The H/Q-correspondence and a generalization of the supergravity c-map

Vicente Cortés and Kazuyuki Hasegawa

July 21, 2022

Abstract

Given a hypercomplex manifold with a rotating vector field (and additional data), we construct a conical hypercomplex manifold. As a consequence, we associate a quaternionic manifold to a hypercomplex manifold of the same dimension with a rotating vector field. This is a generalization of the HK/QK-correspondence. As an application, we show that a quaternionic manifold can be associated to a conical special complex manifold of half its dimension. Furthermore, a projective special complex manifold (with a canonical c-projective structure) associates with a quaternionic manifold. The latter is a generalization of the supergravity c-map. We do also show that the tangent bundle of any special complex manifold carries a canonical Ricci-flat hypercomplex structure, thereby generalizing the rigid c-map.

2020 Mathematics Subject Classification : 53C10, 53C56, 53C26. Keywords : conical hypercomplex manifold, H/Q-correspondence, generalized supergravity c-map.

Contents

1	Introduction	2
2	Preliminaries	3
3	Conification of hypercomplex manifolds	5
4	The hypercomplex/quaternionic-correspondence	9
5	Examples of the H/Q -correspondence	13
6	The tangent bundle of a special complex manifold and a generalization of the rigid c-map	19

 $\mathbf{28}$

8 A generalization of the supergravity c-map

1 Introduction

The HK/QK-correspondence is a construction of a (pseudo-)quaternionic Kähler manifold from a (pseudo-)hyper-Kähler manifold of the same dimension with a rotating vector field (see Definition 3.1 and [15, 2, 16, 4]). This correspondence gives also the supergravity c-map, which associates a quaternionic Kähler manifold with a projective special Kähler manifold. The supergravity c-map was introduced in theoretical physics [13].

The inverse construction of the HK/QK-correspondence is called the QK/HK-correspondence. It has been generalized to a Q/H-correspondence, a construction of hypercomplex manifolds from quaternionic manifolds [10]. The purpose of this paper is to construct a quaternionic manifold from a hypercomplex manifold endowed with a rotating vector field and some extra data. We shall call this construction the hypercomplex/quaternionic-correspondence (H/Q-correspondence for short). We briefly explain how we obtain this correspondence. First we define the notion of a conical hypercomplex manifold (Definition 2.1). Next we construct a conical hypercomplex manifold M for every hypercomplex manifold M with a rotating vector field Z (Theorem 3.9) and additional data: a two-form Θ on M, a U(1)-bundle over M whose curvature satisfies (3.1) and a function f on M such that $df = -\iota_Z \Theta$. The manifold M is endowed with a free action of the Lie algebra $\operatorname{Lie} \mathbb{H}^* \cong \mathbb{R} \oplus \mathfrak{su}(2)$ and its quotient space M carries a quaternionic structure, provided that the quotient map $M \to M$ is a submersion. The H/Q-correspondence is then defined as $M \mapsto M$ (Theorems 4.1 and 4.8). In addition, we show that \overline{M} carries not only a quaternionic connection but also an (induced) affine quaternionic vector field (Proposition 4.7). Note that we give an example of our H/Q-correspondence from a hypercomplex Hopf manifold, which does not admit any hyper-Kähler structure (Example 5.3). Therefore the H/Q-correspondence is a proper generalization of the HK/QK-correspondence. Examples like hypercomplex or quaternionic Hopf manifolds show that hypercomplex and quaternionic manifolds arise naturally beyond the context of hyper-Kähler and quaternionic Kähler geometry. We refer to [25, 18, 19] for the theory of quaternionic manifolds and constructions of such manifolds.

The rigid c-map [9] allows to associate with a conical special Kähler manifold its cotangent bundle endowed with a hyper-Kähler structure with a rotating vector field [2]. In the absence of a metric, we show that the tangent bundle of a special complex manifold carries a canonical hypercomplex structure and that its Obata connection is Ricci flat (Theorem 6.5). In this way we establish a generalization of the rigid c-map which assigns a Ricci flat hypercomplex manifold to each special complex manifold. When the special complex manifold is conical, the resulting hypercomplex manifold is shown to admit a canonical rotating vector field (Lemma 8.1). The notion of a (conical) special complex manifold was introduced in [3]. It is a generalization of a (conical)

special Kähler manifold. We give a local example which does not arise as a special Kähler manifold (Example 8.9). In addition, we find many (different) quaternionic structures on the tangent bundle of a conical special complex manifold in this example (Example 8.9), using a generalization of the supergravity c-map.

As an application of our H/Q-correspondence, we indeed generalize the supergravity c-map by associating a quaternionic manifold with every conical special complex manifold and therefore with every projective special complex manifold (using the extra data involved in the H/Q-correspondence), see Theorem 8.3. It is shown in Proposition 7.3 that any projective special complex manifold possesses a canonical c-projective structure and in Theorem 7.10 that its c-projective Weyl curvature is of type (1, 1). So our generalized supergravity c-map can be formulated as associating a quaternionic manifold to a projective special complex manifold endowed with its canonical c-projective structure with c-projective Weyl curvature of type (1, 1). This addresses one of the questions raised in [6], where a different construction of quaternionic manifolds from c-projective structures was obtained, compare Remark 8.5.

In the special case of the HK/QK-correspondence, the two-form Θ , which is part of the data entering the H/Q-correspondence, is the Z-invariant Kähler form ω_1 in the hyper-Kähler-triple ($\omega_1, \omega_2, \omega_3$). However, in general, we have a freedom in the choice of Θ in the H/Q-correspondence (see Section 5). In particular we find two choices of Θ in Example 5.4 which yield different quaternionic structures on the resulting space. This shows that our H/Q-correspondence is not an inverse construction of the Q/Hcorrespondence without a further specification of Θ . It is left for future studies to find a suitable choice of Θ which gives an inverse construction.

We summarize our constructions in this paper as the following commutative diagram.

$$\begin{split} \hat{M}: \text{ conical hypercomplex} \\ \hat{M}: \text{ quaternionic} \\ \text{N: conical special complex} \\ \hat{N}: \text{ projective special complex} \\ & & \downarrow \text{U}(1) \\ & & \begin{pmatrix} N, J, \nabla, \xi \end{pmatrix}_{\text{Theorem 6.5}}^{\text{rigid c-map.}} (M = TN, f, \Theta)_{\text{Theorem 3.9}}^{\text{conification}} \hat{M} = \mathcal{C}_P(M) \\ & & & \ddots \\ \text{Theorem 6.5} \\ \text{Proposition 7.3} \\ & & \downarrow p_N \\ & & & \text{Theorem 8.1 and 4.8} \\ & & \ddots \\ & & & \downarrow \hat{\pi} \\ & & (\bar{N}, \bar{J}, \mathcal{P}_{\bar{\nabla}'}) \\ & & & \text{Theorem 8.3} \\ \end{split}$$

2 Preliminaries

Л Л

Throughout this paper, all manifolds are assumed to be smooth and without boundary and maps are assumed to be smooth unless otherwise mentioned. The space of sections of a vector bundle $E \to M$ is denoted by $\Gamma(E)$. In this section we introduce hypercomplex and quaternionic structures and derive some properties of conical hypercomplex manifolds.

We say that M is a quaternionic manifold with the quaternionic structure Q if Q is a subbundle of End(TM) of rank 3 which at every point $x \in M$ is spanned by endomorphisms $I_1, I_2, I_3 \in \text{End}(T_xM)$ satisfying

(2.1)
$$I_1^2 = I_2^2 = I_3^2 = -id, \ I_1I_2 = -I_2I_1 = I_3,$$

and there exists a torsion-free connection ∇ on M such that ∇ preserves Q, that is, $\nabla_X \Gamma(Q) \subset \Gamma(Q)$ for all $X \in \Gamma(TM)$. Such a torsion-free connection ∇ is called a **quaternionic connection** and the triplet (I_1, I_2, I_3) is called an **admissible frame** of Q at x. Note that we use the same letter ∇ for the connection on $\operatorname{End}(TM)$ induced by ∇ . The dimension of the quaternionic manifold M is denoted by 4n.

An almost hypercomplex manifold is defined to be a manifold M endowed with 3 almost complex structures I_1 , I_2 , I_3 satisfying the quaternionic relations (2.1). If I_1 , I_2 , I_3 are integrable, then M is called a hypercomplex manifold. There exists a unique torsion-free connection on a hypercomplex manifold for which the hypercomplex structures are parallel. It is called the Obata connection [22]. Obviously, hypercomplex manifolds are quaternionic manifolds with $Q = \langle I_1, I_2, I_3 \rangle$.

Definition 2.1. We say that a hypercomplex manifold $(M, (I_1, I_2, I_3))$ with a vector field V is conical if $\nabla^0 V = \text{id holds}$, where ∇^0 is the Obata connection. The vector field V is called the Euler vector field.

We state some lemmas for conical hypercomplex manifolds, which will be used later.

Lemma 2.2. Let $(M, (I_1, I_2, I_3), V)$ be a conical hypercomplex manifold. Then we have $L_V I_{\alpha} = 0$, $L_{I_{\alpha}V} I_{\alpha} = 0$ for $\alpha \in \{1, 2, 3\}$ and $L_{I_{\alpha}V} I_{\beta} = -2I_{\gamma}$ for any cyclic permutation (α, β, γ) .

Proof. The formulas follow immediately from $L_V = \nabla_V^0 - \nabla^0 V = \nabla_V^0 - \text{id}$ and $L_{I_{\alpha}V} = \nabla_{I_{\alpha}V}^0 - I_{\alpha}$.

For a connection ∇ and $X \in \Gamma(TM)$, we define

(2.2)
$$(L_X \nabla)_Y Z := L_X (\nabla_Y Z) - \nabla_{L_X Y} Z - \nabla_Y (L_X Z),$$

where $Y, Z \in \Gamma(TM)$. Note that $L_X \nabla$ is a tensor.

Lemma 2.3. Let $(M, (I_1, I_2, I_3), V)$ be a conical hypercomplex manifold. Then we have $L_V \nabla^0 = 0$ and $L_{I_{\alpha V}} \nabla^0 = 0$.

Proof. By Lemma 2.2, V and $I_{\alpha}V$ are quaternionic vector fields, namely $L_V\Gamma(Q) \subset \Gamma(Q)$ and $L_{I_{\alpha}V}\Gamma(Q) \subset \Gamma(Q)$, where $Q = \langle I_1, I_2, I_3 \rangle$. By [10, Proposition 4.2], it is enough to check $Ric^{\nabla^0}(V, \cdot) = 0$ and $Ric^{\nabla^0}(I_{\alpha}V, \cdot) = 0$. We have

$$Ric^{\nabla^0}(V,Y) = -Ric^{\nabla^0}(Y,V) = -\operatorname{Tr} R^{\nabla^0}(\cdot,Y)V = 0.$$

Here we used the skew-symmetry of the Ricci tensor of the Obata connection. It follows that also $Ric^{\nabla^0}(I_{\alpha}V, \cdot) = -Ric^{\nabla^0}(V, I_{\alpha} \cdot) = 0$, by the hermitian property of the Ricci tensor of the Obata connection.

Alternatively we could have used Lemma 2.2 and the explicit form of the Obata connection to check $L_{I_{\alpha V}}\nabla^0 = 0$. Note that $L_V\nabla^0 = 0$ follows from the uniqueness of the Obata connection, since the vector field V preserves the hypercomplex structure.

Example 2.4 (The Swann bundle). The principal $\mathbb{R}^{>0} \times SO(3)$ bundle over a quaternionic manifold, whose fibers consist of all volume elements and admissible frames at each point, possesses a hypercomplex structure (see [24, 10]). It is conical and is called the Swann bundle. The fundamental vector field generated by $c(\neq 0) \in T_1 \mathbb{R}^{>0} = \mathbb{R}$ is the Euler vector field, as can be easily checked from the explicit representation of the Obata connection (see [5] for example). In the notation of [10] with $\varepsilon = -1$ and c = -4(n+1), a basis of fundamental vector fields for the principal action is given by the vector fields $V = Z_0$ and $Z_{\alpha} = -I_{\alpha}Z_0$ with non-trivial commutators $[Z_{\alpha}, Z_{\beta}] = -2Z_{\gamma}$ and Lie derivatives $L_{Z_{\alpha}}I_{\beta} = -2I_{\gamma}$ for any cyclic permutation of $\{1, 2, 3\}$, where we have denoted by (I_1, I_2, I_3) the hypercomplex structure of the Swann bundle. Specializing to the Swann bundle $\mathbb{H}^*/\{\pm 1\}$ of a point, we see that Z_0 corresponds to 1 and (Z_1, Z_2, Z_3) to (i, j, k) in $T_1(\mathbb{H}^*/\{\pm 1\}) = T_1\mathbb{H} = \mathbb{H}$.

Lemma 2.5. On any conical hypercomplex manifold $(M, (I_1, I_2, I_3), V)$, the distribution $\mathcal{D} := \langle V, I_1 V, I_2 V, I_3 V \rangle$ on $\{x \in M \mid V_x \neq 0\}$ is integrable.

Proof. This follows from Lemma 2.2.

3 Conification of hypercomplex manifolds

The main result of this section is a construction of conical hypercomplex manifolds \hat{M} of dimension dim $\hat{M} = \dim M + 4$ from hypercomplex manifolds M with a rotating vector field.

Let M be a hypercomplex manifold of dimension 4n with a hypercomplex structure $H = (I_1, I_2, I_3)$.

Definition 3.1. A vector field Z on a hypercomplex manifold $(M, (I_1, I_2, I_3))$ is called rotating if $L_Z I_1 = 0$ and $L_Z I_2 = -2I_3$.

Note that if Z is rotating, then $L_Z I_3 = 2I_2$. In this section we will essentially show that by choosing a (local) primitive of the one-form $\iota_Z \Theta$ we can construct a conical hypercomplex manifold (\hat{M}, \hat{H}, V) for a hypercomplex manifold (M, H) with a rotating vector field Z and a closed two-form Θ such that $L_Z \Theta = 0$.

Let f be a smooth function on M such that $df = -\iota_Z \Theta$ and $f_1 := f - (1/2)\Theta(Z, I_1Z)$ is nowhere vanishing. Consider a principal U(1)-bundle $\pi : P \to M$ with a connection form η whose curvature form is

(3.1)
$$d\eta = \pi^* \left(\Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1) \right).$$

Since the curvature $d\eta$ is a basic form, we will usually identify it with its projection $\Theta - \frac{1}{2}d((\iota_Z \Theta) \circ I_1)$ on M. With this understood we have the following lemma, which follows immediately from the definition of f_1 .

 \Box

Lemma 3.2. $df_1 = -\iota_Z d\eta$.

Define a vector field Z_1 on P by $Z_1 = Z^{h_\eta} + (\pi^* f_1) X_P$, where Z^{h_η} is the η -horizontal lift and X_P is the fundamental vector field such that $\eta(X_P) = 1$. We will write f_1 for $\pi^* f_1$.

Remark 3.3. Note that $[X_P, Z_1] = 0$. Therefore if Z_1 generates a U(1)-action on P, then its action commutes with the principal action of $\pi : P \to M$.

Set $\tilde{M} = \mathbb{H}^* \times P$. Let $(e_0^R, e_1^R, e_2^R, e_3^R)$ (resp. $(e_0^L, e_1^L, e_2^L, e_3^L)$) be the right-invariant (resp. the left-invariant) frame of \mathbb{H}^* which coincides with (1, i, j, k) at $1 \in \mathbb{H}^*$. Note that $[e_1^R, e_2^R] = -2e_3^R$. We will use the same letter for vectors or vector fields canonically lifted to the product $\tilde{M} = \mathbb{H}^* \times P$ as for those on the factors \mathbb{H}^* and P. Set

$$V_1 := e_1^L - Z_1.$$

We denote the space of integral curves of V_1 by \hat{M} . We assume that the quotient map $\tilde{\pi}: \tilde{M} \to \hat{M}$ is a submersion. Note that "submersion" requires that the quotient space \hat{M} is smooth.

Lemma 3.4. We assume that the equation (3.1) holds. If $L_Z I_1 = 0$ and $L_Z \Theta = 0$, we have

$$L_{V_1}Y^{h_\eta} = -[Z,Y]^{h_\eta}$$

for all $Y \in \Gamma(TM)$.

Proof.

$$-L_{V_1}Y^{h_{\eta}} = -[e_1^L - Z_1, Y^{h_{\eta}}] = [Z_1, Y^{h_{\eta}}]$$

= $[Z^{h_{\eta}}, Y^{h_{\eta}}] + [f_1X_P, Y^{h_{\eta}}] = [Z^{h_{\eta}}, Y^{h_{\eta}}] - (Y^{h_{\eta}}f_1)X_P$
= $[Z, Y]^{h_{\eta}} + \eta([Z^{h_{\eta}}, Y^{h_{\eta}}])X_P - (Y^{h_{\eta}}f_1)X_P$
= $[Z, Y]^{h_{\eta}} - d\eta(Z, Y)X_P - (Yf_1)X_P$
= $[Z, Y]^{h_{\eta}},$

where we have used Lemma 3.2.

Note that

$$T_{(z,p)}\tilde{M} \cong T_z \mathbb{H}^* \oplus T_p P = \langle e_0^R, e_1^R, e_2^R, e_3^R \rangle_z \oplus \langle X_P \rangle_p \oplus \operatorname{Ker} \eta_p$$
$$= \langle V_1 \rangle_{(z,p)} \oplus \langle e_0^R, e_1^R, e_2^R, e_3^R \rangle_z \oplus \operatorname{Ker} \eta_p$$

for $(z, p) \in \mathbb{H}^* \times P$. We define three endomorphisms fields $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ on \tilde{M} of rank $\tilde{I}_{\alpha} = 4n + 4$ ($\alpha = 1, 2, 3$) as follows:

$$\widetilde{I}_{\alpha}V_{1} = 0, \ \widetilde{I}_{\alpha}e_{0}^{R} = e_{\alpha}^{R}, \ \widetilde{I}_{\alpha}e_{\alpha}^{R} = -e_{0}^{R}, \ \widetilde{I}_{\alpha}e_{\beta}^{R} = e_{\gamma}^{R}, \ \widetilde{I}_{\alpha}e_{\gamma}^{R} = -e_{\beta}^{R},
(\widetilde{I}_{\alpha})_{(z,p)}((Y^{h_{\eta}})_{(z,p)}) = ((I_{\alpha}')_{\pi(p)}(\pi_{*}Y))^{h_{\eta}}{}_{(z,p)}$$

for $Y \in T_p M$. Here I'_{α} is defined by

(3.2)
$$I'_{\alpha} = \sum_{\beta=1}^{3} A_{\alpha\beta} I_{\beta}$$

where $A = (A_{\alpha\beta}) \in \text{SO}(3)$ is the representation matrix of $\text{Ad}_z|_{\text{Im}\mathbb{H}}$ with respect to the basis (i, j, k). Note that $\text{Ker } \tilde{I}_{\alpha} = \langle V_1 \rangle$, $\text{Im } \tilde{I}_{\alpha} = T\mathbb{H}^* \oplus \text{Ker } \eta$ ($\alpha = 1, 2, 3$) and that $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ satisfy the quaternionic relations on $T\mathbb{H}^* \oplus \text{Ker } \eta$.

