}-Z_{1}, Y^{h_{\eta}}\right]=\left[Z_{1}, Y^{h_{\eta}}\right] \\
& =\left[Z^{h_{\eta}}, Y^{h_{\eta}}\right]+\left[f_{1} X_{P}, Y^{h_{\eta}}\right]=\left[Z^{h_{\eta}}, Y^{h_{\eta}}\right]-\left(Y^{h_{\eta}} f_{1}\right) X_{P} \\
& =[Z, Y]^{h_{\eta}}+\eta\left(\left[Z^{h_{\eta}}, Y^{h_{\eta}}\right]\right) X_{P}-\left(Y^{h_{\eta}} f_{1}\right) X_{P} \\
& =[Z, Y]^{h_{\eta}}-d \eta(Z, Y) X_{P}-\left(Y f_{1}\right) X_{P} \\
& =[Z, Y]^{h_{\eta}},
\end{aligned}
$$

where we have used Lemma 3.2.
Note that

$$
\begin{aligned}
T_{(z, p)} \tilde{M} & \cong T_{z} \mathbb{H}^{*} \oplus T_{p} P=\left\langle e_{0}^{R}, e_{1}^{R}, e_{2}^{R}, e_{3}^{R}\right\rangle_{z} \oplus\left\langle X_{P}\right\rangle_{p} \oplus \operatorname{Ker} \eta_{p} \\
& =\left\langle V_{1}\right\rangle_{(z, p)} \oplus\left\langle e_{0}^{R}, e_{1}^{R}, e_{2}^{R}, e_{3}^{R}\right\rangle_{z} \oplus \operatorname{Ker} \eta_{p}
\end{aligned}
$$

for $(z, p) \in \mathbb{H}^{*} \times P$. We define three endomorphisms fields $\tilde{I}_{1}, \tilde{I}_{2}, \tilde{I}_{3}$ on $\tilde{M}$ of $\operatorname{rank} \tilde{I}_{\alpha}=$ $4 n+4(\alpha=1,2,3)$ as follows:

$$
\begin{aligned}
& \tilde{I}_{\alpha} V_{1}=0, \tilde{I}_{\alpha} e_{0}^{R}=e_{\alpha}^{R}, \tilde{I}_{\alpha} e_{\alpha}^{R}=-e_{0}^{R}, \tilde{I}_{\alpha} e_{\beta}^{R}=e_{\gamma}^{R}, \tilde{I}_{\alpha} e_{\gamma}^{R}=-e_{\beta}^{R}, \\
& \left(\tilde{I}_{\alpha}\right)_{(z, p)}\left(\left(Y^{h_{\eta}}\right)_{(z, p)}\right)=\left(\left(I_{\alpha}^{\prime}\right)_{\pi(p)}\left(\pi_{*} Y\right)\right)^{h_{\eta}}(z, p)
\end{aligned}
$$

for $Y \in T_{p} M$. Here $I_{\alpha}^{\prime}$ is defined by

$$
\begin{equation*}
I_{\alpha}^{\prime}=\sum_{\beta=1}^{3} A_{\alpha \beta} I_{\beta}, \tag{3.2}
\end{equation*}
$$

where $A=\left(A_{\alpha \beta}\right) \in \mathrm{SO}(3)$ is the representation matrix of $\left.\mathrm{Ad}_{z}\right|_{\text {ImHiH }}$ with respect to the basis $(i, j, k)$. Note that $\operatorname{Ker} \tilde{I}_{\alpha}=\left\langle V_{1}\right\rangle, \operatorname{Im} \tilde{I}_{\alpha}=T \mathbb{H}^{*} \oplus \operatorname{Ker} \eta(\alpha=1,2,3)$ and that $\tilde{I}_{1}, \tilde{I}_{2}, \tilde{I}_{3}$ satisfy the quaternionic relations on $T \mathbb{H}^{*} \oplus \operatorname{Ker} \eta$.

Lemma 3.5. $L_{e_{0}^{R}} \tilde{I}_{\alpha}=0$.
Proof. The flow $\varphi_{t}:(z, p) \mapsto\left(e^{t} z, p\right)$ of $e_{0}^{R}$ preserves the decomposition $\tilde{M}=\mathbb{H}^{*} \times P$ and acts trivially on the second factor. In particular, it preserves the distribution Ker $\eta$. The action on the first factor is tri-holomorphic with respect to the (standard) hypercomplex structure induced by $\left(\tilde{I}_{\alpha}\right)$ on $\mathbb{H}^{*}$. Since $\operatorname{Ad}_{z}=\operatorname{Ad}_{r z}$ for all $r>0$, we also see that $\varphi_{t}$ preserves the tensors $\left.\tilde{I}_{\alpha}\right|_{\text {Ker } \eta}$.
Lemma 3.6. If $Z$ is rotating and $L_{Z} \Theta=0$, then we have $L_{V_{1}} \tilde{I}_{\alpha}=0$.
Proof. By the definition of $\tilde{I}_{\alpha}$, it is easy to obtain $\left(L_{V_{1}} \tilde{I}_{\alpha}\right) V_{1}=0$ and $\left(L_{V_{1}} \tilde{I}_{\alpha}\right) e_{\delta}^{R}=0$ $(\delta=0, \ldots, 3)$. Moreover, by Lemma 3.4, we have

$$
\begin{aligned}
& \left(L_{V_{1}} \tilde{I}_{\alpha}\right)_{(z, p)}\left(Y^{h_{\eta}}\right) \\
= & {\left[V_{1}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}-\tilde{I}_{\alpha}\left[V_{1}, Y^{h_{\eta}}\right]_{(z, p)} } \\
= & {\left[e_{1}^{L}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}-\left[Z_{1}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}+\tilde{I}_{\alpha}[Z, Y]^{h_{\eta}}{ }_{(z, p)} } \\
= & {\left[e_{1}^{L}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}-\left[Z^{h}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}-\left[f_{1} X_{P}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}+\left(I_{\alpha}^{\prime}[Z, Y]\right)^{h_{\eta}}{ }_{(z, p)} } \\
= & {\left[e_{1}^{L}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}-\left(\left(L_{Z} I_{\alpha}^{\prime}\right) Y\right)^{h_{\eta}}{ }_{(z, p)}, }
\end{aligned}
$$

where we have used that $\left[Z_{\eta}^{h}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]+\left[f_{1} X_{P}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]=\left[Z, I_{\alpha}^{\prime} Y\right]^{h_{\eta}}+\eta\left(\left[Z, I_{\alpha}^{\prime} Y\right]\right) X_{P}-$ $\left(I_{\alpha}^{\prime} Y\right)\left(f_{1}\right) X_{P}=\left[Z, I_{\alpha}^{\prime} Y\right]^{h_{\eta}}$ at the point $(z, p)$, by Lemma 3.2. Taking the flow $\varphi_{t}$ generated by $e_{1}^{L}$, we have

$$
\left[e_{1}^{L}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]_{(z, p)}=\sum_{\beta=1}^{3}\left(\left.\frac{d}{d t}\right|_{t=0} A_{\alpha \beta}(t)\right)\left(I_{\beta} Y\right)^{h_{\eta}}(z, p),
$$

where

$$
A(t)=\left(A_{\alpha \beta}(t)\right)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos 2 t & \sin 2 t \\
0 & -\sin 2 t & \cos 2 t
\end{array}\right) \in \mathrm{SO}(3)
$$

is the matrix associated with $\varphi_{t}(z)$. On the other hand, we see that

$$
\begin{aligned}
L_{Z} I_{1}^{\prime} & =-2 A_{12} I_{3}+2 A_{13} I_{2} \\
L_{Z} I_{2}^{\prime} & =-2 A_{22} I_{3}+2 A_{23} I_{2} \\
L_{Z} I_{3}^{\prime} & =-2 A_{32} I_{3}+2 A_{33} I_{2}
\end{aligned}
$$

and hence

$$
L_{Z}\left(I_{1}^{\prime}, I_{2}^{\prime}, I_{3}^{\prime}\right)=\left(L_{Z} I_{1}^{\prime}, L_{Z} I_{2}^{\prime}, L_{Z} I_{3}^{\prime}\right)=\left(I_{1}, I_{2}, I_{3}\right)\left(\frac{d}{d t} A(t)\right)
$$

Therefore we have $\left(L_{V_{1}} \tilde{I}_{\alpha}\right)_{(z, p)}\left(Y^{h_{\eta}}\right)=0$.
By Lemma 3.6, we can define an almost hypercomplex structure $\left(\hat{I}_{1}, \hat{I}_{2}, \hat{I}_{3}\right)$ on $\hat{M}$ satisfying $\tilde{\pi}_{*} \circ \tilde{I}_{\alpha}=\hat{I}_{\alpha} \circ \tilde{\pi}_{*}$.
Lemma 3.7. The almost hypercomplex structure $\hat{H}=\left(\hat{I}_{1}, \hat{I}_{2}, \hat{I}_{3}\right)$ is integrable, that is, $(\hat{M}, \hat{H})$ is a hypercomplex manifold.

Proof. Let $\tilde{X}$ and $\tilde{Y}$ be projectable vector fields on the total space of the submersion $\tilde{\pi}: \tilde{M} \rightarrow \hat{M}$ and denote by $X=\tilde{\pi}_{*} \tilde{X}, Y=\tilde{\pi}_{*} \tilde{Y}$ their projections. Then we have $\tilde{\pi}_{*}\left(N^{\tilde{I}_{\alpha}}(\tilde{X}, \tilde{Y})\right)=N^{\hat{I}_{\alpha}}(X, Y)$, where $N^{\tilde{I}_{\alpha}}$ and $N^{\hat{I}_{\alpha}}$ are the Nijenhuis tensors of $\tilde{I}_{\alpha}$ and $\hat{I}_{\alpha}$, respectively. Using that $\tilde{I}_{\alpha} V_{1}=0$ and $L_{V_{1}} \tilde{I}_{\alpha}=0$ (Lemma3.6) we see that $N^{\tilde{I}_{\alpha}}\left(V_{1}, \cdot\right)=0$. Since $N^{\tilde{I}_{\alpha}}$ and $N^{\hat{I}_{\alpha}}$ are tensors, it is sufficient to show that the horizontal component of $N^{\tilde{I}_{\alpha}}(A, B)$ vanishes for sections $A$ and $B$ of $\left\langle e_{0}^{R}, e_{1}^{R}, e_{2}^{R}, e_{3}^{R}\right\rangle \oplus \operatorname{Ker} \eta$. It is easy to see that $N^{\tilde{I}_{\alpha}}\left(e_{a}^{R}, e_{b}^{R}\right)=0$ and $N^{\tilde{I}_{\alpha}}\left(e_{a}^{R}, X^{h_{\eta}}\right)=0$, for all $a, b \in\{0, \ldots, 3\}$. So we only need to show that the horizontal component of $N^{\tilde{I}_{\alpha}}\left(X^{h_{\eta}}, Y^{h_{\eta}}\right)$ vanishes, i.e. the component in $\left\langle e_{0}^{R}, e_{1}^{R}, e_{2}^{R}, e_{3}^{R}\right\rangle \oplus \operatorname{Ker} \eta$. It is given by

$$
\left([X, Y]+I_{\alpha}^{\prime}\left[X, I_{\alpha}^{\prime} Y\right]+I_{\alpha}^{\prime}\left[I_{\alpha}^{\prime} X, Y\right]-\left[I_{\alpha}^{\prime} X, I_{\alpha}^{\prime} Y\right]\right)^{h_{\eta}}=0
$$

since $\left(I_{1}^{\prime}, I_{2}^{\prime}, I_{3}^{\prime}\right)$ is a hypercomplex structure on $M$, for every $z \in \mathbb{H}^{*}$.
Since $L_{V_{1}} e_{0}^{R}=0$, we can define a vector field $V=\tilde{\pi}_{*} e_{0}^{R}$ on $\hat{M}$. Let $\hat{\nabla}^{0}$ be the Obata connection with respect to $\hat{H}$.

Lemma 3.8. We have $\hat{\nabla}^{0} V=$ id.
Proof. Using the explicit representation of the Obata connection (see [5] for example) and Lemma 3.5, we have

$$
12\left(\hat{\nabla}_{\tilde{\pi}_{*} Y}^{0} \tilde{\pi}_{*} e_{0}^{R}\right)=\tilde{\pi}_{*}\left(\sum_{(\alpha, \beta, \gamma)}\left(\tilde{I}_{\alpha}\left[\tilde{I}_{\beta} Y, e_{\gamma}^{R}\right]+\tilde{I}_{\alpha}\left[e_{\beta}^{R}, \tilde{I}_{\gamma} Y\right]\right)+2 \sum_{\alpha=1}^{3} \tilde{I}_{\alpha}\left[e_{\alpha}^{R}, Y\right]\right)
$$

where $(\alpha, \beta, \gamma)$ indicates sum over cyclic permutations of $(1,2,3)$ and $Y$ is a projectable vector field on $\tilde{M}$ commuting with $e_{0}^{R}$. Evaluating the expression on $Y=e_{a}^{R}$ and $Y=U^{h_{\eta}}$, we obtain $12 \tilde{\pi}_{*} Y$.

As a consequence, by Lemmas 3.7 and 3.8, we can conclude

Theorem 3.9 (Conification). Let $M$ be a hypercomplex manifold with a hypercomplex structure $H=\left(I_{1}, I_{2}, I_{3}\right)$, a closed two-form $\Theta$ and a rotating vector field $Z$ such that $L_{Z} \Theta=0$. Let $f$ be a smooth function on $M$ such that $d f=-\iota_{Z} \Theta$ and assume $f_{1}:=$ $f-(1 / 2) \Theta\left(Z, I_{1} Z\right)$ does nowhere vanish. Consider a principal $\mathrm{U}(1)$-bundle $\pi: P \rightarrow M$ with a connection form $\eta$ whose curvature form is

$$
d \eta=\pi^{*}\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z} \Theta\right) \circ I_{1}\right)\right)
$$

If the quotient map $\tilde{\pi}: \tilde{M} \rightarrow \hat{M}$ is a submersion, then $(\hat{M}, \hat{H})$ is a conical hypercomplex manifold with the Euler vector field $V=\tilde{\pi}_{*} e_{0}^{R}$.

Remark 3.10. The assumption that $\tilde{\pi}$ is a submersion is always satisfied locally by considering local 1-parameter subgroup generated by $V_{1}$, since the vector field $V_{1}$ has no zeros. Note that "submersion" requires that the quotient space is a smooth manifold.

We say that $(\hat{M}, \hat{H}, V)$ is the conification of $(M, H, Z, f, \Theta)$ associated with $(P, \eta)$ and denote it by $(\hat{M}, \hat{H}, V)=\mathcal{C}_{(P, \eta)}(M, H, Z, f, \Theta)$ (or simply $\hat{M}=\mathcal{C}_{P}(M)$ if there is no confusion).

## 4 The hypercomplex/quaternionic-correspondence

Building on the conification construction of the last section we will now construct a quaternionic manifold $\bar{M}$ of dimension $\operatorname{dim} \bar{M}=\operatorname{dim} M$ from a hypercomplex manifold $M$ with rotating vector field. The resulting quaternionic manifold is endowed with a torsion-free quaternionic connection and an affine quaternionic vector field $X$.

The space of leaves of the integrable distribution $\mathcal{D}:=\left\langle V, \hat{I}_{1} V, \hat{I}_{2} V, \hat{I}_{3} V\right\rangle$ on $\hat{M}$ is denoted by $\bar{M}$. We shall show that $\bar{M}=\mathcal{C}_{P}(M) / \mathcal{D}$ is a quaternionic manifold, which is the main theorem of this paper. In addition, we show that $\bar{M}$ has a natural quaternionic connection $\bar{\nabla}$ and an affine quaternionic vector field $X$ induced from the fundamental vector field $X_{P}$ of $P \rightarrow M$.

Using Theorem 3.9 and a similar argument as in [24, Theorem 2.1], we prove Theorem 4.1.

Theorem 4.1 (H/Q-correspondence). Let $M$ be a hypercomplex manifold with a hypercomplex structure $H=\left(I_{1}, I_{2}, I_{3}\right)$, a closed two-form $\Theta$ and a rotating vector field $Z$ such that $L_{Z} \Theta=0$. Let $f$ be a smooth function on $M$ such that $d f=-\iota_{Z} \Theta$ and assume that $f_{1}:=f-(1 / 2) \Theta\left(Z, I_{1} Z\right)$ does nowhere vanish. Consider a principal $\mathrm{U}(1)$-bundle $\pi: P \rightarrow M$ with a connection form $\eta$ whose curvature form is

$$
d \eta=\pi^{*}\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z} \Theta\right) \circ I_{1}\right)\right)
$$

If both quotient maps $\tilde{\pi}: \tilde{M} \rightarrow \hat{M}$ and $\hat{\pi}: \hat{M} \rightarrow \bar{M}$ defined above are submersions, then there exists an induced quaternionic structure $\bar{Q}$ on $\bar{M}$.

Proof. As we proved in Theorem 3.9, $\hat{M}=\mathcal{C}_{P}(M)$ is a conical hypercomplex manifold with the hypercomplex structure $\hat{H}=\left(\hat{I}_{1}, \hat{I}_{2}, \hat{I}_{3}\right)$. Let $\varphi=\sum_{a=0}^{3} \varphi_{a} i_{a}\left(\left(i_{0}, i_{1}, i_{2}, i_{3}\right)=\right.$ $(1, i, j, k))$ be the right-invariant Maurer-Cartan form on $\mathbb{H}^{*}$ and extend it with the same letter to $\tilde{M}$ as $\left.\varphi\right|_{T P_{\tilde{\sim}}}=0$. Set $\tilde{\theta}_{0}=\varphi_{0}$. Since $L_{V_{1}} \tilde{\theta}_{0}=0$, we can define the one-form $\hat{\theta}_{0}$ on $\hat{M}$ such that $\tilde{\theta}_{0}=\tilde{\pi}^{*} \hat{\theta}_{0}$. We define $\hat{\theta}^{\prime}=\hat{\theta}_{0}+\sum_{\alpha=1}^{3}\left(\hat{\theta}_{0} \circ \hat{I}_{\alpha}\right) i_{\alpha}$ and take the Euler vector field $V$ on $\hat{M}$ as in Theorem 3.9, Here define an $\hat{I}_{\alpha}$-invariant distribution

$$
\hat{\mathcal{H}}:=\operatorname{Ker} \hat{\theta}^{\prime}
$$

It holds that $T \hat{M}=\mathcal{D} \oplus \hat{\mathcal{H}}$. Since $L_{V} \hat{\theta}^{\prime}=0$ and $L_{\hat{I}_{\alpha} V} \hat{\theta}^{\prime}=2\left(\hat{\theta}_{0} \circ \hat{I}_{\beta}\right) i_{\gamma}-2\left(\hat{\theta}_{0} \circ \hat{I}_{\gamma}\right) i_{\beta}$ for any cyclic permutation $(\alpha, \beta, \gamma)$ (these are checked by straightforward calculations), the distribution $\hat{\mathcal{H}}$ is invariant along leaves of $\mathcal{D}$. Since $\hat{\pi}$ is a submersion, there exist a neighborhood $\mathcal{U} \subset \bar{M}$ of $x \in \bar{M}$ and a section $s: \mathcal{U} \rightarrow \hat{M}$. Then we can define

$$
\bar{I}_{\alpha}(Y):=\hat{\pi}_{*}\left(\hat{I}_{\alpha}\left(Y_{s(y)}^{h_{\hat{\theta}^{\prime}}}\right)\right)
$$

for $y \in \mathcal{U}$, where $Y \in T_{y} \bar{M}$ and $Y^{h_{\hat{\theta}^{\prime}}}$ is the $\hat{\theta}^{\prime}$-horizontal lift of $Y$ with respect to $\hat{\mathcal{H}}$. Although each $\bar{I}_{\alpha}$ depends on the sections, the subbundle $\bar{Q}=\left\langle\bar{I}_{1}, \bar{I}_{2}, \bar{I}_{3}\right\rangle \subset \operatorname{End}(T \bar{M})$ is independent of the section by Lemma 2.2. This means that $(\bar{M}, \bar{Q})$ is an almost quaternionic manifold.

