THE HAMILTONIAN STRUCTURE OF THE SECOND PAINLEVÉ HIERARCHY.

MARTA MAZZOCCO AND MAN YUE MO

ABSTRACT. In this paper we study the Hamiltonian structure of the second Painlevé hierarchy, an infinite sequence of nonlinear ordinary differential equations containing PII as its simplest equation. The n-th element of the hierarchy is a non linear ODE of order $2n$ in the independent variable z depending on n parameters denoted by $t_1, \dots, t_{n-1}$ and α_n. We introduce new canonical coordinates and obtain Hamiltonians for the z and $t_1, \dots, t_{n-1}$ evolutions. We give explicit formulae for these Hamiltonians showing that they are polynomials in our canonical coordinates.

CONTENTS

1. INTRODUCTION

In this paper we study the Hamiltonian structure of the second Painlevé hierarchy, an infinite sequence of nonlinear ordinary differential equations containing

$$\mathrm{P}_{\mathrm{II}}: \qquad w_{zz} = 2w^3 + z\,w + \alpha_1,$$

as its simplest equation. The n-th element of the hierarchy is a non linear ODE of order $2n$, depending on n parameters denoted by $t_1, \dots, t_{n-1}$ and α_n:

$$\mathrm{P}_{\mathrm{II}}^{(n)}: \qquad \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_n\left[w_z - w^2\right] + \sum_{l=1}^{n-1} t_l \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_l\left[w_z - w^2\right] = zw + \alpha_n, \qquad n \geq 1,$$

where $\mathcal{L}_n$ is the operator defined by the recursion relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{L}_{n+1} = \left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + 4(w_z - w^2)\frac{\mathrm{d}}{\mathrm{d}z} + 2(w_z - w^2)_z\right)\mathcal{L}_n, \qquad \mathcal{L}_0[w_z - w^2] = \tfrac{1}{2}. \tag{1}$$

The second Painlevé equation and its hierarchy appear in several applications including Hele-Shaw geometry [17], nonlinear optics [19] and random matrix theory [38, 12] to name only a few.

The Hamiltonian structure of the classical six Painlevé equations was discovered long ago by Okamoto [36], Jimbo and Miwa [29]. In the case of $n = 1$, i.e. P_{II}, the Hamiltonian is

$$\mathcal{H}^{(1)} = 4P^2 + \frac{1}{4}Q + \frac{1}{4}PQ^2 + 2Pz - \frac{1}{2}Q\alpha_1.$$

where

$$Q = 4w, \qquad P = \frac{1}{2}\left(w_z - w^2 - \frac{z}{2}\right).$$

1991 *Mathematics Subject Classification.* Primary 34M55. Secondary 37K20, 35Q53.
Key words and phrases. Painlevé equations, isomonodromy problems, flows on coadjoint orbits, mKdV.

Using such formulation Okamoto was able to describe his initial conditions space, to characterize the action of the Bäcklund transformations found by Gambier [18] and Lukashevich [31] in terms of affine Weyl groups and to produce immediately the so called Riccati-type classical solutions of the second Painelvé equation [18, 21, 5]. Also several properties of the Yablonskii-Vorob'ev polynomials [41, 40] describing the rational solutions were proved using the Hamiltonian formulation [11]. Umemura and Watanabe [39] used the Hamiltonian structure in to prove the irreducibility of $\mathrm{P_{II}}$.

In this paper we introduce canonical coordinates $P_1,\dots,P_n,Q_1,\dots,Q_n$ and a Hamiltonian function $\mathcal{H}^{(n)}$ such that $\mathrm{P_{II}^{(n)}}$ is equivalent to

$$\frac{\partial Q_i}{\partial z}=\frac{\partial \mathcal{H}^{(n)}}{\partial P_i},\qquad \frac{\partial P_i}{\partial z}=-\frac{\partial \mathcal{H}^{(n)}}{\partial Q_i},\quad i=1,\dots,n. \tag{2}$$

In particular we show that $\mathcal{H}^{(n)}$ is a polynomial in $P_1,\dots,P_n,Q_1,\dots,Q_n$ and that the Hamiltonian equations satisfy the Painlevé property.

Observe that starting from the second element of the hierarchy, the parameters $t_1,\dots,t_{n-1}$ appear in $\mathrm{P_{II}^{(n)}}$. The solutions w will depend on the *times* $t_1,\dots,t_{n-1}$ according to the equation

$$(2k+1)\frac{\partial w}{\partial t_k}+\partial_z\left(\partial_z+2w\right)\mathcal{L}_k\left[w_z-w^2\right]=0,\qquad k=1,\dots,n-1, \tag{3}$$

which is actually the k-th element of the mKdV hierarchy. In fact, the second Painlevé hierarchy was discovered as self–similarity reduction of the mKdV hierarchy [2, 5, 14] (details on this derivation are recalled in section 2 below).