Lemma 3.5. $L_{e_0^R} \tilde{I}_{\alpha} = 0.$

Proof. The flow $\varphi_t : (z, p) \mapsto (e^t z, p)$ of e_0^R preserves the decomposition $\tilde{M} = \mathbb{H}^* \times P$ and acts trivially on the second factor. In particular, it preserves the distribution Ker η . The action on the first factor is tri-holomorphic with respect to the (standard) hypercomplex structure induced by (\tilde{I}_{α}) on \mathbb{H}^* . Since $\operatorname{Ad}_z = \operatorname{Ad}_{rz}$ for all r > 0, we also see that φ_t preserves the tensors $\tilde{I}_{\alpha}|_{\operatorname{Ker}\eta}$.

Lemma 3.6. If Z is rotating and $L_Z \Theta = 0$, then we have $L_{V_1} \tilde{I}_{\alpha} = 0$.

Proof. By the definition of \tilde{I}_{α} , it is easy to obtain $(L_{V_1}\tilde{I}_{\alpha})V_1 = 0$ and $(L_{V_1}\tilde{I}_{\alpha})e_{\delta}^R = 0$ $(\delta = 0, ..., 3)$. Moreover, by Lemma 3.4, we have

$$(L_{V_{1}}\tilde{I}_{\alpha})_{(z,p)}(Y^{h_{\eta}})$$

$$=[V_{1}, \tilde{I}_{\alpha}Y^{h_{\eta}}]_{(z,p)} - \tilde{I}_{\alpha}[V_{1}, Y^{h_{\eta}}]_{(z,p)}$$

$$=[e_{1}^{L}, \tilde{I}_{\alpha}Y^{h_{\eta}}]_{(z,p)} - [Z_{1}, \tilde{I}_{\alpha}Y^{h_{\eta}}]_{(z,p)} + \tilde{I}_{\alpha}[Z, Y]^{h_{\eta}}_{(z,p)}$$

$$=[e_{1}^{L}, \tilde{I}_{\alpha}Y^{h_{\eta}}]_{(z,p)} - [Z^{h}, \tilde{I}_{\alpha}Y^{h_{\eta}}]_{(z,p)} - [f_{1}X_{P}, \tilde{I}_{\alpha}Y^{h_{\eta}}]_{(z,p)} + (I'_{\alpha}[Z, Y])^{h_{\eta}}_{(z,p)}$$

$$=[e_{1}^{L}, \tilde{I}_{\alpha}Y^{h_{\eta}}]_{(z,p)} - ((L_{Z}I'_{\alpha})Y)^{h_{\eta}}_{(z,p)},$$

where we have used that $[Z_{\eta}^{h}, \tilde{I}_{\alpha}Y^{h_{\eta}}] + [f_{1}X_{P}, \tilde{I}_{\alpha}Y^{h_{\eta}}] = [Z, I_{\alpha}'Y]^{h_{\eta}} + \eta([Z, I_{\alpha}'Y])X_{P} - (I_{\alpha}'Y)(f_{1})X_{P} = [Z, I_{\alpha}'Y]^{h_{\eta}}$ at the point (z, p), by Lemma 3.2. Taking the flow φ_{t} generated by e_{1}^{L} , we have

$$[e_1^L, \tilde{I}_{\alpha} Y^{h_{\eta}}]_{(z,p)} = \sum_{\beta=1}^3 \left(\frac{d}{dt} \Big|_{t=0} A_{\alpha\beta}(t) \right) (I_{\beta} Y)^{h_{\eta}}{}_{(z,p)},$$

where

$$A(t) = (A_{\alpha\beta}(t)) = \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos 2t & \sin 2t\\ 0 & -\sin 2t & \cos 2t \end{pmatrix} \in SO(3),$$

is the matrix associated with $\varphi_t(z)$. On the other hand, we see that

$$L_Z I'_1 = -2A_{12}I_3 + 2A_{13}I_2,$$

$$L_Z I'_2 = -2A_{22}I_3 + 2A_{23}I_2,$$

$$L_Z I'_3 = -2A_{32}I_3 + 2A_{33}I_2$$

and hence

$$L_Z(I'_1, I'_2, I'_3) = (L_Z I'_1, L_Z I'_2, L_Z I'_3) = (I_1, I_2, I_3) \left(\frac{d}{dt} A(t)\right).$$

Therefore we have $(L_{V_1}\tilde{I}_{\alpha})_{(z,p)}(Y^{h_{\eta}}) = 0.$

By Lemma 3.6, we can define an almost hypercomplex structure $(\hat{I}_1, \hat{I}_2, \hat{I}_3)$ on \hat{M} satisfying $\tilde{\pi}_* \circ \tilde{I}_{\alpha} = \hat{I}_{\alpha} \circ \tilde{\pi}_*$.

Lemma 3.7. The almost hypercomplex structure $\hat{H} = (\hat{I}_1, \hat{I}_2, \hat{I}_3)$ is integrable, that is, (\hat{M}, \hat{H}) is a hypercomplex manifold.

Proof. Let \tilde{X} and \tilde{Y} be projectable vector fields on the total space of the submersion $\tilde{\pi}: \tilde{M} \to \hat{M}$ and denote by $X = \tilde{\pi}_* \tilde{X}, Y = \tilde{\pi}_* \tilde{Y}$ their projections. Then we have $\tilde{\pi}_*(N^{\tilde{I}_{\alpha}}(\tilde{X}, \tilde{Y})) = N^{\tilde{I}_{\alpha}}(X, Y)$, where $N^{\tilde{I}_{\alpha}}$ and $N^{\hat{I}_{\alpha}}$ are the Nijenhuis tensors of \tilde{I}_{α} and \hat{I}_{α} , respectively. Using that $\tilde{I}_{\alpha}V_1 = 0$ and $L_{V_1}\tilde{I}_{\alpha} = 0$ (Lemma 3.6) we see that $N^{\tilde{I}_{\alpha}}(V_1, \cdot) = 0$. Since $N^{\tilde{I}_{\alpha}}$ and $N^{\hat{I}_{\alpha}}$ are tensors, it is sufficient to show that the horizontal component of $N^{\tilde{I}_{\alpha}}(A, B)$ vanishes for sections A and B of $\langle e_0^R, e_1^R, e_2^R, e_3^R \rangle \oplus \text{Ker } \eta$. It is easy to see that $N^{\tilde{I}_{\alpha}}(e_a^R, e_b^R) = 0$ and $N^{\tilde{I}_{\alpha}}(e_a^R, X^{h_{\eta}}) = 0$, for all $a, b \in \{0, \ldots, 3\}$. So we only need to show that the horizontal component of $N^{\tilde{I}_{\alpha}}(R_0, e_1^R, e_2^R, e_3^R) \oplus \text{Ker } \eta$. It is given by

$$([X,Y] + I'_{\alpha}[X,I'_{\alpha}Y] + I'_{\alpha}[I'_{\alpha}X,Y] - [I'_{\alpha}X,I'_{\alpha}Y])^{h_{\eta}} = 0,$$

since (I'_1, I'_2, I'_3) is a hypercomplex structure on M, for every $z \in \mathbb{H}^*$.

Since $L_{V_1}e_0^R = 0$, we can define a vector field $V = \tilde{\pi}_*e_0^R$ on \hat{M} . Let $\hat{\nabla}^0$ be the Obata connection with respect to \hat{H} .

Lemma 3.8. We have $\hat{\nabla}^0 V = \mathrm{id}$.

Proof. Using the explicit representation of the Obata connection (see [5] for example) and Lemma 3.5, we have

$$12(\hat{\nabla}^0_{\tilde{\pi}_*Y}\tilde{\pi}_*e^R_0) = \tilde{\pi}_*\left(\sum_{(\alpha,\beta,\gamma)} (\tilde{I}_{\alpha}[\tilde{I}_{\beta}Y,e^R_{\gamma}] + \tilde{I}_{\alpha}[e^R_{\beta},\tilde{I}_{\gamma}Y]) + 2\sum_{\alpha=1}^3 \tilde{I}_{\alpha}[e^R_{\alpha},Y]\right),$$

where (α, β, γ) indicates sum over cyclic permutations of (1, 2, 3) and Y is a projectable vector field on \tilde{M} commuting with e_0^R . Evaluating the expression on $Y = e_a^R$ and $Y = U^{h_\eta}$, we obtain $12\tilde{\pi}_*Y$.

As a consequence, by Lemmas 3.7 and 3.8, we can conclude

Theorem 3.9 (Conification). Let M be a hypercomplex manifold with a hypercomplex structure $H = (I_1, I_2, I_3)$, a closed two-form Θ and a rotating vector field Z such that $L_Z \Theta = 0$. Let f be a smooth function on M such that $df = -\iota_Z \Theta$ and assume $f_1 :=$ $f - (1/2)\Theta(Z, I_1Z)$ does nowhere vanish. Consider a principal U(1)-bundle $\pi : P \to M$ with a connection form η whose curvature form is

$$d\eta = \pi^* \left(\Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1) \right).$$

If the quotient map $\tilde{\pi} : \tilde{M} \to \hat{M}$ is a submersion, then (\hat{M}, \hat{H}) is a conical hypercomplex manifold with the Euler vector field $V = \tilde{\pi}_* e_0^R$.

Remark 3.10. The assumption that $\tilde{\pi}$ is a submersion is always satisfied locally by considering local 1-parameter subgroup generated by V_1 , since the vector field V_1 has no zeros. Note that "submersion" requires that the quotient space is a smooth manifold.

We say that (M, H, V) is the confication of (M, H, Z, f, Θ) associated with (P, η) and denote it by $(\hat{M}, \hat{H}, V) = C_{(P,\eta)}(M, H, Z, f, \Theta)$ (or simply $\hat{M} = C_P(M)$ if there is no confusion).

4 The hypercomplex/quaternionic-correspondence

Building on the confication construction of the last section we will now construct a quaternionic manifold \overline{M} of dimension dim $\overline{M} = \dim M$ from a hypercomplex manifold M with rotating vector field. The resulting quaternionic manifold is endowed with a torsion-free quaternionic connection and an affine quaternionic vector field X.

The space of leaves of the integrable distribution $\mathcal{D} := \langle V, \hat{I}_1 V, \hat{I}_2 V, \hat{I}_3 V \rangle$ on \hat{M} is denoted by \overline{M} . We shall show that $\overline{M} = \mathcal{C}_P(M)/\mathcal{D}$ is a quaternionic manifold, which is the main theorem of this paper. In addition, we show that \overline{M} has a natural quaternionic connection $\overline{\nabla}$ and an affine quaternionic vector field X induced from the fundamental vector field X_P of $P \to M$.

Using Theorem 3.9 and a similar argument as in [24, Theorem 2.1], we prove Theorem 4.1.

Theorem 4.1 (H/Q-correspondence). Let M be a hypercomplex manifold with a hypercomplex structure $H = (I_1, I_2, I_3)$, a closed two-form Θ and a rotating vector field Z such that $L_Z \Theta = 0$. Let f be a smooth function on M such that $df = -\iota_Z \Theta$ and assume that $f_1 := f - (1/2)\Theta(Z, I_1Z)$ does nowhere vanish. Consider a principal U(1)-bundle $\pi: P \to M$ with a connection form η whose curvature form is

$$d\eta = \pi^* \left(\Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1) \right).$$

If both quotient maps $\tilde{\pi}: \tilde{M} \to \hat{M}$ and $\hat{\pi}: \hat{M} \to \bar{M}$ defined above are submersions, then there exists an induced quaternionic structure \bar{Q} on \bar{M} .

Proof. As we proved in Theorem 3.9, $\hat{M} = C_P(M)$ is a conical hypercomplex manifold with the hypercomplex structure $\hat{H} = (\hat{I}_1, \hat{I}_2, \hat{I}_3)$. Let $\varphi = \sum_{a=0}^{3} \varphi_a i_a$ $((i_0, i_1, i_2, i_3) =$ (1, i, j, k)) be the right-invariant Maurer-Cartan form on \mathbb{H}^* and extend it with the same letter to \tilde{M} as $\varphi|_{TP} = 0$. Set $\tilde{\theta}_0 = \varphi_0$. Since $L_{V_1}\tilde{\theta}_0 = 0$, we can define the one-form $\hat{\theta}_0$ on \hat{M} such that $\tilde{\theta}_0 = \tilde{\pi}^* \hat{\theta}_0$. We define $\hat{\theta}' = \hat{\theta}_0 + \sum_{\alpha=1}^{3} (\hat{\theta}_0 \circ \hat{I}_\alpha) i_\alpha$ and take the Euler vector field V on \hat{M} as in Theorem 3.9. Here define an \hat{I}_{α} -invariant distribution

$$\hat{\mathcal{H}} := \operatorname{Ker} \hat{\theta}'.$$

It holds that $T\hat{M} = \mathcal{D} \oplus \hat{\mathcal{H}}$. Since $L_V \hat{\theta}' = 0$ and $L_{\hat{I}_{\alpha V}} \hat{\theta}' = 2(\hat{\theta}_0 \circ \hat{I}_{\beta})i_{\gamma} - 2(\hat{\theta}_0 \circ \hat{I}_{\gamma})i_{\beta}$ for any cyclic permutation (α, β, γ) (these are checked by straightforward calculations), the distribution $\hat{\mathcal{H}}$ is invariant along leaves of \mathcal{D} . Since $\hat{\pi}$ is a submersion, there exist a neighborhood $\mathcal{U} \subset \overline{M}$ of $x \in \overline{M}$ and a section $s : \mathcal{U} \to \hat{M}$. Then we can define

$$\bar{I}_{\alpha}(Y) := \hat{\pi}_*(\hat{I}_{\alpha}(Y^{h_{\hat{\theta}'}}_{s(y)}))$$

for $y \in \mathcal{U}$, where $Y \in T_y \overline{M}$ and $Y^{h_{\hat{\theta}'}}$ is the $\hat{\theta}'$ -horizontal lift of Y with respect to $\hat{\mathcal{H}}$. Although each \overline{I}_{α} depends on the sections, the subbundle $\overline{Q} = \langle \overline{I}_1, \overline{I}_2, \overline{I}_3 \rangle \subset \operatorname{End}(T\overline{M})$ is independent of the section by Lemma 2.2. This means that $(\overline{M}, \overline{Q})$ is an almost quaternionic manifold.

Next we show that there exists a torsion-free connection which preserves \bar{Q} . We define a connection $\bar{\nabla}$ on \bar{M} by

(4.1)
$$\bar{\nabla}_Y W = \hat{\pi}_* (\hat{\nabla}^0_{Y^h_{\hat{\theta}'}} W^{h_{\hat{\theta}'}}), \quad Y, W \in \Gamma(T\bar{M}),$$

where $\hat{\nabla}^0$ is the Obata connection of \hat{M} . Note that $\bar{\nabla}$ is well-defined by Lemma 2.3. Since the Obata connection is torsion-free, then so is $\bar{\nabla}$. To show that $\bar{\nabla}$ preserves \bar{Q} , we consider $I \in \Gamma(\bar{Q})$. Then $(IW)^{h_{\hat{\theta}'}} = \sum_{\alpha=1}^{3} a_{\alpha} \hat{I}_{\alpha} W^{h_{\hat{\theta}'}}$ for some functions a_{α} with $\sum_{\alpha=1}^{3} a_{\alpha}^2 = 1$, which implies

$$(\bar{\nabla}_Y I)W = \hat{\pi}_* (\sum_{\alpha=1}^3 (Y^{h_{\hat{\theta}'}} a_\alpha) \hat{I}_\alpha W^{h_{\hat{\theta}'}}),$$

showing that $\overline{\nabla}$ preserves \overline{Q} . Therefore $(\overline{M}, \overline{Q})$ is a quaternionic manifold.

Remark 4.2. The assumption that $\hat{\pi}$ is a submersion is always satisfied locally.

Next we shall show that our construction induces a vector field X which is an affine quaternionic vector field of $(\overline{M}, \overline{Q}, \overline{\nabla})$, where $\overline{\nabla}$ is given by (4.1).

Lemma 4.3. We have $L_{V_1}X_P = 0$ and $L_{X_P}I_{\alpha} = 0$.

Proof. The first equation can be checked by a straightforward calculation. The second follows from $[X_P, \tilde{I}_{\alpha}Y^{h_{\eta}}] = [X_P, (I'_{\alpha}Y)^{h_{\eta}}] = 0.$

By Lemma 4.3, we can define a vector field $\widehat{X_P} := \widetilde{\pi}_* X_P$ on \widehat{M} . Moreover $\widehat{X_P}$ satisfies the following.

Lemma 4.4. We have $L_{\widehat{X_P}}\hat{I}_{\alpha} = 0$, in addition, $L_{\widehat{X_P}}\hat{\nabla}^0 = 0$.

Proof. The first claim follows from Lemma 4.3, as $(L_{\widehat{X_P}}\hat{I}_{\alpha}) \circ \tilde{\pi}_* = \tilde{\pi}_* \circ (L_{X_P}\tilde{I}_{\alpha})$. Since the Obata connection is uniquely determined by the hypercomplex structure, we have $L_{\widehat{X_P}}\hat{\nabla}^0 = 0$ by the invariance of the hypercomplex structure $(\hat{I}_1, \hat{I}_2, \hat{I}_3)$ under $\widehat{X_P}$.

The next two lemmas follow respectively from $[e_a^R, X_P] = 0$ and $L_{X_P} \tilde{\theta}_0 = 0$ by projection.

Lemma 4.5. We have $L_V \widehat{X_P} = 0$ and $L_{\widehat{I}_{\alpha V}} \widehat{X_P} = 0$.

Lemma 4.6. We have $L_{\widehat{X_P}}\hat{\theta}_0 = 0$ on \hat{M} .

Lemma 4.5 allows us to define a vector field $X := \hat{\pi}_* \widehat{X_P}$ on \overline{M} .

Proposition 4.7. Let (\bar{M}, \bar{Q}) be a quaternionic manifold obtained from a hypercomplex manifold M satisfying the assumptions in Theorem 4.1 and $\bar{\nabla}$ the quaternionic connection defined by (4.1). The vector field X is an affine quaternionic vector field of $(\bar{M}, \bar{Q}, \bar{\nabla})$, that is, satisfies $L_X \Gamma(\bar{Q}) \subset \Gamma(\bar{Q})$ and $L_X \bar{\nabla} = 0$.