Next we show that there exists a torsion-free connection which preserves $\bar{Q}$. We define a connection $\bar{\nabla}$ on $\bar{M}$ by

$$
\begin{equation*}
\bar{\nabla}_{Y} W=\hat{\pi}_{*}\left(\hat{\nabla}_{Y^{h} \hat{\theta}^{\prime}}^{0} W^{h_{\hat{\theta}^{\prime}}}\right), \quad Y, W \in \Gamma(T \bar{M}) \tag{4.1}
\end{equation*}
$$

where $\hat{\nabla}^{0}$ is the Obata connection of $\hat{M}$. Note that $\bar{\nabla}$ is well-defined by Lemma 2.3, Since the Obata connection is torsion-free, then so is $\bar{\nabla}$. To show that $\bar{\nabla}$ preserves $\bar{Q}$, we consider $I \in \Gamma(\bar{Q})$. Then $(I W)^{h_{\hat{\theta}^{\prime}}}=\sum_{\alpha=1}^{3} a_{\alpha} \hat{I}_{\alpha} W^{h_{\hat{\theta}^{\prime}}}$ for some functions $a_{\alpha}$ with $\sum_{\alpha=1}^{3} a_{\alpha}^{2}=1$, which implies

$$
\left(\bar{\nabla}_{Y} I\right) W=\hat{\pi}_{*}\left(\sum_{\alpha=1}^{3}\left(Y^{h_{\hat{\theta^{\prime}}}} a_{\alpha}\right) \hat{I}_{\alpha} W^{h_{\hat{\theta}^{\prime}}}\right)
$$

showing that $\bar{\nabla}$ preserves $\bar{Q}$. Therefore $(\bar{M}, \bar{Q})$ is a quaternionic manifold.
Remark 4.2. The assumption that $\hat{\pi}$ is a submersion is always satisfied locally.
Next we shall show that our construction induces a vector field $X$ which is an affine quaternionic vector field of $(\bar{M}, \bar{Q}, \bar{\nabla})$, where $\bar{\nabla}$ is given by (4.1).
Lemma 4.3. We have $L_{V_{1}} X_{P}=0$ and $L_{X_{P}} \tilde{I}_{\alpha}=0$.
Proof. The first equation can be checked by a straightforward calculation. The second follows from $\left[X_{P}, \tilde{I}_{\alpha} Y^{h_{\eta}}\right]=\left[X_{P},\left(I_{\alpha}^{\prime} Y\right)^{h_{\eta}}\right]=0$.

By Lemma 4.3, we can define a vector field $\widehat{X_{P}}:=\tilde{\pi}_{*} X_{P}$ on $\hat{M}$. Moreover $\widehat{X_{P}}$ satisfies the following.

Lemma 4.4. We have $L_{\widehat{X_{P}}} \hat{I}_{\alpha}=0$, in addition, $L_{\widehat{X_{P}}} \hat{\nabla}^{0}=0$.
Proof. The first claim follows from Lemma 4.3, as $\left(L_{\widehat{X_{P}}} \hat{I}_{\alpha}\right) \circ \tilde{\pi}_{*}=\tilde{\pi}_{*} \circ\left(L_{X_{P}} \tilde{I}_{\alpha}\right)$. Since the Obata connection is uniquely determined by the hypercomplex structure, we have $L_{\widehat{X_{P}}} \hat{\nabla}^{0}=0$ by the invariance of the hypercomplex structure $\left(\hat{I}_{1}, \hat{I}_{2}, \hat{I}_{3}\right)$ under $\widehat{X_{P}}$.

The next two lemmas follow respectively from $\left[e_{a}^{R}, X_{P}\right]=0$ and $L_{X_{P}} \tilde{\theta}_{0}=0$ by projection.
Lemma 4.5. We have $L_{V} \widehat{X_{P}}=0$ and $L_{\hat{I}_{\alpha} V} \widehat{X_{P}}=0$.
Lemma 4.6. We have $L_{\widehat{X_{P}}} \hat{\theta}_{0}=0$ on $\hat{M}$.
Lemma 4.5 allows us to define a vector field $X:=\hat{\pi}_{*} \widehat{X_{P}}$ on $\bar{M}$.
Proposition 4.7. Let $(\bar{M}, \bar{Q})$ be a quaternionic manifold obtained from a hypercomplex manifold $M$ satisfying the assumptions in Theorem 4.1 and $\bar{\nabla}$ the quaternionic connection defined by (4.1). The vector field $X$ is an affine quaternionic vector field of $(\bar{M}, \bar{Q}, \bar{\nabla})$, that is, satisfies $L_{X} \Gamma(\bar{Q}) \subset \Gamma(\bar{Q})$ and $L_{X} \bar{\nabla}=0$.
Proof. It follows easily from Lemma 4.4 that $X$ preserves the quaternionic structure $\bar{Q}$. From Lemma 4.4, Lemma 4.6 and the closure of $\hat{\theta}_{0}$ we do also obtain that $p_{v}\left[\widehat{X_{P}}, Y^{h_{\hat{\theta}^{\prime}}}\right]=$ 0 , where $p_{h}$ and $p_{v}$ denote the projections from $T \hat{M}$ onto the horizontal and vertical subbundles, respectively. Using this, for any vector fields $Y$ and $W$ on $\bar{M}$, we compute

$$
\begin{aligned}
\left(L_{X} \bar{\nabla}\right)_{Y} W & =\hat{\pi}_{*}\left(\left[\widehat{X_{P}}, \hat{\nabla}_{Y^{h_{\hat{\theta}}}}^{0} W^{\hat{H}}\right]-\hat{\nabla}_{p_{h}\left[\widehat{X_{P}}, Y^{h_{\hat{\theta}^{\prime}}}\right.}^{0} W^{h_{\hat{\theta}^{\prime}}}-\hat{\nabla}_{Y^{h_{\hat{\theta}}}}^{0} p_{h}\left[\widehat{X_{P}}, W^{h_{\hat{\theta}^{\prime}}}\right]\right) \\
& =\hat{\pi}_{*}\left(\left(L_{\widehat{X_{P}}} \hat{\nabla}^{0}\right)_{Y^{h_{\hat{\theta}}}} W^{h_{\hat{\theta}^{\prime}}}+\hat{\nabla}_{p_{v}\left[\widehat{X_{P}}, Y^{\left.h_{\hat{\theta}^{\prime}}\right]}\right.}^{0} W^{h_{\hat{\theta}^{\prime}}}+\hat{\nabla}_{Y^{h}}^{0}{ }_{\hat{\theta^{\prime}}} p_{v}\left[\widehat{X_{P}}, W^{h_{\hat{\theta}^{\prime}}}\right]\right)=0 .
\end{aligned}
$$

We call the correspondence from a hypercomplex manifold $(M, H, Z, f, \Theta)$ to a quaternionic manifold ( $\bar{M}, \bar{Q}, \bar{\nabla}, X$ ) described in Theorem 4.1 (and Proposition 4.7 for the additional structure $X$ ) the hypercomplex/quaternionic-correspondence (H/Qcorrespondence for short). As we mentioned in Remarks 3.10 and 4.2, the global assumption in Theorem4.1(H/Q-correspondence) that $\tilde{\pi}$ and $\hat{\pi}$ are submersions is always satisfied locally. Under stronger assumptions and by considering Swann's twist [27], we have the following global result. We use the notation $\zeta_{A}$ for the action induced from the group $\langle A\rangle$ generated by a vector field $A$ to distinguish $\mathrm{U}(1)$-actions.
Theorem 4.8 (H/Q-correspondence, second version). Let $M$ be a hypercomplex manifold with a hypercomplex structure $H=\left(I_{1}, I_{2}, I_{3}\right)$, a closed two-form $\Theta$ and a rotating vector field $Z$ such that $L_{Z} \Theta=0$. Let $f$ be a smooth function on $M$ such that $d f=-\iota_{Z} \Theta$ and assume that $f_{1}:=f-(1 / 2) \Theta\left(Z, I_{1} Z\right)$ does nowhere vanish. Consider a principal $\mathrm{U}(1)$-bundle $\pi: P \rightarrow M$ with a connection form $\eta$ whose curvature form is

$$
d \eta=\pi^{*}\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z} \Theta\right) \circ I_{1}\right)\right)
$$

If $Z_{1}=Z^{h_{\eta}}+f_{1} X_{P}$ generates a free $\mathrm{U}(1)$-action on $P$, then the conification $\hat{M}$ of $M$ is $\mathbb{H}^{*} \times{ }_{\left\langle V_{1}\right\rangle} P$ and the quaternionic manifold $\bar{M}$ coincides with the twist of $M$ given by the twist data $\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z} \Theta\right) \circ I_{1}\right), Z, f_{1}\right)$ as manifolds.

Proof. By Lemma 3.2, we see $\iota_{Z} d \eta=-d f_{1}$. It follows that $L_{Z} d \eta=0$ from the assumptions $L_{Z} \Theta=0$ and $L_{Z} I_{1}=0$. Therefore we obtain a twist $M^{\prime}:=P /\left\langle Z_{1}\right\rangle$ of $M$ with the twist data $\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z} \Theta\right) \circ I_{1}\right), Z, f_{1}\right)$ since $Z_{1}=Z^{h_{\eta}}+f_{1} X_{P}$ generates a free $\mathrm{U}(1)$-action. Let $\pi^{\prime}: P \rightarrow M^{\prime}$ be the quotient map by the action of $\left\langle Z_{1}\right\rangle$. We define an action of $\left\langle V_{1}\right\rangle(\cong \mathrm{U}(1)) \subset\left\langle e_{1}^{L}\right\rangle \times\left\langle Z_{1}\right\rangle$ on $\mathbb{H}^{*} \times P$ by

$$
\zeta_{V_{1}}(u)(z, p)=\left(\zeta_{e_{1}^{L}}(u) z, \zeta_{Z_{1}}\left(u^{-1}\right) p\right)
$$

for $(z, p) \in \mathbb{H}^{*} \times P$. We see that the conification $\hat{M}$ of $M$ is a fiber bundle $\left(\mathbb{H}^{*} \times P\right) /\left\langle V_{1}\right\rangle$ over $M^{\prime}$, which is associated with $\pi^{\prime}: P \rightarrow M^{\prime}$ and usually denoted by $\mathbb{H}^{*} \times{ }_{\left\langle V_{1}\right\rangle} P$. Moreover the quotient of $\hat{M}$ by $\mathbb{H}^{*}$ is $M^{\prime}$, that is, $\bar{M}=M^{\prime}$.


In the above diagram, $\pi_{2}$ is the projection onto the second factor $P$.
Remark 4.9. Note that the bundle $\hat{\pi}: \hat{M} \rightarrow \bar{M}$ is associated to the principal $\mathrm{U}(1)-$ bundle $P \rightarrow \bar{M}=M^{\prime}=P /\left\langle Z_{1}\right\rangle$. Therefore sections of $\hat{\pi}$ are in one-to-one correspondence with equivariant maps $P \rightarrow \mathbb{H}^{*}$. Let $\lambda: P \rightarrow \mathbb{H}^{*}$ be such that $\lambda\left(\zeta_{Z_{1}}(u) p\right)=$ $\zeta_{e_{1}^{L}}\left(u^{-1}\right) \lambda(p)$ for all $u \in \mathrm{U}(1)$ and $p \in P$ and set $F_{\lambda}:=[\lambda, \mathrm{id}]_{\left\langle V_{1}\right\rangle}: P \rightarrow \hat{M}$. If we consider a local section $s: U\left(\subset \bar{M}=M^{\prime}\right) \rightarrow P$, then $s^{\prime}:=F_{\lambda} \circ s: U \rightarrow \hat{M}$ is a local section of $\hat{\pi}: \hat{M} \rightarrow \bar{M}$ and the equivariance of $\lambda$ implies that $s^{\prime}$ is independent of $s$. As we observed in the proof of Theorem 4.1, the quaternionic structure $\bar{Q}=\left\langle\bar{I}_{1}, \bar{I}_{2}, \bar{I}_{3}\right\rangle$ on $\bar{M}$ is induced from the hypercomplex structure on $\hat{M}$ and a local section $s^{\prime}$. For $Y \in T_{x} \bar{M}$, we have

$$
\bar{I}_{\alpha}(Y)=\hat{\pi}_{*}\left(\hat{I}_{\alpha} Y_{s^{\prime}(x)}^{h_{\hat{\theta}^{\prime}}}\right)=\hat{\pi}_{*}\left(\hat{I}_{\alpha} s_{*}^{\prime} Y\right),
$$

since the decomposition $T \hat{M}=\mathcal{D} \oplus \hat{\mathcal{H}}$ is $\hat{I}_{\alpha}$-invariant. From $s^{\prime}=F_{\lambda} \circ s=[\lambda \circ s, s]_{\left\langle V_{1}\right\rangle}=$ $\tilde{\pi} \circ(\lambda \circ s, s)$, it holds that

$$
\begin{align*}
\bar{I}_{\alpha}(Y) & =\hat{\pi}_{*}\left(\hat{I}_{\alpha} s_{*}^{\prime} Y\right)  \tag{4.2}\\
& =\hat{\pi}_{*}\left(\hat{I}_{\alpha}\left(\tilde{\pi}_{*}\left((\lambda \circ s)_{*}(Y)+s_{*} Y\right)\right)\right. \\
& =\hat{\pi}_{*}\left(\tilde{\pi}_{*}\left(\tilde{I}_{\alpha}\left((\lambda \circ s)_{*}(Y)+s_{*} Y\right)\right)\right. \\
& =\pi^{\prime}{ }_{*}\left(\pi_{2 *}\left(\tilde{I}_{\alpha}\left((\lambda \circ s)_{*}(Y)+s_{*} Y\right)\right)\right. \\
& =\pi^{\prime}\left(\pi_{2 *}\left(\tilde{I}_{\alpha} s_{*} Y\right)\right) .
\end{align*}
$$

Note that $(\lambda \circ s)_{*}(Y)+s_{*} Y \in T_{(\lambda(s(x)), s(x))} \tilde{M}$.
Next we consider the decomposition $\left.T P\right|_{s(U)}=\left\langle Z_{1}\right\rangle \oplus s_{*}(T U)$. Let $p^{\vee}$ be the projection from $\left.T P\right|_{s(U)}$ onto $s_{*}(T U)$. Note that $s_{*}\left(T_{x} U\right)$ is generated by the tangent vectors of the form $p^{\vee}\left(W_{s(x)}^{h_{\eta}}\right)=$ : $W^{\vee}$ at each point $s(x)$, where $W$ is a tangent vector of $M$ at $\pi(s(x))$ and $\eta$ is the connection form on $P$. We define (an almost hypercomplex structure) $I_{\alpha}^{\vee}$ on $s(U)$ by $I_{\alpha}^{\vee}\left(W_{\tilde{\sim}}^{\vee}\right)=\left(I_{\alpha}^{\prime} W\right)^{\vee}$ for each $W^{\vee} \in_{\tilde{\sim}} s_{*}\left(T_{x} U\right)$, where $I_{\alpha}^{\prime}$ is given by (3.2) for $z=\lambda(s(x))$. Since $\tilde{I}_{\alpha}\left(Z_{1}\right)=\tilde{I}_{\alpha}\left(e_{1}^{L}\right) \in T \mathbb{H}^{*}\left(\right.$ by $\left.\tilde{I}_{\alpha} V_{1}=0\right)$, we have

$$
\begin{align*}
p^{\vee}\left(\pi_{2 *}\left(\tilde{I}_{\alpha}\left(W^{\vee}\right)\right)\right) & =p^{\vee}\left(\pi_{2 *}\left(\tilde{I}_{\alpha}\left(W^{h_{\eta}}+a Z_{1}\right)\right)\right)=p^{\vee}\left(\tilde{I}_{\alpha} W^{h_{\eta}}\right)=p^{\vee}\left(\left(I_{\alpha}^{\prime} W\right)^{h_{\eta}}\right)  \tag{4.3}\\
& =\left(I_{\alpha}^{\prime} W\right)^{\vee}=I_{\alpha}^{\vee}\left(W^{\vee}\right),
\end{align*}
$$

where $a \in \mathbb{R}$. Then it holds that

$$
\bar{I}_{\alpha}(Y)=\pi_{*}^{\prime}\left(\pi_{2 *}\left(\tilde{I}_{\alpha} s_{*} Y\right)\right)=\pi_{*}^{\prime}\left(p^{\vee}\left(\pi_{2 *}\left(\tilde{I}_{\alpha}\left(s_{*} Y\right)\right)\right)=\pi_{*}^{\prime}\left(I_{\alpha}^{\vee}\left(s_{*} Y\right)\right)\right.
$$

from (4.2) and (4.3). Therefore $\bar{Q}$ can be identified with $\left\langle I_{1}^{\vee}, I_{2}^{\vee}, I_{3}^{\vee}\right\rangle$ on $s(U)$. Note that $\left\langle I_{1}^{\vee}, I_{2}^{\vee}, I_{3}^{\vee}\right\rangle$ is independent of the choice of $\lambda$, and hence it is shown again that $\bar{Q}$ is independent of the choice of $\lambda$, which is identified with a section of $\hat{M}$.

Note that a quaternionic Kähler metric obtained by the HK/QK-correspondence is described directly in terms of the objects on $P($ instead of $\hat{M})$ in [4, 21].

