THE HADAMARD THREE-CIRCLES THEOREMS FOR PARTIAL DIFFERENTIAL EQUATIONS

BY R. VÝBORNÝ[1]

Communicated by Hans Weinberger, July 7, 1973

1. The famous Hadamard three-circles theorem of the complex function theory has been generalized to solutions of elliptic and parabolic equations. For references as well as for some interesting applications we refer to [3]. The purpose of this note is to show that (a) three circles (spheres)-theorems lead naturally to a sharpened version of the boundary point maximum principle (see [1], [2]), and (b) to prove a Hadamard type theorem for a quasilinear equation.

2. Let $G = \{x; x \in R^n, |x| < a\}$, u be a nonconstant solution of $\Delta u \geq 0$, which is of class C^2 in G and continuous in $\bar{G}$. Let

$$(1) \qquad M(r) = \text{Max}\{u(x) : |x| = r\}$$

for $0 < r \leq a$. The strong maximum principle implies that $M(r)$ is a strictly increasing function of r. The Hadamard theorem states that $M(r)$ is a convex function of s, where $s = \log r$ for $n = 2$ and $s = -r^{2-n}$ for $n > 2$. Define $f(s) = M(r)$. Since f is a convex function it possesses a left-hand derivative $f'_-(s)$ on $(0, s(a)]$, and since s is a differentiable function of r M has the left-hand derivative $M'_-(r) = f'_-(s)(ds/dr)$. (Note that although the chain rule is not generally valid for one-sided derivatives it can be used here. Note also that $f'_-(a)$ can be infinite.) It will be proved now that $M'_-(a) > 0$. Assume contrary to what one wishes to prove that $M'_-(a) \leq 0$. Then we would have $f'_-(s) \leq 0$ at $s = s(a)$ and hence for all $s \in (0, s(a)]$ since f'_- is increasing. Hence $M'_-(r) \leq 0$ for $r \in (0, a]$. However $M'_-(r) \geq 0$ since M is increasing; and it would follow that $M'_-(r) = 0$ and $M(r) = \text{const}$. By the strong maximum principle u would be constant—contradiction. Hence we have proved $M'_-(a) > 0$.

Let y be that point on $S=\{x; |x|=a\}$ where u reaches its maximum and x a point on the normal to S at y. Then

$$\frac{u(x) - u(y)}{|x - y|} \leqq \frac{M(r) - M(a)}{a - r}.$$

Hence

(2) $$\limsup_{x \to y; x \in n} \frac{u(x) - u(y)}{|x - y|} \leqq -M'_-(a) < 0.$$

We have arrived at the following sharpened version of the boundary point maximum principle.

THEOREM. *If*

(i) *u is a solution of $\Delta u \geqq 0$ which is of class C^2 in G and which is continuous in $\bar{G}$,*

(ii) *$M(r)$ is defined by (1), and*

(iii) *$u(x) < u(y)$ for $|x| < a$, $|y| = a$,*

then (2) *holds.*

This theorem can be easily generalized to linear elliptic partial differential inequalities, since the proof hinges only on the strong maximum principle and on the convexity obtained from the Hadamard theorem.

3.   Consider now the quasilinear operator

$$E(u) = \sum a_{ij}(x, u, \operatorname{grad} u) D_i D_j u - c(x, u, \operatorname{grad} u)$$

where the functions $a_{ij}(x, u, p)$ and $c(x, u, p)$ are defined for all x in $D=\{x; x \in R^n, a < |x| < b,\}$, all $u \in R'$ and all $p \in R^n$. We shall assume that a_{ij} and c are Lipschitzian in p at 0; more precisely we shall assume that there is a constant L such that

(3) $$|a_{ij}(x, u, p) - a_{ij}(x, u, 0)| \leqq L |p|,$$

(4) $$c(x, u, p) - c(x, u, 0) \geqq -L |p|.$$

The C^2 solutions to the inequality $E(u) \geqq 0$ then satisfy the strong maximum principle (see e.g. [4]) if

(5) $$\sum_{i,j=1}^{n} a_{ij}(x, u, 0)\lambda_i\lambda_j \geqq m \sum_{1}^{n} \lambda_i^2, \qquad m > 0,$$

and

(6) $$c(x, u, 0) \geqq 0$$

for $x \in D$, $u \in R^1$ (or just for $u=u(x)$ the solution in question).

Since the solution to $E(u) \geq 0$ cannot have a maximum inside D, the function $M(r)$ is either increasing or decreasing or first decreasing and then increasing. To obtain a Hadamard type theorem one has to consider separately the intervals where $M(r)$ increases or decreases (cf. [3, p. 134]).