We refer to the evolution in the times $t_1,\dots,t_{n-1}$, as *time-flows.* We prove that the time-flows are Hamiltonian and we compute the Hamiltonians $\mathcal{H}_1^{(n)},\dots,\mathcal{H}_{n-1}^{(n)}$ such that the system

$$\frac{\partial Q_i}{\partial t_k}=\frac{\partial \mathcal{H}_k^{(n)}}{\partial P_i},\qquad \frac{\partial P_i}{\partial t_k}=-\frac{\partial \mathcal{H}_k^{(n)}}{\partial Q_i},\quad i=1,\dots,n,$$

is equivalent to (3). These Hamiltonians are also polynomials in $P_1,\dots,P_n,Q_1,\dots,Q_n$ and again the Painlevé property is satisfied. Amazingly, we obtain explicit formulae for $\mathcal{H}^{(n)}$ and $\mathcal{H}_k^{(n)}$, $k=1,\dots,n-1$, in terms of $P_1,\dots,P_n,Q_1,\dots,Q_n$ without need of recursion.

Our construction is based on the fact that the second Painlevé hierarchy can be interpreted as monodromy preserving deformation equation[1] of an auxiliary linear system

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}=\mathcal{A}^{(n)}(\lambda;z,t_1,\dots,t_{n-1})\Psi$$

where $\mathcal{A}^{(n)}$ is a matrix function of λ holomorphic in $\mathbb{C}^*$, having a simple pole at 0 and a pole of order $2n+2$ at infinity. The isomonodromic condition is expressed by the zero–curvature conditions

$$\frac{\partial \mathcal{A}^{(n)}}{\partial z}-\frac{\partial \mathcal{B}}{\partial \lambda}=[\mathcal{B},\mathcal{A}^{(n)}],\qquad \mathcal{B}=-\left(\frac{\mathcal{A}^{(n)}\lambda^{1-2n}}{4^n}\right)_+, \tag{4a}$$

$$\partial_\lambda \hat{M}^{(k)}-(2k+1)\partial_{t_k}\mathcal{A}^{(n)}=-[\hat{M}^{(k)},\mathcal{A}^{(n)}], \tag{4b}$$

where given any Laurent series L of λ, $(L)_+$ denotes its non-negative part and the relation between $\hat{M}_k$ and $\mathcal{A}^{(n)}$ is explained in Section 3 below.

We interpret equation (4a) as flow on the dual space of the following twisted loop algebra:

$$\begin{aligned}\mathfrak{g}_- &= \left\{X(\lambda)=\sum_{-\infty}^{-1}X_i\lambda^i | X(\lambda)\sigma_1=\sigma_1X(-\lambda)\right\}/\mathfrak{g}_{2n+1},\qquad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ \mathfrak{g}_{2n+1} &= \left\{X(\lambda)=\sum_{-\infty}^{-2n-2}X_i\lambda^i | X(\lambda)\sigma_1=\sigma_1X(-\lambda)\right\}\end{aligned}$$

[1]The isomonodromic deformation problem for the second Painlevé hierarchy with $t_1=\cdots=t_{n-1}=0$ was derived in [10] and in [28] following the approach proposed in [1] starting form the isomonodromy deformation problem given in [14] for the second Painlevé equation. Here we generalize the construction of [10] to the case of generic values $t_1,\dots,t_{n-1}$.

Its Lie bracket and a precise description of the corresponding loop group can be found in Section 4. The dual space $\mathfrak{g}_-^*$ of $\mathfrak{g}_-$ can be identified with

$$\begin{aligned}\mathfrak{g}_-^* &= \left\{ \Xi(\lambda) = \sum_0^\infty \Xi_i \lambda^i \,|\Xi_i \in \mathfrak{sl}(2,\mathbb{C}),\, \Xi(\lambda)\sigma_1 = -\sigma_1 \Xi(-\lambda) \right\} / \mathfrak{g}_{2n+1}^*, \\ \mathfrak{g}_{2n+1}^* &= \left\{ \Xi(\lambda) = \sum_{2n+1}^\infty \Xi_i \lambda^i \,|\Xi_i \in \mathfrak{sl}(2,\mathbb{C}),\, \Xi(\lambda)\sigma_1 = -\sigma_1 \Xi(-\lambda) \right\} / \mathfrak{g}_{2n+1}^*,\end{aligned}$$

by the killing form in the loop algebra $\tilde{\mathfrak{sl}}(2,\mathbb{C})$ (see (23) below).