Remark 4.10. The conification $\hat{M}$ of $M$ is locally isomorphic to the Swann bundle of $\bar{M}$, which is conical as discussed in Example 2.4. Note that the Swann bundle is an $\mathbb{H}^{*} /\{ \pm 1\}$-bundle over a quaternionic manifold whereas $\bar{M}$ is the quotient of $\hat{M}$ by $\mathbb{H}^{*}$ as above. Indeed, take an open set $U$ of $\bar{M}$ and local sections $s: U \rightarrow \hat{M}$, $s^{\prime}: U \rightarrow \mathcal{U}(\bar{M})$, where $\pi^{S w}: \mathcal{U}(\bar{M}) \rightarrow \bar{M}$ is the Swann bundle of $\bar{M}$. For a local trivialization $\Phi: \hat{\pi}^{-1}(U) \rightarrow U \times \mathbb{H}^{*}$ associated to $s$ and given by $\Phi(x)=(\hat{\pi}(x), \phi(x))$, we can define a double covering $F: \hat{\pi}^{-1}(U) \rightarrow\left(\pi^{S w}\right)^{-1}(U)$ by

$$
F(x)=\Phi^{\prime-1}\left(s^{\prime}(\hat{\pi}(x)), p(\phi(x))\right) .
$$

Here $\Phi^{\prime}:\left(\pi^{S w}\right)^{-1}(U) \rightarrow U \times \mathbb{H}^{*} /\{ \pm 1\}$ is a local trivialization associated to $s^{\prime}$ and $p: \mathbb{H}^{*} \rightarrow \mathbb{H}^{*} /\{ \pm 1\}$ is the projection. See [24, [6] for the (twisted) Swann bundle.

## 5 Examples of the H/Q-correspondence

In this section, we give examples of the $\mathrm{H} / \mathrm{Q}$-correspondence.
Example 5.1 (HK/QK-correspondence). Let $\left(M, g, H=\left(I_{1}, I_{2}, I_{3}\right)\right)$ be a (possibly indefinite) hyper-Kähler manifold with a rotating Killing vector field $Z$ and $f$ a nowhere vanishing smooth function such that $d f=-\iota_{Z} \Theta$, where $\Theta$ is the Kähler form with respect to $g$ and $I_{1}$. Set $f_{1}=f-(1 / 2) g(Z, Z)$ and assume that the functions $g(Z, Z)$ and $f_{1}$ are nowhere zero. From these data, we can obtain a (possibly indefinite) quaternionic Kähler manifold $(\bar{M}, \bar{g})[15,2,4]$. The metric $\bar{g}$ is positive definite under the assumptions specified in [2, Corollary 2] for the signs of the functions $f, f_{1}$ and for the signature of $g$. Also the sign of the scalar curvature of $\bar{M}$ is determined by these choices.

In the HK/QK-correspondence, the initial data $\Theta$ is a non-degenerate 2-form. In our more general setting, we may also choose $\Theta=0$, like in the following example.

Example 5.2 (Conical hypercomplex manifold). Let $\left(M,\left(I_{1}, I_{2}, I_{3}\right), V\right)$ be a conical hypercomplex manifold with the Euler vector field $V$. Choose $f_{1}=f=1, \Theta=0$, and consider the trivial principal bundle $P=M \times \mathrm{U}(1)$ with the connection $\eta=d t$, where $t$ is the angular coordinate of $\mathrm{U}(1)$ such that $d t\left(X_{P}\right)=1$ on the fundamental vector field $X_{P}$. We assume that $Z:=I_{1} V$ generates a free $\mathrm{U}(1)$-action on $M$ and that the periods of $Z, X_{P}$ and $e_{1}^{L}$ are the same. It holds that $Z$ is rotating from Lemma 2.2, Then $V_{1}$ generates a free $\mathrm{U}(1)$-action on $\tilde{M}=\mathbb{H}^{*} \times P=\mathbb{H}^{*} \times M \times \mathrm{U}(1)$ of the same period. Therefore

$$
\hat{M}\left(=\left(\mathbb{H}^{*} \times M \times \mathrm{U}(1)\right) /\left\langle V_{1}\right\rangle\right) \ni[z, p, q]=\left[z q, \zeta_{Z}\left(q^{-1}\right) p, 1\right] \mapsto\left(z q, \zeta_{Z}\left(q^{-1}\right) p\right) \in \mathbb{H}^{*} \times M
$$

gives a diffeomorphism $\hat{M} \cong \mathbb{H}^{*} \times M$, and hence $\bar{M} \cong M$ as smooth manifolds. In fact, we can define a diffeomorphism $\varphi^{\prime}: M \rightarrow M^{\prime}(=\bar{M})$ by $\varphi^{\prime}(x)=\pi^{\prime}(x, 1)$. A global section $\bar{M} \rightarrow \hat{M}$ gives rise to a hypercomplex structure $\left(\bar{I}_{1}, \bar{I}_{2}, \bar{I}_{3}\right)$ on $\bar{M}$ but the latter does not coincide with $\left(I_{1}, I_{2}, I_{3}\right)$ in general (under the diffeomorphism $\varphi^{\prime}$ ). The quaternionic structure $\bar{Q}$ on $\bar{M}$, however, coincides with $\left\langle I_{1}, I_{2}, I_{3}\right\rangle$. Note that $\bar{Q}$ is independent of the section, as shown in the proof of Theorem 4.1 and Remark 4.9. More explicitly, considering $\lambda_{z}: M \times \mathrm{U}(1) \rightarrow \mathbb{H}^{*}$ defined by $\lambda_{z}(x, u)=z \cdot u^{-1}\left(z \in \mathbb{H}^{*}\right)$ and the section $s: \bar{M} \rightarrow P$ defined by $s(x)=\left(\left(\varphi^{\prime}\right)^{-1}(x), 1\right)$, we see that the section $F_{\lambda_{1}} \circ s$ gives the hypercomplex structure $\left(I_{1}, I_{2}, I_{3}\right)$ and, hence, the quaternionic structure $\left\langle I_{1}, I_{2}, I_{3}\right\rangle$ on $\bar{M} \cong M$.

The next example shows that our $\mathrm{H} / \mathrm{Q}$-correspondence is a proper generalization of the HK/QK-correspondence.

Example 5.3 (Hypercomplex Hopf manifold). Consider $\mathbb{H}^{n} \cong \mathbb{R}^{4 n}$ as a right-vector space over the quaternions with the standard hypercomplex structure

$$
\tilde{H}=\left(\tilde{I}_{1}=R_{i}, \tilde{I}_{2}=R_{j}, \tilde{I}_{3}=\tilde{I}_{1} \tilde{I}_{2}=-R_{k}\right)
$$

and the standard flat hyper-Kähler metric $\tilde{g}$ and set $\tilde{M}=\mathbb{H}^{n} \backslash\{0\}$. Take $A \in \operatorname{Sp}(n) \operatorname{Sp}(1)$ and $\lambda>1$. Then $\langle\lambda A\rangle$ is a group of homotheties which acts freely and properly discontinuously on the simply connected manifold $\tilde{M}$. The quotient space $\tilde{M} /\langle\lambda A\rangle$ inherits a quaternionic structure $Q$ and a quaternionic connection $\nabla$ which are invariant under the centralizer $G^{Q}$ of $\lambda A$ in $\operatorname{GL}(n, \mathbb{H}) \operatorname{Sp}(1)$. In fact, the quaternionic structure $\tilde{Q}$ on $\tilde{M}$ is $\operatorname{GL}(n, \mathbb{H}) \operatorname{Sp}(1)$-invariant and induces therefore an almost quaternionic structure $Q$ on $\tilde{M} /\langle\lambda A\rangle$, since $\langle\lambda A\rangle \subset \mathrm{GL}(n, \mathbb{H}) \operatorname{Sp}(1)$. Moreover, the Levi-Civita connection $\tilde{\nabla}$ on $(\tilde{M}, \tilde{g})$, which coincides with the Obata connection with respect to $\tilde{H}$, is invariant under all homotheties of $\tilde{M}$. Since $\langle\lambda A\rangle$ acts by homotheties, we see that $\tilde{\nabla}$ induces a torsion-free connection $\nabla$ on $\tilde{M} /\langle\lambda A\rangle$, which preserves $Q$. This means that $Q$ is a quaternionic structure on $\tilde{M} /\langle\lambda A\rangle$. In particular, if $A \in \operatorname{Sp}(n)$, then the quotient $\tilde{M} /\langle\lambda A\rangle$ inherits an induced hypercomplex structure $H=\left(I_{1}, I_{2}, I_{3}\right)$ from $\tilde{H}$, which is invariant under the centralizer $G^{H}$ of $\lambda A$ in $\mathrm{GL}(n, \mathbb{H})$, since $\langle\lambda A\rangle$ preserves $\tilde{H}$. We say
that $(\tilde{M} /\langle\lambda A\rangle, Q)$ (resp. $(\tilde{M} /\langle\lambda A\rangle, H))$ is a quaternionic (resp. hypercomplex) Hopf manifold. See [23, 10].

We start with a hypercomplex Hopf manifold $M:=\tilde{M} /\langle\lambda A\rangle$, where $A \in \operatorname{Sp}(n)$. Take $q \in \operatorname{Sp}(1)$ such that $q \neq \pm 1$. The centralizer of $q$ in $\operatorname{Sp}(1)$ is isomorphic to $\mathrm{U}(1)$, which is denoted by $\mathrm{U}_{q}(1)$. We consider a $\mathrm{U}(1)$-action : $z \mapsto z e^{-i t}$ on $\tilde{M}$ defined by the right multiplication of $\mathrm{U}(1) \cong \mathrm{U}_{q}(1) \subset \operatorname{Sp}(n) \mathrm{U}_{q}(1) \subset \operatorname{Sp}(n) \operatorname{Sp}(1)$. This action induces one on $M$ and the corresponding vector field $Z$ is rotating. Therefore we can apply the same procedure as in Example 5.2 under the setting $P=M \times \mathrm{U}_{q}(1)$ (resp. $\left.\tilde{\sim}=\tilde{M} \times \mathrm{U}_{q}(1)\right)$ and $\Theta=0$, and we have the quaternionic manifold $\bar{M}\left(=M^{\prime}\right)$ (resp. $\left.\tilde{M}\left(=\tilde{M}^{\prime}\right)\right)$ by the $\mathrm{H} / \mathrm{Q}$-correspondence. In the following, the quotient map of an action by a group $G$ is denoted by $\pi_{G}$ and the objects associated with $\tilde{M}$ are denoted by the corresponding letters for $M$ with ~, for example, the projection of the twist from $M \times \mathrm{U}_{q}(1)$ is denoted as $\tilde{\pi}^{\prime}$, where we use the notation of Theorem 4.8. Let $R_{q}$ be the right multiplication by $q$.


Since $\pi^{\prime} \circ \pi_{\langle\lambda A\rangle}=\pi_{\left\langle\lambda A R_{q}\right\rangle} \circ \tilde{\pi}^{\prime}$ and $\tilde{M}^{\prime}=\tilde{M}$ is a manifold with an invariant quaternionic structure under the action of $\left\langle\lambda A R_{q}\right\rangle$ (Example 5.2 and Proposition 4.7), we have

$$
\bar{M}=M^{\prime}=\tilde{M} /\left\langle\lambda A R_{q}\right\rangle
$$

Therefore it holds that

$$
M=\tilde{M} /\langle\lambda A\rangle \stackrel{\mathrm{H} / \mathrm{Q}}{\longmapsto} \bar{M}=\tilde{M} /\left\langle\lambda A R_{q}\right\rangle .
$$

In particular, we can choose $A=E_{n} \in \operatorname{Sp}(n)$. Then the centralizer $G^{H}$ of $\lambda=\lambda E_{n}$ is $\mathbb{R}^{>0} \times \operatorname{SL}(n, \mathbb{H})$. We take the subgroup $\mathbb{R}^{>0} \times \operatorname{Sp}(n)$ of $G^{H}$, which acts transitively on $M$. Then

$$
M=\left(\mathbb{R}^{>0} /\langle\lambda\rangle\right) \times \frac{\operatorname{Sp}(n)}{\operatorname{Sp}(n-1)}
$$

On the other hand, considering the subgroup $\mathbb{R}^{>0} \times \operatorname{Sp}(n) \mathrm{U}_{q}(1)$ of the centralizer $G^{Q}$ of $\lambda R_{q}$, we see that

$$
\left(\mathbb{R}^{>0} /\langle\lambda\rangle\right) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)} \stackrel{\mathrm{H} / \mathrm{Q}}{\longmapsto}\left(\mathbb{R}^{>0} /\langle\lambda\rangle\right) \times \frac{\mathrm{Sp}(n) \mathrm{U}(1)}{\mathrm{Sp}(n-1) \triangle_{\mathrm{U}(1)}},
$$

where $\triangle_{\mathrm{U}(1)}$ is a diagonally embedded subgroup of $\operatorname{Sp}(n) \mathrm{U}(1) \subset \operatorname{Sp}(n) \operatorname{Sp}(1)$ which is isomorphic to $\mathrm{U}(1)$. Considering the case of $n=2$, we have an invariant quaternionic structure on the homogeneous space

$$
\bar{M}=\mathbb{R}^{>0} /\langle\lambda\rangle \times \frac{\mathrm{Sp}(2) \mathrm{U}(1)}{\mathrm{Sp}(1) \triangle_{\mathrm{U}(1)}}=\frac{T^{2} \cdot \mathrm{Sp}(2)}{\mathrm{U}(2)}
$$

by the H/Q-correspondence. Note that $T^{2} \times \mathrm{Sp}(2)$ carries a hypercomplex structure and $\left(T^{2} \times \mathrm{Sp}(2)\right) / \mathrm{U}(2)$ is a homogeneous quaternionic manifold considered in [19].

Since $M$ is diffeomorphic to $S^{1} \times S^{4 n-1}, M$ can not admit any hyper-Kähler structure. Therefore the HK/QK-correspondence can not be applied to the hypercomplex Hopf manifold $M$. The H/Q-correspondence is thus a proper generalization of the HK/QK one.

In the following example, the closed form $\Theta$ is non-zero and degenerate.
Example 5.4 (Lie group with left-invariant hypercomplex structure). Consider $G=$ $\mathrm{SU}(3)$. The Lie algebra $\mathfrak{g}$ of $G$ is decomposed as $\mathfrak{g}=\mathfrak{g}_{0}+\mathfrak{g}_{1}$, where $\mathfrak{g}_{0}=\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \cong$ $\mathfrak{u}(1) \oplus \mathfrak{s u}(2) \cong \mathbb{H}$ and $\mathfrak{g}_{1}$ is the unique complementary $\mathfrak{g}_{0}$-module with the action of $\mathbb{H}$ obtained from the adjoint action of $\mathfrak{g}_{0}$ [19]. Denote by $V \in \mathfrak{g}_{0}$ the vector which corresponds to $1 \in \mathbb{H}$. We use the same letters for left-invariant vector fields and corresponding elements of $\mathfrak{g}$ in this example. Three complex structures $I_{1}, I_{2}, I_{3}$ on $\mathfrak{g}$ can be defined as follows. They preserve the decomposition $\mathfrak{g}=\mathfrak{g}_{0}+\mathfrak{g}_{1}$ and act on $\mathfrak{g}_{0}=\mathbb{H}$ by the standard hypercomplex structure $\left(R_{i}, R_{j}, R_{i} R_{j}=-R_{k}\right)$. On $\mathfrak{g}_{1}$ they are defined by

$$
\begin{equation*}
\left.I_{\alpha}\right|_{\mathfrak{g}_{1}}=-\left.\operatorname{ad}_{I_{\alpha} V}\right|_{\mathfrak{g}_{1}}, \quad \alpha=1,2,3 \tag{5.1}
\end{equation*}
$$

These structures extend to a left-invariant hypercomplex structure on $G$ [19], which we denote again by $\left(I_{1}, I_{2}, I_{3}\right)$.

Let $G_{0} \cong(\mathrm{U}(1) \times \mathrm{SU}(2)) /\{ \pm 1\} \cong \mathrm{U}(2)$ be the subgroup of $G$ corresponding to $\mathfrak{g}_{0}$. Note that $G_{0} \subset G$ is a hypercomplex submanifold and therefore totally geodesic with respect to the Obata connection $\nabla^{G}$ of $G$ [24]. The Obata connection $\nabla^{G_{0}}$ of $G_{0}$ is given by $\nabla_{X}^{G_{0}} Y=X Y$ for $X, Y \in \mathfrak{g}_{0}=\mathbb{H}$, where $X Y$ denotes the product of the quaternions $X$ and $Y$. Indeed, $\nabla^{G_{0}}$ is torsion-free and $I_{1}, I_{2}, I_{3}$ are parallel with respect to $\nabla^{G_{0}}$. Then it holds $\nabla_{X}^{G} V=\nabla_{X}^{G_{0}} V=X$ for $X \in \mathfrak{g}_{0}$. For $X \in \mathfrak{g}_{1}$, by (5.1) and the explicit expression of the Obata connection (see [5]), we also find that $\nabla_{X}^{G} V=X$. Hence the hypercomplex manifold $\left(G,\left(I_{1}, I_{2}, I_{3}\right)\right)$ is conical with the Euler vector field $V$ (see also [26]).