THEOREM. *If*
(i) *u is continuous in $\bar{D}$ and of class C^2 in D,*
(ii) *there exists a constant M such that $|D_i D_j u| \leq M$ in D,*
(iii) *$E(u) \geq 0$ in D,*
(iv) *$M(r)$ is defined by (1) and is a strictly increasing function of r for $a \leq r \leq b$,*
(v) *$\sum_{i=1}^{n} a_{ii}(x, u, 0) < C$,*
(vi) *the inequalities (3), (4), (5) and (6) are satisfied for all $x \in D$ and for all real λ_i in the case of (5),*
then there exists a strictly increasing function $v = v(r)$ such that $M(r)$ is a convex function of v.

REMARK. The theorem remains valid if the phrase 'strictly increasing' in (iv) and in the conclusion of the theorem is replaced by the phrase 'strictly decreasing'.

The following elementary lemma is easy to prove.

LEMMA. *Let M and v be continuous strictly increasing (decreasing) functions on $[a, b]$. Then M is a convex function of v if and only if the following condition is satisfied: for every $\gamma \geq 0$ and every interval $[\alpha, \beta] \subset [a, b]$ the function $M(r) - \gamma v(r)$ attains its maximum either at α or at β.*

PROOF OF THE THEOREM. Let v be a solution of $v'' = -((A/r) + B)v'$ with $A > 0$, $B > 0$ and such that $v' > 0$. (Such a solution can be found by quadrature.) The constants A and B will be chosen later. Let us assume, contrary to what we want to prove, that $M(r) - \gamma v(r)$ for some positive γ attains its maximum over $[\alpha, \beta]$ in (α, β). Then the function $w(x) = u(x) - \gamma v(r)$ attains its maximum over $\{x; \alpha \leq |x| \leq \beta\}$ at an interior point x_0. Define an auxiliary linear operator $E_0(z) = \sum_{i,j=1}^{n} a_{ij}(x, u, 0) D_i D_j z$. Then $E_0(w) \leq 0$ at x_0. The proof will be completed by showing $E_0(w) > 0$ at x_0. We have at $x = x_0$

$$
\begin{aligned}
E_0(w) &\geq E_0(u) - E(u) - c(x, u, 0) - \gamma E_0(v) \\
&\geq -L(n^2 M + 1)\,|\text{grad } u| \\
&\quad - \gamma \sum_{i,j=1}^{n} a_{ij}(x, u, 0) \left[v'' \frac{x_i x_j}{r^2} + \delta_{ij} \frac{v'}{r} - \frac{x_i x_j}{r^3} v' \right] \\
&= -L(n^2 M + 1)\gamma v' \\
&\quad - \gamma \sum_{i,j}^{n} a_{ij}(x, u, 0) \left[v'' \frac{x_i x_j}{r^2} + \delta_{ij} \frac{v'}{r} - \frac{x_i x_j}{r^3} v' \right].
\end{aligned}
$$

Since $v'' < 0$, $v' > 0$ we have

$$E_0(w) > \gamma\left\{ -\frac{m}{2} v'' - v' \left[L(n^2 M + 1) + \frac{1}{r} \sum_{i=1}^n a_{ii}(x, u, 0) \right. \right.$$
$$\left. \left. - \frac{1}{r} \sum a_{ij}(x, u, 0) \frac{x_i x_j}{r^2} \right] \right\}$$

$$E_0(w) > (m\gamma/2)\{-v'' - (2v'/m)[L(n^2 M + 1) + (C - m)/r]\}$$
$$= (m\gamma/2)[-v'' - v'((A/r) + B)] = 0.$$

REMARK. The function $v(r)$ depends on the bound for the second derivatives; it is an interesting problem whether or not this dependence can be removed. It is fairly obvious from the proof that the assumption (ii) is superfluous if the operator is linear.

REMARK. The theorem of this paragraph leads to a sharpened version of the boundary point maximum principle for the operator $E(u)$ analogous to that proved in paragraph 2 for subharmonic functions.

REFERENCES

1. E. Hopf, *A remark on linear elliptic differential equations of the second order*, Proc. Amer. Math. Soc. **3** (1952), 791–793. MR **14**, 280.

2. O. A. Oleĭnik, *On properties of solutions of some boundary problems for equations of elliptic type*, Mat. Sb. **30** (72) (1952), 695–702. (Russian) MR **14**, 280.

3. M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967. MR **36** #2935.

4. R. Výborný, *On a certain extension of the maximum principle*, Differential Equations and their Applications (Proc. Conf., Prague, 1962), Publ. House Czechoslovak Acad. Sci., Prague; Academic Press, New York, 1963, pp. 223–228. MR **30** #4068a.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF QUEENSLAND, QUEENSLAND, AUSTRALIA