More precisely, to interpret (4a) as flow on the $\mathfrak{g}_-^*$, we denote by A the dynamical part of $\mathcal{A}^{(n)}$, i.e. $A = \left(\mathcal{A}^{(n)}\right)_+$, and we define

$$B = \left(\frac{\mathcal{A}^{(n)} \lambda^{1-2n}}{4^n} \right)_-,$$

where given any Laurent series L of λ, $(L)_-$ denotes its strictly negative part. In this way $A \in \mathfrak{g}_-^*$ and $B \in \mathfrak{g}_-$. Then we show that equation (4a) is equivalent to

$$\frac{\mathrm{d}A}{\mathrm{d}z} - \frac{\partial A}{\partial z} = [B, A] = \mathrm{ad}_B^* A, \tag{5}$$

so that the r.h.s. defines a vector field on the coadjoint orbit $\mathcal{O}_A$ of A obtained by fixing the values of the parameters $t_1, \dots, t_{n-1}$ which are the Casimirs of the standard Poisson bracket on $\mathfrak{g}_-^*$. We then prove that the vector field defined by $\mathrm{ad}_B^* A$, is Hamiltonian with Hamiltonian given by

$$H^{(n)} := -\frac{1}{2\,4^n} \mathrm{TrRes} \left(\lambda^{1-2n} \left(\mathcal{A}^{(n)} \right)^2 \right),$$

where Res denotes the formal residue at 0, i.e. the coefficient of the term in λ^{-1}.

Let us now describe our canonical coordinates $P_1, \dots, P_n, Q_1, \dots, Q_n$ on the coadjoint orbit $\mathcal{O}_A$. Our construction is based on the so–called algebro geometric Darboux coordinates. The latter are the projections of the points in the divisor of a line–bundle on the spectral curve [13, 35, 4]. However this classical construction in our case is not straightforward: on one side there are too many points in the divisor, on the other side the dependence of the matrix entries of $\mathcal{A}^{(n)}$ on w and its z–derivatives is very complicated as it involves the recursive relation (1). In particular, this means that the matrix entries are dependent on each other in a complicated way. We have resolved both these problems by expressing the Lenard recursion operator $\mathcal{L}_n$ in terms of $\mathcal{L}_0, \dots, \mathcal{L}_{n-1}$ and their derivatives (see (54) below) to obtain the matrix entries as polynomials of the canonical coordinates (see Theorem 6.1 below). It is worth noting that in [15] and [16], a different set of canonical coordinates on the coadjoint orbits of $\tilde{sl}(2,\mathbb{C})$ was found. Although these preserve the Painlevé property, it is difficult to apply their construction to our case because of the complicated dependencies between the matrix entries of $\mathcal{A}^{(n)}$.

Resuming we prove that the map

$$O_A \to (P_1, \dots, P_n, Q_1, \dots, Q_n) \tag{6}$$

gives a system of rational Darboux coordinates on the coadjoint orbit. However this map depends on $z, t_1, \dots, t_{n-1}$ explicitly. This produces a shift $\delta H^{(n)}$ in the Hamiltonian $H^{(n)}$, so that the Hamiltonian $\mathcal{H}^{(n)}$ appearing in (2) is given by

$$\mathcal{H}^{(n)} = H^{(n)} + \delta H^{(n)}.$$

Thanks to the fact that all formulae are explicit, we can compute this shift (see (66) below).

The idea of interpreting the isomonodromic deformation equations as Hamiltonian flows on the coadjoint orbits of the loop group $\tilde{SL}(2,\mathbb{C})$ was already used by Harnad and Routhier [23] to study the Hamiltonian structure of the six classical Painlevé equations (see also [22]). Later Krichever [27] used the Lax representation approach to construct the isomonodromy equations for meromorphic connections with irregular and regular singularities on algebraic curves.

On the other hand, the coadjoint orbits of the dual loop algebras can also be thought of as moduli spaces of meromorphic connections on Riemann surfaces. Audin [6] generalized the Aityah-Bott symplectic structure on the moduli space of holomorphic connections to the case where the Riemann-surfaces can have boundaries. This symplectic structure was used by Hitchin to study the Schlesinger system [25]. Later, Boalch [7] further generalized this symplectic structure to the moduli space of generic meromorphic connections. He then showed that these moduli spaces are isomorphic to the coadjoint orbits as symplectic manifolds, and that isomonodromic flows induce symplectomorphism between coadjoint orbits at different times. This abstractly indicates that one could express a general isomonodromic deformation as a time-dependent Hamiltonian flow on the dual loop algebra. Woodhouse [42] showed that the isomonodromic deformations with poles fixed are autonomous Hamiltonian systems w.r.t. the Konstant Kirillov Poisson structure on a central extension of the dual loop algebra (which is different from what we are considering in this paper).