Consider the right-action of $\mathrm{U}(2)$ on $\mathrm{SU}(3)$ given by

$$
A B:=A\left(\begin{array}{cc}
B & 0 \\
0 & \operatorname{det}(B)^{-1}
\end{array}\right)
$$

for $A \in \mathrm{SU}(3)$ and $B \in \mathrm{U}(2)$. Let $l: \mathrm{SU}(3) \rightarrow \mathrm{SU}(3) / \mathrm{U}(2) \cong \mathbb{C} P^{2}$ be the projection and $k: S^{5} \rightarrow \mathbb{C} P^{2}$ the Hopf fibration. The pullback bundle $P:=l^{\#} S^{5}$ of $k: S^{5} \rightarrow \mathbb{C} P^{2}$
by $l$ is a $\mathrm{U}(1)$-bundle over $\mathrm{SU}(3)$. The usual identification between the Stiefel manifold $V_{2}\left(\mathbb{C}^{3}\right)$ and $\mathrm{SU}(3)$ is given by

$$
V_{2}\left(\mathbb{C}^{3}\right) \ni\left(a_{1}, a_{2}\right) \leftrightarrow A=\left(a_{1}, a_{2}, \bar{a}_{1} \times \bar{a}_{2}\right) \in \mathrm{SU}(3) .
$$

We can write

$$
\begin{aligned}
P & =\left\{(A, u) \in \mathrm{SU}(3) \times S^{5} \mid l(A)=k(u)\right\} \\
& =\left\{(A, u) \in \mathrm{SU}(3) \times S^{5} \mid\left\langle c_{3}(A)\right\rangle=\langle u\rangle \in \mathbb{C} P^{2}\right\} \\
& =\left\{\left(A, \alpha c_{3}(A)\right) \in \mathrm{SU}(3) \times S^{5} \mid \alpha \in \mathrm{U}(1)\right\} \\
& \cong \mathrm{SU}(3) \times \mathrm{U}(1),
\end{aligned}
$$

where $c_{3}(A)$ denotes the third column of $A$. This shows that $P$ is a trivial bundle. Let $l_{\#}: P \rightarrow S^{5}$ be the bundle map given by $l_{\#}(A, \alpha)=\alpha\left(\bar{a}_{1} \times \bar{a}_{2}\right)=\alpha c_{3}(A)$. Consider the pullback connection $l_{\#}^{*} \eta$ on $P$ from the standard one $\eta$ of $k$ and take $\Theta=l^{*} \omega$, where $\omega$ is the Kähler form on $\mathbb{C} P^{2}$. Set $Z:=I_{1} V$. We see that $Z$ generates a $\mathrm{U}(1)$-action on $\mathrm{SU}(3)$ and is rotating by Lemma 2.2. Since

$$
\langle Z\rangle \subset \mathrm{SU}(2) \subset \mathrm{U}(2)
$$

$Z$ is tangent to the fiber of $l$. Hence, we have $\iota_{Z} \Theta=0, L_{Z} \Theta=0$, and also have $d \Theta=0$ by $d \omega=0$. So we can choose $f=f_{1}=1$ (see Section 3 for the notation) and then see that $Z_{1}$ generates a free $\mathrm{U}(1)$-action on $P$ given by

$$
\zeta_{Z_{1}}(u)(A, \alpha)=\left(\zeta_{Z}(u)(A), u \alpha\right), \quad u \in \mathrm{U}(1)
$$

To see this, it is sufficient to check that $Z$ is horizontal with respect to the pull back connection. The vector field $Z$ is lifted to $\mathrm{SU}(3) \times \mathrm{U}(1)$ as $Z_{(A, \alpha)}=\left(Z_{A}, 0\right) \in T \mathrm{SU}(3) \times$ $T \mathrm{U}(1)$ for $A \in \mathrm{SU}(3)$ and $\alpha \in \mathrm{U}(1)$ with the same letter $Z$. From $Z \in \mathfrak{s u}(2)$, it holds that

$$
\begin{aligned}
l_{\# *} Z_{(A, \alpha)} & =\left.\frac{d}{d t} l_{\#}\left(\zeta_{Z}\left(e^{i t}\right)(A, \alpha)\right)\right|_{t=0} \\
& =\left.\frac{d}{d t} l_{\#}\left(\left(\zeta_{Z}\left(e^{i t}\right)(A), \alpha\right)\right)\right|_{t=0} \\
& =\left.\frac{d}{d t} \alpha c_{3}\left(\zeta_{Z}\left(e^{i t}\right)(A)\right)\right|_{t=0} \\
& =\left.\frac{d}{d t} \alpha c_{3}(A)\right|_{t=0}=0
\end{aligned}
$$

In particular, $\left(l_{\#}^{*} \eta\right)(Z)=0$, that is, $Z$ is horizontal with respect to the pullback connection. So we see that $Z_{1}=Z+X_{P}$. Therefore, by applying the H/Q-correspondence to $G=\mathrm{SU}(3)$, we have a quaternionic manifold

$$
\bar{G}=P /\left\langle Z_{1}\right\rangle=(\mathrm{SU}(3) \times \mathrm{U}(1)) / \mathrm{U}(1) \cong \mathrm{SU}(3)
$$

The identification is given by

$$
(\mathrm{SU}(3) \times \mathrm{U}(1)) / \mathrm{U}(1) \ni[(A, \alpha)]_{\left\langle Z_{1}\right\rangle}=\left[\left(\zeta_{Z}\left(\alpha^{-1}\right) A, 1\right)\right]_{\left\langle Z_{1}\right\rangle} \cong \zeta_{Z}\left(\alpha^{-1}\right) A \in \mathrm{SU}(3)
$$

Note that there exists no Riemannian metric $g$ on $G$ such that $g$ is hyper-Kählerian with respect to $\left(I_{1}, I_{2}, I_{3}\right)$ since $G$ is compact. The situation is summarized in the following diagram.


Note also that $\mathrm{SU}(3) \times \mathrm{U}(1)$ is a three-fold covering of $\mathrm{U}(3):(A, \alpha) \mapsto \alpha A$. The kernel is the cyclic group $\left\{\left(\zeta 1, \zeta^{-1}\right) \mid \zeta^{3}=1\right\}$. The principal bundle $P \rightarrow \mathrm{SU}(3)$ induces a principal bundle $\mathrm{U}(3)=P / \mathbb{Z}_{3} \rightarrow \operatorname{PSU}(3)=\mathrm{SU}(3) / \mathbb{Z}_{3}$. The actions generated by $Z_{1}$ and $Z$ commutes with that of $\mathbb{Z}_{3}$. The vector field $Z$ (resp. $Z_{1}$ ) on $\mathrm{SU}(3)$ (resp. $\mathrm{SU}(3) \times \mathrm{U}(1))$ induces one on $\mathrm{PSU}(3)$ (resp. $\mathrm{U}(3)$ ), which is denoted by the same letter $Z$ (resp. $Z_{1}$ ). We obtain the following diagram.


We can apply the $\mathrm{H} / \mathrm{Q}$-correspondence to the Lie group $G_{1}=\mathrm{PSU}(3)$ with the induced left-invariant hypercomplex structure and see that its resulting space is $\operatorname{SU}(3) / \mathbb{Z}^{3}$. In fact, since the action of $\left\langle Z_{1}\right\rangle$ on $\mathrm{U}(3)$ is given by $\zeta_{Z_{1}}(u)(\alpha A)=(u \alpha)\left(\zeta_{Z}(u)(A)\right)$ and its orbit $\left\{(u \alpha)\left(\zeta_{Z}(u)(A)\right) \mid u \in\left\langle Z_{1}\right\rangle\right\}$ of $\alpha A \in \mathrm{U}(3)$ intersects $\mathrm{SU}(3)$ at exactly three
points, then the resulting space $\mathrm{U}(3) /\left\langle Z_{1}\right\rangle$ is $\mathrm{SU}(3) / \mathbb{Z}^{3}$. Consequently, we have $\bar{G}_{1} \cong G_{1}$ again.

Next we compare the quaternionic structures on the resulting space(s) derived from the pullback connection $\eta_{1}$, which is not flat, and the trivial connection $\eta_{0}$ as in Example 5.2. Recall the notation in Remark 4.9, We claim that the two quaternionic structures are different. We label the objects obtained from $\eta_{i}$ by the symbol $\eta_{i}$ or just by the letter $i(i=0,1)$, when no confusion is possible. Since $Z^{h_{\eta_{0}}}=Z^{h_{\eta_{1}}}$, $\iota_{Z} \Theta_{0}=\iota_{Z} 0=0$ and $\iota_{Z} \Theta_{1}=0$, the vector field $Z_{1}$ on $P$ is $Z_{1}=Z+X_{P}$ for both connections $\eta_{0}$ and $\eta_{1}$. Then the resulting spaces $\bar{G}^{0}$ and $\bar{G}^{1}$ coincide and we simply write $\bar{G}$ for both. Let $a$ be the 1 -form on $\bar{G}$ such that $\eta_{1}-\eta_{0}=\pi^{*} a$. Consider a local section $s: \bar{G} \rightarrow P$. Since $W^{h_{\eta_{1}}}-W^{h_{\eta_{0}}}=-a(W) X_{P}$ for a tangent vector $W$ at $\pi(s(x)) \in \bar{G}$ (we omit the reference points of tangent vectors), we have

$$
W^{\vee 1}-W^{\vee 0}=-a(W) \mathfrak{X}
$$

where $\mathfrak{X}=p^{\vee}\left(X_{P}\right)$ and we recall that $W^{\vee i}=p^{\vee}\left(W^{h_{\eta_{i}}}\right)$. Therefore we see that

$$
\begin{aligned}
I_{\alpha}^{\vee 1}\left(W^{\vee 1}\right) & =\left(I_{\alpha}^{\prime} W\right)^{\vee 1} \\
& =\left(I_{\alpha}^{\prime} W\right)^{\vee 0}-a\left(I_{\alpha}^{\prime} W\right) \mathfrak{X} \\
& =I_{\alpha}^{\vee 0}\left(W^{\vee 0}\right)-a\left(I_{\alpha}^{\prime} W\right) \mathfrak{X} \\
& =I_{\alpha}^{\vee}\left(W^{\vee 1}\right)+a(W) I_{\alpha}^{\vee 0} \mathfrak{X}-a\left(I_{\alpha}^{\prime} W\right) \mathfrak{X} .
\end{aligned}
$$

On the other hand, since $W^{\vee 1}=W^{h_{\eta_{1}}}+c Z_{1}=W^{h_{\eta_{1}}}+c\left(Z^{h_{\eta_{1}}}+X_{P}\right)$, we have $\eta_{1}\left(W^{\vee 1}\right)=c$ and $\pi_{*}\left(W^{\vee 1}\right)=W+c Z$. It holds that

$$
\left(\pi^{*} a\right)\left(W^{\vee 1}\right)=a(W)+c a(Z)=a(W)+a(Z) \eta_{1}\left(W^{\vee 1}\right)
$$

Hence we have

$$
I_{\alpha}^{\vee 1}=I_{\alpha}^{\vee 0}+\left(\pi^{*} a-a(Z) \eta_{1}\right) \otimes\left(I_{\alpha}^{\vee 0} \mathfrak{X}\right)-\left(\left(\pi^{*} a-a(Z) \eta_{1}\right) \circ I_{\alpha}^{\vee 1}\right) \otimes \mathfrak{X} .
$$

Set $\rho:=\pi^{*} a-a(Z) \eta_{1}$ and $A:=\rho \otimes\left(I_{\alpha}^{\vee 0} \mathfrak{X}\right)-\left(\rho \circ I_{\alpha}^{\vee 1}\right) \otimes \mathfrak{X}$. If $Q^{\vee 0}\left(:=\left\langle I_{1}^{\vee 0}, I_{2}^{\vee 0}, I_{3}^{\vee 0}\right\rangle\right)=$ $Q^{\vee 1}\left(:=\left\langle I_{1}^{\vee 1}, I_{2}^{\vee 1}, I_{3}^{\vee 1}\right\rangle\right)$, then $A^{2}=-|A|^{2} \mathrm{id}$, where $|\cdot|$ is the norm induced from the metric on $Q^{\vee 0}$ such that $I_{1}^{\vee 0}, I_{2}^{\vee 0}, I_{3}^{\vee 0}$ are orthonormal. As the rank of $A$ is at most 2 , this is only possible if $A=0$. This implies $\rho=\pi^{*} a-a(Z) \eta_{1}=0$, which is equivalent to $a=0$. By Remark 4.9, the quaternionic structure $\bar{Q}^{i}$ can be identified with $Q^{\vee i}$ $(i=0,1)$. Then we see that $\bar{Q}^{0} \neq \bar{Q}^{1}$ since $\eta_{0} \neq \eta_{1}$. This proves the claim.

## 6 The tangent bundle of a special complex manifold and a generalization of the rigid c-map

In this section, we consider a generalization of the rigid c-map [9, 14, 3]. The generalization associates a hypercomplex manifold $M$, the Obata connection of which is Ricci-flat, with a special complex manifold. In the case of a conical special complex manifold, we
shall show that the hypercomplex manifold carries a canonical rotating vector field $Z^{M}$ (Lemma 8.1), such that we can apply our H/Q correspondence. Consequently, we shall construct a quaternionic manifold from a conical special complex manifold as the generalized supergravity c-map (Theorem 8.3). We start with defining a class of manifolds generalizing conical special Kähler manifolds [3, 21].
Definition 6.1. A special complex manifold $(N, J, \nabla)$ is a complex manifold $(N, J)$ endowed with a torsion-free flat connection $\nabla$ such that the $(1,1)$-tensor field $\nabla J$ is symmetric. A conical special complex manifold $(N, J, \nabla, \xi)$ is a special complex manifold $(N, J, \nabla)$ endowed with a vector field $\xi$ such that

- $\nabla \xi=\mathrm{id}$ and
- $L_{\xi} J=0$ or, equivalently, $\nabla_{\xi} J=0$.

The connection $\nabla$ is called the special connection. To see that $L_{\xi} J=0$ is equivalent to $\nabla_{\xi} J=0$ it suffices to write $L_{\xi}=\nabla_{\xi}-\nabla \xi=\nabla_{\xi}$-id, using that $\nabla$ is torsion-free and $\nabla \xi=\mathrm{id}$. We also note that the integrability of $J$ follows from the symmetry of $\nabla J$ since $\nabla$ is torsion-free. We set $A:=\nabla J$.
Lemma 6.2. For every conical special complex manifold, we have $L_{J \xi} J=A_{J \xi}=0$.
Proof. Based on the symmetry of $\nabla J$, we compute

$$
A_{J \xi}=A(J \xi)=-J(A \xi)=-J A_{\xi}=0
$$

Using this and the properties listed in Definition 6.1, we then obtain

$$
\left(L_{J \xi} J\right) X=-A_{J X} \xi+J A_{X} \xi=0
$$

for all $X \in \Gamma(T N)$. Note that in the last step we have used the symmetry of $A=\nabla J$.
Next we consider the tangent bundle $T N=: M$ of a special complex manifold $(N, J, \nabla)$. We can define the $\nabla$-horizontal lift $X^{h} \nabla$ and the vertical lift $X^{v}$ of $X \in$ $\Gamma(T N)$. See [7] for example. The $C^{\infty}(M)$-module $\Gamma(T M)$ is generated by vector fields of the form $X^{h_{\nabla}}+Y^{v}$, where $X, Y \in \Gamma(T N)$. On $M$, we define a triple of (1,1)-tensors $\left(I_{1}, I_{2}, I_{3}\right)$ by

$$
\begin{align*}
& I_{1}\left(X^{h} \nabla+Y^{v}\right)=(J X)^{h} \nabla-(J Y)^{v}  \tag{6.1}\\
& I_{2}\left(X^{h} \nabla+Y^{v}\right)=Y^{h}-X^{v}  \tag{6.2}\\
& I_{3}\left(X^{h} \nabla+Y^{v}\right)=(J Y)^{h}+(J X)^{v} \tag{6.3}
\end{align*}
$$

for $X^{h} \nabla+Y^{v} \in T M$. Note that $\left(I_{1}, I_{2}, I_{3}\right)$ is an almost hypercomplex structure. In fact, it is easy to see $I_{\alpha}^{2}=-\mathrm{id}$ and

$$
\begin{aligned}
& \left(I_{1} \circ I_{2}\right)\left(X^{h_{\nabla}}+Y^{v}\right)=I_{1}\left(Y^{h_{\nabla}}-X^{v}\right)=(J Y)^{h_{\nabla}}+(J X)^{v}=I_{3}\left(X^{h_{\nabla}}+Y^{v}\right), \\
& \left(I_{2} \circ I_{1}\right)\left(X^{h_{\nabla}}+Y^{v}\right)=I_{2}\left((J X)^{h_{\nabla}}-(J Y)^{v}\right)=-(J Y)^{h_{\nabla}}-(J X)^{v}=-I_{3}\left(X^{h_{\nabla}}+Y^{v}\right)
\end{aligned}
$$

for $X^{h_{\nabla}}+Y^{v} \in T M$. Note that it holds

$$
\begin{equation*}
\left[X^{h_{\nabla}}, Y^{h \nabla}\right]=[X, Y]^{h_{\nabla}},\left[X^{h_{\nabla}}, Y^{v}\right]=\left(\nabla_{X} Y\right)^{v},\left[X^{v}, Y^{v}\right]=0 \tag{6.4}
\end{equation*}
$$

for $X, Y \in \Gamma(T N)$.

Lemma 6.3. For every special complex manifold $(N, J, \nabla)$, the canonical almost hypercomplex structure $\left(I_{1}, I_{2}, I_{3}\right)$ on $M=T N$ is integrable, that is, $\left(M,\left(I_{1}, I_{2}, I_{3}\right)\right)$ is a hypercomplex manifold.

Proof. Thanks to (6.4), the Nijenhuis tensors of $I_{1}$ and $I_{2}$ can be easily calculated and we find the following. Using that $J$ is integrable, $\nabla$ is flat and $\nabla J$ is symmetric, we see that $I_{1}$ is integrable. Because $\nabla$ is flat and torsion-free, $I_{2}$ is integrable. The integrability of $I_{3}$ follows from that of $I_{1}$ and $I_{2}$ [5, Theorem 3.2].

We define a connection $\nabla^{\prime}$ by

$$
\nabla^{\prime}:=\nabla-\frac{1}{2} J(\nabla J)=\nabla-\frac{1}{2} J A .
$$

Then we see that $\nabla^{\prime} J=0$ and $\nabla^{\prime}$ is torsion-free for every special complex manifold. Moreover, when the special complex manifold is conical, it holds that $\nabla^{\prime} \xi=\nabla \xi=\mathrm{id}$.
Lemma 6.4. For every special complex manifold $(N, J, \nabla)$, we have

$$
R_{X, Y}^{\nabla^{\prime}}=-\frac{1}{4}\left[A_{X}, A_{Y}\right]
$$

for $X, Y \in T N$.
Proof. Set $S=-(1 / 2) J(\nabla J)$. Since $\nabla$ is flat, we see that the curvature $R^{\nabla^{\prime}}$ of $\nabla^{\prime}$ is given by

$$
R_{X, Y}^{\nabla^{\prime}}=\left(\nabla_{X} S\right)_{Y}-\left(\nabla_{Y} S\right)_{X}+\left[S_{X}, S_{Y}\right]
$$

for $X, Y \in T N$. By

$$
\begin{aligned}
\left(\nabla_{X} S\right)_{Y}-\left(\nabla_{Y} S\right)_{X} & =-\frac{1}{2}\left[A_{X}, A_{Y}\right]-\frac{1}{2} J\left(R_{X, Y}^{\nabla} J\right), \\
{\left[S_{X}, S_{Y}\right] } & =\frac{1}{4}\left[A_{X}, A_{Y}\right]
\end{aligned}
$$

we have the conclusion.
Hence a special complex manifold admits the complex connection $\nabla^{\prime}$ such that $R^{\nabla^{\prime}}$ is of type $(1,1)$. In fact, it follows from $A_{J X}=-J A_{X}$ for all $X \in T N$. The following theorem is a generalization of the rigid c-map in the absence of a metric.

Theorem 6.5 (Generalized rigid c-map). The tangent bundle of any special complex manifold $(N, J, \nabla)$ carries a canonical hypercomplex structure, defined by (6.1)-(6.3), and the Obata connection of the hypercomplex manifold $\left(M=T N,\left(I_{1}, I_{2}, I_{3}\right)\right)$ is Ricci flat.