Proof. It follows easily from Lemma 4.4 that X preserves the quaternionic structure \bar{Q} . From Lemma 4.4, Lemma 4.6 and the closure of $\hat{\theta}_0$ we do also obtain that $p_v[\widehat{X_P}, Y^{h_{\hat{\theta}'}}] = 0$, where p_h and p_v denote the projections from $T\hat{M}$ onto the horizontal and vertical subbundles, respectively. Using this, for any vector fields Y and W on \bar{M} , we compute

$$(L_X \bar{\nabla})_Y W = \hat{\pi}_* \left([\widehat{X_P}, \hat{\nabla}^0_{Y^h_{\hat{\theta}'}} W^{\hat{\mathcal{H}}}] - \hat{\nabla}^0_{p_h[\widehat{X_P}, Y^h_{\hat{\theta}'}]} W^{h_{\hat{\theta}'}} - \hat{\nabla}^0_{Y^h_{\hat{\theta}'}} p_h[\widehat{X_P}, W^{h_{\hat{\theta}'}}] \right)$$

= $\hat{\pi}_* \left((L_{\widehat{X_P}} \hat{\nabla}^0)_{Y^h_{\hat{\theta}'}} W^{h_{\hat{\theta}'}} + \hat{\nabla}^0_{p_v[\widehat{X_P}, Y^h_{\hat{\theta}'}]} W^{h_{\hat{\theta}'}} + \hat{\nabla}^0_{Y^h_{\hat{\theta}'}} p_v[\widehat{X_P}, W^{h_{\hat{\theta}'}}] \right) = 0. \ \Box$

We call the correspondence from a hypercomplex manifold (M, H, Z, f, Θ) to a quaternionic manifold $(\overline{M}, \overline{Q}, \overline{\nabla}, X)$ described in Theorem 4.1 (and Proposition 4.7 for the additional structure X) the hypercomplex/quaternionic-correspondence (H/Qcorrespondence for short). As we mentioned in Remarks 3.10 and 4.2, the global assumption in Theorem 4.1 (H/Q-correspondence) that $\tilde{\pi}$ and $\hat{\pi}$ are submersions is always satisfied locally. Under stronger assumptions and by considering Swann's twist [27], we have the following global result. We use the notation ζ_A for the action induced from the group $\langle A \rangle$ generated by a vector field A to distinguish U(1)-actions.

Theorem 4.8 (H/Q-correspondence, second version). Let M be a hypercomplex manifold with a hypercomplex structure $H = (I_1, I_2, I_3)$, a closed two-form Θ and a rotating vector field Z such that $L_Z \Theta = 0$. Let f be a smooth function on M such that $df = -\iota_Z \Theta$ and assume that $f_1 := f - (1/2)\Theta(Z, I_1Z)$ does nowhere vanish. Consider a principal U(1)-bundle $\pi : P \to M$ with a connection form η whose curvature form is

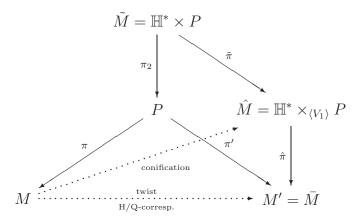
$$d\eta = \pi^* \left(\Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1) \right).$$

If $Z_1 = Z^{h_{\eta}} + f_1 X_P$ generates a free U(1)-action on P, then the confication \hat{M} of M is $\mathbb{H}^* \times_{\langle V_1 \rangle} P$ and the quaternionic manifold \bar{M} coincides with the twist of M given by the twist data $(\Theta - \frac{1}{2}d((\iota_Z \Theta) \circ I_1), Z, f_1)$ as manifolds.

Proof. By Lemma 3.2, we see $\iota_Z d\eta = -df_1$. It follows that $L_Z d\eta = 0$ from the assumptions $L_Z \Theta = 0$ and $L_Z I_1 = 0$. Therefore we obtain a twist $M' := P/\langle Z_1 \rangle$ of M with the twist data $(\Theta - \frac{1}{2}d((\iota_Z \Theta) \circ I_1), Z, f_1)$ since $Z_1 = Z^{h_\eta} + f_1 X_P$ generates a free U(1)-action. Let $\pi' : P \to M'$ be the quotient map by the action of $\langle Z_1 \rangle$. We define an action of $\langle V_1 \rangle (\cong U(1)) \subset \langle e_1^L \rangle \times \langle Z_1 \rangle$ on $\mathbb{H}^* \times P$ by

$$\zeta_{V_1}(u)(z,p) = (\zeta_{e_1^L}(u)z, \zeta_{Z_1}(u^{-1})p)$$

for $(z, p) \in \mathbb{H}^* \times P$. We see that the conification \hat{M} of M is a fiber bundle $(\mathbb{H}^* \times P)/\langle V_1 \rangle$ over M', which is associated with $\pi' : P \to M'$ and usually denoted by $\mathbb{H}^* \times_{\langle V_1 \rangle} P$. Moreover the quotient of \hat{M} by \mathbb{H}^* is M', that is, $\bar{M} = M'$.



In the above diagram, π_2 is the projection onto the second factor P.

Remark 4.9. Note that the bundle $\hat{\pi} : \hat{M} \to \bar{M}$ is associated to the principal U(1)bundle $P \to \bar{M} = M' = P/\langle Z_1 \rangle$. Therefore sections of $\hat{\pi}$ are in one-to-one correspondence with equivariant maps $P \to \mathbb{H}^*$. Let $\lambda : P \to \mathbb{H}^*$ be such that $\lambda(\zeta_{Z_1}(u)p) = \zeta_{e_1^L}(u^{-1})\lambda(p)$ for all $u \in U(1)$ and $p \in P$ and set $F_{\lambda} := [\lambda, \mathrm{id}]_{\langle V_1 \rangle} : P \to \hat{M}$. If we consider a local section $s : U(\subset \bar{M} = M') \to P$, then $s' := F_{\lambda} \circ s : U \to \hat{M}$ is a local section of $\hat{\pi} : \hat{M} \to \bar{M}$ and the equivariance of λ implies that s' is independent of s. As we observed in the proof of Theorem 4.1, the quaternionic structure $\bar{Q} = \langle \bar{I}_1, \bar{I}_2, \bar{I}_3 \rangle$ on \bar{M} is induced from the hypercomplex structure on \hat{M} and a local section s'. For $Y \in T_x \bar{M}$, we have

$$\bar{I}_{\alpha}(Y) = \hat{\pi}_{*}(\hat{I}_{\alpha}Y^{h_{\hat{\theta}'}}_{s'(x)}) = \hat{\pi}_{*}(\hat{I}_{\alpha}s'_{*}Y),$$

since the decomposition $T\hat{M} = \mathcal{D} \oplus \hat{\mathcal{H}}$ is \hat{I}_{α} -invariant. From $s' = F_{\lambda} \circ s = [\lambda \circ s, s]_{\langle V_1 \rangle} = \tilde{\pi} \circ (\lambda \circ s, s)$, it holds that

(4.2)

$$\bar{I}_{\alpha}(Y) = \hat{\pi}_{*}(\hat{I}_{\alpha}s'_{*}Y) \\
= \hat{\pi}_{*}(\hat{I}_{\alpha}(\tilde{\pi}_{*}((\lambda \circ s)_{*}(Y) + s_{*}Y))) \\
= \hat{\pi}_{*}(\tilde{\pi}_{*}(\tilde{I}_{\alpha}((\lambda \circ s)_{*}(Y) + s_{*}Y))) \\
= \pi'_{*}(\pi_{2*}(\tilde{I}_{\alpha}((\lambda \circ s)_{*}(Y) + s_{*}Y))) \\
= \pi'_{*}(\pi_{2*}(\tilde{I}_{\alpha}s_{*}Y)).$$

Note that $(\lambda \circ s)_*(Y) + s_*Y \in T_{(\lambda(s(x)),s(x))}\tilde{M}$.

Next we consider the decomposition $TP|_{s(U)} = \langle Z_1 \rangle \oplus s_*(TU)$. Let p^{\vee} be the projection from $TP|_{s(U)}$ onto $s_*(TU)$. Note that $s_*(T_xU)$ is generated by the tangent vectors of the form $p^{\vee}(W_{s(x)}^{h_{\eta}}) =: W^{\vee}$ at each point s(x), where W is a tangent vector of M at $\pi(s(x))$ and η is the connection form on P. We define (an almost hypercomplex structure) I_{α}^{\vee} on s(U) by $I_{\alpha}^{\vee}(W^{\vee}) = (I'_{\alpha}W)^{\vee}$ for each $W^{\vee} \in s_*(T_xU)$, where I'_{α} is given by (3.2) for $z = \lambda(s(x))$. Since $\tilde{I}_{\alpha}(Z_1) = \tilde{I}_{\alpha}(e_1^L) \in T\mathbb{H}^*$ (by $\tilde{I}_{\alpha}V_1 = 0$), we have

(4.3)
$$p^{\vee}(\pi_{2*}(\tilde{I}_{\alpha}(W^{\vee}))) = p^{\vee}(\pi_{2*}(\tilde{I}_{\alpha}(W^{h_{\eta}} + aZ_{1}))) = p^{\vee}(\tilde{I}_{\alpha}W^{h_{\eta}}) = p^{\vee}((I'_{\alpha}W)^{h_{\eta}})$$

 $= (I'_{\alpha}W)^{\vee} = I^{\vee}_{\alpha}(W^{\vee}),$

where $a \in \mathbb{R}$. Then it holds that

$$\bar{I}_{\alpha}(Y) = \pi'_{*}(\pi_{2*}(\tilde{I}_{\alpha}s_{*}Y)) = \pi'_{*}(p^{\vee}(\pi_{2*}(\tilde{I}_{\alpha}(s_{*}Y))) = \pi'_{*}(I^{\vee}_{\alpha}(s_{*}Y))$$

from (4.2) and (4.3). Therefore \bar{Q} can be identified with $\langle I_1^{\vee}, I_2^{\vee}, I_3^{\vee} \rangle$ on s(U). Note that $\langle I_1^{\vee}, I_2^{\vee}, I_3^{\vee} \rangle$ is independent of the choice of λ , and hence it is shown again that \bar{Q} is independent of the choice of λ , which is identified with a section of \hat{M} .

Note that a quaternionic Kähler metric obtained by the HK/QK-correspondence is described directly in terms of the objects on P (instead of \hat{M}) in [4, 21].

Remark 4.10. The confication \hat{M} of M is locally isomorphic to the Swann bundle of \bar{M} , which is conical as discussed in Example 2.4. Note that the Swann bundle is an $\mathbb{H}^*/\{\pm 1\}$ -bundle over a quaternionic manifold whereas \bar{M} is the quotient of \hat{M} by \mathbb{H}^* as above. Indeed, take an open set U of \bar{M} and local sections $s : U \to \hat{M}$, $s' : U \to \mathcal{U}(\bar{M})$, where $\pi^{Sw} : \mathcal{U}(\bar{M}) \to \bar{M}$ is the Swann bundle of \bar{M} . For a local trivialization $\Phi : \hat{\pi}^{-1}(U) \to U \times \mathbb{H}^*$ associated to s and given by $\Phi(x) = (\hat{\pi}(x), \phi(x))$, we can define a double covering $F : \hat{\pi}^{-1}(U) \to (\pi^{Sw})^{-1}(U)$ by

$$F(x) = \Phi'^{-1}(s'(\hat{\pi}(x)), p(\phi(x))).$$

Here $\Phi' : (\pi^{Sw})^{-1}(U) \to U \times \mathbb{H}^* / \{\pm 1\}$ is a local trivialization associated to s' and $p : \mathbb{H}^* \to \mathbb{H}^* / \{\pm 1\}$ is the projection. See [24, 6] for the (twisted) Swann bundle.

5 Examples of the H/Q-correspondence

In this section, we give examples of the H/Q-correspondence.

Example 5.1 (HK/QK-correspondence). Let $(M, g, H = (I_1, I_2, I_3))$ be a (possibly indefinite) hyper-Kähler manifold with a rotating Killing vector field Z and f a nowhere vanishing smooth function such that $df = -\iota_Z \Theta$, where Θ is the Kähler form with respect to g and I_1 . Set $f_1 = f - (1/2)g(Z, Z)$ and assume that the functions g(Z, Z) and f_1 are nowhere zero. From these data, we can obtain a (possibly indefinite) quaternionic Kähler manifold $(\overline{M}, \overline{g})$ [15, 2, 4]. The metric \overline{g} is positive definite under the assumptions specified in [2, Corollary 2] for the signs of the functions f, f_1 and for the signature of \overline{M} is determined by these choices.

In the HK/QK-correspondence, the initial data Θ is a non-degenerate 2-form. In our more general setting, we may also choose $\Theta = 0$, like in the following example.

Example 5.2 (Conical hypercomplex manifold). Let $(M, (I_1, I_2, I_3), V)$ be a conical hypercomplex manifold with the Euler vector field V. Choose $f_1 = f = 1$, $\Theta = 0$, and consider the trivial principal bundle $P = M \times U(1)$ with the connection $\eta = dt$, where t is the angular coordinate of U(1) such that $dt(X_P) = 1$ on the fundamental vector field X_P . We assume that $Z := I_1 V$ generates a free U(1)-action on M and that the periods of Z, X_P and e_1^L are the same. It holds that Z is rotating from Lemma 2.2. Then V_1 generates a free U(1)-action on $\tilde{M} = \mathbb{H}^* \times P = \mathbb{H}^* \times M \times U(1)$ of the same period. Therefore

$$\hat{M}(=(\mathbb{H}^* \times M \times \mathrm{U}(1))/\langle V_1 \rangle) \ni [z, p, q] = [zq, \zeta_Z(q^{-1})p, 1] \mapsto (zq, \zeta_Z(q^{-1})p) \in \mathbb{H}^* \times M$$

gives a diffeomorphism $\hat{M} \cong \mathbb{H}^* \times M$, and hence $\bar{M} \cong M$ as smooth manifolds. In fact, we can define a diffeomorphism $\varphi' : M \to M'(=\bar{M})$ by $\varphi'(x) = \pi'(x, 1)$. A global section $\bar{M} \to \hat{M}$ gives rise to a hypercomplex structure $(\bar{I}_1, \bar{I}_2, \bar{I}_3)$ on \bar{M} but the latter does not coincide with (I_1, I_2, I_3) in general (under the diffeomorphism φ'). The quaternionic structure \bar{Q} on \bar{M} , however, coincides with $\langle I_1, I_2, I_3 \rangle$. Note that \bar{Q} is independent of the section, as shown in the proof of Theorem 4.1 and Remark 4.9. More explicitly, considering $\lambda_z : M \times \mathrm{U}(1) \to \mathbb{H}^*$ defined by $\lambda_z(x, u) = z \cdot u^{-1}$ ($z \in \mathbb{H}^*$) and the section $s : \bar{M} \to P$ defined by $s(x) = ((\varphi')^{-1}(x), 1)$, we see that the section $F_{\lambda_1} \circ s$ gives the hypercomplex structure $\langle I_1, I_2, I_3 \rangle$ and, hence, the quaternionic structure $\langle I_1, I_2, I_3 \rangle$ on $\bar{M} \cong M$.

The next example shows that our H/Q-correspondence is a proper generalization of the HK/QK-correspondence.

Example 5.3 (Hypercomplex Hopf manifold). Consider $\mathbb{H}^n \cong \mathbb{R}^{4n}$ as a right-vector space over the quaternions with the standard hypercomplex structure

$$\tilde{H} = (\tilde{I}_1 = R_i, \tilde{I}_2 = R_j, \tilde{I}_3 = \tilde{I}_1 \tilde{I}_2 = -R_k)$$

and the standard flat hyper-Kähler metric \tilde{g} and set $\tilde{M} = \mathbb{H}^n \setminus \{0\}$. Take $A \in \mathrm{Sp}(n)\mathrm{Sp}(1)$ and $\lambda > 1$. Then $\langle \lambda A \rangle$ is a group of homotheties which acts freely and properly discontinuously on the simply connected manifold \tilde{M} . The quotient space $\tilde{M}/\langle \lambda A \rangle$ inherits a quaternionic structure Q and a quaternionic connection ∇ which are invariant under the centralizer G^Q of λA in $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. In fact, the quaternionic structure \tilde{Q} on \tilde{M} is $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$ -invariant and induces therefore an almost quaternionic structure Q on $\tilde{M}/\langle \lambda A \rangle$, since $\langle \lambda A \rangle \subset \mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. Moreover, the Levi-Civita connection $\tilde{\nabla}$ on (\tilde{M}, \tilde{g}) , which coincides with the Obata connection with respect to \tilde{H} , is invariant under all homotheties of \tilde{M} . Since $\langle \lambda A \rangle$ acts by homotheties, we see that $\tilde{\nabla}$ induces a torsion-free connection ∇ on $\tilde{M}/\langle \lambda A \rangle$, which preserves Q. This means that Q is a quaternionic structure on $\tilde{M}/\langle \lambda A \rangle$. In particular, if $A \in \mathrm{Sp}(n)$, then the quotient $\tilde{M}/\langle \lambda A \rangle$ inherits an induced hypercomplex structure $H = (I_1, I_2, I_3)$ from \tilde{H} , which is invariant under the centralizer G^H of λA in $\mathrm{GL}(n, \mathbb{H})$, since $\langle \lambda A \rangle$ preserves \tilde{H} . We say that $(\tilde{M}/\langle \lambda A \rangle, Q)$ (resp. $(\tilde{M}/\langle \lambda A \rangle, H)$) is a quaternionic (resp. hypercomplex) Hopf manifold. See [23, 10].

We start with a hypercomplex Hopf manifold $M := \tilde{M}/\langle \lambda A \rangle$, where $A \in \operatorname{Sp}(n)$. Take $q \in \operatorname{Sp}(1)$ such that $q \neq \pm 1$. The centralizer of q in Sp(1) is isomorphic to U(1), which is denoted by $\operatorname{U}_q(1)$. We consider a U(1)-action : $z \mapsto ze^{-it}$ on \tilde{M} defined by the right multiplication of U(1) $\cong \operatorname{U}_q(1) \subset \operatorname{Sp}(n)\operatorname{U}_q(1) \subset \operatorname{Sp}(n)\operatorname{Sp}(1)$. This action induces one on M and the corresponding vector field Z is rotating. Therefore we can apply the same procedure as in Example 5.2 under the setting $P = M \times \operatorname{U}_q(1)$ (resp. $\tilde{P} = \tilde{M} \times \operatorname{U}_q(1)$) and $\Theta = 0$, and we have the quaternionic manifold $\bar{M}(=M')$ (resp. $\tilde{M}(=\tilde{M'})$) by the H/Q-correspondence. In the following, the quotient map of an action by a group G is denoted by π_G and the objects associated with \tilde{M} are denoted by the corresponding letters for M with $\tilde{}$, for example, the projection of the twist from $\tilde{M} \times \operatorname{U}_q(1)$ is denoted as $\tilde{\pi'}$, where we use the notation of Theorem 4.8. Let R_q be the right multiplication by q.