Our construction is very explicit and allows us to go further to interpret the time–flows (4b) as flows on the dual loop algebra $\mathfrak{g}_-^*$. This is not trivial because the term $\partial_\lambda \hat{M}^{(k)}$ is not tangent to the coadjoint orbit $\mathcal{O}_A$. To overcome this difficulty we introduce new coordinates $u_1,\dots,u_{2n}$ on the coajoint orbit $\mathcal{O}_A$ and new times $s_1,\dots,s_{n-1}$ such that

$$\partial_{s_k}\mathcal{A}^{(n)} = [L_k,\mathcal{A}^{(n)}] + \partial_\lambda L_k, \qquad L_k = \left(\mathcal{A}^{(n)}\lambda^{1-2k}\right)_+, \quad \partial^u_{s_k}\mathcal{A}^{(n)} = \partial_\lambda L_k, \quad k=1,\dots,n.$$

where $\partial^u_{s_k}$ denotes the partial derivative with respect to s_k when u is fixed.

We then interpret the equation

$$\partial_{s_k}\mathcal{A}^{(n)} - \partial_\lambda L_k = (\partial_{s_k} - \partial^u_{s_k})\mathcal{A}^{(n)} = [L_k,\mathcal{A}^{(n)}]. \tag{7}$$

as flow on the coadjoint orbit $\mathcal{O}_A$ and compute the corresponding Hamiltonians $h_1^{(n)},\dots,h_{n-1}^{(n)}$

$$h_k^{(n)} = \frac{1}{2}\mathrm{TrRes}\left(\lambda^{1-2k}\left(\mathcal{A}^{(n)}\right)^2\right), \qquad k=1,\dots,n,$$

so that in particular $H^{(n)} = -\frac{h_n^{(n)}}{4^n}$. Finally we show that the time–flows Hamiltonians $\mathcal{H}_1^{(n)},\dots,\mathcal{H}_{n-1}^{(n)}$ are given in terms of $h_1^{(n)},\dots,h_{n-1}^{(n)}$ (and their shifts due to the explicit dependence of (6) on $z,t_1,\dots,t_{n-1}$) by a simple formula (see Corollary 8.5 below).

This result gives an insight into how one could express a general isomonodromic deformation as a non–autonomous Hamiltonian system.

Remark 1.1. *Note that $h_1^{(n)},\dots,h_n^{(n)}$ are spectral invariants. In the context of iso–spectral deformations, this can be used to show that the algebro–geometric Darboux coordinates are separated for the isospectral system (see, for example* [37]*). However, in the isomonodromic case, all the functions $h_1^{(n)},\dots,h_n^{(n)}$ are non–autonomous, i.e. they involve the variables $t_1,\dots,t_{n-1}$ and z explicitly. Therefore we don't have separability in the sense of classical mechanics.*

This paper is organized as follows. In Section 2, we recall the derivation of the second Painlevé hierarchy as self–similarity reduction of the mKdV hierarchy. In Section 3, we describe the monodromy problem associated to the second Painlevé hierarchy. In Section 4, we introduce our twisted loop algebra, interpret equation (4a) as flow on its dual space $\mathfrak{g}_-^*$ study the Poisson bracket on $\mathfrak{g}_-^*$. We prove that the parameters $t_1,\dots,t_{n-1}$ belong to the kernel of such Poisson bracket and characterize the symplectic leaves. In Section 5, we introduce our coordinates $P_1,\dots,P_n,Q_1,\dots,Q_n$ and prove that they are canonical with respect to the symplectic form on the coadjoint orbit. In Section 6, we obtain formula (54) expressing the Lenard recursion operator $\mathcal{L}_n$ in terms of $\mathcal{L}_0,\dots,\mathcal{L}_{n-1}$ and their derivatives and we give the explicit formulae for the matrix entries of $\mathcal{A}^{(n)}$ in terms of our canonical coordinates. We give an explicit example to illustrate our procedure in detail. In Section 7, we compute the Hamiltonians $\mathcal{H}^{(n)}$. In Section 8, we interpret equation (7) as flow on the coadjoint orbit, compute the corresponding Hamiltonians $h_1^{(n)},\dots,h_{n-1}^{(n)}$ and show that they are spectral invariants. Finally we obtain the Hamiltonians $\mathcal{H}_1^{(n)},\dots,\mathcal{H}_{n-1}^{(n)}$. We follow all details of our construction in an explicit example.

Acknowledgments. The authors are grateful to H. Flaschka, M. Talon, J. Harnad, A. Hone and to V. Kuznetsov for helpful conversations. This research was sponsored by the ESF grant MISGAM, the Marie Curie network ENIGMA and by EPSRC.