Proof. The integrability of the canonical almost hypercomplex structure defined by (6.1)- (6.3) was proven in Lemma 6.3, Let $\tilde{\nabla}^{0}$ be its Obata connection. Using the explicit expression of the Obata connection, we have

$$
\begin{aligned}
& \tilde{\nabla}_{X^{h} \nabla}^{0} Y^{h_{\nabla}}=\left(\nabla_{X}^{\prime} Y\right)^{h_{\nabla}}, \quad \tilde{\nabla}_{U^{v}}^{0} X^{h_{\nabla}}=-\frac{1}{2}\left(J A_{X} U\right)^{v}=-\frac{1}{2}\left(J A_{U} X\right)^{v} \\
& \tilde{\nabla}_{X^{h} \nabla}^{0} U^{v}=\left(\nabla_{X}^{\prime} U\right)^{v}, \quad \tilde{\nabla}_{U^{v}}^{0} V^{v}=\frac{1}{2}\left(J A_{V} U\right)^{h_{\nabla}}=\frac{1}{2}\left(J A_{U} V\right)^{h_{\nabla}}
\end{aligned}
$$

for $X, Y, U, V \in \Gamma(T N)$. It can be also checked directly, using by (6.1)-(6.4), that the above formulas for $\tilde{\nabla}^{0}$ on horizontal and vertical lifts extend uniquely to a torsion-free connection $\tilde{\nabla}^{0}$ for which $I_{1}, I_{2}, I_{3}$ are parallel. We see that the bundle projection from $\left(T N, \tilde{\nabla}^{0}\right)$ onto $\left(N, \nabla^{\prime}\right)$ is an affine submersion [1]. Again, a straightforward calculation (or the fundamental equations of an affine submersion) gives

$$
\begin{aligned}
R_{U^{v}, V^{v}}^{\tilde{\nabla}^{0}} W^{v} & =-\frac{1}{4}\left(A_{U} A_{V} W\right)^{v}+\frac{1}{4}\left(A_{V} A_{U} W\right)^{v}=\left(R_{U, V}^{\nabla^{\prime}} W\right)^{v}, \\
R_{U^{v}, V^{v}}^{\tilde{\sigma}^{0}} X^{h} & =-\frac{1}{4}\left(A_{U} A_{V} X\right)^{h}+\frac{1}{4}\left(A_{V} A_{U} X\right)^{h_{\nabla}}=\left(R_{U, V}^{\nabla^{\prime}} X\right)^{h \nabla}, \\
R_{U^{v}, X^{h} \nabla}^{\tilde{\nabla}^{0}} V^{v} & =-\frac{1}{2}\left(J\left(H_{U, V}^{\nabla} J\right) X\right)^{h_{\nabla}}-\frac{1}{4}\left(A_{X} A_{U} V\right)^{h_{\nabla}}-\frac{1}{4}\left(A_{U} A_{X} V\right)^{h \nabla}, \\
R_{U^{v}, X^{h} \nabla}^{\tilde{\sigma}^{0}} Y^{h_{\nabla}} & =\frac{1}{2}\left(J\left(H_{X, Y}^{\nabla} J\right) U\right)^{v}+\frac{1}{4}\left(A_{X} A_{Y} U\right)^{v}+\frac{1}{4}\left(A_{U} A_{X} Y\right)^{v}, \\
R_{X^{h_{\nabla}}, Y^{h} \nabla}^{\tilde{\sigma}^{0}} U^{v} & =\left(R_{X, Y}^{\nabla^{\prime}} U\right)^{v}, \\
R_{X^{h_{\nabla, Y}}{ }^{n} \nabla}^{\tilde{\sigma}^{0}} Z^{h_{\nabla}} & =\left(R_{X, Y}^{\nabla^{\prime}} Z\right)^{h_{\nabla}}
\end{aligned}
$$

for $X, Y, Z, U, V, W \in T N$, where $H^{\nabla}$ is the Hessian (the second covariant derivative) with respect to $\nabla$ and we have used Lemma 6.4. Note that $\left(H_{X, Y}^{\nabla} J\right)(Z)=\left(H_{X, Z}^{\nabla} J\right)(Y)$ for all $X, Y, Z \in T N$, since $\nabla J$ is symmetric. Hence the flatness of $\nabla$ means that the Hessian of $J$ with respect to $\nabla$ is totally symmetric. By these equations, the Ricci tensor of $\tilde{\nabla}^{0}$ satisfies

$$
\begin{aligned}
\operatorname{Ric}^{\tilde{\nabla}^{0}}\left(X^{h_{\nabla}}, Y^{h_{\nabla}}\right) & =\frac{1}{2} \operatorname{Tr} J\left(H_{X, Y}^{\nabla} J\right)+\frac{1}{2} \operatorname{Tr} A_{X} A_{Y}, \\
\operatorname{Ric}^{\tilde{\nabla}^{0}}\left(X^{h_{\nabla}}, U^{v}\right) & =\operatorname{Ric}^{\tilde{\nabla}^{0}}\left(U^{v}, X^{h}\right)=0, \\
\operatorname{Ric}^{\tilde{\nabla}^{0}}\left(U^{v}, V^{v}\right) & =\frac{1}{2} \operatorname{Tr} J\left(H_{U, V}^{\nabla} J\right)+\frac{1}{2} \operatorname{Tr} A_{U} A_{V}
\end{aligned}
$$

for $X, Y, U, V \in T N$. From $(\nabla J) J=-J(\nabla J)$, it holds that

$$
\operatorname{Tr} J\left(H_{X, Y}^{\nabla} J\right)+\operatorname{Tr} A_{X} A_{Y}=0
$$

for all $X, Y \in T N$. Therefore the Obata connection of the manifolds obtained from our hypercomplex version of the c-map is Ricci flat.

Remark 6.6. The horizontal distribution on $M$ is integrable by (6.4) and each leaf is totally geodesic with respect to the Obata connection $\tilde{\nabla}^{0}$, since $\tilde{\nabla}_{X^{h} \nabla}^{0} Y^{h}{ }_{\nabla}=\left(\nabla_{X}^{\prime} Y\right)^{h}$ for $X, Y \in \Gamma(T N)$.

Remark 6.7. In [12, Theorem A], a hypercomplex structure was obtained on a neighborhood of the zero section of the tangent bundle of a complex manifold with a complex connection whose curvature is of type $(1,1)$. By contrast, our generalized rigid c-map gives a hypercomplex structure on the whole tangent bundle when the manifold is special complex.

## 7 The c-projective structure on a projective special complex manifold

In this section, we discuss projective special complex manifolds and obtain the cprojective Weyl curvature of a canonically induced c-projective structure. Let ( $N, J, \nabla, \xi$ ) be a conical special complex manifold. Since $L_{\xi} J=0$ and $L_{J \xi} J=0$, we obtain a complex structure $\bar{J}$ on the quotient $\bar{N}:=N /\langle\xi, J \xi\rangle$ if $\bar{N}$ is a smooth manifold.

Lemma 7.1. We have $L_{\xi} \nabla^{\prime}=0$ and $L_{J \xi} \nabla^{\prime}=0$.
Proof. By Lemmas 6.4 and 6.2, we have

$$
\begin{aligned}
\left(L_{\xi} \nabla^{\prime}\right)_{X} Y & =\left[\xi, \nabla_{X}^{\prime} Y\right]-\nabla_{[\xi, X]}^{\prime} Y-\nabla_{X}^{\prime}[\xi, Y] \\
& =\nabla_{\xi}^{\prime} \nabla_{X}^{\prime} Y-\nabla_{\nabla_{X}^{\prime} Y}^{\prime} \xi-\nabla_{[\xi, X]}^{\prime} Y-\nabla_{X}^{\prime} \nabla_{\xi}^{\prime} Y+\nabla_{X}^{\prime} \nabla_{Y}^{\prime} \xi \\
& =R_{\xi, X}^{\nabla^{\prime}} Y=0
\end{aligned}
$$

and

$$
\begin{aligned}
\left(L_{J \xi} \nabla^{\prime}\right)_{X} Y & =\left[J \xi, \nabla_{X}^{\prime} Y\right]-\nabla_{[J \xi, X]}^{\prime} Y-\nabla_{X}^{\prime}[J \xi, Y] \\
& =\nabla_{J \xi}^{\prime} \nabla_{X}^{\prime} Y-\nabla_{\nabla_{X}^{\prime} Y}^{\prime} J \xi-\nabla_{[J \xi, X]}^{\prime} Y-\nabla_{X}^{\prime} \nabla_{J \xi}^{\prime} Y+\nabla_{X}^{\prime} \nabla_{Y}^{\prime} J \xi \\
& =R_{J \xi, X}^{\nabla^{\prime}} Y=0
\end{aligned}
$$

for all $X, Y \in \Gamma(T N)$.
Recall [17] that a smooth curve $t \mapsto c(t)$ on a complex manifold $(M, J)$ is called $J$-planar with respect to a connection $\nabla$ if $\nabla_{c^{\prime}} c^{\prime} \in \operatorname{span}\left\{c^{\prime}, J c^{\prime}\right\}$. We say that torsion-free complex connections $\nabla^{1}$ and $\nabla^{2}$ on a complex manifold $(M, J)$ are c-projectively related [8] if they have the same $J$-planar curves. It is known that $\nabla^{1}$ and $\nabla^{2}$ are c-projectively related if and only if there exists a one-form $\theta$ on $M$ such that

$$
\nabla_{X}^{1} Y=\nabla_{X}^{2} Y+\theta(X) Y+\theta(Y) X-\theta(J X) J Y-\theta(J Y) J X
$$

for $X, Y \in \Gamma(T M)$. See [17] for example. This defines an equivalence relation on the space of torsion-free complex connections on $M$. The equivalence classes are called c-projective structures.

Definition 7.2. We call the complex manifold $(\bar{N}, \bar{J})$ a projective special complex manifold if $p_{N}:(N, J, \nabla, \xi) \rightarrow(\bar{N}, \bar{J})$ is a principal $\mathbb{C}^{*}$-bundle, where the principal $\mathbb{C}^{*}$-action is generated by the holomorphic vector field $\xi-\sqrt{-1} J \xi$.

Note that a projective special Kähler manifold is a Kähler quotient of a conical special Kähler manifold. Similarly, a projective special complex manifold carries an induced c-projective structure as follows.

Proposition 7.3. Any projective special complex manifold $(\bar{N}, \bar{J})$ carries a canonical c-projective structure.

Proof. Consider a connection form $\hat{\alpha}=\alpha-\sqrt{-1}(\alpha \circ J)$ of type $(1,0)$ on the principal $\mathbb{C}^{*}$-bundle $p_{N}: N \rightarrow \bar{N}$. (Note that any $\mathbb{C}^{*}$-invariant real one-form $\alpha$ such that $\alpha(\xi)=1$ is the real part of such a connection.) We have $T N=\operatorname{Ker} \hat{\alpha} \oplus\langle\xi, J \xi\rangle$, where $\operatorname{Ker} \hat{\alpha}$ is $J$-invariant. We denote the $\hat{\alpha}$-horizontal lift of $X \in \Gamma(T \bar{N})$ by $X^{h_{\alpha}}$. By Lemma 7.1, we can define $\bar{\nabla}^{\prime \alpha}$ by

$$
\begin{equation*}
\bar{\nabla}_{X}^{\prime \alpha} Y=p_{N *}\left(\nabla_{X^{h_{\alpha}}}^{\prime} Y^{h_{\alpha}}\right) \tag{7.1}
\end{equation*}
$$

for $X, Y \in \Gamma(T \bar{N})$. We claim that $\bar{\nabla}^{\prime \alpha} \bar{J}=0$. In fact, using that $J Y^{h_{\alpha}}=(\bar{J} Y)^{h_{\alpha}}$ for $Y \in T \bar{N}$ we have

$$
\bar{\nabla}_{X}^{\prime \alpha}(\bar{J} Y)=p_{N *}\left(\nabla_{X^{h_{\alpha}}}^{\prime} J Y^{h_{\alpha}}\right)=p_{N *} J\left(\nabla_{X^{h_{\alpha}}}^{\prime} Y^{h_{\alpha}}\right)=\bar{J} p_{N *}\left(\nabla_{X^{h_{\alpha}}}^{\prime} Y^{h_{\alpha}}\right)
$$

To show that the c-projective structure $\left[\bar{\nabla}^{\prime \alpha}\right]$ does not depend on $\alpha$, we consider another connection form $\hat{\beta}=\beta-\sqrt{-1}(\beta \circ J)$ of type $(1,0)$. Then there exist one-forms $\theta_{0}$ and $\theta_{1}$ on $\bar{N}$ such that

$$
\hat{\beta}-\hat{\alpha}=\left(p_{N}^{*} \theta_{0}\right)+\left(p_{N}^{*} \theta_{1}\right) \sqrt{-1}
$$

On the other hand, we can write $X^{h_{\alpha}}-X^{h_{\beta}}=a \xi+b J \xi$ for some functions $a, b$ on $N$. It is easy to see that

$$
a=\theta_{0}(X) \circ p_{N}, b=-\theta_{0}(\bar{J} X) \circ p_{N}, \theta_{1}=-\theta_{0} \circ \bar{J}
$$

for $X \in T \bar{N}$. By the definition (17.1) of the induced connection on $\bar{N}$, we have

$$
\begin{aligned}
\bar{\nabla}_{X}^{\prime \alpha} Y= & p_{N *}\left(\nabla_{X^{h_{\alpha}}}^{\prime} Y^{h_{\alpha}}\right) \\
= & p_{N *}\left(\nabla_{X^{h_{\beta}}+\theta_{0}(X) \xi-\theta_{0}(\bar{J} X) J \xi}\left(Y^{h_{\beta}}+\theta_{0}(Y) \xi-\theta_{0}(\bar{J} Y) J \xi\right)\right) \\
= & p_{N *}\left(\nabla_{X^{\prime}}^{\prime} Y^{h_{\beta}}+\nabla_{X^{h_{\beta}}}^{\prime} \theta_{0}(Y) \xi-\nabla_{X^{h_{\beta}}}^{\prime} \theta_{0}(\bar{J} Y) J \xi\right. \\
& \quad+\theta_{0}(X)\left(\nabla_{\xi}^{\prime} Y^{h_{\beta}}+\nabla_{\xi}^{\prime} \theta_{0}(Y) \xi-\nabla_{\xi}^{\prime} \theta_{0}(\bar{J} Y) J \xi\right) \\
& \quad-\theta_{0}(\bar{J} X)\left(\nabla_{J \xi}^{\prime} Y^{h_{\beta}}+\nabla_{J \xi}^{\prime} \theta_{0}(Y) \xi-\nabla_{J \xi}^{\prime} \theta_{0}(\bar{J} Y) J \xi\right) \\
& =\bar{\nabla}_{X}^{\prime \beta} Y+ \\
& \theta_{0}(Y) X+\theta_{0}(X) Y-\theta_{0}(\bar{J} Y) \bar{J} X-\theta_{0}(\bar{J} X) \bar{J} Y
\end{aligned}
$$

for $X, Y \in \Gamma(T \bar{N})$, which means that $\bar{\nabla}^{\prime \alpha}$ and $\bar{\nabla}^{\prime \beta}$ are c-projectively related. Here we write $\theta_{0}(X)$ for $\theta_{0}(X) \circ p_{N}$ etc.

We denote the induced c-projective structure given in Proposition 7.3 by $\mathcal{P}_{\bar{\nabla}^{\prime}}$ (without a label $\alpha$ ). Next we prove that the c-projective Weyl curvature of $\mathcal{P}_{\bar{\nabla}^{\prime}}$ is of type $(1,1)$ (see Theorem 7.10).

Note that $\xi, J \xi$ are the fundamental vector fields generated by $1, \sqrt{-1} \in \mathbb{C}=$ Lie $\mathbb{C}^{*}$, respectively. Recall that $A=\nabla J$ and $A_{\xi}=A_{J \xi}=0$. We also have that $L_{\xi} A=0$, since $L_{\xi} \nabla=0$ and $L_{\xi} J=0$.

Lemma 7.4. $L_{J \xi} \nabla=A, L_{J \xi} A=-2 J A$ and $L_{J \xi}(J A)=2 A$.

Let $\eta$ be a connection form on the principal bundle $p_{N}: N \rightarrow \bar{N}$. As before, we assume that Ker $\eta$ is $J$-invariant or, equivalently, that $\eta$ is of type $(1,0)$ (but not necessarily holomorphic). Using $\eta$ we can project the connection $\nabla^{\prime}$ on $N$ to a connection $\bar{\nabla}^{\prime \eta}$ on $\bar{N}$, which is complex with respect to $\bar{J}$, as shown in the proof of Proposition 7.3, Note that the quotient $p_{N}:\left(N, \nabla^{\prime}\right) \rightarrow\left(\bar{N}, \bar{\nabla}^{\prime \eta}\right)$ is an affine submersion with the horizontal distribution $\mathcal{H}:=\operatorname{Ker} \eta$ (in the sense defined in [1]). From now on the $\eta$-horizontal lift of $X \in T \bar{N}$ is denoted by $\tilde{X}$. Note that our sign convention for the curvature tensor is different from the one in [1]. Let $h: T N \rightarrow \mathcal{H}$ and $v: T N \rightarrow \mathcal{V}$ be the projections with respect to the decomposition $T N=\mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}=\operatorname{Ker} p_{N *}$. We define the fundamental tensors $\mathcal{A}^{\nabla^{\prime}}$ and $\mathcal{T}^{\nabla^{\prime}}$ by

$$
\mathcal{A}_{E}^{\nabla^{\prime}} F=v\left(\nabla_{h E}^{\prime} h F\right)+h\left(\nabla_{h E}^{\prime} v F\right)
$$

and

$$
\mathcal{T}_{E}^{\nabla^{\prime}} F=h\left(\nabla_{v E}^{\prime} v F\right)+v\left(\nabla_{v E}^{\prime} h F\right)
$$

for $E, F \in \Gamma(T N)$.
Lemma 7.5. We have $\mathcal{T}^{\nabla^{\prime}}=0, \mathcal{A}_{X}^{\nabla^{\prime}} \xi=X$ and $\mathcal{A}_{X}^{\nabla^{\prime}} J \xi=J X$ for any horizontal vector $X$.

Let $a$ and $b$ be ( 0,2 )-tensors defined by

$$
\mathcal{A}_{X}^{\nabla^{\prime}} Y=a(X, Y) \xi+b(X, Y) J \xi
$$

for horizontal vectors $X$ and $Y$. Since $\nabla^{\prime}$ and the projections $v, h$ are $\mathbb{C}^{*}$-invariant, $\mathcal{A}^{\nabla^{\prime}}$ is $\mathbb{C}^{*}$-invariant, and hence, $a=p_{N}^{*} \bar{a}$ and $b=p_{N}^{*} \bar{b}$ for some tensors $\bar{a}$ and $\bar{b}$ on $\bar{N}$. For any ( 0,2 )-tensor $k$ on a complex manifold with a complex structure $J$, define the $(0,2)$-tensor $k_{J}$ by $k_{J}(X, Y):=k(X, J Y)$.