Since $\pi' \circ \pi_{\langle \lambda A \rangle} = \pi_{\langle \lambda A R_q \rangle} \circ \tilde{\pi}'$ and $\tilde{M}' = \tilde{M}$ is a manifold with an invariant quaternionic structure under the action of $\langle \lambda A R_q \rangle$ (Example 5.2 and Proposition 4.7), we have

$$\bar{M} = M' = \tilde{M} / \langle \lambda A R_q \rangle$$

Therefore it holds that

$$M = \tilde{M} / \langle \lambda A \rangle \stackrel{\text{\tiny H/Q}}{\longmapsto} \bar{M} = \tilde{M} / \langle \lambda A R_q \rangle.$$

In particular, we can choose $A = E_n \in \operatorname{Sp}(n)$. Then the centralizer G^H of $\lambda = \lambda E_n$ is $\mathbb{R}^{>0} \times \operatorname{SL}(n, \mathbb{H})$. We take the subgroup $\mathbb{R}^{>0} \times \operatorname{Sp}(n)$ of G^H , which acts transitively on M. Then

$$M = (\mathbb{R}^{>0} / \langle \lambda \rangle) \times \frac{\operatorname{Sp}(n)}{\operatorname{Sp}(n-1)}$$

On the other hand, considering the subgroup $\mathbb{R}^{>0} \times \operatorname{Sp}(n) \operatorname{U}_q(1)$ of the centralizer G^Q of λR_q , we see that

$$(\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\operatorname{Sp}(n)}{\operatorname{Sp}(n-1)} \stackrel{\text{H/Q}}{\longmapsto} (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\operatorname{Sp}(n)\operatorname{U}(1)}{\operatorname{Sp}(n-1)\bigtriangleup_{\operatorname{U}(1)}},$$

where $\triangle_{\mathrm{U}(1)}$ is a diagonally embedded subgroup of $\mathrm{Sp}(n)\mathrm{U}(1) \subset \mathrm{Sp}(n)\mathrm{Sp}(1)$ which is isomorphic to U(1). Considering the case of n = 2, we have an invariant quaternionic structure on the homogeneous space

$$\bar{M} = \mathbb{R}^{>0} / \langle \lambda \rangle \times \frac{\operatorname{Sp}(2)\operatorname{U}(1)}{\operatorname{Sp}(1)\Delta_{\operatorname{U}(1)}} = \frac{T^2 \cdot \operatorname{Sp}(2)}{\operatorname{U}(2)}$$

by the H/Q-correspondence. Note that $T^2 \times \text{Sp}(2)$ carries a hypercomplex structure and $(T^2 \times \text{Sp}(2))/\text{U}(2)$ is a homogeneous quaternionic manifold considered in [19].

Since M is diffeomorphic to $S^1 \times S^{4n-1}$, M can not admit any hyper-Kähler structure. Therefore the HK/QK-correspondence can not be applied to the hypercomplex Hopf manifold M. The H/Q-correspondence is thus a proper generalization of the HK/QK one.

In the following example, the closed form Θ is non-zero and degenerate.

Example 5.4 (Lie group with left-invariant hypercomplex structure). Consider G = SU(3). The Lie algebra \mathfrak{g} of G is decomposed as $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \cong \mathfrak{u}(1) \oplus \mathfrak{su}(2) \cong \mathbb{H}$ and \mathfrak{g}_1 is the unique complementary \mathfrak{g}_0 -module with the action of \mathbb{H} obtained from the adjoint action of \mathfrak{g}_0 [19]. Denote by $V \in \mathfrak{g}_0$ the vector which corresponds to $1 \in \mathbb{H}$. We use the same letters for left-invariant vector fields and corresponding elements of \mathfrak{g} in this example. Three complex structures I_1, I_2, I_3 on \mathfrak{g} can be defined as follows. They preserve the decomposition $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ and act on $\mathfrak{g}_0 = \mathbb{H}$ by the standard hypercomplex structure $(R_i, R_j, R_i R_j = -R_k)$. On \mathfrak{g}_1 they are defined by

(5.1)
$$I_{\alpha}|_{\mathfrak{g}_1} = -\mathrm{ad}_{I_{\alpha}V}|_{\mathfrak{g}_1}, \quad \alpha = 1, 2, 3.$$

These structures extend to a left-invariant hypercomplex structure on G [19], which we denote again by (I_1, I_2, I_3) .

Let $G_0 \cong (\mathrm{U}(1) \times \mathrm{SU}(2))/\{\pm 1\} \cong \mathrm{U}(2)$ be the subgroup of G corresponding to \mathfrak{g}_0 . Note that $G_0 \subset G$ is a hypercomplex submanifold and therefore totally geodesic with respect to the Obata connection ∇^G of G [24]. The Obata connection ∇^{G_0} of G_0 is given by $\nabla_X^{G_0}Y = XY$ for $X, Y \in \mathfrak{g}_0 = \mathbb{H}$, where XY denotes the product of the quaternions X and Y. Indeed, ∇^{G_0} is torsion-free and I_1, I_2, I_3 are parallel with respect to ∇^{G_0} . Then it holds $\nabla_X^G V = \nabla_X^{G_0}V = X$ for $X \in \mathfrak{g}_0$. For $X \in \mathfrak{g}_1$, by (5.1) and the explicit expression of the Obata connection (see [5]), we also find that $\nabla_X^G V = X$. Hence the hypercomplex manifold $(G, (I_1, I_2, I_3))$ is conical with the Euler vector field V (see also [26]).

Consider the right-action of U(2) on SU(3) given by

$$AB := A \left(\begin{array}{cc} B & 0\\ 0 & \det(B)^{-1} \end{array} \right)$$

for $A \in SU(3)$ and $B \in U(2)$. Let $l : SU(3) \to SU(3)/U(2) \cong \mathbb{C}P^2$ be the projection and $k : S^5 \to \mathbb{C}P^2$ the Hopf fibration. The pullback bundle $P := l^{\#}S^5$ of $k : S^5 \to \mathbb{C}P^2$ by l is a U(1)-bundle over SU(3). The usual identification between the Stiefel manifold $V_2(\mathbb{C}^3)$ and SU(3) is given by

$$V_2(\mathbb{C}^3) \ni (a_1, a_2) \leftrightarrow A = (a_1, a_2, \bar{a}_1 \times \bar{a}_2) \in \mathrm{SU}(3).$$

We can write

$$P = \{(A, u) \in SU(3) \times S^5 \mid l(A) = k(u)\}$$

= $\{(A, u) \in SU(3) \times S^5 \mid \langle c_3(A) \rangle = \langle u \rangle \in \mathbb{C}P^2\}$
= $\{(A, \alpha c_3(A)) \in SU(3) \times S^5 \mid \alpha \in U(1)\}$
 $\cong SU(3) \times U(1),$

where $c_3(A)$ denotes the third column of A. This shows that P is a trivial bundle. Let $l_{\#}: P \to S^5$ be the bundle map given by $l_{\#}(A, \alpha) = \alpha(\bar{a}_1 \times \bar{a}_2) = \alpha c_3(A)$. Consider the pullback connection $l_{\#}^* \eta$ on P from the standard one η of k and take $\Theta = l^* \omega$, where ω is the Kähler form on $\mathbb{C}P^2$. Set $Z := I_1 V$. We see that Z generates a U(1)-action on SU(3) and is rotating by Lemma 2.2. Since

$$\langle Z \rangle \subset \mathrm{SU}(2) \subset \mathrm{U}(2),$$

Z is tangent to the fiber of l. Hence, we have $\iota_Z \Theta = 0$, $L_Z \Theta = 0$, and also have $d\Theta = 0$ by $d\omega = 0$. So we can choose $f = f_1 = 1$ (see Section 3 for the notation) and then see that Z_1 generates a free U(1)-action on P given by

$$\zeta_{Z_1}(u)(A,\alpha) = (\zeta_Z(u)(A), u\alpha), \quad u \in \mathcal{U}(1).$$

To see this, it is sufficient to check that Z is horizontal with respect to the pull back connection. The vector field Z is lifted to $SU(3) \times U(1)$ as $Z_{(A,\alpha)} = (Z_A, 0) \in TSU(3) \times TU(1)$ for $A \in SU(3)$ and $\alpha \in U(1)$ with the same letter Z. From $Z \in \mathfrak{su}(2)$, it holds that

$$l_{\#*}Z_{(A,\alpha)} = \frac{d}{dt} l_{\#} \left(\zeta_Z(e^{it})(A,\alpha) \right) \Big|_{t=0}$$
$$= \frac{d}{dt} l_{\#} \left(\left(\zeta_Z(e^{it})(A),\alpha \right) \right) \Big|_{t=0}$$
$$= \frac{d}{dt} \alpha c_3 \left(\zeta_Z(e^{it})(A) \right) \Big|_{t=0}$$
$$= \frac{d}{dt} \alpha c_3 (A) \Big|_{t=0} = 0.$$

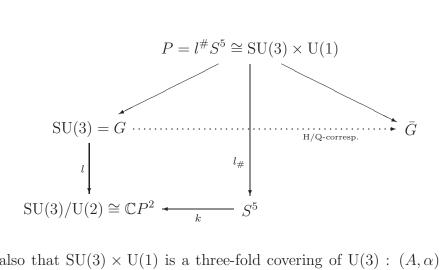
In particular, $(l_{\#}^*\eta)(Z) = 0$, that is, Z is horizontal with respect to the pullback connection. So we see that $Z_1 = Z + X_P$. Therefore, by applying the H/Q-correspondence to G = SU(3), we have a quaternionic manifold

$$\overline{G} = P/\langle Z_1 \rangle = (\mathrm{SU}(3) \times \mathrm{U}(1)) / \mathrm{U}(1) \cong \mathrm{SU}(3).$$

The identification is given by

$$\left(\mathrm{SU}(3) \times \mathrm{U}(1)\right) / \mathrm{U}(1) \ni \left[(A, \alpha)\right]_{\langle Z_1 \rangle} = \left[\left(\zeta_Z(\alpha^{-1})A, 1\right)\right]_{\langle Z_1 \rangle} \cong \zeta_Z(\alpha^{-1})A \in \mathrm{SU}(3).$$

Note that there exists no Riemannian metric g on G such that g is hyper-Kählerian with respect to (I_1, I_2, I_3) since G is compact. The situation is summarized in the following diagram.



Note also that $SU(3) \times U(1)$ is a three-fold covering of $U(3) : (A, \alpha) \mapsto \alpha A$. The kernel is the cyclic group $\{(\zeta 1, \zeta^{-1}) | \zeta^3 = 1\}$. The principal bundle $P \to SU(3)$ induces a principal bundle $U(3) = P/\mathbb{Z}_3 \to PSU(3) = SU(3)/\mathbb{Z}_3$. The actions generated by Z_1 and Z commutes with that of \mathbb{Z}_3 . The vector field Z (resp. Z_1) on SU(3) (resp. $SU(3) \times U(1)$) induces one on PSU(3) (resp. U(3)), which is denoted by the same letter Z (resp. Z_1). We obtain the following diagram.

$$P/\langle Z_1 \rangle = \bar{G} \cong \mathrm{SU}(3) \xrightarrow{\mathbb{Z}_3} \mathrm{U}(3)/\langle Z_1 \rangle = \bar{G}_1 \cong \mathrm{SU}(3)/\mathbb{Z}^3$$

$$/\langle Z_1 \rangle \qquad /\langle Z_1 \rangle \qquad /\langle$$

We can apply the H/Q-correspondence to the Lie group $G_1 = \text{PSU}(3)$ with the induced left-invariant hypercomplex structure and see that its resulting space is $\text{SU}(3)/\mathbb{Z}^3$. In fact, since the action of $\langle Z_1 \rangle$ on U(3) is given by $\zeta_{Z_1}(u)(\alpha A) = (u\alpha)(\zeta_Z(u)(A))$ and its orbit $\{(u\alpha)(\zeta_Z(u)(A)) \mid u \in \langle Z_1 \rangle\}$ of $\alpha A \in U(3)$ intersects SU(3) at exactly three

points, then the resulting space $U(3)/\langle Z_1 \rangle$ is $SU(3)/\mathbb{Z}^3$. Consequently, we have $\bar{G}_1 \cong G_1$ again.

Next we compare the quaternionic structures on the resulting space(s) derived from the pullback connection η_1 , which is not flat, and the trivial connection η_0 as in Example 5.2. Recall the notation in Remark 4.9. We claim that the two quaternionic structures are different. We label the objects obtained from η_i by the symbol η_i or just by the letter i (i = 0, 1), when no confusion is possible. Since $Z^{h_{\eta_0}} = Z^{h_{\eta_1}}$, $\iota_Z \Theta_0 = \iota_Z 0 = 0$ and $\iota_Z \Theta_1 = 0$, the vector field Z_1 on P is $Z_1 = Z + X_P$ for both connections η_0 and η_1 . Then the resulting spaces \bar{G}^0 and \bar{G}^1 coincide and we simply write \bar{G} for both. Let a be the 1-form on \bar{G} such that $\eta_1 - \eta_0 = \pi^* a$. Consider a local section $s: \bar{G} \to P$. Since $W^{h_{\eta_1}} - W^{h_{\eta_0}} = -a(W)X_P$ for a tangent vector W at $\pi(s(x)) \in \bar{G}$ (we omit the reference points of tangent vectors), we have

$$W^{\vee 1} - W^{\vee 0} = -a(W)\mathfrak{X}$$

where $\mathfrak{X} = p^{\vee}(X_P)$ and we recall that $W^{\vee i} = p^{\vee}(W^{h_{\eta_i}})$. Therefore we see that

$$\begin{split} I_{\alpha}^{\vee 1}(W^{\vee 1}) &= (I_{\alpha}'W)^{\vee 1} \\ &= (I_{\alpha}'W)^{\vee 0} - a(I_{\alpha}'W)\mathfrak{X} \\ &= I_{\alpha}^{\vee 0}(W^{\vee 0}) - a(I_{\alpha}'W)\mathfrak{X} \\ &= I_{\alpha}^{\vee 0}(W^{\vee 1}) + a(W)I_{\alpha}^{\vee 0}\mathfrak{X} - a(I_{\alpha}'W)\mathfrak{X}. \end{split}$$

On the other hand, since $W^{\vee 1} = W^{h_{\eta_1}} + cZ_1 = W^{h_{\eta_1}} + c(Z^{h_{\eta_1}} + X_P)$, we have $\eta_1(W^{\vee 1}) = c$ and $\pi_*(W^{\vee 1}) = W + cZ$. It holds that

$$(\pi^* a)(W^{\vee 1}) = a(W) + ca(Z) = a(W) + a(Z)\eta_1(W^{\vee 1}).$$

Hence we have

$$I_{\alpha}^{\vee 1} = I_{\alpha}^{\vee 0} + (\pi^* a - a(Z)\eta_1) \otimes (I_{\alpha}^{\vee 0}\mathfrak{X}) - ((\pi^* a - a(Z)\eta_1) \circ I_{\alpha}^{\vee 1}) \otimes \mathfrak{X}$$

Set $\rho := \pi^* a - a(Z)\eta_1$ and $A := \rho \otimes (I_{\alpha}^{\vee 0}\mathfrak{X}) - (\rho \circ I_{\alpha}^{\vee 1}) \otimes \mathfrak{X}$. If $Q^{\vee 0}(:= \langle I_1^{\vee 0}, I_2^{\vee 0}, I_3^{\vee 0} \rangle) = Q^{\vee 1}(:= \langle I_1^{\vee 1}, I_2^{\vee 1}, I_3^{\vee 1} \rangle)$, then $A^2 = -|A|^2$ id, where $|\cdot|$ is the norm induced from the metric on $Q^{\vee 0}$ such that $I_1^{\vee 0}, I_2^{\vee 0}, I_3^{\vee 0}$ are orthonormal. As the rank of A is at most 2, this is only possible if A = 0. This implies $\rho = \pi^* a - a(Z)\eta_1 = 0$, which is equivalent to a = 0. By Remark 4.9, the quaternionic structure \bar{Q}^i can be identified with $Q^{\vee i}$ (i = 0, 1). Then we see that $\bar{Q}^0 \neq \bar{Q}^1$ since $\eta_0 \neq \eta_1$. This proves the claim.

6 The tangent bundle of a special complex manifold and a generalization of the rigid c-map

In this section, we consider a generalization of the rigid c-map [9, 14, 3]. The generalization associates a hypercomplex manifold M, the Obata connection of which is Ricci-flat, with a special complex manifold. In the case of a *conical* special complex manifold, we

shall show that the hypercomplex manifold carries a canonical rotating vector field Z^M (Lemma 8.1), such that we can apply our H/Q correspondence. Consequently, we shall construct a quaternionic manifold from a conical special complex manifold as the generalized supergravity c-map (Theorem 8.3). We start with defining a class of manifolds generalizing conical special Kähler manifolds [3, 21].

Definition 6.1. A special complex manifold (N, J, ∇) is a complex manifold (N, J)endowed with a torsion-free flat connection ∇ such that the (1, 1)-tensor field ∇J is symmetric. A conical special complex manifold (N, J, ∇, ξ) is a special complex manifold (N, J, ∇) endowed with a vector field ξ such that

- $\nabla \xi = \text{id } and$
- $L_{\xi}J = 0$ or, equivalently, $\nabla_{\xi}J = 0$.

The connection ∇ is called the **special connection**. To see that $L_{\xi}J = 0$ is equivalent to $\nabla_{\xi}J = 0$ it suffices to write $L_{\xi} = \nabla_{\xi} - \nabla_{\xi} = \nabla_{\xi} - id$, using that ∇ is torsion-free and $\nabla_{\xi} = id$. We also note that the integrability of J follows from the symmetry of ∇J since ∇ is torsion-free. We set $A := \nabla J$.

Lemma 6.2. For every conical special complex manifold, we have $L_{J\xi}J = A_{J\xi} = 0$.

Proof. Based on the symmetry of ∇J , we compute

$$A_{J\xi} = A(J\xi) = -J(A\xi) = -JA_{\xi} = 0.$$

Using this and the properties listed in Definition 6.1, we then obtain

$$(L_{J\xi}J)X = -A_{JX}\xi + JA_X\xi = 0$$

for all $X \in \Gamma(TN)$. Note that in the last step we have used the symmetry of $A = \nabla J$.