2. The second $\mathrm{P_{II}}$ hierarchy

The second Painlevé hierarchy is obtained as *self-similarity reduction* of the modified Korteweg-de Vries (mKdV) hierarchy (see [2, 5, 14]):

$$\frac{\partial}{\partial T_{n+1}}v + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{R}_n\left[v_x - v^2\right] = 0, \qquad n=0,1,2,3,\dots \tag{8}$$

where $\mathcal{R}_n$ satisfies the Lenard recursion relation [30]

$$\frac{\partial}{\partial x}\mathcal{R}_{n+1} = \left(\frac{\partial^3}{\partial x^3} + 4(v_x - v^2)\frac{\partial}{\partial x} + 2(v_x - v^2)_x\right)\mathcal{R}_n, \qquad \mathcal{R}_0[u] = \tfrac{1}{2}. \tag{9}$$

Each equation in the mKdV hierarchy defines a Hamiltonian flow and can be viewed as a symmetry for all others. Consider the space of stationary solutions w.r.t. the symmetry defined by the n-th mKdV equation, i.e. the space of solutions $v(x; T_1, T_2, \dots)$ such that $\frac{\partial v}{\partial T_{n+1}} = 0$. Due to the fact that all Hamiltonian flows commute, all other elements of the hierarchy can be restricted to this space.

There are also other symmetries acting on the mKdV hierarchy. They are called Virasoro symmetries. The n-th Virasoro symmetry is given by the following infinitesimal generator

$$\frac{\mathrm{d}}{\mathrm{d}s_n} := \sum_{l=0}^{n} (2l+1) T_{l+1} \frac{\partial}{\partial T_{l+1}}.$$

The stationary solutions w.r.t. this generator are by definition such that $\frac{\mathrm{d}v}{\mathrm{d}s_n} \equiv 0$. They satisfy

$$\frac{\mathrm{d}v}{\mathrm{d}s_n} = -\sum_{l=0}^{n} (2l+1) T_{l+1} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{R}_l \left[v_x - v^2 \right] = 0,$$

and after integration

$$-\sum_{l=0}^{n} (2l+1) T_{l+1} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{R}_l \left[v_x - v^2 \right] = \alpha_n, \tag{10}$$

where α_n is some constant.[2] From the $n = 0$ equation of the mKdV hierarchy, we can set $T_1 = -x$ so that (10) is an ODE in the variable x depending on some extra parameters $T_2, \dots, T_{n+1}$. The parameter T_{n+1} can be absorbed by the following symmetry reduction (see [9] for details):

$$v(x, T_{n+1}) = \frac{w(z)}{\left[(2n+1)T_{n+1}\right]^{1/(2n+1)}}, \qquad z = \frac{x}{\left[(2n+1)T_{n+1}\right]^{1/(2n+1)}}, \tag{11a}$$

$$\mathcal{R}_l \left[v_x - v^2 \right] = \frac{1}{\left[(2n+1)T_{n+1}\right]^{2l/(2n+1)}} \mathcal{L}_l [w_z - w^2], \tag{11b}$$

$$t_0 = -z, \qquad t_l := \frac{(2l+1)T_{l+1}}{\left[(2n+1)T_{n+1}\right]^{(2l+1)/(2n+1)}}, \quad l = 1, \dots, n, \quad t_n = 1. \tag{11c}$$

In this way we obtain the Second Painleve Hierarchy:[3]

$$\mathrm{P}_{\mathrm{II}}^{(n)}: \quad \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w \right) \mathcal{L}_n \left[w_z - w^2 \right] + \sum_{l=1}^{n-1} t_l \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w \right) \mathcal{L}_l \left[w_z - w^2 \right] = zw + \alpha_n, \qquad n \geq 1, \tag{12}$$

where α_n are constants and $\mathcal{L}_n$ is the operator defined by

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathcal{L}_{n+1} = \left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + 4(w_z - w^2) \frac{\mathrm{d}}{\mathrm{d}z} + 2(w_z - w^2)_z \right) \mathcal{L}_n, \qquad \mathcal{L}_0[w_z - w^2] = \tfrac{1}{2}. \tag{13}$$

Example 2.1. For $n = 1$, equation (12) is P_{II}:

$$w_{zz} - 2w^3 = zw + \alpha_1.$$

For $n = 2$, it is:

$$t_1(w_{zz} - 2w^3) + (w_{zzzz} - 10ww_z^2 - 10w^2 w_{zz} + 6w^5) = zw + \alpha_2.$$

In the case of the Virasoro symmetries, only the first n flows of the mKdV hierarchy can be restricted to the space of such stationary solutions. These give the flows in $t_1, \dots, t_{n-1}$:

$$(2k+1) \frac{\partial w}{\partial t_k} + \partial_z \left(\partial_z + 2w \right) \mathcal{L}_k \left[w_z - w^2 \right] = 0, \qquad k = 1, \dots, n-1. \tag{14}$$

We shall call the flows in $t_1, \dots, t_{n-1}$ *time-flows.*

---

[2]Recently S. Kakei [26], proposed a new way to obtain the second Painlevé equation directly as a reduction of mKdV, without integration. In his difference–operator formulation, the constant α_1 appears as a parameter in the symmetry reduction, and not as integration constant. It would be interesting to see whether this construction can be used to produce the whole P_{II} hierarchy.