Lemma 7.6. We have

$$
v\left(\left(\nabla_{\tilde{X}}^{\prime} J\right) \tilde{Y}\right)=(\bar{a}(X, \bar{J} Y)+\bar{b}(X, Y)) \xi+(\bar{b}(X, \bar{J} Y)-\bar{a}(X, Y)) J \xi
$$

for $X, Y \in T \bar{N}$.
Lemma 7.7. We have $\bar{b}(X, Y)=-\bar{a}(X, \bar{J} Y)=-\bar{a}_{\bar{J}}(X, Y)$ for tangent vectors $X$ and $Y$ on $\bar{N}$. Consequently, the fundamental tensor $\mathcal{A}^{\nabla^{\prime}}$ satisfies

$$
\begin{equation*}
\mathcal{A}_{\tilde{X}}^{\nabla^{\prime}} \tilde{Y}=\bar{a}(X, Y) \xi-\bar{a}_{\bar{J}}(X, Y) J \xi \tag{7.2}
\end{equation*}
$$

for tangent vectors $X, Y$ on $\bar{N}$.
Proof. By $\nabla^{\prime} J=0$ and Lemma 7.6, we have the conclusion.
Let $(r, \theta)$ be the polar coordinates with respect to a (smooth) local trivialization $p_{N}^{-1}(\bar{U}) \cong \bar{U} \times \mathbb{C}^{*}$ of the principal $\mathbb{C}^{*}$-bundle $p_{N}: N \rightarrow \bar{N}$ such that $\xi=r \partial / \partial r$ and $J \xi=\partial / \partial \theta$. A principal connection $\eta$ is locally given by

$$
\eta:=\eta_{1} \otimes 1+\eta_{2} \otimes \sqrt{-1}=p_{N}^{*}\left(\gamma_{1} \otimes 1+\gamma_{2} \otimes \sqrt{-1}\right)+\left(\frac{d r}{r} \otimes 1+d \theta \otimes \sqrt{-1}\right)
$$

for a $\mathbb{C}$-valued one-form $\gamma_{1} \otimes 1+\gamma_{2} \otimes \sqrt{-1}$ on $\bar{U} \subset \bar{N}$. For each local trivialization $p_{N}^{-1}(\bar{U}) \cong \bar{U} \times \mathbb{C}^{*}$, we set

$$
B:=e^{2 \theta J} A\left(e^{2 \theta J}=(\cos 2 \theta) \mathrm{id}+(\sin 2 \theta) J\right)
$$

The symmetric (1,2)-tensor field $B$ is defined locally and $B$ is projectable by Lemma 7.4, i.e. horizontal (i.e. $B_{\xi}=B_{J \xi}=0$ ) and $\mathbb{C}^{*}$-invariant. Therefore we obtain an induced locally defined symmetric tensor field $\bar{B}$ on $\bar{N}$.

Lemma 7.8. The tensor $B^{2}:(X, Y) \mapsto B_{X} \circ B_{Y}$ is a globally defined tensor field on $N$, in particular, $[B, B]$ is so. As a consequence, we have the globally defined tensor fields $\bar{B}^{2}$ and $[\bar{B}, \bar{B}]$ on $\bar{N}$.
Proof. It follows from $B^{2}=A^{2}$.
For a ( 0,2 )-tensor $a$ and a (1,1)-tensor $K$, we define an $\operatorname{End}(T N)$-valued 2-form $a \wedge K$ by

$$
(a \wedge K)_{X, Y} Z=a(X, Z) K Y-a(Y, Z) K X
$$

for tangent vectors $X, Y$ and $Z$.
Proposition 7.9. The curvature $R^{\bar{\nabla}^{\prime \eta}}$ of $\bar{\nabla}^{\prime \eta}$ is of the form

$$
R^{\bar{\nabla}^{\prime \eta}}=-\frac{1}{4}[\bar{B}, \bar{B}]+2 \bar{a}^{a} \otimes I d-2\left(\bar{a}_{\bar{J}}\right)^{a} \otimes \bar{J}+\bar{a} \wedge I d-\bar{a}_{\bar{J}} \wedge \bar{J}
$$

where $(\cdot)^{a}$ denotes anti-symmetrization. Moreover we have $d \gamma_{1}=-2 \bar{a}^{a}$ and $d \gamma_{2}=$ $2\left(\bar{a}_{\bar{J}}\right)^{a}$.

Proof. By the fundamental equation for an affine submersion [1], we have

$$
\left(R_{X, Y}^{\bar{\nabla}^{\prime \prime} \eta} Z\right)^{\sim}=h\left(R_{\tilde{X}, \tilde{Y}}^{\nabla^{\prime}} \tilde{Z}\right)+h\left(\nabla_{v[\tilde{X}, \tilde{Y}]}^{\prime} \tilde{Z}\right)+\mathcal{A}_{\tilde{Y}}^{\nabla^{\prime}} \mathcal{A}_{\tilde{X}}^{\nabla^{\prime}} \tilde{Z}-\mathcal{A}_{\tilde{X}}^{\nabla^{\prime}} \mathcal{A}_{\tilde{Y}}^{\nabla^{\prime}} \tilde{Z}
$$

for $X, Y, Z \in \Gamma(T \bar{N})$. Since

$$
\begin{aligned}
v[\tilde{X}, \tilde{Y}] & =\eta_{1}([\tilde{X}, \tilde{Y}]) \xi+\eta_{2}([\tilde{X}, \tilde{Y}]) J \xi \\
& =-\left(d \eta_{1}\right)(\tilde{X}, \tilde{Y}) \xi-\left(d \eta_{2}\right)(\tilde{X}, \tilde{Y}) J \xi \\
& =-\left(d \gamma_{1}\right)(X, Y) \xi-\left(d \gamma_{2}\right)(X, Y) J \xi
\end{aligned}
$$

we have

$$
\begin{aligned}
h\left(\nabla_{v[\tilde{X}, \tilde{Y}]}^{\prime} \tilde{Z}\right) & =h\left(\nabla_{\tilde{Z}}^{\prime} v[\tilde{X}, \tilde{Y}]\right) \\
& =h\left(\nabla_{\tilde{Z}}^{\prime}\left(-\left(d \gamma_{1}\right)(X, Y) \xi-\left(d \gamma_{2}\right)(X, Y) J \xi\right)\right) \\
& =-\left(d \gamma_{1}\right)(X, Y) \tilde{Z}-\left(d \gamma_{2}\right)(X, Y)(\bar{J} Z)
\end{aligned}
$$

Moreover, by
we have $d \gamma_{1}=-2 \bar{a}^{a}$ and $d \gamma_{2}=2\left(\bar{a}_{\bar{J}}\right)^{a}$.

Now we set $\operatorname{dim} N=2(n+1)$. By Proposition 7.9 and $\operatorname{Tr} \bar{B}_{X}=0$ for all $X \in T \bar{N}$, we obtain

$$
\begin{align*}
\operatorname{Ric}^{\bar{\nabla}^{\prime \eta}}(Y, Z)= & \frac{1}{4} \operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}+(\bar{a}(Z, Y)-\bar{a}(Y, Z))  \tag{7.3}\\
& -(\bar{a}(\bar{J} Y, \bar{J} Z)+\bar{a}(Y, Z))-2 n \bar{a}(Y, Z)+\bar{a}(Y, Z)-\bar{a}(\bar{J} Y, \bar{J} Z) \\
= & \frac{1}{4} \operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}-(2 n+1) \bar{a}(Y, Z)+\bar{a}(Z, Y) \\
& -\bar{a}(\bar{J} Y, \bar{J} Z)-\bar{a}(\bar{J} Z, \bar{J} Y) .
\end{align*}
$$

We define a $(0,2)$-tensor $P^{D}$ on a complex manifold $(M, J)$, which is called the Rho tensor of a connection $D$, by

$$
P^{D}(X, Y)=\frac{1}{m+1}\left(\operatorname{Ric}^{D}(X, Y)+\frac{1}{m-1}\left(\left(\operatorname{Ric}^{D}\right)^{s}(X, Y)-\left(\operatorname{Ric}^{D}\right)^{s}(J X, J Y)\right)\right)
$$

for $X, Y \in T M$, where $2 m=\operatorname{dim} M \geq 4, R i c^{D}$ is the Ricci tensor of $D$ and $(\cdot)^{s}$ is the symmetrization of a $(0,2)$-tensor. The c-projective Weyl curvature $W^{c,[\bar{D}]}$ of a c-projective structure $[\bar{D}]$ is given by

$$
\begin{equation*}
W^{c,[\bar{D}]}=R^{\bar{D}}+\left(P^{\bar{D}}\right)^{a} \otimes I d-\left(P_{\bar{J}}^{\bar{D}}\right)^{a} \otimes \bar{J}+\frac{1}{2} P^{\bar{D}} \wedge I d-\frac{1}{2} P_{\bar{J}}^{\bar{D}} \wedge \bar{J} . \tag{7.4}
\end{equation*}
$$

See [8]. We shall compute the c-projective Weyl curvature of $\left[\bar{\nabla}^{\prime \eta}\right]$. From (7.3) it holds

$$
\begin{aligned}
\left(\operatorname{Ric}^{\overline{\bar{~}}^{\prime \eta}}\right)^{s}(Y, Z) & =\frac{1}{4} \operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}-2 n \bar{a}^{s}(Y, Z)-2 \bar{a}^{s}(\bar{J} Y, \bar{J} Z), \\
\left(\operatorname{Ric}^{\bar{\nabla}^{\prime \eta}}\right)^{s}(\bar{J} Y, \bar{J} Z) & =\frac{1}{4} \operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}-2 n \bar{a}^{s}(\bar{J} Y, \bar{J} Z)-2 \bar{a}^{s}(Y, Z)
\end{aligned}
$$

and hence

$$
\left(\operatorname{Ric}^{\bar{\nabla}^{\prime \prime} \eta}\right)^{s}(Y, Z)-\left(\operatorname{Ric}^{\bar{\nabla}^{\prime \eta}}\right)^{s}(\bar{J} Y, \bar{J} Z)=-2(n-1)\left(\bar{a}^{s}(Y, Z)-\bar{a}^{s}(\bar{J} Y, \bar{J} Z)\right)
$$

From these equations, it follows that

$$
\begin{aligned}
(n+1) P^{\overline{\bar{~}}^{\prime \prime}}(Y, Z)= & \frac{1}{4} \operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}-(2 n+1) \bar{a}(Y, Z)+\bar{a}(Z, Y)-\bar{a}(\bar{J} Y, \bar{J} Z)-\bar{a}(\bar{J} Z, \bar{J} Y) \\
& -2\left(\bar{a}^{s}(Y, Z)-\bar{a}^{s}(\bar{J} Y, \bar{J} Z)\right) \\
= & \frac{1}{4} \operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}-(2 n+1) \bar{a}(Y, Z)+\bar{a}(Z, Y)-(\bar{a}(Y, Z)+\bar{a}(Z, Y)) \\
= & \frac{1}{4} \operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}-2(n+1) \bar{a}(Y, Z)
\end{aligned}
$$

Setting $\overline{\mathcal{B}}(Y, Z)=\operatorname{Tr} \bar{B}_{Y} \bar{B}_{Z}$, which is a symmetric, $\bar{J}$-hermitian globally defined (0,2)tensor on $\bar{N}$, we have

$$
\begin{equation*}
\bar{a}=\frac{1}{8(n+1)} \overline{\mathcal{B}}-\frac{1}{2} P^{\bar{\nabla}^{\prime} \eta} \tag{7.5}
\end{equation*}
$$

Therefore the coefficients of the curvature form $d \eta=d \gamma_{1}+\sqrt{-1} d \gamma_{2}=-2 \bar{a}^{a}+2 \sqrt{-1}\left(\bar{a}_{\bar{J}}\right)^{a}$ are determined by

$$
\begin{align*}
\bar{a}^{a} & =-\frac{1}{2}\left(P^{\bar{\nabla}^{\prime} \eta}\right)^{a}\left(=-\frac{1}{2(n+1)}\left(\operatorname{Ric}^{\bar{\nabla}^{\prime} \eta}\right)^{a}\right),  \tag{7.6}\\
\left(\bar{a}_{\bar{J}}\right)^{a} & =\frac{1}{8(n+1)} \overline{\mathcal{B}}_{\bar{J}}-\frac{1}{2}\left(P_{\bar{J}}^{\overline{\bar{J}}^{\prime \eta}}\right)^{a}\left(=\frac{1}{8(n+1)} \overline{\mathcal{B}}_{\bar{J}}-\frac{1}{2(n+1)}\left(\operatorname{Ric}_{\bar{J}}^{\overline{\bar{J}}^{\prime \eta}}\right)^{a}\right) . \tag{7.7}
\end{align*}
$$

By the above calculations we arrive at the following theorem.
Theorem 7.10. Let $(N, J, \nabla, \xi)$ be a conical special complex manifold which is the total space of a (holomorphic) principal $\mathbb{C}^{*}$-bundle $p_{N_{-}}: N \rightarrow \bar{N}$, the base of which is a projective special complex manifold $\bar{N}$ with $\operatorname{dim} \bar{N}=2 n \geq 4$. The c-projective Weyl curvature $W^{c, \mathcal{P}_{\bar{\nabla}^{\prime}}}$ of the canonically induced c-projective structure $\mathcal{P}_{\bar{\nabla}^{\prime}}$ is given by

$$
W^{c, \mathcal{P}_{\bar{\nabla}^{\prime}}}=-\frac{1}{4}[\bar{B}, \bar{B}]-\frac{1}{4(n+1)} \overline{\mathcal{B}}_{\bar{J}} \otimes \bar{J}+\frac{1}{8(n+1)} \overline{\mathcal{B}} \wedge \operatorname{Id}-\frac{1}{8(n+1)} \overline{\mathcal{B}}_{\bar{J}} \wedge \bar{J}
$$

In particular, $W_{\bar{J}(\cdot), \overline{\mathcal{V}}^{\prime}(\cdot)}^{c, \mathcal{P}_{\bar{\prime}}}=W^{c, \mathcal{P}_{\bar{\nabla}^{\prime}}}$, that is, $W^{c, \mathcal{P}_{\bar{\nabla}^{\prime}}}$ is of type $(1,1)$ as an $\operatorname{End}(T \bar{N})$-valued two-form.

Proof. Take a principal connection $\eta$ of type (1, 0). By Proposition 7.3, the canonically induced c-projective structure is $\left[\bar{\nabla}^{\prime \eta}\right]$. From Proposition [7.9, equation (7.4) and the symmetry of $\overline{\mathcal{B}}$, it holds that

$$
W^{c,\left[\bar{\nabla}^{\prime \eta}\right]}=-\frac{1}{4}[\bar{B}, \bar{B}]-\frac{1}{4(n+1)} \overline{\mathcal{B}}_{\bar{J}} \otimes \bar{J}+\frac{1}{8(n+1)} \overline{\mathcal{B}} \wedge \mathrm{Id}-\frac{1}{8(n+1)} \overline{\mathcal{B}}_{\bar{J}} \wedge \bar{J}
$$

Since $\overline{\mathcal{B}}_{\bar{J}},[\bar{B}, \bar{B}]$ and $\overline{\mathcal{B}} \wedge \operatorname{Id}-\overline{\mathcal{B}}_{\bar{J}} \wedge \bar{J}$ are of type $(1,1), W^{c, \mathcal{P}_{\bar{\nabla}}}$ is of type $(1,1)$.
The following corollary is a direct consequence of Theorem 7.10.
Corollary 7.11. Any complex manifold $(\bar{N}, \bar{J})$ with a c-projective structure $\mathcal{P}$ such that $W^{c, \mathcal{P}}$ is not of type $(1,1)$ can not be realized as a projective special complex manifold whose canonical c-projective structure is $\mathcal{P}$.

## 8 A generalization of the supergravity c-map

The supergravity c-map associates a (pseudo-)quternionic Kähler manifold with any projective special Kähler manifold. In this section, we give a generalization of the supergravity c-map by using the results in previous sections. Let $(N, J, \nabla, \xi)$ be a conical special complex manifold and set $Z:=J \xi$.

Lemma 8.1. $2 Z^{h} \nabla$ is a rotating vector field on $T N$.

Proof. Since $L_{Z} J=0$ and $\nabla_{Z} J=0$ (cf. Lemma 6.2), we have $L_{Z^{h} \nabla} I_{1}=0$. Moreover we have

$$
\begin{aligned}
\left(L_{Z^{h_{\nabla}} I_{2}}\right)\left(X^{h_{\nabla}}+Y^{v}\right) & =[Z, Y]^{h_{\nabla}}-\left(\nabla_{Z} X\right)^{v}-\left(\nabla_{Z} Y\right)^{h_{\nabla}}+[Z, X]^{v} \\
& =-\left(\nabla_{Y} Z\right)^{h_{\nabla}}-\left(\nabla_{X} Z\right)^{v} \\
& =-(J Y)^{h_{\nabla}}-(J X)^{v} \\
& =-I_{3}\left(X^{h_{\nabla}}+Y^{v}\right)
\end{aligned}
$$

for all $X, Y \in \Gamma(T N)$.
Remark 8.2. By the equations for $\tilde{\nabla}^{0}$ in the proof of Theorem 6.5, we have

$$
\begin{aligned}
\tilde{\nabla}_{X^{h} \nabla}^{0} \xi^{h \nabla} & =\left(\nabla_{X}^{\prime} \xi\right)^{h_{\nabla}}=\left(\nabla_{X} \xi-\frac{1}{2} J A_{X} \xi\right)^{h_{\nabla}}=X^{h_{\nabla}} \\
\tilde{\nabla}_{X^{v}}^{0} \xi^{h_{\nabla}} & =-\frac{1}{2}\left(J A_{\xi} X\right)^{v}=0
\end{aligned}
$$

for $X \in T N$, when $(N, J, \nabla, \xi)$ is a conical special complex manifold.
We have the following theorem.
Theorem 8.3 (Generalized supergravity c-map). Let $(N, J, \nabla, \xi)$ be a $2(n+1)$-dimensional conical special complex manifold. Let $\Theta$ be a closed two-form on $M=T N$ such that $L_{Z^{M}} \Theta=0$, where $Z^{M}=2 Z^{h}$. Consider a $\mathrm{U}(1)$-bundle $\pi: P \rightarrow M$ over $M$ and $\eta$ a connection form whose curvature form is

$$
d \eta=\pi^{*}\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z^{M}} \Theta\right) \circ I_{1}\right)\right)
$$

Let $f$ be a smooth function on $M$ such that $d f=-\iota_{Z^{M}} \Theta$ and $f_{1}:=f-(1 / 2) \Theta\left(Z^{M}, I_{1} Z^{M}\right)$ does nowhere vanish. If $\tilde{\pi}: \tilde{M} \rightarrow \hat{M}$ and $\hat{\pi}: \hat{M} \rightarrow \bar{M}$ are submersions, we have an assignment from a 2n-dimensional projective special complex manifold ( $\bar{N}, \bar{J}, \mathcal{P}_{\bar{\nabla}^{\prime}}$ ) whose c-projective Weyl curvature is of type $(1,1)$ to a $4(n+1)$-dimensional quaternionic manifold

$$
\bar{M}(=\overline{T N})=\mathcal{C}_{(P, \eta)}\left(M,\left\langle I_{1}, I_{2}, I_{3}\right\rangle, Z^{M}, f, \Theta\right) / \mathcal{D}
$$

foliated by $(2 n+4)$-dimensional leaves such that $\bar{N}$ coincides with the space of its leaves.
Proof. By Theorem 4.1, Lemma 8.1 and Proposition 7.3, we have an assignment from a $2 n$-dimensional projective special complex manifold $\left(\bar{N}, \bar{J}, \mathcal{P}_{\bar{\nabla}^{\prime}}\right)$ to a $4(n+1)$-dimensional quaternionic manifold $\overline{T N}$. By virtue of Theorem 7.10, the c-projective Weyl curvature of $\mathcal{P}_{\bar{\nabla}}$, is of type $(1,1)$. Next we give a foliation on $\overline{T N}$ whose leaves space is $\bar{N}$. Set $\mathcal{L}:=\mathcal{V} \oplus\left\langle\xi^{h} \nabla, Z^{h} \nabla\right\rangle$, where $\mathcal{V}$ is the vertical distribution of $T(T N) \rightarrow T N$. The distribution $\mathcal{L}$ is $Z^{M}=2 Z^{h} \nabla_{\text {-invariant }}$ and integrable by (6.4). Therefore each leaf $L$ of $\mathcal{L}$ is a $Z^{M}=2 Z^{h_{\nabla}}$-invariant submanifold of $T N$. Consider the pull-back $\iota^{\#} P$ of $P$ by the inclusion $\iota: L \rightarrow T N$ with the bundle map $\iota \#: \iota^{\#} P \rightarrow P$ and $\tilde{L}:=\mathbb{H}^{*} \times \iota^{\#} P$. Since $V_{1}$ is tangent to $\tilde{L}$, then $\hat{L}:=\tilde{L} /\left\langle V_{1}\right\rangle$ is a submanifold $\hat{M}$. Moreover $V, \hat{I}_{1}(V), \hat{I}_{2}(V)$,
$\hat{I}_{3}(V)$ are tangent to $\hat{L}$ because $V$ is induced by $e_{0}^{R}$. Taking the quotient again, we obtain a submanifold $\bar{L}:=\hat{L} /\left\langle V, \hat{I}_{1}(V), \hat{I}_{2}(V), \hat{I}_{3}(V)\right\rangle$ on a quaternionic manifold $\overline{T N}$. Therefore the quaternionic manifold $\overline{T N}$ is foliated by $(2 n+4)$-dimensional leaves such that the space of its leaves $\bar{L}$ is the projective special complex manifold $\bar{N}$.

