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The Hadamard's Inequality for s-Convex Function

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Abstract

A monotone nondecreasing mapping connected with Hadamard–type inequality for s–convex function and some applications are given.

Keywords: s–Hadamard's inequality, s–Convex function, Jensen's inequality

1 Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as Hadamard's inequality for convex mappings.

In [1] Hudzik and Maligrada considered among others the class of functions which are s–convex in the second sense. This class is defined in the following way: a function $f : [0,\infty) \to \mathbf{R}$ is said to be *s–convex in the second sense* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{2}$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. It can be easily seen that every s–convex function is convex when $s = 1$.

In [2] Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s–convex functions in the second sense; which is so called s–Hadamard–type inequality for s–convex function in 2^{nd} sense.

Theorem 1.1 *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s–convex function in the second sense, where $s \in (0,1)$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L^1[0,1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1} \tag{3}$$

the constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3). The above inequalities are sharp.

In [6], Yang and Hong established the following theorem which is a refinement of the second inequality of (1).

Theorem 1.2 *Suppose that $f : [a,b] \to \boldsymbol{R}$ is convex on $[a,b]$ and the mapping $F : [0,1] \to \boldsymbol{R}$ is defined by*

$$F(t) = \frac{1}{2(b-a)} \times \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\right] dx$$

Then

(i) F is an convex on $[0,1]$.

(ii) F is monotone increasing on $[0,1]$.

(iii) One has the bounds

$$\inf_{t\in[0,1]} F(t) = F(0) = \frac{1}{(b-a)}\int_a^b f(x)\,dx,$$

and

$$\sup_{t\in[0,1]} F(t) = F(1) = \frac{f(a)+f(b)}{2}.$$

For more refinements, counterparts and generalization see [3–6].

2 Hadamard's Inequality

Lemma 2.1 *Let $f : [a, b] \to \boldsymbol{R}$ be s–convex function and let $a \le y_1 \le x_1 \le x_2 \le y_2 \le b$ with $x_1 + x_2 = y_1 + y_2$. Then*

$$f(x_1) + f(x_2) \le f(y_1) + f(y_2) \tag{4}$$

Proof.

First we show that $f(x_1) + f(x_2) \le f(y_1) + f(y_2)$. If $y_1 = y_2$ then we are done. Suppose $y_1 \ne y_2$ and write

$$x_1 = \frac{y_2 - x_1}{y_2 - y_1} y_1 + \frac{x_1 - y_1}{y_2 - y_1} y_2, \quad x_2 = \frac{y_2 - x_2}{y_2 - y_1} y_1 + \frac{x_2 - y_1}{y_2 - y_1} y_2,$$

since f is s–convex, we have

$$\begin{aligned} f(x_1) + f(x_2) &\le \frac{y_2 - x_1}{y_2 - y_1} f(y_1) + \frac{x_1 - y_1}{y_2 - y_1} f(y_2) \\ &+ \frac{y_2 - x_2}{y_2 - y_1} f(y_1) + \frac{x_2 - y_1}{y_2 - y_1} f(y_2) \\ &= \frac{2y_2 - (x_1 + x_2)}{y_2 - y_1} f(y_1) + \frac{(x_1 + x_2) - 2y_1}{y_2 - y_1} f(y_2) \\ &= f(y_1) + f(y_2). \end{aligned} \tag{5}$$

which completes the proof.

The following inequality is considered the mapping connected with the inequality (3).

Theorem 2.2 *Suppose that $f : [a, b] \to \boldsymbol{R}$ is s–convex on $[a, b]$ and the mapping $F : [0, 1] \to \boldsymbol{R}$ is defined by*

$$\begin{aligned} F(t) = {} & \frac{1}{(s+1)(b-a)} \\ & \times \int_a^b \left[f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) x \right) + f\left(\left(\frac{1+t}{2}\right) b + \left(\frac{1-t}{2}\right) x \right) \right] dx \end{aligned}$$

Then

(i) F is an s–convex on $[0,1]$.

(ii) F is monotone increasing on $[0,1]$.

(iii) One has the bounds

$$\begin{aligned}\inf_{t\in[0,1]} F(t) &= F(0)\\ &= \frac{2}{(s+1)(b-a)}\int_a^b f(x)\,dx,\\ \sup_{t\in[0,1]} F(t) &= F(1)\\ &= \frac{f(a)+f(b)}{s+1}.\end{aligned}$$

Proof.

(i) For all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$, we have:

$$\begin{aligned}F(\alpha t_1+\beta t_2) &= \frac{1}{(b-a)}\int_a^b f\left(\frac{1+(\alpha t_1+\beta t_2)}{2}a+\frac{1-(\alpha t_1+\beta t_2)}{2}x\right)dx\\ &\quad+\frac{1}{(b-a)}\int_a^b f\left(\frac{1+(\alpha t_1+\beta t_2)}{2}b+\frac{1-(\alpha t_1+\beta t_2)}{2}x\right)dx\end{aligned}$$

$$\begin{aligned}&= \frac{1}{(b-a)}\int_a^b f\left(\alpha\frac{(1+t_1)a+(1-t_1)x}{2}+\beta\frac{(1+t_2)a+(1-t_2)x}{2}\right)dx\\ &\quad+\frac{1}{(b-a)}\int_a^b f\left(\alpha\frac{(1+t_1)b+(1-t_1)x}{2}+\beta\frac{(1+t_2)b+(1-t_2)x}{2}\right)dx\\ &\leq \frac{\alpha^s}{(b-a)}\int_a^b\left[f\left(\frac{(1+t_1)}{2}a+\frac{(1-t_1)}{2}x\right)+f\left(\frac{(1+t_1)}{2}b+\frac{(1-t_1)}{2}x\right)\right]dx\\ &\quad+\frac{\beta^s}{(b-a)}\int_a^b\left[f\left(\frac{(1+t_2)}{2}a+\frac{(1-t_2)}{2}x\right)+f\left(\frac{(1+t_2)}{2}b+\frac{(1-t_2)}{2}x\right)\right]dx\\ &= \alpha^s F(t_1)+\beta^s F(t_2).\end{aligned}$$

Therefore, F is s–convex function on $[0,1]$.