[3]To obtain exact solutions of the n-th mKdV equation, one fixes the values of $t_1, \dots, t_{n-1}$. As a consequence, often in the literature the second Painlevé hierarchy is presented with $t_1 = \dots = t_{n-1} = 0$. We will leave $t_1, \dots, t_{n-1}$ free to vary instead.

Remark 2.2. *Another Painlevé hierarchy containing* $\mathrm{P_{II}}$ *as its first element has been introduced by Gordoa, Joshi and Pickering* [20] *by generalizing the isomonodromic deformation equations by Jimbo Miwa. However it is not clear if their hierarchy is different or not from the one studied in this paper. We know that Koike from Kyoto University is building the Hamiltonian structure of Gordoa, Joshi and Pickering hierarchy as confluence procedure of the Garnier systems. Once he is successful, we may try to see whether our Hamiltonian system and Koike's are related by a canonical transformation. If not, the question of whether the* $\mathrm{P_{II}}$ *hierarchy considered in this paper may or may not arise as confluence limit of the Garnier system remains open.*

3. Isomonodromic Problem for the $\mathrm{P_{II}}$ Hierarchy

The isomonodromic deformation problem for the second Painlevé hierarchy with $t_1 = \cdots = t_{n-1} = 0$ was derived in [10] and in [28] following the approach proposed in [1] starting form the isomonodromy deformation problem given in [14] for the second Painlevé equation. Here we generalize the construction of [10] to the case of generic values $t_1, \ldots, t_{n-1}$ (for details see the Appendix A).

The isomonodromic deformation problem for the $\mathrm{P_{II}}$ Hierarchy is the following:

$$\frac{\partial \Psi}{\partial z} = \mathcal{B}\Psi = \begin{pmatrix} -\lambda & w \\ w & \lambda \end{pmatrix} \Psi, \tag{15a}$$

$$\frac{\partial \Psi}{\partial \lambda} = \mathcal{A}^{(n)}\Psi = \frac{1}{\lambda}\left[\begin{pmatrix} -\lambda z & -\alpha_n \\ -\alpha_n & \lambda z \end{pmatrix} + M^{(n)} + \sum_{l=1}^{n-1} t_l M^{(l)}\right]\Psi, \tag{15b}$$

$$(2k+1)\frac{\partial \Psi}{\partial t_k} = \left(M^{(k)} - \begin{pmatrix} 0 & (\partial_z + 2w)\mathcal{L}_k \\ (\partial_z + 2w)\mathcal{L}_k & 0 \end{pmatrix}\right)\Psi, \tag{15c}$$

where

$$M^{(l)} = \begin{pmatrix} \sum_{j=1}^{2l+1} A_j^{(l)}\lambda^j & \sum_{j=1}^{2l} B_j^{(l)}\lambda^j \\ \sum_{j=1}^{2l} C_j^{(l)}\lambda^j & -\sum_{j=1}^{2l+1} A_j^{(l)}\lambda^j \end{pmatrix},$$

with

$$A_{2l+1}^{(l)} = 4^l, \qquad A_{2k} = 0, \quad \forall\, k = 0, \ldots, l, \tag{16a}$$

$$A_{2k+1}^{(l)} = \frac{4^{k+1}}{2}\left\{\mathcal{L}_{l-k}\left[w_z - w^2\right] - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_{l-k-1}\left[w_z - w^2\right]\right\}, \quad k = 0, \ldots, l-1, \tag{16b}$$

$$B_{2k+1}^{(l)} = \frac{4^{k+1}}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_{l-k-1}\left[w_z - w^2\right], \quad k = 0, \ldots, l-1, \tag{16c}$$

$$B_{2k}^{(l)} = -4^k\left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_{l-k}\left[w_z - w^2\right], \quad k = 1, \ldots, l, \tag{16d}$$

$$C_{2k+1}^{(l)} = -B_{2k+1}, \quad k = 0, \ldots, l-1, \tag{16e}$$

$$C_{2k}^{(l)} = B_{2k}, \quad k = 0, \ldots, l. \tag{16f}$$

The compatibility between (15a) and (15b) gives

$$\frac{\partial \mathcal{A}^{(n)}}{\partial z} - \frac{\partial \mathcal{B}}{\partial \lambda} = [\mathcal{B}, \mathcal{A}^{(n)}], \tag{17}$$

which gives (12) (see [10]). Equation (14) is obtained as compatibility between (15a) and (15c):

$$(1+2k)\partial_{t_k}\mathcal{B} - \partial_z \hat{M}^{(k)} = [\mathcal{B}, \hat{M}^{(k)}], \tag{18}$$

where for brevity we put

$$\hat{M}_k = M^{(k)} - \begin{pmatrix} 0 & (\partial_z + 2w)\mathcal{L}_k \\ (\partial_z + 2w)\mathcal{L}_k & 0 \end{pmatrix}.$$