Remark 8.4. If we assume that $Z_{1}=\left(Z^{M}\right)^{h_{\eta}}+f_{1} X_{P}$ generates a free $\mathrm{U}(1)$-action on $P$ instead of assuming that $\tilde{\pi}: \tilde{M} \rightarrow \hat{M}$ and $\hat{\pi}: \hat{M} \rightarrow \bar{M}$ are submersions, we obtain the same result as in Theorem 8.3 (see Theorem 4.8).

Remark 8.5. Borówka and Calderbank have given a construction of a quaternionic manifold from a complex manifold of half the dimension with a c-projective structure, known as the quaternionic Feix-Kaledin construction [6]. Their construction generalizes the original construction [11, 20, which yields a hyper-Kähler structure on a neighborhood of the zero setion of any Kähler manifold. They also point out that this construction is a generalization of [12, Theorem A] (see [6, Proposition 5.4]). More precisely, the initial data of the quaternionic Feix-Kaledin construction are a complex manifold with a c-projective structure of type $(1,1)$ and a complex line bundle with a connection of type $(1,1)$. Note that this construction is different from our generalization of the supergravity c-map, in which the real dimension of the quaternionic manifold $\overline{T N}$ is related to the real dimension of the projective special complex manifold $\bar{N}$ by $\operatorname{dim}(\overline{T N})=2 \operatorname{dim}(\bar{N})+4$.

We consider a conical special complex manifold $(N, J, \nabla, \xi)$, which we endow now with an additional structure. Let $\psi$ be a $J$-hermitian, $\nabla$-parallel two-form on $(N, J, \nabla, \xi)$. We consider a function $\mu=(1 / 2) \psi(\xi, J \xi)$ on $N$. Then we see $d \mu=-\iota_{Z} \psi$. Set

$$
\begin{align*}
\Theta & =-\pi_{T N}^{*} \psi  \tag{8.1}\\
f & =-2 \pi_{T N}^{*} \mu+c \tag{8.2}
\end{align*}
$$

for some constant $c$. Then it holds that

$$
d f=-\iota_{Z^{M}} \Theta, f_{1}=f-\frac{1}{2} \Theta\left(Z^{M}, I_{1} Z^{M}\right)=2 \pi_{T N}^{*} \mu+c
$$

where $\pi_{T N}: T N \rightarrow N$ is the bundle projection. In fact, we have

$$
d f=-2 d\left(\pi_{T N}^{*} \mu\right)=2 \pi_{T N}^{*}\left(\iota_{Z} \psi\right)=-\iota_{Z^{M}} \Theta
$$

and

$$
\begin{aligned}
f_{1} & =f-\frac{1}{2} \Theta\left(Z^{M}, I_{1} Z^{M}\right) \\
& =-\psi(\xi, J \xi) \circ \pi_{T N}-2 \Theta\left(Z^{h_{\nabla}}, I_{1} Z^{h_{\nabla}}\right)+c \\
& =\psi(J \xi, \xi) \circ \pi_{T N}+c=2 \pi_{T N}^{*} \mu+c .
\end{aligned}
$$

Corollary 8.6. Let $(N, J, \nabla, \xi)$ be a $2 n$-dimensional conical special complex manifold and $\psi$ a J-hermitian, $\nabla$-parallel two-form on $N$. Consider a $\mathrm{U}(1)$-bundle $\pi: P \rightarrow M$ over $M=T N$ and $\eta$ a connection form whose curvature form is

$$
d \eta=\left(\pi_{T N} \circ \pi\right)^{*} \psi
$$

If $\tilde{\pi}: \hat{M} \rightarrow \hat{M}$ and $\hat{\pi}: \hat{M} \rightarrow \bar{M}$ are submersions and $\mu^{-1}(-c / 2)=\emptyset$, then the generalized supergravity c-map of Theorem 8.3 can be specialized to this setting such that the data $\Theta$ and $f$ are related to $\psi$ by equations (8.1) and (8.2).

Proof. By a straightforward calculation, we have $d\left(\left(\iota_{Z} \psi\right) \circ J\right)=2 \psi$. Then it is easy to check

$$
\begin{aligned}
d \eta & =\left(\pi_{T N} \circ \pi\right)^{*} \psi \\
& =\left(\pi_{T N} \circ \pi\right)^{*}\left(-\psi+d\left(\left(\iota_{Z} \psi\right) \circ J\right)\right) \\
& =\left(\pi_{T N} \circ \pi\right)^{*}\left(-\psi+\frac{1}{2} d\left(\left(\iota_{2 Z} \psi\right) \circ J\right)\right) \\
& =\pi^{*}\left(\Theta-\frac{1}{2} d\left(\left(\iota_{Z^{M}} \Theta\right) \circ I_{1}\right)\right)
\end{aligned}
$$

where $\Theta$ is the two-form given by (8.1). Since $d \psi=0$ and $\iota_{Z} \psi=-d \mu$, it holds $L_{Z^{M}} \Theta=$ 0 . The function $f_{1}=f-(1 / 2) \Theta\left(Z^{M}, I_{1} Z^{M}\right)$ does nowhere vanish by $\mu^{-1}(-c / 2)=\emptyset$. Therefore Theorem 8.3 leads to the conclusion.

Therefore a conical special complex manifold $(N, J, \nabla, \xi)$ with a $J$-hermitian, $\nabla$ parallel two-form $\psi$ such that $(1 / 2 \pi)[\psi] \in H_{D R}^{2}(N, \mathbb{Z})$ and $\mu=(1 / 2) \psi(\xi, J \xi)$ is not surjective gives rise to a quaternionic manifold of dimension $2 \operatorname{dim} N$ under a suitable choice of the constant $c$.

For $t \in \mathbb{R} / \pi \mathbb{Z}$, we define a connection $\nabla^{t}$ by $\nabla^{t}=e^{t J} \circ \nabla \circ e^{-t J}$, which is a special complex connection by [3, Proposition 1]. Moreover, by

$$
\nabla^{t}=\nabla-(\sin t) e^{t J}(\nabla J)
$$

([3, Lemma 1]), we see that $\nabla^{t}$ satisfies $\nabla^{t} \xi=\mathrm{id}$. . Therefore $\left\{\nabla^{t}\right\}_{t \in \mathbb{R} / \pi \mathbb{Z}}$ is a family of conical special complex connections if $\nabla J \neq 0$.

Lemma 8.7. If $\psi$ is J-hermitian and $\nabla$-parallel, then $\psi$ is $\nabla^{t}$-parallel.
Proof. Since $\nabla^{t}-\nabla$ is a linear combination of $\nabla J$ and $J(\nabla J)=-(\nabla J) J$, it suffices to remark that $\nabla \psi=0, J \cdot \psi=0$ and, hence, $\left(\nabla_{X} J\right) \cdot \psi=0$ for all $X$. Here the dot stands for the action on the tensor algebra by derivations.

Hence, Corollary 8.6 and Lemma 8.7 imply
Corollary 8.8. If $A(=\nabla J) \neq 0$, there exists an $(\mathbb{R} / \pi \mathbb{Z})$-family of quaternionic manifolds obtained from a conical special complex manifold with $\psi$ under the same assumptions of Corollary 8.6 by the $H / Q$-correspondence (for any chosen function $f$ in the construction).

Proof. By Lemma 8.7, $\nabla_{X}^{t} \psi=0$. Since $\left(N, J, \nabla^{t}, \xi\right)$ are conical special complex manifolds, we have the conclusion.

To give an example, we recall the (local) characterization of a conical special complex manifold [3]. Let $\left(\mathbb{C}^{n+1}, J\right)$ be the standard complex vector space and $U$ an open subset in $\mathbb{C}^{n+1}$ with the standard coordinate system $\left(z_{0}, \ldots, z_{n}\right)$. We consider a holomorphic one-form $\alpha=\sum F_{i} d z_{i}$ on $U$, which is also viewed as a holomorphic map $\phi=\phi_{\alpha}$ from $U$ to $\left(T^{*} U=U \times \mathbb{C}^{n+1} \subset\right) \mathbb{C}^{2(n+1)}$. If $\operatorname{Re} \phi: U \rightarrow \mathbb{R}^{2(n+1)}$ is an immersion, which is equivalent to $\phi$ being totally complex [3], then we can find an affine connection $\nabla$ such that $(U, J, \nabla)$ is a special complex manifold. In fact, we can take a local coordinate system

$$
\left(x_{0}:=\operatorname{Re} z_{0}, \ldots, x_{n}:=\operatorname{Re} z_{n}, y_{0}:=\operatorname{Re} F_{0}, \ldots, y_{n}:=\operatorname{Re} F_{n}\right)
$$

on $U$ induced by $\phi$ and a connection $\nabla$ defined by the condition that $\left(x_{0}, \ldots, x_{n}, y_{0}, \ldots, y_{n}\right)$ is affine. Moreover $\sum_{i=0}^{n} d x_{i} \wedge d y_{i}$ is $\nabla$-parallel symplectic form on $U$. In particular, if $\alpha=-\sum_{i=0}^{n} \sqrt{-1} z_{i} d z_{i}$, then the induced affine coordinate system coincides with the real coordinate system underlying the holomorphic coordinate system $\left(z_{0}, \ldots, z_{n}\right)$, hence $(U, J, \nabla)$ is trivial $(\nabla J=0)$ in that special case. In addition to being holomorphic and totally complex, we assume that $\phi$ is conical, which is equivalent to the condition that functions $F_{0}, \ldots, F_{n}$ are homogeneous of degree one, i.e. $F_{i}(\lambda z)=\lambda F_{i}(z)$ for all $\lambda$ near $1 \in \mathbb{C}^{*}$ and $z \in U$. Then $U$ is conical, that is, any conical holomorphic one-form $\phi$ such that $\operatorname{Re} \phi$ is an immersion on $U$ defines a conical special complex (and symplectic) manifold structure of complex dimension $n$. Conversely, any such manifold can be locally obtained in this way (see [3, Corollary 5]).

If we choose $\alpha=-\sum_{i=0}^{n} \sqrt{-1} z^{i} d z^{i}$ on $\mathbb{C}^{n+1} \backslash\{0\}$, then the generalized c-map associates an open submanifold of $\left(\mathbb{H}^{n+1}, Q\right)$ with the standard quaternionic structure $Q$ to the complex projective space $\left(\mathbb{C} P^{n}, J^{s t},\left[\nabla^{F S}\right]\right)$, where $J^{s t}$ is the standard complex structure and $\nabla^{F S}$ is the Levi-Civita connection of the Fubini-Study metric. Here we have chosen $\Theta=0$. We can also apply Corollary 8.6 by choosing the standard symplectic form as $\psi$. More generally, we have the following example.

Example 8.9. For a holomorphic function $g$ of homogeneous degree one, we consider the holomorphic 1-form

$$
\alpha=g d z_{0}-\sqrt{-1} \sum_{i=1}^{n} z_{i} d z_{i}
$$

on $U:=\left\{\left(z_{0}, z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n+1} \mid \operatorname{Im} g_{0} \neq 0\right\}$, where $g_{i}=\frac{\partial g}{\partial z_{i}}(i=0,1, \ldots, n)$. [Comment Vicente: we should perhaps use a different symbol for $F$ to avoid Note that $d \alpha \neq 0$ if there exists $i$ such that $g_{i} \neq 0(i \geq 1)$. Setting $z_{i}=u_{i}+\sqrt{-1} v_{i}(i=0,1, \ldots, n)$, we have

$$
\begin{aligned}
\left(x_{0}, \ldots, x_{n}, y_{0}, y_{1}, \ldots, y_{n}\right) & =\operatorname{Re} \phi\left(u_{0}, \ldots, u_{n}, v_{0}, \ldots, v_{n}\right) \\
& =\left(\operatorname{Re} z_{0}, \ldots, \operatorname{Re} z_{1}, \operatorname{Re} g, \operatorname{Re}\left(-\sqrt{-1} z_{1}\right), \ldots, \operatorname{Re}\left(-\sqrt{-1} z_{n}\right)\right) \\
& =\left(u_{0}, \ldots, u_{n}, \operatorname{Re} g, v_{1}, \ldots, v_{n}\right)
\end{aligned}
$$

Since its Jacobian matrix is given by

$$
\frac{\partial\left(x_{0}, \ldots, y_{n}\right)}{\partial\left(u_{0}, \ldots, v_{n}\right)}=\left(\begin{array}{cccccccc}
1 & & & 0 & 0 & \cdots & \cdots & 0 \\
& \ddots & & \vdots & \vdots & & & \vdots \\
& & 1 & 0 & 0 & \ldots & \ldots & 0 \\
\operatorname{Re} g_{0} & \ldots & \operatorname{Re} g_{n} & -\operatorname{Im} g_{0} & -\operatorname{Im} g_{1} & \ldots & \ldots & -\operatorname{Im} g_{n} \\
0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & & \vdots & \vdots & 0 & 1 & & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \ddots & \\
0 & \ldots & 0 & 0 & 0 & & & 1
\end{array}\right)
$$

we see that $\operatorname{Re} \phi$ is an immersion and we obtain a conical special complex structure on $U$. The coordinate vector fields of $\left(x_{0}, \ldots, y_{n}\right)$ are given by

$$
\begin{aligned}
\frac{\partial}{\partial x_{i}} & =\frac{\partial}{\partial u_{i}}+\frac{\operatorname{Re} g_{i}}{\operatorname{Im} g_{0}} \frac{\partial}{\partial v_{0}}(i \geq 0), \\
\frac{\partial}{\partial y_{0}} & =-\frac{1}{\operatorname{Im} g_{0}} \frac{\partial}{\partial v_{0}}, \quad \frac{\partial}{\partial y_{j}}=-\frac{\operatorname{Im} g_{j}}{\operatorname{Im} g_{0}} \frac{\partial}{\partial v_{0}}+\frac{\partial}{\partial v_{j}}(j \geq 1)
\end{aligned}
$$

on $U$. Let $\nabla$ (resp. $\nabla^{\text {st }}$ ) be the flat affine connection on $U$ such that $\left(x_{0}, \ldots, y_{n}\right)$ (resp. $\left.\left(u_{0}, \ldots, v_{n}\right)\right)$ is a $\nabla\left(\right.$ resp. $\left.\nabla^{\text {st }}\right)$-affine coordinate system. We define $S$ by $\nabla=\nabla^{\text {st }}+S$. Then we calculate

$$
\begin{aligned}
0 & =\nabla_{X} \frac{\partial}{\partial x_{i}}=\left(\nabla_{X}^{\mathrm{st}}+S_{X}\right)\left(\frac{\partial}{\partial u_{i}}+\frac{\operatorname{Re} g_{i}}{\operatorname{Im} g_{0}} \frac{\partial}{\partial v_{0}}\right) \\
& =X\left(\frac{\operatorname{Re} g_{i}}{\operatorname{Im} g_{0}}\right) \frac{\partial}{\partial v_{0}}+S_{X} \frac{\partial}{\partial u_{i}}+\frac{\operatorname{Re} g_{i}}{\operatorname{Im} g_{0}} S_{X} \frac{\partial}{\partial v_{0}}(i \geq 0)
\end{aligned}
$$

and similarly we have

$$
\begin{aligned}
& -X\left(\frac{1}{\operatorname{Im} g_{0}}\right) \frac{\partial}{\partial v_{0}}-\frac{1}{\operatorname{Im} g_{0}} S_{X} \frac{\partial}{\partial v_{0}}=0 \\
& -X\left(\frac{\operatorname{Im} g_{j}}{\operatorname{Im} g_{0}}\right) \frac{\partial}{\partial v_{0}}-\frac{\operatorname{Im} g_{j}}{\operatorname{Im} g_{0}} S_{X} \frac{\partial}{\partial v_{0}}+S_{X} \frac{\partial}{\partial v_{j}}=0(j>0)
\end{aligned}
$$

From these equations, it holds that

$$
\begin{equation*}
S_{X} \frac{\partial}{\partial u_{i}}=-\frac{X \operatorname{Re} g_{i}}{\operatorname{Im} g_{0}} \frac{\partial}{\partial v_{0}}, S_{X} \frac{\partial}{\partial v_{i}}=\frac{X \operatorname{Im} g_{i}}{\operatorname{Im} g_{0}} \frac{\partial}{\partial v_{0}}(i \geq 0) \tag{8.3}
\end{equation*}
$$

Using $A_{X} Y=\left(\nabla_{X} J\right)(Y)=S_{X} J Y-J S_{X} Y$ and (8.3), we have the matrix representation

$$
A=\nabla J=\frac{1}{\operatorname{Im} g_{0}}\left(\begin{array}{ccc}
A_{0} & \cdots & A_{n}  \tag{8.4}\\
0_{2} & \cdots & 0_{2} \\
\vdots & \ddots & \vdots \\
0_{2} & \cdots & 0_{2}
\end{array}\right)
$$

of $A$ with respect to the frame

$$
\left(\frac{\partial}{\partial u_{0}}, \frac{\partial}{\partial v_{0}}, \ldots, \frac{\partial}{\partial u_{n}}, \frac{\partial}{\partial v_{n}}\right),
$$

where

$$
A_{i}=\left(\begin{array}{cc}
-d \operatorname{Re} g_{i} & d \operatorname{Im} g_{i} \\
d \operatorname{Im} g_{i} & d \operatorname{Re} g_{i}
\end{array}\right) \quad \text { and } \quad 0_{2}=\left(\begin{array}{cc}
0 & 0 \\
0 & 0
\end{array}\right) .
$$