Next we consider the tangent bundle TN =: M of a special complex manifold (N, J, ∇) . We can define the ∇ -horizontal lift $X^{h_{\nabla}}$ and the vertical lift X^v of $X \in \Gamma(TN)$. See [7] for example. The $C^{\infty}(M)$ -module $\Gamma(TM)$ is generated by vector fields of the form $X^{h_{\nabla}} + Y^v$, where $X, Y \in \Gamma(TN)$. On M, we define a triple of (1, 1)-tensors (I_1, I_2, I_3) by

(6.1)
$$I_1(X^{h_{\nabla}} + Y^v) = (JX)^{h_{\nabla}} - (JY)^v,$$

(6.2)
$$I_2(X^{h_{\nabla}} + Y^v) = Y^{h_{\nabla}} - X^v,$$

(6.3)
$$I_3(X^{h_{\nabla}} + Y^v) = (JY)^{h_{\nabla}} + (JX)^v$$

for $X^{h_{\nabla}} + Y^{v} \in TM$. Note that (I_1, I_2, I_3) is an almost hypercomplex structure. In fact, it is easy to see $I_{\alpha}^2 = -id$ and

$$(I_1 \circ I_2)(X^{h_{\nabla}} + Y^v) = I_1(Y^{h_{\nabla}} - X^v) = (JY)^{h_{\nabla}} + (JX)^v = I_3(X^{h_{\nabla}} + Y^v),$$

$$(I_2 \circ I_1)(X^{h_{\nabla}} + Y^v) = I_2((JX)^{h_{\nabla}} - (JY)^v) = -(JY)^{h_{\nabla}} - (JX)^v = -I_3(X^{h_{\nabla}} + Y^v)$$

for $X^{h_{\nabla}} + Y^{v} \in TM$. Note that it holds

(6.4)
$$[X^{h_{\nabla}}, Y^{h_{\nabla}}] = [X, Y]^{h_{\nabla}}, [X^{h_{\nabla}}, Y^{v}] = (\nabla_{X}Y)^{v}, [X^{v}, Y^{v}] = 0$$

for $X, Y \in \Gamma(TN)$.

Lemma 6.3. For every special complex manifold (N, J, ∇) , the canonical almost hypercomplex structure (I_1, I_2, I_3) on M = TN is integrable, that is, $(M, (I_1, I_2, I_3))$ is a hypercomplex manifold.

Proof. Thanks to (6.4), the Nijenhuis tensors of I_1 and I_2 can be easily calculated and we find the following. Using that J is integrable, ∇ is flat and ∇J is symmetric, we see that I_1 is integrable. Because ∇ is flat and torsion-free, I_2 is integrable. The integrability of I_3 follows from that of I_1 and I_2 [5, Theorem 3.2].

We define a connection ∇' by

$$\nabla' := \nabla - \frac{1}{2}J(\nabla J) = \nabla - \frac{1}{2}JA.$$

Then we see that $\nabla' J = 0$ and ∇' is torsion-free for every special complex manifold. Moreover, when the special complex manifold is conical, it holds that $\nabla' \xi = \nabla \xi = id$.

Lemma 6.4. For every special complex manifold (N, J, ∇) , we have

$$R_{X,Y}^{\nabla'} = -\frac{1}{4}[A_X, A_Y]$$

for $X, Y \in TN$.

Proof. Set $S = -(1/2)J(\nabla J)$. Since ∇ is flat, we see that the curvature $R^{\nabla'}$ of ∇' is given by

$$R_{X,Y}^{\nabla'} = (\nabla_X S)_Y - (\nabla_Y S)_X + [S_X, S_Y]$$

for $X, Y \in TN$. By

$$(\nabla_X S)_Y - (\nabla_Y S)_X = -\frac{1}{2} [A_X, A_Y] - \frac{1}{2} J(R_{X,Y}^{\nabla} J),$$

 $[S_X, S_Y] = \frac{1}{4} [A_X, A_Y],$

we have the conclusion.

Hence a special complex manifold admits the complex connection ∇' such that $R^{\nabla'}$ is of type (1,1). In fact, it follows from $A_{JX} = -JA_X$ for all $X \in TN$. The following theorem is a generalization of the rigid c-map in the absence of a metric.

Theorem 6.5 (Generalized rigid c-map). The tangent bundle of any special complex manifold (N, J, ∇) carries a canonical hypercomplex structure, defined by (6.1)-(6.3), and the Obata connection of the hypercomplex manifold $(M = TN, (I_1, I_2, I_3))$ is Ricci flat.

Proof. The integrability of the canonical almost hypercomplex structure defined by (6.1)-(6.3) was proven in Lemma 6.3. Let $\tilde{\nabla}^0$ be its Obata connection. Using the explicit expression of the Obata connection, we have

$$\tilde{\nabla}^{0}_{X^{h_{\nabla}}}Y^{h_{\nabla}} = (\nabla'_{X}Y)^{h_{\nabla}}, \quad \tilde{\nabla}^{0}_{U^{v}}X^{h_{\nabla}} = -\frac{1}{2}(JA_{X}U)^{v} = -\frac{1}{2}(JA_{U}X)^{v}$$
$$\tilde{\nabla}^{0}_{X^{h_{\nabla}}}U^{v} = (\nabla'_{X}U)^{v}, \quad \tilde{\nabla}^{0}_{U^{v}}V^{v} = \frac{1}{2}(JA_{V}U)^{h_{\nabla}} = \frac{1}{2}(JA_{U}V)^{h_{\nabla}}$$

for X, Y, U, $V \in \Gamma(TN)$. It can be also checked directly, using by (6.1)-(6.4), that the above formulas for $\tilde{\nabla}^0$ on horizontal and vertical lifts extend uniquely to a torsion-free connection $\tilde{\nabla}^0$ for which I_1 , I_2 , I_3 are parallel. We see that the bundle projection from $(TN, \tilde{\nabla}^0)$ onto (N, ∇') is an affine submersion [1]. Again, a straightforward calculation (or the fundamental equations of an affine submersion) gives

$$\begin{split} R_{U^{v},V^{v}}^{\tilde{\nabla}^{0}}W^{v} &= -\frac{1}{4}(A_{U}A_{V}W)^{v} + \frac{1}{4}(A_{V}A_{U}W)^{v} = (R_{U,V}^{\nabla'}W)^{v}, \\ R_{U^{v},V^{v}}^{\tilde{\nabla}^{0}}X^{h_{\nabla}} &= -\frac{1}{4}(A_{U}A_{V}X)^{h_{\nabla}} + \frac{1}{4}(A_{V}A_{U}X)^{h_{\nabla}} = (R_{U,V}^{\nabla'}X)^{h_{\nabla}}, \\ R_{U^{v},X^{h_{\nabla}}}^{\tilde{\nabla}^{0}}V^{v} &= -\frac{1}{2}(J(H_{U,V}^{\nabla}J)X)^{h_{\nabla}} - \frac{1}{4}(A_{X}A_{U}V)^{h_{\nabla}} - \frac{1}{4}(A_{U}A_{X}V)^{h_{\nabla}}, \\ R_{U^{v},X^{h_{\nabla}}}^{\tilde{\nabla}^{0}}Y^{h_{\nabla}} &= \frac{1}{2}(J(H_{X,Y}^{\nabla}J)U)^{v} + \frac{1}{4}(A_{X}A_{Y}U)^{v} + \frac{1}{4}(A_{U}A_{X}Y)^{v}, \\ R_{X^{h_{\nabla}},Y^{h_{\nabla}}}^{\tilde{\nabla}^{0}}U^{v} &= (R_{X,Y}^{\nabla'}U)^{v}, \\ R_{X^{h_{\nabla}},Y^{h_{\nabla}}}^{\tilde{\nabla}^{0}}Z^{h_{\nabla}} &= (R_{X,Y}^{\nabla'}Z)^{h_{\nabla}} \end{split}$$

for $X, Y, Z, U, V, W \in TN$, where H^{∇} is the Hessian (the second covariant derivative) with respect to ∇ and we have used Lemma 6.4. Note that $(H_{X,Y}^{\nabla}J)(Z) = (H_{X,Z}^{\nabla}J)(Y)$ for all $X, Y, Z \in TN$, since ∇J is symmetric. Hence the flatness of ∇ means that the Hessian of J with respect to ∇ is totally symmetric. By these equations, the Ricci tensor of $\tilde{\nabla}^0$ satisfies

$$Ric^{\tilde{\nabla}^{0}}(X^{h_{\nabla}}, Y^{h_{\nabla}}) = \frac{1}{2} \operatorname{Tr} J(H_{X,Y}^{\nabla}J) + \frac{1}{2} \operatorname{Tr} A_{X} A_{Y},$$
$$Ric^{\tilde{\nabla}^{0}}(X^{h_{\nabla}}, U^{v}) = Ric^{\tilde{\nabla}^{0}}(U^{v}, X^{h_{\nabla}}) = 0,$$
$$Ric^{\tilde{\nabla}^{0}}(U^{v}, V^{v}) = \frac{1}{2} \operatorname{Tr} J(H_{U,V}^{\nabla}J) + \frac{1}{2} \operatorname{Tr} A_{U} A_{V}$$

for X, Y, U, $V \in TN$. From $(\nabla J)J = -J(\nabla J)$, it holds that

$$\operatorname{Tr} J(H_{X,Y}^{\nabla}J) + \operatorname{Tr} A_X A_Y = 0$$

for all $X, Y \in TN$. Therefore the Obata connection of the manifolds obtained from our hypercomplex version of the c-map is Ricci flat.

Remark 6.6. The horizontal distribution on M is integrable by (6.4) and each leaf is totally geodesic with respect to the Obata connection $\tilde{\nabla}^0$, since $\tilde{\nabla}^0_{X^{h_{\nabla}}}Y^{h_{\nabla}} = (\nabla'_X Y)^{h_{\nabla}}$ for $X, Y \in \Gamma(TN)$.

Remark 6.7. In [12, Theorem A], a hypercomplex structure was obtained on a neighborhood of the zero section of the tangent bundle of a complex manifold with a complex connection whose curvature is of type (1, 1). By contrast, our generalized rigid c-map gives a hypercomplex structure on the whole tangent bundle when the manifold is special complex.

7 The c-projective structure on a projective special complex manifold

In this section, we discuss projective special complex manifolds and obtain the cprojective Weyl curvature of a canonically induced c-projective structure. Let (N, J, ∇, ξ) be a conical special complex manifold. Since $L_{\xi}J = 0$ and $L_{J\xi}J = 0$, we obtain a complex structure \bar{J} on the quotient $\bar{N} := N/\langle \xi, J\xi \rangle$ if \bar{N} is a smooth manifold.

Lemma 7.1. We have $L_{\xi}\nabla' = 0$ and $L_{J\xi}\nabla' = 0$.

Proof. By Lemmas 6.4 and 6.2, we have

$$(L_{\xi}\nabla')_X Y = [\xi, \nabla'_X Y] - \nabla'_{[\xi,X]} Y - \nabla'_X [\xi, Y]$$

= $\nabla'_{\xi} \nabla'_X Y - \nabla'_{\nabla'_X Y} \xi - \nabla'_{[\xi,X]} Y - \nabla'_X \nabla'_{\xi} Y + \nabla'_X \nabla'_Y \xi$
= $R^{\nabla'}_{\xi,X} Y = 0$

and

$$(L_{J\xi}\nabla')_X Y = [J\xi, \nabla'_X Y] - \nabla'_{[J\xi,X]} Y - \nabla'_X [J\xi,Y]$$

= $\nabla'_{J\xi} \nabla'_X Y - \nabla'_{\nabla'_X Y} J\xi - \nabla'_{[J\xi,X]} Y - \nabla'_X \nabla'_{J\xi} Y + \nabla'_X \nabla'_Y J\xi$
= $R_{J\xi,X}^{\nabla'} Y = 0$

for all $X, Y \in \Gamma(TN)$.

Recall [17] that a smooth curve $t \mapsto c(t)$ on a complex manifold (M, J) is called *J*-planar with respect to a connection ∇ if $\nabla_{c'}c' \in \text{span}\{c', Jc'\}$. We say that torsion-free complex connections ∇^1 and ∇^2 on a complex manifold (M, J) are c-projectively related [8] if they have the same *J*-planar curves. It is known that ∇^1 and ∇^2 are c-projectively related if and only if there exists a one-form θ on M such that

$$\nabla_X^1 Y = \nabla_X^2 Y + \theta(X)Y + \theta(Y)X - \theta(JX)JY - \theta(JY)JX$$

for $X, Y \in \Gamma(TM)$. See [17] for example. This defines an equivalence relation on the space of torsion-free complex connections on M. The equivalence classes are called **c-projective structures**.

Definition 7.2. We call the complex manifold (\bar{N}, \bar{J}) a projective special complex manifold if $p_N : (N, J, \nabla, \xi) \to (\bar{N}, \bar{J})$ is a principal \mathbb{C}^* -bundle, where the principal \mathbb{C}^* -action is generated by the holomorphic vector field $\xi - \sqrt{-1}J\xi$.

Note that a projective special Kähler manifold is a Kähler quotient of a conical special Kähler manifold. Similarly, a projective special complex manifold carries an induced c-projective structure as follows.

Proposition 7.3. Any projective special complex manifold (\bar{N}, \bar{J}) carries a canonical *c*-projective structure.

Proof. Consider a connection form $\hat{\alpha} = \alpha - \sqrt{-1}(\alpha \circ J)$ of type (1,0) on the principal \mathbb{C}^* -bundle $p_N : N \to \overline{N}$. (Note that any \mathbb{C}^* -invariant real one-form α such that $\alpha(\xi) = 1$ is the real part of such a connection.) We have $TN = \text{Ker } \hat{\alpha} \oplus \langle \xi, J\xi \rangle$, where $\text{Ker } \hat{\alpha}$ is J-invariant. We denote the $\hat{\alpha}$ -horizontal lift of $X \in \Gamma(T\overline{N})$ by $X^{h_{\alpha}}$. By Lemma 7.1, we can define $\overline{\nabla}'^{\alpha}$ by

(7.1)
$$\bar{\nabla}_X^{\prime \alpha} Y = p_{N*} (\nabla_{X^{h_\alpha}}^{\prime} Y^{h_\alpha})$$

for $X, Y \in \Gamma(T\bar{N})$. We claim that $\bar{\nabla}'^{\alpha}\bar{J} = 0$. In fact, using that $JY^{h_{\alpha}} = (\bar{J}Y)^{h_{\alpha}}$ for $Y \in T\bar{N}$ we have

$$\bar{\nabla}_X^{\prime\alpha}(\bar{J}Y) = p_{N*}(\nabla_{X^{h_\alpha}}^{\prime}JY^{h_\alpha}) = p_{N*}J(\nabla_{X^{h_\alpha}}^{\prime}Y^{h_\alpha}) = \bar{J}p_{N*}(\nabla_{X^{h_\alpha}}^{\prime}Y^{h_\alpha}).$$

To show that the c-projective structure $[\bar{\nabla}^{\prime \alpha}]$ does not depend on α , we consider another connection form $\hat{\beta} = \beta - \sqrt{-1}(\beta \circ J)$ of type (1,0). Then there exist one-forms θ_0 and θ_1 on \bar{N} such that

$$\hat{\beta} - \hat{\alpha} = (p_N^* \theta_0) + (p_N^* \theta_1) \sqrt{-1}.$$

On the other hand, we can write $X^{h_{\alpha}} - X^{h_{\beta}} = a\xi + bJ\xi$ for some functions a, b on N. It is easy to see that

$$a = \theta_0(X) \circ p_N, \ b = -\theta_0(\bar{J}X) \circ p_N, \ \theta_1 = -\theta_0 \circ \bar{J}$$

for $X \in T\overline{N}$. By the definition (7.1) of the induced connection on \overline{N} , we have

$$\begin{split} \bar{\nabla}_X^{\prime \alpha} Y =& p_{N*} (\nabla_{X^{h_\alpha}}^{\prime} Y^{h_\alpha}) \\ =& p_{N*} (\nabla_{X^{h_\beta} + \theta_0(X)\xi - \theta_0(\bar{J}X)J\xi} (Y^{h_\beta} + \theta_0(Y)\xi - \theta_0(\bar{J}Y)J\xi)) \\ =& p_{N*} (\nabla_{X^{h_\beta}}^{\prime} Y^{h_\beta} + \nabla_{X^{h_\beta}}^{\prime} \theta_0(Y)\xi - \nabla_{X^{h_\beta}}^{\prime} \theta_0(\bar{J}Y)J\xi \\ &\quad + \theta_0(X) (\nabla_\xi^{\prime} Y^{h_\beta} + \nabla_\xi^{\prime} \theta_0(Y)\xi - \nabla_\xi^{\prime} \theta_0(\bar{J}Y)J\xi) \\ &\quad - \theta_0(\bar{J}X) (\nabla_{J\xi}^{\prime} Y^{h_\beta} + \nabla_{J\xi}^{\prime} \theta_0(Y)\xi - \nabla_{J\xi}^{\prime} \theta_0(\bar{J}Y)J\xi) \\ =& \bar{\nabla}_X^{\prime \beta} Y + \theta_0(Y) X + \theta_0(X) Y - \theta_0(\bar{J}Y)\bar{J}X - \theta_0(\bar{J}X)\bar{J}Y \end{split}$$

for $X, Y \in \Gamma(T\bar{N})$, which means that $\bar{\nabla}'^{\alpha}$ and $\bar{\nabla}'^{\beta}$ are c-projectively related. Here we write $\theta_0(X)$ for $\theta_0(X) \circ p_N$ etc.

We denote the induced c-projective structure given in Proposition 7.3 by $\mathcal{P}_{\bar{\nabla}'}$ (without a label α). Next we prove that the c-projective Weyl curvature of $\mathcal{P}_{\bar{\nabla}'}$ is of type (1, 1) (see Theorem 7.10).

Note that ξ , $J\xi$ are the fundamental vector fields generated by $1, \sqrt{-1} \in \mathbb{C} = \text{Lie } \mathbb{C}^*$, respectively. Recall that $A = \nabla J$ and $A_{\xi} = A_{J\xi} = 0$. We also have that $L_{\xi}A = 0$, since $L_{\xi}\nabla = 0$ and $L_{\xi}J = 0$.

Lemma 7.4. $L_{J\xi}\nabla = A$, $L_{J\xi}A = -2JA$ and $L_{J\xi}(JA) = 2A$.