Finally the compatibility between (15b) and (15c) is

$$\partial_\lambda \hat{M}^{(k)} - (2k+1)\partial_{t_k}\mathcal{A}^{(n)} = -[\hat{M}^{(k)}, \mathcal{A}^{(n)}]. \tag{19}$$

The proof of the fact that equations (17), (18) and (19) are indeed consistent is sketched in the Appendix A.

To simplify our computations, it is convenient to introduce some new notations. We define

$$a_{2k+1}^{(n)} = \sum_{l=1}^{n} t_l A_{2k+1}^{(l)}, \qquad k=1,\dots,n, \qquad a_1^{(n)} = \sum_{l=1}^{n} t_l A_1^{(l)} - z,$$
$$b_{2k+1}^{(n)} = \sum_{l=1}^{n} t_l B_{2k+1}^{(l)}, \qquad k=0,\dots,n-1,$$
$$b_{2k}^{(n)} = \sum_{l=1}^{n} t_l B_{2k}^{(l)}, \qquad k=1,\dots,n, \qquad b_0^{(n)} = -\alpha_n,$$

where $t_n = 1$, so that we can write

$$\mathcal{A}^{(n)} := \begin{pmatrix} \sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k} & \sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1} + \sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k} \\ \sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1} - \sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k} & -\sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k} \end{pmatrix}. \tag{20}$$

4. Coadjoint orbit interpretation

In this section we show that the Hamiltonian structure of the second Painlevé hierarchy can be derived from the one on an appropriate dual loop algebra.

Since our matrices $\mathcal{A}^{(n)}$ depend on z and λ and the variable z appears both implicitly, through $w(z)$ and its derivatives, and explicitly, we need to introduce some notation. Given any function f of $\lambda, z, w, w_z, \dots$, let us denote the partial derivative of f w.r.t. z as follows:

$$\partial_z f := \frac{\partial f}{\partial z} + \frac{\partial f}{\partial w} w_z + \frac{\partial f}{\partial w_z} w_{zz} + \dots,$$

and $\partial_z^w f$ the partial derivative of f considered as a differential polynomial of w depending on z, λ:

$$\partial_z^w f := \frac{\partial f}{\partial z}.$$

Analogously $\partial_\lambda f$ denotes the partial derivative of f w.r.t. λ. Then given the matrices $\mathcal{B}$ and $\mathcal{A}^{(n)}$ as in (15) and (16), one has

$$\partial_z^w \mathcal{A}^{(n)} = \partial_\lambda \mathcal{B}, \tag{21}$$

so that equation (17) is equivalent to

$$(\partial_z - \partial_z^w)\mathcal{A}^{(n)} = [\mathcal{B}, \mathcal{A}^{(n)}]. \tag{22}$$

Remark 4.1. *This phenomenon, i.e. equation (21), is a common feature of all Painlevé equations and it was used in* [23] *to find the algebro–geometric Darboux coordinates for the six Painlevé equations. As far as we know, a proof of the fact that (21) is a common feature of all the isomonodromic deformations equations is still missing.*

We are now going to interpret the evolution along $(\partial_z - \partial_z^w)$ as a vector field on a coadjoint orbit of an element of an appropriate twisted loop algebra. Let LG be the group of smooth maps f from S^1 to SL_2 such that

$$f(\lambda)\sigma_1(f(-\lambda))^{-1} = I, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and λ is considered as a parameter on S^1. Denote by $L_{2n+2}G$ the subgroup of maps of the form $f = I + \lambda^{-2n-2} f_\infty$, where f_∞ is holomorphic outside S^1 and let $\mathfrak{g}_{2n+2}$ be its Lie algebra:

$$\mathfrak{g}_{2n+2} = \left\{ X(\lambda) = \sum_{-\infty}^{-2n-2} X_i \lambda^i | X_i \in \mathfrak{sl}(2,\mathbb{C}),\ X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right\}.$$