Note that we change the order of the frame for simplicity. This means that $A \neq 0$ if there exists $i$ such that $g_{i} \neq$ constant. By Lemma 7.8 and (8.4), $A^{2}=(\nabla J)^{2}$ induces a globally defined tensor on $\bar{U}$, in particular

$$
\operatorname{Tr} A^{2}=\operatorname{Tr} A_{0}^{2}=\frac{2}{\left(\operatorname{Im} g_{0}\right)^{2}}\left(d \operatorname{Re} g_{0} \otimes d \operatorname{Re} g_{0}+d \operatorname{Im} g_{0} \otimes d \operatorname{Im} g_{0}\right)
$$

also induces the the symmetric tensor $\overline{\mathcal{B}}$ on $\bar{U}$. By Lemma 6.4 and (8.4), we see that

$$
R^{\nabla^{\prime}}=-\frac{1}{4} A \wedge A=-\frac{1}{4\left(\operatorname{Im} g_{0}\right)^{2}}\left(\begin{array}{cccc}
A_{0} \wedge A_{0} & A_{0} \wedge A_{1} & \ldots & A_{0} \wedge A_{n} \\
0_{2} & 0_{2} & \ldots & 0_{2} \\
\vdots & \vdots & \ddots & \vdots \\
0_{2} & 0_{2} & \cdots & 0_{2}
\end{array}\right)
$$

as the matrix representation.
Since

$$
\begin{aligned}
& d x_{i}=d u_{i} \quad(i \geq 0), \quad d y_{0}=\sum_{i=0}^{n} \operatorname{Re} g_{i} d u_{i}-\operatorname{Im} g_{i} d v_{i}, \\
& d y_{j}=d v_{j} \quad(j>0)
\end{aligned}
$$

a 2-form $\psi=\sum_{i=1}^{n} d x_{i} \wedge d y_{i}\left(=\sum_{i=1}^{n} d u_{i} \wedge d v_{i}\right)$ is $J$-hermitian and $\nabla$-parallel. Note that $\sum_{i=0}^{n} d x_{i} \wedge d y_{i}=d x_{0} \wedge d y_{0}+\psi$ is not $J$-hermitian, that is, $\left(U, J, \nabla, d x_{0} \wedge d y_{0}+\psi\right)$ is not special Kählerian if there exists $i>0$ such that $\operatorname{Re} g_{i} \neq 0$. However it is a special symplectic manifold. In fact, it holds

$$
\left(d x_{0} \wedge d y_{0}\right)\left(\frac{\partial}{\partial u_{0}}, \frac{\partial}{\partial u_{i}}\right)=\operatorname{Re} g_{i} \text { and }\left(d x_{0} \wedge d y_{0}\right)\left(J \frac{\partial}{\partial u_{0}}, J \frac{\partial}{\partial u_{i}}\right)=0
$$

Moreover since

$$
\begin{aligned}
\xi & =\sum_{i=0}^{n} x_{i} \frac{\partial}{\partial x_{i}}+y_{i} \frac{\partial}{\partial y_{i}}=\cdots+\sum_{i=1}^{n} u_{i} \frac{\partial}{\partial u_{i}}+v_{i} \frac{\partial}{\partial v_{i}} \\
J \xi & =\cdots+\sum_{i=1}^{n} u_{i} \frac{\partial}{\partial v_{i}}-v_{i} \frac{\partial}{\partial u_{i}}
\end{aligned}
$$

we have $\mu=\psi(\xi, J \xi)=(1 / 2) \sum_{i=1}^{n}\left(u_{i}^{2}+v_{i}^{2}\right)$. Take a $\mathrm{U}(1)$-bundle $\pi: T U \times \mathrm{U}(1) \rightarrow T U$ with a connection form

$$
\eta=\left(\pi_{T U} \circ \pi\right)^{*}\left(\sum_{i=1}^{n} u_{i} d v_{i}\right)+d \theta
$$

where $\theta$ is the angular coordinate of $\mathrm{U}(1)$. The special case Corollary 8.6 of Theorem 8.3 can be applied and then we obtain a quternionic manifold.

We consider the horizontal subbundle of $p_{U}: U \rightarrow \bar{U}$ given by the kernel of $\kappa=$ $-(1 / 2 s) d \mu \circ J$ on each level set $\mu^{-1}(s) \subset U(s \neq 0)$. We retake $U$ as an open set in $\cup_{s>0} \mu^{-1}(s)$. For horizontal vector fields $X$ and $Y$ tangent to each level set $\mu^{-1}(s)$, $X Y \mu=0$ means that

$$
\left(p_{U}^{*} \bar{a}\right)(X, Y)=\frac{1}{2 s} \psi(J X, Y),
$$

where $\bar{a}$ is the $\xi$-component of the fundamental tensor of $\mathcal{A}^{\nabla^{\prime}}$ as in Section 7 . Here we used $d \kappa=\psi / s$. This means that $\bar{a}$ is symmetric and $\bar{J}$-hermitian, and hence the Ricci tensor of the connection $\bar{\nabla}^{\prime \kappa}$ on $\bar{U}$ induced from $\kappa$ is symmetric and $\bar{J}$-hermitian. Therefore it holds

$$
p_{U}^{*} \bar{a}=-\frac{1}{\sum_{i=1}^{n}\left(u_{i}^{2}+v_{i}^{2}\right)} \sum_{i=1}^{n} d u_{i} \otimes d u_{i}+d v_{i} \otimes d v_{i} .
$$

Hence the Ricci tensor $\operatorname{Ric}^{\bar{\nabla} / \kappa}$ of $\bar{\nabla}^{\prime \kappa}$ satisfies

$$
\begin{aligned}
& -\frac{1}{\sum_{i=1}^{n}\left(u_{i}^{2}+v_{i}^{2}\right)} \sum_{i=1}^{n} d u_{i} \otimes d u_{i}+d v_{i} \otimes d v_{i} \\
& =\frac{1}{4(n+1)\left(\operatorname{Im} g_{0}\right)^{2}}\left(d \operatorname{Re} g_{0} \otimes d \operatorname{Re} g_{0}+d \operatorname{Im} g_{0} \otimes d \operatorname{Im} g_{0}\right)-\frac{1}{2(n+1)} p_{\bar{U}}^{*}\left(R i c^{\bar{\nabla}^{\prime \kappa}}\right)
\end{aligned}
$$

by (7.5). In particular, we see that $R i c^{\bar{\nabla}^{\prime k}} \geq 0$. For example, when we choose $g=$ $-\sqrt{-1} z_{1}^{l} / z_{0}^{l-1}$ for $l(\neq 1) \in \mathbb{Z}$, we obtain

$$
\begin{aligned}
d \operatorname{Re} g_{0} & =\frac{\sqrt{-1}}{2}(-l+1) l\left(-w^{l-1} d w+\bar{w}^{l-1} d \bar{w}\right) \\
d \operatorname{Im} g_{0} & =-\frac{1}{2}(-l+1) l\left(w^{l-1} d w+\bar{w}^{l-1} d \bar{w}\right) \\
d \operatorname{Re} g_{1} & =\frac{\sqrt{-1}}{2}(-l+1) l\left(w^{l-2} d w-\bar{w}^{l-1} d \bar{w}\right) \\
d \operatorname{Im} g_{1} & =\frac{1}{2}(-l+1) l\left(w^{l-2} d w+\bar{w}^{l-2} d \bar{w}\right) \\
d \operatorname{Re} g_{j} & =d \operatorname{Im} g_{j}=0 \quad(j>1)
\end{aligned}
$$

where $w=z_{1} / z_{0}$. We denote the corresponding objects with subscript $l$ for ones given by $g=-\sqrt{-1} z_{1}^{l} / z_{0}^{l-1}$. It holds that

$$
R^{\nabla^{l \prime}}=-\frac{\sqrt{-1} l^{2}|w|^{2(l-2)}}{\left(w^{l}+\bar{w}^{l}\right)^{2}}\left(\begin{array}{ccccccc}
0 & -|w|^{2} & -\operatorname{Im} w & \operatorname{Re} w & 0 & \ldots & 0  \tag{8.5}\\
|w|^{2} & 0 & -\operatorname{Re} w & -\operatorname{Im} w & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0
\end{array}\right) d w \wedge d \bar{w}
$$

and

$$
\begin{equation*}
\operatorname{Tr}\left(A^{l}\right)^{2}=\operatorname{Tr}\left(\nabla^{l} J\right)^{2}=\frac{4 l^{2}|w|^{2(l-1)}}{\left(w^{l}+\bar{w}^{l}\right)^{2}}(d w \otimes d \bar{w}+d \bar{w} \otimes d w) \tag{8.6}
\end{equation*}
$$

Finally we consider the quaternionic Weyl curvature of $T U$. Let $W^{q}$ be the quaternionic Weyl curvature of the quaternionic structure $Q=\left\langle I_{1}, I_{2}, I_{3}\right\rangle$. In [5], the explicit expression of $W^{q}$ is given and it is shown that $W^{q}$ is independent of the choice of the quaternionic connection. Since the Obata connection of the c-map is Ricci flat by Theorem [6.5, we have $W^{q, l}=R^{\tilde{\nabla}^{0, l}}$ for $g=-\sqrt{-1} z_{1}^{l} / z_{0}^{l-1}$. If $l \neq 1$, then we see that

$$
W_{X^{v}, Y^{v}}^{q, l} Z^{v}=R_{X^{v}, Y^{v}}^{\tilde{\nabla}^{0, l}} Z^{v}=\left(R_{X, Y}^{\nabla^{l \prime}} Z\right)^{v}
$$

Because the vertical lift is determined by a differential manifold structure (not by a connection), we see that $W^{q, l} \neq W^{q, k}$ on $T\left(U_{k} \cap U_{l}\right)$ if $l \neq k$, where $U_{j}=\left\{\left(z_{0}, z_{1}, \ldots, z_{n}\right) \in\right.$ $\left.\mathbb{C}^{n+1} \mid \operatorname{Im} g_{0} \neq 0\right\}=\left\{\left(z_{0}, z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n+1} \mid \operatorname{Re}\left(z_{1} / z_{0}\right)^{j} \neq 0\right\}$ for $g=-\sqrt{-1} z_{1}^{j} / z_{0}^{j-1}$. Here we used (8.5). So we can find different quaternionic structures $Q^{\alpha_{1}}, \ldots, Q^{\alpha_{t}}$ on $T\left(\bigcap_{i=1}^{t} U_{\alpha_{i}}\right)$, where $1 \neq \alpha_{i} \in \mathbb{Z}$. Note that $Q^{0}$ is the flat quaternionic structure.

Remark 8.10. Since $d \alpha \neq 0$ except the trivial case $g=-\sqrt{-1} z_{0}$, Example 8.9 with $g=-\sqrt{-1} z_{1}^{l} / z_{0}^{l-1}(l \neq 0)$, which is local one, is not given by a local special Kählerian one.

Remark 8.11. For a conical special Kähler manifold $N$, the particular twist data which yields the quaternionic Kähler structure of the supergravity c-map on $T^{*} N \cong T N$ is given in [21, Lemma 5.1] in consistency with [4]. As we noted in the introduction, we also have a freedom in the choice of the data $\Theta$ etc. for our generalized supergravity c-map. For instance, the two form $\Theta$ can be chosen as trivial $(\Theta=0)$ or as in equation (8.1). For illustration, we can give yet another possible choice of $\Theta$. Assume that $\operatorname{dim} N \geq 6$. Let $\left\{\bar{U}_{\alpha}\right\}_{\alpha \in \Lambda}$ be an open covering of $\bar{N}$ with local trivializations $U_{\alpha}:=p_{N}^{-1}\left(\bar{U}_{\alpha}\right) \cong \bar{U}_{\alpha} \times \mathbb{C}^{*}$ and $g_{\alpha \beta}: \bar{U}_{\alpha} \cap \bar{U}_{\beta} \rightarrow \mathbb{C}^{*}$ be the corresponding transition functions. Let $\left(r_{\alpha}, \theta_{\alpha}\right)$ be the polar coordinates with respect to a (smooth) local trivialization $p_{N}^{-1}\left(\bar{U}_{\alpha}\right) \cong \bar{U}_{\alpha} \times \mathbb{C}^{*}$ for each $\alpha \in \Lambda$. A principal connection $\eta$ is locally given by

$$
\eta=p_{N}^{*}\left(\gamma_{1}^{\alpha} \otimes 1+\gamma_{2}^{\alpha} \otimes \sqrt{-1}\right)+\left(\frac{d r_{\alpha}}{r_{\alpha}} \otimes 1+d \theta_{\alpha} \otimes \sqrt{-1}\right)
$$

for a $\mathbb{C}$-valued one-form $\gamma_{1}^{\alpha} \otimes 1+\gamma_{2}^{\alpha} \otimes \sqrt{-1}$ on $\bar{U}_{\alpha} \subset \bar{N}$ for each $\alpha \in \Lambda$. If we write $g_{\alpha \beta}=e^{f_{\alpha \beta}^{1}+f_{\alpha \beta}^{2} \sqrt{-1}}$, then

$$
\begin{aligned}
& f_{\alpha \beta}^{1}+f_{\beta \gamma}^{1}-f_{\alpha \gamma}^{1}=0, \\
& f_{\alpha \beta}^{2}+f_{\beta \gamma}^{2}-f_{\alpha \gamma}^{2} \in 2 \pi \mathbb{Z}, \\
& \gamma_{\beta}^{1}-\gamma_{\alpha}^{1}=d f_{\alpha \beta}^{1} \\
& \gamma_{\beta}^{2}-\gamma_{\alpha}^{2}=d f_{\alpha \beta}^{2}
\end{aligned}
$$

Therefore we obtain a principal $\mathrm{U}(1)$-bundle $p_{S}: S \rightarrow \bar{N}$ with transition functions $e^{f_{\alpha \beta}^{2} \sqrt{-1}}: \bar{U}_{\alpha} \cap \bar{U}_{\beta} \rightarrow \mathrm{U}(1)$ and connection $\eta_{S}$ locally given by

$$
p_{S}^{*}\left(\gamma_{2}^{\alpha} \otimes \sqrt{-1}\right)+d \theta_{\alpha} \otimes \sqrt{-1}
$$

In fact, the collection $\left\{e^{f_{\alpha \beta}^{2} \sqrt{-1}}\right\}$ of local $U(1)$-valued functions satisfies the cocycle condtion and the collection $\left\{\gamma_{\alpha}\right\}$ of local $\sqrt{-1} \mathbb{R}$-valued one-forms satisfying $\gamma_{\beta}^{2}-\gamma_{\alpha}^{2}=d f_{\alpha \beta}^{2}$ defines a connection form $\eta_{S}$. By Proposition 7.9 and (7.7), its curvature $d \eta_{S}\left(=p_{S}^{*}\left(d \gamma_{2}^{\alpha}\right)\right)$ is $2\left(\bar{a}_{\bar{J}}\right)^{a}$, where $\left(\bar{a}_{\bar{J}}\right)^{a}$ is given by

$$
\left(\bar{a}_{\bar{J}}\right)^{a}=\frac{1}{8(n+1)} \overline{\mathcal{B}}_{\bar{J}}-\frac{1}{2}\left(P_{\bar{J}}^{\overline{\bar{J}}^{\prime}}\right)^{a}
$$

On $T N$, we choose the two-form $\Theta=2\left(p_{N} \circ \pi_{T N}\right)^{*}\left(\left(\bar{a}_{\bar{J}}\right)^{a}\right)$ and consider the pull-back connection $\left(p_{N \#} \circ \pi_{T N \#}\right)^{*} \eta_{S}$ on the pull-back bundle $P=\pi_{T N}{ }^{\#} p_{N}^{\#} S$. Since $\iota_{Z^{M}} \Theta=0$, we can see that the assumptions in Theorem 8.3 hold. It is left for future studies to find a canonical choice of $\Theta$ for the generalized supergravity c-map, which allows to invert the $\mathrm{H} / \mathrm{Q}$-correspondence of [10].

As an application of Theorem8.3, we have the following corollary by patching quaternionic manifolds locally constructed by the generalized supergravity c-maps.

Corollary 8.12. Let $(M, J,[\nabla])$ be a complex manifold with a c-projective structure $[\nabla]$ and $\operatorname{dim} M=2 n$. If $2 n=\operatorname{dim} M \geq 4$ and the harmonic curvature of its normal Cartan connection vanishes, then there exists a $4(n+1)$-dimensional quaternionic manifold $(\check{M}, Q)$ with the vanishing quaternionic Weyl curvature foliated by $(n+2)$-dimensional complex manifolds whose leaves space is $M$.

Proof. Since $\operatorname{dim} M \geq 4$ and the harmonic curvature of its normal Cartan connection vanishes, $(M, J,[\nabla])$ is locally isomorphic to ( $\left.\mathbb{C} P^{n}, J^{s t},\left[\nabla^{F S}\right]\right)$ (see [8 for example). So we may assume that $M=\bigcup_{\alpha} U_{\alpha}$, where $U_{\alpha}$ is an open subset $\mathbb{C} P^{n}$. Set $V_{\alpha}:=$ $p^{-1}\left(U_{\alpha}\right)$, where $p: \mathbb{C}^{n+1} \backslash\{0\} \rightarrow \mathbb{C} P^{n}$ is the projection. We consider the standard complex structure and the standard flat connection induced from $\mathbb{C}^{n+1}$ on each $V_{\alpha}$. By Theorem 8.3, we have a quaternionic manifold $W_{\alpha}:=\varphi^{\prime}\left(T V_{\alpha}\right) \subset \mathbb{H}^{n+1}$, where $\varphi^{\prime}$ is the diffeomorphism given in Example 5.2. Here we have chosen the two-form $\Theta=0$ and $f=f_{1}=1$ on $T V_{\alpha}$ for each $\alpha$. We set $\check{M}:=\bigcup_{\alpha} W_{\alpha}$. The induced quaternionic structure on each $W_{\alpha}$ coincides with the standard one from $\mathbb{H}^{n+1}$. Hence an almost quaternion structure $Q$ on $\check{M}$ can be obtained. Since there exists a quaternionic connection on each $W_{\alpha}$, one can obtain a quaternionic connection on $\bar{M}$ by the partition of unity, that is, $Q$ is a quaternionic structure with vanishing quaternionic Weyl curvature. For each $p \in T V_{\alpha} \cap T V_{\beta}$, the leaf of $\mathcal{L}$ through $p$ in $T V_{\alpha}$ is denoted by $L^{\alpha}$ and corresponding leaf in $W_{\alpha}$ is denoted by $\hat{L}^{\alpha}$, that is $\hat{L}^{\alpha}=\varphi^{\prime}\left(L^{\alpha}\right)$. Since $\hat{L}^{\alpha}=\hat{L}^{\beta}$ in $\check{M}$, we obtain leaves in $\check{M}$ and see that its leaves space is $M$. Since the subbundle $\mathcal{L}$ is an $I_{1}$-invariant in $T\left(T V_{\alpha}\right)$, each leaf $L$ is a complex manifold with $I:=\left.I_{1}\right|_{L}$. Each leaf $\hat{L}$ on $\check{M}$ is obtained by the Swann's twist with an almost complex structure $\hat{I}$. By [27, Proposition 3.8] and $\Theta=0, \hat{I}$ is integrable.

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