Let η be a connection form on the principal bundle $p_N : N \to \overline{N}$. As before, we assume that Ker η is *J*-invariant or, equivalently, that η is of type (1,0) (but not necessarily holomorphic). Using η we can project the connection ∇' on N to a connection $\overline{\nabla}'^{\eta}$ on \overline{N} , which is complex with respect to \overline{J} , as shown in the proof of Proposition 7.3. Note that the quotient $p_N : (N, \nabla') \to (\overline{N}, \overline{\nabla}'^{\eta})$ is an affine submersion with the horizontal distribution $\mathcal{H} := \operatorname{Ker} \eta$ (in the sense defined in [1]). From now on the η -horizontal lift of $X \in T\overline{N}$ is denoted by \widetilde{X} . Note that our sign convention for the curvature tensor is different from the one in [1]. Let $h: TN \to \mathcal{H}$ and $v: TN \to \mathcal{V}$ be the projections with respect to the decomposition $TN = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \operatorname{Ker} p_{N*}$. We define the fundamental tensors $\mathcal{A}^{\nabla'}$ and $\mathcal{T}^{\nabla'}$ by

$$\mathcal{A}_E^{\nabla'}F = v(\nabla'_{hE}hF) + h(\nabla'_{hE}vF)$$

and

$$\mathcal{T}_E^{\nabla'}F = h(\nabla'_{vE}vF) + v(\nabla'_{vE}hF)$$

for $E, F \in \Gamma(TN)$.

Lemma 7.5. We have $\mathcal{T}^{\nabla'} = 0$, $\mathcal{A}_X^{\nabla'} \xi = X$ and $\mathcal{A}_X^{\nabla'} J \xi = J X$ for any horizontal vector X.

Let a and b be (0, 2)-tensors defined by

$$\mathcal{A}_X^{\nabla'}Y = a(X,Y)\xi + b(X,Y)J\xi$$

for horizontal vectors X and Y. Since ∇' and the projections v, h are \mathbb{C}^* -invariant, $\mathcal{A}^{\nabla'}$ is \mathbb{C}^* -invariant, and hence, $a = p_N^* \bar{a}$ and $b = p_N^* \bar{b}$ for some tensors \bar{a} and \bar{b} on \bar{N} . For any (0, 2)-tensor k on a complex manifold with a complex structure J, define the (0, 2)-tensor k_J by $k_J(X, Y) := k(X, JY)$.

Lemma 7.6. We have

$$v((\nabla'_{\tilde{X}}J)\tilde{Y}) = \left(\bar{a}(X,\bar{J}Y) + \bar{b}(X,Y)\right)\xi + \left(\bar{b}(X,\bar{J}Y) - \bar{a}(X,Y)\right)J\xi$$

for $X, Y \in T\bar{N}$.

Lemma 7.7. We have $\bar{b}(X, Y) = -\bar{a}(X, \bar{J}Y) = -\bar{a}_{\bar{J}}(X, Y)$ for tangent vectors X and Y on \bar{N} . Consequently, the fundamental tensor $\mathcal{A}^{\nabla'}$ satisfies

(7.2)
$$\mathcal{A}_{\tilde{X}}^{\nabla'}\tilde{Y} = \bar{a}(X,Y)\xi - \bar{a}_{\bar{J}}(X,Y)J\xi$$

for tangent vectors X, Y on \overline{N} .

Proof. By $\nabla' J = 0$ and Lemma 7.6, we have the conclusion.

Let (r, θ) be the polar coordinates with respect to a (smooth) local trivialization $p_N^{-1}(\bar{U}) \cong \bar{U} \times \mathbb{C}^*$ of the principal \mathbb{C}^* -bundle $p_N : N \to \bar{N}$ such that $\xi = r\partial/\partial r$ and $J\xi = \partial/\partial \theta$. A principal connection η is locally given by

$$\eta := \eta_1 \otimes 1 + \eta_2 \otimes \sqrt{-1} = p_N^*(\gamma_1 \otimes 1 + \gamma_2 \otimes \sqrt{-1}) + \left(\frac{dr}{r} \otimes 1 + d\theta \otimes \sqrt{-1}\right)$$

for a \mathbb{C} -valued one-form $\gamma_1 \otimes 1 + \gamma_2 \otimes \sqrt{-1}$ on $\overline{U} \subset \overline{N}$. For each local trivialization $p_N^{-1}(\overline{U}) \cong \overline{U} \times \mathbb{C}^*$, we set

$$B := e^{2\theta J} A \ (e^{2\theta J} = (\cos 2\theta) \mathrm{id} + (\sin 2\theta) J).$$

The symmetric (1, 2)-tensor field B is defined *locally* and B is projectable by Lemma 7.4, i.e. horizontal (i.e. $B_{\xi} = B_{J\xi} = 0$) and \mathbb{C}^* -invariant. Therefore we obtain an induced *locally* defined symmetric tensor field \overline{B} on \overline{N} .

Lemma 7.8. The tensor $B^2 : (X, Y) \mapsto B_X \circ B_Y$ is a globally defined tensor field on N, in particular, [B, B] is so. As a consequence, we have the globally defined tensor fields \overline{B}^2 and $[\overline{B}, \overline{B}]$ on \overline{N} .

Proof. It follows from $B^2 = A^2$.

For a (0, 2)-tensor a and a (1, 1)-tensor K, we define an $\operatorname{End}(TN)$ -valued 2-form $a \wedge K$ by

$$(a \wedge K)_{X,Y}Z = a(X,Z)KY - a(Y,Z)KX$$

for tangent vectors X, Y and Z.

Proposition 7.9. The curvature $R^{\overline{\nabla}'\eta}$ of $\overline{\nabla}'^{\eta}$ is of the form

$$R^{\bar{\nabla}'^{\eta}} = -\frac{1}{4}[\bar{B},\bar{B}] + 2\bar{a}^a \otimes Id - 2(\bar{a}_{\bar{J}})^a \otimes \bar{J} + \bar{a} \wedge Id - \bar{a}_{\bar{J}} \wedge \bar{J}_{\bar{J}}$$

where $(\cdot)^a$ denotes anti-symmetrization. Moreover we have $d\gamma_1 = -2\bar{a}^a$ and $d\gamma_2 = 2(\bar{a}_{\bar{J}})^a$.

Proof. By the fundamental equation for an affine submersion [1], we have

$$(R_{X,Y}^{\bar{\nabla}'\eta}Z)^{\tilde{}} = h(R_{\tilde{X},\tilde{Y}}^{\nabla'}\tilde{Z}) + h(\nabla_{v[\tilde{X},\tilde{Y}]}^{\prime}\tilde{Z}) + \mathcal{A}_{\tilde{Y}}^{\nabla'}\mathcal{A}_{\tilde{X}}^{\nabla'}\tilde{Z} - \mathcal{A}_{\tilde{X}}^{\nabla'}\mathcal{A}_{\tilde{Y}}^{\nabla'}\tilde{Z}$$

for $X, Y, Z \in \Gamma(T\overline{N})$. Since

$$v[\tilde{X}, \tilde{Y}] = \eta_1([\tilde{X}, \tilde{Y}])\xi + \eta_2([\tilde{X}, \tilde{Y}])J\xi$$

= $-(d\eta_1)(\tilde{X}, \tilde{Y})\xi - (d\eta_2)(\tilde{X}, \tilde{Y})J\xi$
= $-(d\gamma_1)(X, Y)\xi - (d\gamma_2)(X, Y)J\xi$

we have

$$h(\nabla'_{v[\tilde{X},\tilde{Y}]}\tilde{Z}) = h(\nabla'_{\tilde{Z}}v[\tilde{X},\tilde{Y}])$$

= $h(\nabla'_{\tilde{Z}}(-(d\gamma_1)(X,Y)\xi - (d\gamma_2)(X,Y)J\xi))$
= $-(d\gamma_1)(X,Y)\tilde{Z} - (d\gamma_2)(X,Y)(\bar{J}Z)$.

Moreover, by

$$\mathcal{A}_{\tilde{X}}^{\nabla'}\tilde{Y} - \mathcal{A}_{\tilde{Y}}^{\nabla'}\tilde{X} = v[\tilde{X}, \tilde{Y}] = -d\gamma_1(X, Y)\xi - d\gamma_2(X, Y)J\xi,$$

we have $d\gamma_1 = -2\bar{a}^a$ and $d\gamma_2 = 2(\bar{a}_{\bar{J}})^a$.

Now we set dim N = 2(n+1). By Proposition 7.9 and $\text{Tr}\bar{B}_X = 0$ for all $X \in T\bar{N}$, we obtain

(7.3)
$$Ric^{\bar{\nabla}'^{\eta}}(Y,Z) = \frac{1}{4} \operatorname{Tr}\bar{B}_{Y}\bar{B}_{Z} + (\bar{a}(Z,Y) - \bar{a}(Y,Z)) - (\bar{a}(\bar{J}Y,\bar{J}Z) + \bar{a}(Y,Z)) - 2n\bar{a}(Y,Z) + \bar{a}(Y,Z) - \bar{a}(\bar{J}Y,\bar{J}Z) = \frac{1}{4} \operatorname{Tr}\bar{B}_{Y}\bar{B}_{Z} - (2n+1)\bar{a}(Y,Z) + \bar{a}(Z,Y) - \bar{a}(\bar{J}Y,\bar{J}Z) - \bar{a}(\bar{J}Z,\bar{J}Y).$$

We define a (0, 2)-tensor P^D on a complex manifold (M, J), which is called the **Rho** tensor of a connection D, by

$$P^{D}(X,Y) = \frac{1}{m+1} \left(Ric^{D}(X,Y) + \frac{1}{m-1} \left((Ric^{D})^{s}(X,Y) - (Ric^{D})^{s}(JX,JY) \right) \right),$$

for $X, Y \in TM$, where $2m = \dim M \ge 4$, Ric^D is the Ricci tensor of D and $(\cdot)^s$ is the symmetrization of a (0,2)-tensor. The c-projective Weyl curvature $W^{c,[\bar{D}]}$ of a c-projective structure $[\bar{D}]$ is given by

(7.4)
$$W^{c,[\bar{D}]} = R^{\bar{D}} + (P^{\bar{D}})^a \otimes Id - (P^{\bar{D}}_{\bar{J}})^a \otimes \bar{J} + \frac{1}{2}P^{\bar{D}} \wedge Id - \frac{1}{2}P^{\bar{D}}_{\bar{J}} \wedge \bar{J}.$$

See [8]. We shall compute the c-projective Weyl curvature of $[\bar{\nabla}'^{\eta}]$. From (7.3) it holds

$$(Ric^{\bar{\nabla}'^{\eta}})^{s}(Y,Z) = \frac{1}{4} \operatorname{Tr}\bar{B}_{Y}\bar{B}_{Z} - 2n\bar{a}^{s}(Y,Z) - 2\bar{a}^{s}(\bar{J}Y,\bar{J}Z),$$
$$(Ric^{\bar{\nabla}'^{\eta}})^{s}(\bar{J}Y,\bar{J}Z) = \frac{1}{4} \operatorname{Tr}\bar{B}_{Y}\bar{B}_{Z} - 2n\bar{a}^{s}(\bar{J}Y,\bar{J}Z) - 2\bar{a}^{s}(Y,Z)$$

and hence

$$(Ric^{\bar{\nabla}'^{\eta}})^{s}(Y,Z) - (Ric^{\bar{\nabla}'^{\eta}})^{s}(\bar{J}Y,\bar{J}Z) = -2(n-1)\left(\bar{a}^{s}(Y,Z) - \bar{a}^{s}(\bar{J}Y,\bar{J}Z)\right).$$

From these equations, it follows that

$$(n+1)P^{\bar{\nabla}'^{\eta}}(Y,Z) = \frac{1}{4} \operatorname{Tr} \bar{B}_Y \bar{B}_Z - (2n+1)\bar{a}(Y,Z) + \bar{a}(Z,Y) - \bar{a}(\bar{J}Y,\bar{J}Z) - \bar{a}(\bar{J}Z,\bar{J}Y) - 2(\bar{a}^s(Y,Z) - \bar{a}^s(\bar{J}Y,\bar{J}Z)) = \frac{1}{4} \operatorname{Tr} \bar{B}_Y \bar{B}_Z - (2n+1)\bar{a}(Y,Z) + \bar{a}(Z,Y) - (\bar{a}(Y,Z) + \bar{a}(Z,Y)) = \frac{1}{4} \operatorname{Tr} \bar{B}_Y \bar{B}_Z - 2(n+1)\bar{a}(Y,Z).$$

Setting $\bar{\mathcal{B}}(Y,Z) = \text{Tr}\bar{B}_Y\bar{B}_Z$, which is a symmetric, \bar{J} -hermitian globally defined (0,2)-tensor on \bar{N} , we have

(7.5)
$$\bar{a} = \frac{1}{8(n+1)}\bar{\mathcal{B}} - \frac{1}{2}P^{\bar{\nabla}'^{\eta}}.$$

Therefore the coefficients of the curvature form $d\eta = d\gamma_1 + \sqrt{-1}d\gamma_2 = -2\bar{a}^a + 2\sqrt{-1}(\bar{a}_{\bar{J}})^a$ are determined by

(7.6)
$$\bar{a}^a = -\frac{1}{2} (P^{\bar{\nabla}'^\eta})^a \left(= -\frac{1}{2(n+1)} (Ric^{\bar{\nabla}'^\eta})^a \right),$$

(7.7)
$$(\bar{a}_{\bar{J}})^a = \frac{1}{8(n+1)} \bar{\mathcal{B}}_{\bar{J}} - \frac{1}{2} (P_{\bar{J}}^{\bar{\nabla}^{\prime \eta}})^a \left(= \frac{1}{8(n+1)} \bar{\mathcal{B}}_{\bar{J}} - \frac{1}{2(n+1)} (Ric_{\bar{J}}^{\bar{\nabla}^{\prime \eta}})^a \right).$$

By the above calculations we arrive at the following theorem.

Theorem 7.10. Let (N, J, ∇, ξ) be a conical special complex manifold which is the total space of a (holomorphic) principal \mathbb{C}^* -bundle $p_N : N \to \overline{N}$, the base of which is a projective special complex manifold \overline{N} with dim $\overline{N} = 2n \ge 4$. The c-projective Weyl curvature $W^{c,\mathcal{P}_{\overline{\nabla}'}}$ of the canonically induced c-projective structure $\mathcal{P}_{\overline{\nabla}'}$ is given by

$$W^{c,\mathcal{P}_{\bar{\nabla}'}} = -\frac{1}{4}[\bar{B},\bar{B}] - \frac{1}{4(n+1)}\bar{\mathcal{B}}_{\bar{J}} \otimes \bar{J} + \frac{1}{8(n+1)}\bar{\mathcal{B}} \wedge \mathrm{Id} - \frac{1}{8(n+1)}\bar{\mathcal{B}}_{\bar{J}} \wedge \bar{J}.$$

In particular, $W^{c,\mathcal{P}_{\bar{\nabla}'}}_{\bar{J}(\cdot),\bar{J}(\cdot)} = W^{c,\mathcal{P}_{\bar{\nabla}'}}$, that is, $W^{c,\mathcal{P}_{\bar{\nabla}'}}$ is of type (1,1) as an End($T\bar{N}$)-valued two-form.

Proof. Take a principal connection η of type (1,0). By Proposition 7.3, the canonically induced c-projective structure is $[\bar{\nabla}'^{\eta}]$. From Proposition 7.9, equation (7.4) and the symmetry of $\bar{\mathcal{B}}$, it holds that

$$W^{c,[\bar{\nabla}'^{\eta}]} = -\frac{1}{4}[\bar{B},\bar{B}] - \frac{1}{4(n+1)}\bar{\mathcal{B}}_{\bar{J}} \otimes \bar{J} + \frac{1}{8(n+1)}\bar{\mathcal{B}} \wedge \mathrm{Id} - \frac{1}{8(n+1)}\bar{\mathcal{B}}_{\bar{J}} \wedge \bar{J}.$$

Since $\bar{\mathcal{B}}_{\bar{J}}$, $[\bar{B}, \bar{B}]$ and $\bar{\mathcal{B}} \wedge \mathrm{Id} - \bar{\mathcal{B}}_{\bar{J}} \wedge \bar{J}$ are of type (1, 1), $W^{c, \mathcal{P}_{\bar{\nabla}'}}$ is of type (1, 1).

The following corollary is a direct consequence of Theorem 7.10.

Corollary 7.11. Any complex manifold $(\overline{N}, \overline{J})$ with a c-projective structure \mathcal{P} such that $W^{c,\mathcal{P}}$ is not of type (1,1) can not be realized as a projective special complex manifold whose canonical c-projective structure is \mathcal{P} .

8 A generalization of the supergravity c-map

The supergravity c-map associates a (pseudo-)quternionic Kähler manifold with any projective special Kähler manifold. In this section, we give a generalization of the supergravity c-map by using the results in previous sections. Let (N, J, ∇, ξ) be a conical special complex manifold and set $Z := J\xi$.

Lemma 8.1. $2Z^{h_{\nabla}}$ is a rotating vector field on TN.

Proof. Since $L_Z J = 0$ and $\nabla_Z J = 0$ (cf. Lemma 6.2), we have $L_{Z^{h_{\nabla}}} I_1 = 0$. Moreover we have

$$(L_{Z^{h_{\nabla}}}I_2)(X^{h_{\nabla}} + Y^v) = [Z, Y]^{h_{\nabla}} - (\nabla_Z X)^v - (\nabla_Z Y)^{h_{\nabla}} + [Z, X]^v$$
$$= -(\nabla_Y Z)^{h_{\nabla}} - (\nabla_X Z)^v$$
$$= -(JY)^{h_{\nabla}} - (JX)^v$$
$$= -I_3(X^{h_{\nabla}} + Y^v)$$

 \Box

for all $X, Y \in \Gamma(TN)$.

Remark 8.2. By the equations for $\tilde{\nabla}^0$ in the proof of Theorem 6.5, we have

$$\tilde{\nabla}^0_{X^{h_{\nabla}}}\xi^{h_{\nabla}} = (\nabla'_X\xi)^{h_{\nabla}} = \left(\nabla_X\xi - \frac{1}{2}JA_X\xi\right)^{h_{\nabla}} = X^{h_{\nabla}},$$
$$\tilde{\nabla}^0_{X^v}\xi^{h_{\nabla}} = -\frac{1}{2}(JA_\xi X)^v = 0$$

for $X \in TN$, when (N, J, ∇, ξ) is a conical special complex manifold.

We have the following theorem.

Theorem 8.3 (Generalized supergravity c-map). Let (N, J, ∇, ξ) be a 2(n+1)-dimensional conical special complex manifold. Let Θ be a closed two-form on M = TN such that $L_{Z^M}\Theta = 0$, where $Z^M = 2Z^{h_{\nabla}}$. Consider a U(1)-bundle $\pi : P \to M$ over M and η a connection form whose curvature form is

$$d\eta = \pi^* \left(\Theta - \frac{1}{2} d((\iota_{Z^M} \Theta) \circ I_1) \right).$$

Let f be a smooth function on M such that $df = -\iota_{Z^M}\Theta$ and $f_1 := f - (1/2)\Theta(Z^M, I_1Z^M)$ does nowhere vanish. If $\tilde{\pi} : \tilde{M} \to \hat{M}$ and $\hat{\pi} : \hat{M} \to \bar{M}$ are submersions, we have an assignment from a 2n-dimensional projective special complex manifold $(\bar{N}, \bar{J}, \mathcal{P}_{\bar{\nabla}'})$ whose c-projective Weyl curvature is of type (1, 1) to a 4(n+1)-dimensional quaternionic manifold

 $\overline{M}(=\overline{TN}) = \mathcal{C}_{(P,\eta)}(M, \langle I_1, I_2, I_3 \rangle, Z^M, f, \Theta) / \mathcal{D}$

foliated by (2n+4)-dimensional leaves such that \overline{N} coincides with the space of its leaves.