(ii) Let $0\leq t_1\leq t_2\leq 1$, $a\leq x\leq b$. Since

$$\int_a^b f\left(\frac{(1+t_1)}{2}b+\frac{(1-t_1)}{2}x\right)dx$$

$$= \int_a^b f\left(\frac{(1+t_1)}{2}b + \frac{(1-t_1)}{2}(b+a-x)\right)dx.$$

Thus, we have

$$F(t_1) = \frac{1}{(b-a)}\int_a^b \left[f\left(\frac{(1+t_1)}{2}a + \frac{(1-t_1)}{2}x\right) + f\left(\frac{(1+t_1)}{2}b + \frac{(1-t_1)}{2}(b+a-x)\right)\right]dx$$

and since

$$\begin{aligned}\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x &\leq \frac{(1+t_1)}{2}a + \frac{(1-t_1)}{2}x \\ &\leq \frac{(1+t_1)}{2}b + \frac{(1-t_1)}{2}(b+a-x) \\ &\leq \frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}(b+a-x)\end{aligned}$$

Thus,

$$\begin{aligned}&\left[\frac{(1+t_1)}{2}a + \frac{(1-t_1)}{2}x\right] + \left[\frac{(1+t_1)}{2}b + \frac{(1-t_1)}{2}(b+a-x)\right] \\ &= \left[\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x\right] + \left[\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}(b+a-x)\right]\end{aligned}$$

and since f is s–convex on $[a,b]$, and by Lemma 2.1, we have:

$$F(t_1) \leq \frac{1}{(b-a)}\int_a^b \left[f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x\right) + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}(b+a-x)\right)\right]dx$$

$$= \frac{1}{(b-a)}\int_a^b \left[f\left(\frac{(1+t_2)}{2}a + \frac{(1-t_2)}{2}x\right) + f\left(\frac{(1+t_2)}{2}b + \frac{(1-t_2)}{2}x\right)\right]dx$$
$$= F(t_2).$$

This shows that $F(t)$ is monotone increasing for all $t \in [0,1]$.

(iii) It follows from (ii), that, for all $t \in [0,1]$

$$\begin{aligned} F(t) &\geq F(0) \\ &= \frac{1}{(s+1)(b-a)} \int_a^b \left[f\left(\frac{a+x}{2}\right) + f\left(\frac{b+x}{2}\right) \right] dx \\ &= \frac{2}{(s+1)(b-a)} \int_a^b f(x)\, dx, \end{aligned} \tag{6}$$

and

$$\begin{aligned} F(t) &\leq F(1) \\ &= \frac{1}{(s+1)(b-a)} \int_a^b [f(a) + f(b)]\, dx \\ &= \frac{f(a)+f(b)}{s+1} \end{aligned} \tag{7}$$

Remark 1 : In (6) and (7), set $s = 1$ we get inequality 1. Also, if we set $s = 1$ in (3) we get the same result.

3 Hadamard's Inequality For Lipschitzian Mapping

Theorem 3.1 *Let $f : [a,b] \to \boldsymbol{R}$ satisfy Lipschitzian conditions. That is, for t_1 and $t_2 \in [0,1]$, we have*

$$|f(t_1) - f(t_2)| \leq L |t_1 - t_2|$$

where L is positive constant. Then

$$|F(t_1) - F(t_2)| \leq \frac{L |t_1 - t_2| (b-a)}{s+1} \tag{8}$$

Proof.

For $t_1, t_2 \in [0,1]$, we have

$$F(t_1) \leq \frac{1}{(s+1)(b-a)} \times \int_a^b \left[\left| f\left(\frac{(1+t_1)}{2} a + \frac{(1-t_1)}{2} x \right) - f\left(\frac{(1+t_2)}{2} a + \frac{(1-t_2)}{2} x \right) \right| \right.$$

$$
+\left| f\left(\frac{(1+t_1)}{2}b+\frac{(1-t_1)}{2}x\right)-f\left(\frac{(1+t_2)}{2}b+\frac{(1-t_2)}{2}x\right)\right|\Bigg]\, dx
$$

$$
\leq \frac{1}{(s+1)(b-a)}
$$

$$
\times\int_a^b L\left[\left|\left(\frac{t_1-t_2}{2}\right)a+\left(\frac{t_2-t_1}{2}\right)x\right|+\left|\left(\frac{t_1-t_2}{2}\right)b+\left(\frac{t_2-t_1}{2}\right)x\right|\right]dx
$$

$$
=\frac{L\left|t_1-t_2\right|(b-a)}{(s+1)}
$$

This completes the proof.

Remark 2 : In (8) if we take $t_1 = 0$ and $t_1 = 1$, then (8) reduce to

$$
\left|\frac{f(a)+f(b)}{s+1}-\frac{2}{(s+1)(b-a)}\int_a^b f(x)\,dx\right|\leq\frac{L(b-a)}{(s+1)}. \tag{9}
$$

The inequality (9) is the s–Hadamard–type inequality for Lipschitzian mapping of one variable.

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