Proof. By Theorem 4.1, Lemma 8.1 and Proposition 7.3, we have an assignment from a 2*n*-dimensional projective special complex manifold $(\bar{N}, \bar{J}, \mathcal{P}_{\bar{\nabla}'})$ to a 4(n+1)-dimensional quaternionic manifold \overline{TN} . By virtue of Theorem 7.10, the c-projective Weyl curvature of $\mathcal{P}_{\bar{\nabla}'}$ is of type (1,1). Next we give a foliation on \overline{TN} whose leaves space is \bar{N} . Set $\mathcal{L} := \mathcal{V} \oplus \langle \xi^{h_{\nabla}}, Z^{h_{\nabla}} \rangle$, where \mathcal{V} is the vertical distribution of $T(TN) \to TN$. The distribution \mathcal{L} is $Z^M = 2Z^{h_{\nabla}}$ -invariant and integrable by (6.4). Therefore each leaf L of \mathcal{L} is a $Z^M = 2Z^{h_{\nabla}}$ -invariant submanifold of TN. Consider the pull-back $\iota^{\#}P$ of P by the inclusion $\iota : L \to TN$ with the bundle map $\iota_{\#} : \iota^{\#}P \to P$ and $\tilde{L} := \mathbb{H}^* \times \iota^{\#}P$. Since V_1 is tangent to \tilde{L} , then $\hat{L} := \tilde{L}/\langle V_1 \rangle$ is a submanifold \hat{M} . Moreover V, $\hat{I}_1(V)$, $\hat{I}_2(V)$,

 $\hat{I}_3(V)$ are tangent to \hat{L} because V is induced by e_0^R . Taking the quotient again, we obtain a submanifold $\bar{L} := \hat{L}/\langle V, \hat{I}_1(V), \hat{I}_2(V), \hat{I}_3(V) \rangle$ on a quaternionic manifold \overline{TN} . Therefore the quaternionic manifold \overline{TN} is foliated by (2n+4)-dimensional leaves such that the space of its leaves \bar{L} is the projective special complex manifold \bar{N} .

Remark 8.4. If we assume that $Z_1 = (Z^M)^{h_{\eta}} + f_1 X_P$ generates a free U(1)-action on P instead of assuming that $\tilde{\pi} : \tilde{M} \to \hat{M}$ and $\hat{\pi} : \hat{M} \to \bar{M}$ are submersions, we obtain the same result as in Theorem 8.3 (see Theorem 4.8).

Remark 8.5. Borówka and Calderbank have given a construction of a quaternionic manifold from a complex manifold of half the dimension with a c-projective structure, known as the quaternionic Feix-Kaledin construction [6]. Their construction generalizes the original construction [11, 20], which yields a hyper-Kähler structure on a neighborhood of the zero setion of any Kähler manifold. They also point out that this construction is a generalization of [12, Theorem A] (see [6, Proposition 5.4]). More precisely, the initial data of the quaternionic Feix-Kaledin construction are a complex manifold with a c-projective structure of type (1, 1) and a complex line bundle with a connection of type (1, 1). Note that this construction is different from our generalization of the supergravity c-map, in which the real dimension of the quaternionic manifold \overline{TN} is related to the real dimension of the projective special complex manifold \overline{N} by $\dim(\overline{TN}) = 2 \dim(\overline{N}) + 4$.

We consider a conical special complex manifold (N, J, ∇, ξ) , which we endow now with an additional structure. Let ψ be a *J*-hermitian, ∇ -parallel two-form on (N, J, ∇, ξ) . We consider a function $\mu = (1/2)\psi(\xi, J\xi)$ on *N*. Then we see $d\mu = -\iota_Z \psi$. Set

(8.1)
$$\Theta = -\pi_{TN}^* \psi,$$

$$(8.2) f = -2\pi_{TN}^*\mu + c$$

for some constant c. Then it holds that

$$df = -\iota_{Z^M}\Theta, \ f_1 = f - \frac{1}{2}\Theta(Z^M, I_1Z^M) = 2\pi^*_{TN}\mu + c,$$

where $\pi_{TN}: TN \to N$ is the bundle projection. In fact, we have

$$df = -2d(\pi_{TN}^*\mu) = 2\pi_{TN}^*(\iota_Z\psi) = -\iota_{ZM}\Theta$$

and

$$f_{1} = f - \frac{1}{2}\Theta(Z^{M}, I_{1}Z^{M})$$

= $-\psi(\xi, J\xi) \circ \pi_{TN} - 2\Theta(Z^{h_{\nabla}}, I_{1}Z^{h_{\nabla}}) + c$
= $\psi(J\xi, \xi) \circ \pi_{TN} + c = 2\pi_{TN}^{*}\mu + c.$

Corollary 8.6. Let (N, J, ∇, ξ) be a 2n-dimensional conical special complex manifold and ψ a J-hermitian, ∇ -parallel two-form on N. Consider a U(1)-bundle $\pi : P \to M$ over M = TN and η a connection form whose curvature form is

$$d\eta = (\pi_{TN} \circ \pi)^* \psi.$$

If $\tilde{\pi} : \tilde{M} \to \hat{M}$ and $\hat{\pi} : \hat{M} \to \bar{M}$ are submersions and $\mu^{-1}(-c/2) = \emptyset$, then the generalized supergravity c-map of Theorem 8.3 can be specialized to this setting such that the data Θ and f are related to ψ by equations (8.1) and (8.2).

Proof. By a straightforward calculation, we have $d((\iota_Z \psi) \circ J) = 2\psi$. Then it is easy to check

$$d\eta = (\pi_{TN} \circ \pi)^* \psi$$

= $(\pi_{TN} \circ \pi)^* (-\psi + d((\iota_Z \psi) \circ J))$
= $(\pi_{TN} \circ \pi)^* \left(-\psi + \frac{1}{2}d((\iota_{2Z}\psi) \circ J)\right)$
= $\pi^* \left(\Theta - \frac{1}{2}d((\iota_{Z^M}\Theta) \circ I_1)\right),$

where Θ is the two-form given by (8.1). Since $d\psi = 0$ and $\iota_Z \psi = -d\mu$, it holds $L_{Z^M} \Theta = 0$. The function $f_1 = f - (1/2)\Theta(Z^M, I_1Z^M)$ does nowhere vanish by $\mu^{-1}(-c/2) = \emptyset$. Therefore Theorem 8.3 leads to the conclusion.

Therefore a conical special complex manifold (N, J, ∇, ξ) with a *J*-hermitian, ∇ parallel two-form ψ such that $(1/2\pi)[\psi] \in H^2_{DR}(N,\mathbb{Z})$ and $\mu = (1/2)\psi(\xi, J\xi)$ is not surjective gives rise to a quaternionic manifold of dimension $2 \dim N$ under a suitable choice of the constant c.

For $t \in \mathbb{R}/\pi\mathbb{Z}$, we define a connection ∇^t by $\nabla^t = e^{tJ} \circ \nabla \circ e^{-tJ}$, which is a special complex connection by [3, Proposition 1]. Moreover, by

$$\nabla^t = \nabla - (\sin t)e^{tJ}(\nabla J)$$

([3, Lemma 1]), we see that ∇^t satisfies $\nabla^t \xi = \text{id.}$ Therefore $\{\nabla^t\}_{t \in \mathbb{R}/\pi\mathbb{Z}}$ is a family of conical special complex connections if $\nabla J \neq 0$.

Lemma 8.7. If ψ is J-hermitian and ∇ -parallel, then ψ is ∇^t -parallel.

Proof. Since $\nabla^t - \nabla$ is a linear combination of ∇J and $J(\nabla J) = -(\nabla J)J$, it suffices to remark that $\nabla \psi = 0$, $J \cdot \psi = 0$ and, hence, $(\nabla_X J) \cdot \psi = 0$ for all X. Here the dot stands for the action on the tensor algebra by derivations.

Hence, Corollary 8.6 and Lemma 8.7 imply

Corollary 8.8. If $A(=\nabla J) \neq 0$, there exists an $(\mathbb{R}/\pi\mathbb{Z})$ -family of quaternionic manifolds obtained from a conical special complex manifold with ψ under the same assumptions of Corollary 8.6 by the H/Q-correspondence (for any chosen function f in the construction).

Proof. By Lemma 8.7, $\nabla_X^t \psi = 0$. Since (N, J, ∇^t, ξ) are conical special complex manifolds, we have the conclusion.

To give an example, we recall the (local) characterization of a conical special complex manifold [3]. Let (\mathbb{C}^{n+1}, J) be the standard complex vector space and U an open subset in \mathbb{C}^{n+1} with the standard coordinate system (z_0, \ldots, z_n) . We consider a holomorphic one-form $\alpha = \sum F_i dz_i$ on U, which is also viewed as a holomorphic map $\phi = \phi_\alpha$ from U to $(T^*U = U \times \mathbb{C}^{n+1} \subset) \mathbb{C}^{2(n+1)}$. If $\operatorname{Re} \phi : U \to \mathbb{R}^{2(n+1)}$ is an immersion, which is equivalent to ϕ being totally complex [3], then we can find an affine connection ∇ such that (U, J, ∇) is a special complex manifold. In fact, we can take a local coordinate system

$$(x_0 := \operatorname{Re} z_0, \dots, x_n := \operatorname{Re} z_n, y_0 := \operatorname{Re} F_0, \dots, y_n := \operatorname{Re} F_n)$$

on U induced by ϕ and a connection ∇ defined by the condition that $(x_0, \ldots, x_n, y_0, \ldots, y_n)$ is affine. Moreover $\sum_{i=0}^n dx_i \wedge dy_i$ is ∇ -parallel symplectic form on U. In particular, if $\alpha = -\sum_{i=0}^n \sqrt{-1}z_i dz_i$, then the induced affine coordinate system coincides with the real coordinate system underlying the holomorphic coordinate system (z_0, \ldots, z_n) , hence (U, J, ∇) is trivial $(\nabla J = 0)$ in that special case. In addition to being holomorphic and totally complex, we assume that ϕ is conical, which is equivalent to the condition that functions F_0, \ldots, F_n are homogeneous of degree one, i.e. $F_i(\lambda z) = \lambda F_i(z)$ for all λ near $1 \in \mathbb{C}^*$ and $z \in U$. Then U is conical, that is, any conical holomorphic one-form ϕ such that $\operatorname{Re} \phi$ is an immersion on U defines a conical special complex (and symplectic) manifold structure of complex dimension n. Conversely, any such manifold can be locally obtained in this way (see [3, Corollary 5]).

If we choose $\alpha = -\sum_{i=0}^{n} \sqrt{-1}z^{i}dz^{i}$ on $\mathbb{C}^{n+1}\setminus\{0\}$, then the generalized c-map associates an open submanifold of (\mathbb{H}^{n+1}, Q) with the standard quaternionic structure Q to the complex projective space $(\mathbb{C}P^{n}, J^{st}, [\nabla^{FS}])$, where J^{st} is the standard complex structure and ∇^{FS} is the Levi-Civita connection of the Fubini-Study metric. Here we have chosen $\Theta = 0$. We can also apply Corollary 8.6 by choosing the standard symplectic form as ψ . More generally, we have the following example.

Example 8.9. For a holomorphic function g of homogeneous degree one, we consider the holomorphic 1-form

$$\alpha = gdz_0 - \sqrt{-1}\sum_{i=1}^n z_i dz_i$$

on $U := \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \text{Im } g_0 \neq 0\}$, where $g_i = \frac{\partial g}{\partial z_i}$ $(i = 0, 1, \ldots, n)$. [Comment Vicente: we should perhaps use a different symbol for F to avoid Note that $d\alpha \neq 0$ if there exists i such that $g_i \neq 0$ $(i \geq 1)$. Setting $z_i = u_i + \sqrt{-1}v_i$ $(i = 0, 1, \ldots, n)$, we have

$$(x_0, \dots, x_n, y_0, y_1, \dots, y_n) = \operatorname{Re} \phi(u_0, \dots, u_n, v_0, \dots, v_n)$$

= (Re z_0, \dots , Re z_1 , Re g , Re $(-\sqrt{-1}z_1), \dots$, Re $(-\sqrt{-1}z_n)$)
= $(u_0, \dots, u_n, \operatorname{Re} g, v_1, \dots, v_n)$.

Since its Jacobian matrix is given by

$$\frac{\partial(x_0,\ldots,y_n)}{\partial(u_0,\ldots,v_n)} = \begin{pmatrix} 1 & 0 & 0 & \ldots & \ldots & 0 \\ & \ddots & \vdots & \vdots & & \vdots \\ & 1 & 0 & 0 & \ldots & \ldots & 0 \\ \operatorname{Re} g_0 & \ldots & \operatorname{Re} g_n & -\operatorname{Im} g_0 & -\operatorname{Im} g_1 & \ldots & \ldots & -\operatorname{Im} g_n \\ 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & \ldots & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we see that Re ϕ is an immersion and we obtain a conical special complex structure on U. The coordinate vector fields of (x_0, \ldots, y_n) are given by

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial u_i} + \frac{\operatorname{Re} g_i}{\operatorname{Im} g_0} \frac{\partial}{\partial v_0} \quad (i \ge 0),$$

$$\frac{\partial}{\partial y_0} = -\frac{1}{\operatorname{Im} g_0} \frac{\partial}{\partial v_0}, \quad \frac{\partial}{\partial y_j} = -\frac{\operatorname{Im} g_j}{\operatorname{Im} g_0} \frac{\partial}{\partial v_0} + \frac{\partial}{\partial v_j} \quad (j \ge 1)$$

on U. Let ∇ (resp. ∇^{st}) be the flat affine connection on U such that (x_0, \ldots, y_n) (resp. (u_0, \ldots, v_n)) is a ∇ (resp. ∇^{st})-affine coordinate system. We define S by $\nabla = \nabla^{st} + S$. Then we calculate

$$0 = \nabla_X \frac{\partial}{\partial x_i} = (\nabla_X^{\text{st}} + S_X) \left(\frac{\partial}{\partial u_i} + \frac{\operatorname{Re} g_i}{\operatorname{Im} g_0} \frac{\partial}{\partial v_0} \right)$$
$$= X \left(\frac{\operatorname{Re} g_i}{\operatorname{Im} g_0} \right) \frac{\partial}{\partial v_0} + S_X \frac{\partial}{\partial u_i} + \frac{\operatorname{Re} g_i}{\operatorname{Im} g_0} S_X \frac{\partial}{\partial v_0} \quad (i \ge 0)$$

and similarly we have

$$-X\left(\frac{1}{\operatorname{Im} g_{0}}\right)\frac{\partial}{\partial v_{0}} - \frac{1}{\operatorname{Im} g_{0}}S_{X}\frac{\partial}{\partial v_{0}} = 0,$$

$$-X\left(\frac{\operatorname{Im} g_{j}}{\operatorname{Im} g_{0}}\right)\frac{\partial}{\partial v_{0}} - \frac{\operatorname{Im} g_{j}}{\operatorname{Im} g_{0}}S_{X}\frac{\partial}{\partial v_{0}} + S_{X}\frac{\partial}{\partial v_{j}} = 0 \quad (j > 0).$$

From these equations, it holds that

(8.3)
$$S_X \frac{\partial}{\partial u_i} = -\frac{X \operatorname{Re} g_i}{\operatorname{Im} g_0} \frac{\partial}{\partial v_0}, \ S_X \frac{\partial}{\partial v_i} = \frac{X \operatorname{Im} g_i}{\operatorname{Im} g_0} \frac{\partial}{\partial v_0} \ (i \ge 0).$$

Using $A_X Y = (\nabla_X J)(Y) = S_X JY - JS_X Y$ and (8.3), we have the matrix representation

(8.4)
$$A = \nabla J = \frac{1}{\operatorname{Im} g_0} \begin{pmatrix} A_0 & \dots & A_n \\ 0_2 & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ 0_2 & \cdots & 0_2 \end{pmatrix}$$

of A with respect to the frame

$$\left(\frac{\partial}{\partial u_0}, \frac{\partial}{\partial v_0}, \dots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}\right),\,$$

where

$$A_{i} = \begin{pmatrix} -d\operatorname{Re} g_{i} & d\operatorname{Im} g_{i} \\ d\operatorname{Im} g_{i} & d\operatorname{Re} g_{i} \end{pmatrix} \text{ and } 0_{2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that we change the order of the frame for simplicity. This means that $A \neq 0$ if there exists *i* such that $g_i \neq \text{constant}$. By Lemma 7.8 and (8.4), $A^2 = (\nabla J)^2$ induces a globally defined tensor on \overline{U} , in particular

$$\operatorname{Tr} A^{2} = \operatorname{Tr} A_{0}^{2} = \frac{2}{(\operatorname{Im} g_{0})^{2}} (d\operatorname{Re} g_{0} \otimes d\operatorname{Re} g_{0} + d\operatorname{Im} g_{0} \otimes d\operatorname{Im} g_{0})$$

also induces the the symmetric tensor $\overline{\mathcal{B}}$ on \overline{U} . By Lemma 6.4 and (8.4), we see that

$$R^{\nabla'} = -\frac{1}{4}A \wedge A = -\frac{1}{4(\operatorname{Im} g_0)^2} \begin{pmatrix} A_0 \wedge A_0 & A_0 \wedge A_1 & \dots & A_0 \wedge A_n \\ 0_2 & 0_2 & \dots & 0_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & \dots & 0_2 \end{pmatrix}$$

as the matrix representation.

Since

$$dx_i = du_i \ (i \ge 0), \ dy_0 = \sum_{i=0}^n \operatorname{Re} g_i \, du_i - \operatorname{Im} g_i \, dv_i,$$
$$dy_j = dv_j \ (j > 0),$$

a 2-form $\psi = \sum_{i=1}^{n} dx_i \wedge dy_i (= \sum_{i=1}^{n} du_i \wedge dv_i)$ is *J*-hermitian and ∇ -parallel. Note that $\sum_{i=0}^{n} dx_i \wedge dy_i = dx_0 \wedge dy_0 + \psi$ is not *J*-hermitian, that is, $(U, J, \nabla, dx_0 \wedge dy_0 + \psi)$ is not special Kählerian if there exists i > 0 such that $\operatorname{Re} g_i \neq 0$. However it is a special symplectic manifold. In fact, it holds

$$(dx_0 \wedge dy_0)(\frac{\partial}{\partial u_0}, \frac{\partial}{\partial u_i}) = \operatorname{Re} g_i \text{ and } (dx_0 \wedge dy_0)(J\frac{\partial}{\partial u_0}, J\frac{\partial}{\partial u_i}) = 0.$$

Moreover since

$$\xi = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} = \dots + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} + v_i \frac{\partial}{\partial v_i}$$
$$J\xi = \dots + \sum_{i=1}^{n} u_i \frac{\partial}{\partial v_i} - v_i \frac{\partial}{\partial u_i},$$

we have $\mu = \psi(\xi, J\xi) = (1/2) \sum_{i=1}^{n} (u_i^2 + v_i^2)$. Take a U(1)-bundle $\pi : TU \times U(1) \to TU$ with a connection form

$$\eta = (\pi_{TU} \circ \pi)^* (\sum_{i=1}^n u_i dv_i) + d\theta,$$

where θ is the angular coordinate of U(1). The special case Corollary 8.6 of Theorem 8.3 can be applied and then we obtain a quternionic manifold.

We consider the horizontal subbundle of $p_U : U \to \overline{U}$ given by the kernel of $\kappa = -(1/2s)d\mu \circ J$ on each level set $\mu^{-1}(s) \subset U$ ($s \neq 0$). We retake U as an open set in $\bigcup_{s>0}\mu^{-1}(s)$. For horizontal vector fields X and Y tangent to each level set $\mu^{-1}(s)$, $XY\mu = 0$ means that

$$(p_U^*\bar{a})(X,Y) = \frac{1}{2s}\psi(JX,Y),$$

where \bar{a} is the ξ -component of the fundamental tensor of $\mathcal{A}^{\nabla'}$ as in Section 7. Here we used $d\kappa = \psi/s$. This means that \bar{a} is symmetric and \bar{J} -hermitian, and hence the Ricci tensor of the connection $\bar{\nabla}'^{\kappa}$ on \bar{U} induced from κ is symmetric and \bar{J} -hermitian. Therefore it holds

$$p_U^*\bar{a} = -\frac{1}{\sum_{i=1}^n (u_i^2 + v_i^2)} \sum_{i=1}^n du_i \otimes du_i + dv_i \otimes dv_i.$$

Hence the Ricci tensor $Ric^{\bar{\nabla}'^{\kappa}}$ of $\bar{\nabla}'^{\kappa}$ satisfies

$$-\frac{1}{\sum_{i=1}^{n}(u_{i}^{2}+v_{i}^{2})}\sum_{i=1}^{n}du_{i}\otimes du_{i}+dv_{i}\otimes dv_{i}$$

$$=\frac{1}{4(n+1)(\operatorname{Im} g_{0})^{2}}(d\operatorname{Re} g_{0}\otimes d\operatorname{Re} g_{0}+d\operatorname{Im} g_{0}\otimes d\operatorname{Im} g_{0})-\frac{1}{2(n+1)}p_{\bar{U}}^{*}(Ric^{\bar{\nabla}^{\prime\kappa}})$$

by (7.5). In particular, we see that $Ric^{\bar{\nabla}^{\prime\kappa}} \geq 0$. For example, when we choose $g = -\sqrt{-1}z_1^l/z_0^{l-1}$ for $l(\neq 1) \in \mathbb{Z}$, we obtain

$$d\operatorname{Re} g_{0} = \frac{\sqrt{-1}}{2}(-l+1)l(-w^{l-1}dw + \bar{w}^{l-1}d\bar{w}),$$

$$d\operatorname{Im} g_{0} = -\frac{1}{2}(-l+1)l(w^{l-1}dw + \bar{w}^{l-1}d\bar{w}),$$

$$d\operatorname{Re} g_{1} = \frac{\sqrt{-1}}{2}(-l+1)l(w^{l-2}dw - \bar{w}^{l-1}d\bar{w}),$$

$$d\operatorname{Im} g_{1} = \frac{1}{2}(-l+1)l(w^{l-2}dw + \bar{w}^{l-2}d\bar{w}),$$

$$d\operatorname{Re} g_{i} = d\operatorname{Im} g_{i} = 0 \quad (j > 1),$$

where $w = z_1/z_0$. We denote the corresponding objects with subscript l for ones given by $g = -\sqrt{-1}z_1^l/z_0^{l-1}$. It holds that

$$(8.5) \ R^{\nabla^{l'}} = -\frac{\sqrt{-1} l^2 |w|^{2(l-2)}}{(w^l + \bar{w}^l)^2} \begin{pmatrix} 0 & -|w|^2 & -\operatorname{Im} w & \operatorname{Re} w & 0 & \dots & 0 \\ |w|^2 & 0 & -\operatorname{Re} w & -\operatorname{Im} w & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} dw \wedge d\bar{w}$$

and

(8.6)
$$\operatorname{Tr}(A^{l})^{2} = \operatorname{Tr}(\nabla^{l}J)^{2} = \frac{4 \, l^{2} \, |w|^{2(l-1)}}{(w^{l} + \bar{w}^{l})^{2}} (dw \otimes d\bar{w} + d\bar{w} \otimes dw).$$

Finally we consider the quaternionic Weyl curvature of TU. Let W^q be the quaternionic Weyl curvature of the quaternionic structure $Q = \langle I_1, I_2, I_3 \rangle$. In [5], the explicit expression of W^q is given and it is shown that W^q is independent of the choice of the quaternionic connection. Since the Obata connection of the c-map is Ricci flat by Theorem 6.5, we have $W^{q,l} = R^{\tilde{\nabla}^{0,l}}$ for $g = -\sqrt{-1}z_1^l/z_0^{l-1}$. If $l \neq 1$, then we see that

$$W_{X^{v},Y^{v}}^{q,l}Z^{v} = R_{X^{v},Y^{v}}^{\tilde{\nabla}^{0,l}}Z^{v} = \left(R_{X,Y}^{\nabla^{l}}Z\right)^{v}.$$

Because the vertical lift is determined by a differential manifold structure (not by a connection), we see that $W^{q,l} \neq W^{q,k}$ on $T(U_k \cap U_l)$ if $l \neq k$, where $U_j = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \text{Im } g_0 \neq 0\} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \text{Im } g_0 \neq 0\}$ for $g = -\sqrt{-1}z_1^j/z_0^{j-1}$. Here we used (8.5). So we can find different quaternionic structures $Q^{\alpha_1}, \ldots, Q^{\alpha_t}$ on $T(\bigcap_{i=1}^t U_{\alpha_i})$, where $1 \neq \alpha_i \in \mathbb{Z}$. Note that Q^0 is the flat quaternionic structure.

Remark 8.10. Since $d\alpha \neq 0$ except the trivial case $g = -\sqrt{-1}z_0$, Example 8.9 with $g = -\sqrt{-1}z_1^l/z_0^{l-1}$ $(l \neq 0)$, which is local one, is not given by a local special Kählerian one.

Remark 8.11. For a conical special Kähler manifold N, the particular twist data which yields the quaternionic Kähler structure of the supergravity c-map on $T^*N \cong TN$ is given in [21, Lemma 5.1] in consistency with [4]. As we noted in the introduction, we also have a freedom in the choice of the data Θ etc. for our generalized supergravity c-map. For instance, the two form Θ can be chosen as trivial ($\Theta = 0$) or as in equation (8.1). For illustration, we can give yet another possible choice of Θ . Assume that dim $N \ge 6$. Let $\{\bar{U}_{\alpha}\}_{\alpha\in\Lambda}$ be an open covering of \bar{N} with local trivializations $U_{\alpha} := p_N^{-1}(\bar{U}_{\alpha}) \cong \bar{U}_{\alpha} \times \mathbb{C}^*$ and $g_{\alpha\beta} : \bar{U}_{\alpha} \cap \bar{U}_{\beta} \to \mathbb{C}^*$ be the corresponding transition functions. Let $(r_{\alpha}, \theta_{\alpha})$ be the polar coordinates with respect to a (smooth) local trivialization $p_N^{-1}(\bar{U}_{\alpha}) \cong \bar{U}_{\alpha} \times \mathbb{C}^*$ for each $\alpha \in \Lambda$. A principal connection η is locally given by

$$\eta = p_N^*(\gamma_1^\alpha \otimes 1 + \gamma_2^\alpha \otimes \sqrt{-1}) + \left(\frac{dr_\alpha}{r_\alpha} \otimes 1 + d\theta_\alpha \otimes \sqrt{-1}\right)$$

for a \mathbb{C} -valued one-form $\gamma_1^{\alpha} \otimes 1 + \gamma_2^{\alpha} \otimes \sqrt{-1}$ on $\overline{U}_{\alpha} \subset \overline{N}$ for each $\alpha \in \Lambda$. If we write $g_{\alpha\beta} = e^{f_{\alpha\beta}^1 + f_{\alpha\beta}^2 \sqrt{-1}}$, then

$$\begin{split} f^1_{\alpha\beta} + f^1_{\beta\gamma} - f^1_{\alpha\gamma} &= 0, \\ f^2_{\alpha\beta} + f^2_{\beta\gamma} - f^2_{\alpha\gamma} &\in 2\pi\mathbb{Z}, \\ \gamma^1_{\beta} - \gamma^1_{\alpha} &= df^1_{\alpha\beta}, \\ \gamma^2_{\beta} - \gamma^2_{\alpha} &= df^2_{\alpha\beta}. \end{split}$$

Therefore we obtain a principal U(1)-bundle $p_S : S \to \overline{N}$ with transition functions $e^{f_{\alpha\beta}^2 \sqrt{-1}} : \overline{U}_{\alpha} \cap \overline{U}_{\beta} \to \mathrm{U}(1)$ and connection η_S locally given by

$$p_S^*(\gamma_2^{\alpha} \otimes \sqrt{-1}) + d\theta_{\alpha} \otimes \sqrt{-1}.$$

In fact, the collection $\{e^{f_{\alpha\beta}^2\sqrt{-1}}\}$ of local U(1)-valued functions satisfies the cocycle condtion and the collection $\{\gamma_{\alpha}\}$ of local $\sqrt{-1}\mathbb{R}$ -valued one-forms satisfying $\gamma_{\beta}^2 - \gamma_{\alpha}^2 = df_{\alpha\beta}^2$ defines a connection form η_S . By Proposition 7.9 and (7.7), its curvature $d\eta_S(=p_S^*(d\gamma_2^\alpha))$ is $2(\bar{a}_{\bar{I}})^a$, where $(\bar{a}_{\bar{I}})^a$ is given by

$$(\bar{a}_{\bar{J}})^a = \frac{1}{8(n+1)}\bar{\mathcal{B}}_{\bar{J}} - \frac{1}{2}(P_{\bar{J}}^{\bar{\nabla}'})^a.$$

On TN, we choose the two-form $\Theta = 2(p_N \circ \pi_{TN})^*((\bar{a}_{\bar{J}})^a)$ and consider the pull-back connection $(p_{N\#} \circ \pi_{TN\#})^*\eta_S$ on the pull-back bundle $P = \pi_{TN}^{\#}p_N^{\#}S$. Since $\iota_{Z^M}\Theta = 0$, we can see that the assumptions in Theorem 8.3 hold. It is left for future studies to find a canonical choice of Θ for the generalized supergravity c-map, which allows to invert the H/Q-correspondence of [10].

As an application of Theorem 8.3, we have the following corollary by patching quaternionic manifolds locally constructed by the generalized supergravity c-maps.

Corollary 8.12. Let $(M, J, [\nabla])$ be a complex manifold with a c-projective structure $[\nabla]$ and dim M = 2n. If $2n = \dim M \ge 4$ and the harmonic curvature of its normal Cartan connection vanishes, then there exists a 4(n + 1)-dimensional quaternionic manifold (\check{M}, Q) with the vanishing quaternionic Weyl curvature foliated by (n + 2)-dimensional complex manifolds whose leaves space is M.

Proof. Since dim $M \ge 4$ and the harmonic curvature of its normal Cartan connection vanishes, $(M, J, [\nabla])$ is locally isomorphic to $(\mathbb{C}P^n, J^{st}, [\nabla^{FS}])$ (see [8] for example). So we may assume that $M = \bigcup_{\alpha} U_{\alpha}$, where U_{α} is an open subset $\mathbb{C}P^n$. Set $V_{\alpha} :=$ $p^{-1}(U_{\alpha})$, where $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ is the projection. We consider the standard complex structure and the standard flat connection induced from \mathbb{C}^{n+1} on each V_{α} . By Theorem 8.3, we have a quaternionic manifold $W_{\alpha} := \varphi'(TV_{\alpha}) \subset \mathbb{H}^{n+1}$, where φ' is the diffeomorphism given in Example 5.2. Here we have chosen the two-form $\Theta = 0$ and $f = f_1 = 1$ on TV_α for each α . We set $M := \bigcup_\alpha W_\alpha$. The induced quaternionic structure on each W_{α} coincides with the standard one from \mathbb{H}^{n+1} . Hence an almost quaternion structure Q on M can be obtained. Since there exists a quaternionic connection on each W_{α} , one can obtain a quaternionic connection on M by the partition of unity, that is, Q is a quaternionic structure with vanishing quaternionic Weyl curvature. For each $p \in TV_{\alpha} \cap TV_{\beta}$, the leaf of \mathcal{L} through p in TV_{α} is denoted by L^{α} and corresponding leaf in W_{α} is denoted by \hat{L}^{α} , that is $\hat{L}^{\alpha} = \varphi'(L^{\alpha})$. Since $\hat{L}^{\alpha} = \hat{L}^{\beta}$ in \check{M} , we obtain leaves in \check{M} and see that its leaves space is M. Since the subbundle \mathcal{L} is an I_1 -invariant in $T(TV_{\alpha})$, each leaf L is a complex manifold with $I := I_1|_L$. Each leaf L on M is obtained by the Swann's twist with an almost complex structure \hat{I} . By [27, Proposition 3.8] and $\Theta = 0, \hat{I}$ is integrable. Acknowledgments. We thank Aleksandra Borówka for comments. Research by the first author is partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC 2121 Quantum Universe – 390833306. The second author's research is partially supported by JSPS KAKENHI Grant Number 18K03272.

References

- [1] N. Abe and K. Hasegawa, An affine submersion with horizontal distribution and its applications, Differential Geom. Appl. 14 (2001), 235-250.
- [2] D. Alekseevsky, V. Cortés and T. Mohaupt, Confication of Kähler and hyper-Kähler manifolds, Comm. Math. Phys. 324 (2013), 637-655.
- [3] D. Alekseevsky, V. Cortés and C. Devchand, Special complex manifolds, J. Geom. Phys. 42 (2002), 85-105.
- [4] D. Alekseevsky, V. Cortés, M. Dyckmanns and T. Mohaupt, *Quaternionic Kähler metrics associated with special Kähler manifolds*, J. Geom. Phys. 92 (2015), 271-287.
- [5] D. Alekseevsky and S. Marchiafava, Quaternionic structures on a manifold and subordinated structures, Ann. Mat. Pura Appl. (4) 171 (1996), 205-273.
- [6] A. Borówka and D. Calderbank, Projective geometry and the quaternionic Feix-Kaledin construction, Trans. Amer. Math. Soc., 372 (2019), 4729-4760.
- [7] D. Blair, *Riemannian geometry of contact and symplectic manifolds*, Second edition. Progress in Mathematics, 203. Birkhäuser Boston, Ltd., 2010.
- [8] D. Calderbank, M. Eastwood, V. S. Matveev and K. Neusser, *C-projective geometry*, Mem. Amer. Math. Soc. 267 (2020).
- [9] S. Cecotti, S. Ferrara and L. Girardello, Geometry of type II superstrings and the moduli of superconformal field theories, Journal of Modern Physics A, 4(10) (1989), 2475-2529.
- [10] V. Cortés and K. Hasegawa, The quaternionic/hypercomplex-correspondence, Osaka J. Math. 58 (2021), 213-238.
- B. Feix, Hyperkähler metrics on cotangent bundles, J. Reine Angew. Math. 532 (2001), 33–46.
- [12] B. Feix, Hypercomplex manifolds and hyperholomorphic bundles, Math. Proc. Cambridge Philos. Soc. 133 (2002), 443-457.
- [13] S. Ferrara and S. Sabharwal, Quaternionic manifolds for type II superstring vacua of Calabi Yau spaces, Nuclear Physics B, 332(2) (1990), 317-332.

- [14] D. S. Freed, Special Kähler manifolds, Comm. Math. Phys. 203 (1999), 31-52.
- [15] A. Haydys, Hyper-Kähler and quaternionic Kähler manifolds with S¹-symmetries, J. Geom. Phys. 58 (2008), 293-306.
- [16] N. Hitchin, On the hyperkähler/quaternion Kähler correspondence, Comm. Math. Phys. 324 (2013), 77-106,
- [17] S. Ishihara, Holomorphically projective changes and their groups in an almost complex manifold, Tohoku Math. J. (2) 9 (1957), 273-297.
- [18] D. Joyce, The hypercomplex quotient and the quaternionic quotient, Math. Ann. 290 (1991), 323-340.
- [19] D. Joyce, Compact hypercomplex and quaternionic manifolds, J. Differential Geometry, 35 (1992), 743-761.
- [20] D. Kaledin, A canonical hyperkähler metric on the total space of a cotangent bundle, Quaternionic structures in mathematics and physics (Rome, 1999), 195–230, Univ. Studi Roma "La Sapienza", Rome, 1999.
- [21] O. Macia and A. Swann, Twist geometry of the c-map, Commun. Math. Phys. 336 (2015), 1329-1357.
- [22] M. Obata, Affine connections on manifolds with almost complex, quaternion or Hermitian structure, Jap. J. Math., 26 (1956), 43-79.
- [23] L. Ornea and P. Piccinni, Locally conformal Kähler manifold structures in quaternionic geometry, Trans. Amer. Math. Soc., 349 (1997), 641-355.
- [24] H. Pedersen, Y. Poon and A. Swann, Hypercomplex structures associated to quaternionic manifolds, Differential Geom. Appl. 9 (1998), 273-292.
- [25] S. Salamon, Differential geometry of quaternionic manifolds, Ann. Scient. Ec. Norm. Sup. 19 (1986), 31-55.
- [26] A. Soldatenkov, Holonomy of the Obata connection in SU(3), Int. Math. Res. Not. 15 (2012), 3483-3497.
- [27] A. Swann, Twisting hermitian and hypercomplex geometries, Duke Math. J. 155 (2010), 403-431.

Vicente Cortés Department of Mathematics and Center for Mathematical Physics University of Hamburg Bundesstraße 55, D-20146 Hamburg, Germany. email: vicente.cortes @uni-hamburg.de

Kazuyuki Hasegawa Faculty of teacher education Institute of human and social sciences Kanazawa university Kakuma-machi, Kanazawa, Ishikawa, 920-1192, Japan. e-mail:kazuhase@staff.kanazawa-u.ac.jp