Then let G be the quotient of these 2 groups

$$G = LG / L_{2n+2}G.$$

Its Lie algebra is given by

$$\mathfrak{g} = \left\{ X(\lambda) = \sum_{-\infty}^{\infty} X_i \lambda^i | X_i \in \mathfrak{sl}(2,\mathbb{C}),\ X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right\} / \mathfrak{g}_{2n+2},$$

with Lie bracket defined as:

$$[X(\lambda),\tilde X(\lambda)]=\sum_{i=-2n-1}^{\infty}\left(\sum_{k=-2n-1}^{i+2n+1}[X_k,\tilde X_{i-k}]\right)\lambda^i \mod \mathfrak{g}_{2n+2},$$

which obviously gives $[X(\lambda),\tilde X(\lambda)]\in\mathfrak{g}$ and satisfies the Jacobi identity. The dual space $\mathfrak{g}^*$ can be identified with

$$\begin{aligned}\mathfrak{g}^* &= \left\{\Xi(\lambda)=\sum_{-\infty}^{\infty}\Xi_i\lambda^i \mid N\in\mathbb{N},\, \Xi_i\in\mathfrak{sl}(2,\mathbb{C}),\, \Xi(\lambda)\sigma_1=-\sigma_1\Xi(-\lambda)\right\}/\mathfrak{g}^*_{2n+2},\\ \mathfrak{g}^*_{2n+2} &= \left\{X(\lambda)=\sum_{2n+1}^{\infty}X_i\lambda^i|X_i\in\mathfrak{sl}(2,\mathbb{C}),\, X(\lambda)\sigma_1=\sigma_1X(-\lambda)\right\}.\end{aligned}$$

by the following pairing

$$\langle X(\lambda),\Xi(\lambda)\rangle := \mathrm{Tr}\left(\mathrm{Res}X(\lambda)\Xi(\lambda)\right),\quad \forall\, X(\lambda)\in\mathfrak{g},\, \Xi(\lambda)\in\mathfrak{g}^*, \tag{23}$$

where Res indicates the formal residue, i.e. the coefficient of the λ^{-1} term. Consider the subalgebra

$$\mathfrak{g}_- = \left\{X(\lambda)=\sum_{-\infty}^{-1}X_i\lambda^i|X(\lambda)\sigma_1=\sigma_1X(-\lambda)\right\}/\mathfrak{g}_{2n+2}, \tag{24}$$

its dual space can be identified with

$$\mathfrak{g}^*_- = \left\{\Xi(\lambda)=\sum_{0}^{\infty}\Xi_i\lambda^i|\Xi_i\in\mathfrak{sl}(2,\mathbb{C}),\, \Xi(\lambda)\sigma_1=-\sigma_1\Xi(-\lambda)\right\}/\mathfrak{g}^*_{2n+2}. \tag{25}$$

An element X in the Lie algebra $\mathfrak{g}$ acts on an element $\Xi\in\mathfrak{g}^*$ by the coadjoint action

$$\langle \mathrm{ad}^*_X\Xi, Y\rangle := -\langle\Xi,[X,Y]\rangle = \langle[X,\Xi],Y\rangle \tag{26}$$

for any $Y\in\mathfrak{g}$. This shows that for every $X\in\mathfrak{g}$, $\Xi\in\mathfrak{g}^*$

$$[X,\Xi]=\mathrm{ad}^*_X\Xi\in\mathfrak{g}^*.$$

When we restrict the coadjoint action to the subalgebra $\mathfrak{g}_-$ and to its dual space $\mathfrak{g}^*_-$, we obtain the following identification

$$[X_-,\Xi]_+ = \mathrm{ad}^*_{X_-}\Xi,\quad \Xi\in\mathfrak{g}^*_-,\quad X_-\in\mathfrak{g}_- \tag{27}$$

where $(\cdot)_+$ is the projection from $\mathfrak{g}^*$ onto $\mathfrak{g}^*_-$ and $(\cdot)_-$ denotes the projection onto $\mathfrak{g}_-$.

Lemma 4.2. *Given the matrices $\mathcal{B}$ and $\mathcal{A}^{(n)}$ as in (15) and (16), one has*

$$[\mathcal{B},\mathcal{A}^{(n)}]=\mathrm{ad}^*_BA, \tag{28}$$

where $B=\left(\frac{\mathcal{A}^{(n)}\lambda^{-2n+1}}{4^n}\right)_-\in\mathfrak{g}_-$ and $A=\left(\mathcal{A}^{(n)}\right)_+\in\mathfrak{g}^*_-$, which is the dynamical part of $\mathcal{A}^{(n)}$.

Proof. Using (15) and (16), we notice that:

$$\mathcal{B}=-\left(\frac{\mathcal{A}^{(n)}\lambda^{-2n+1}}{4^n}\right)_+.$$

Then using the Drinfeld–Sokolov trick:

$$\begin{aligned}[\mathcal{B},\mathcal{A}^{(n)}] &= -\left[\left(\frac{\mathcal{A}^{(n)}\lambda^{-2n+1}}{4^n}\right)_+,\mathcal{A}^{(n)}\right]=\\ &= \left[\left(\frac{\mathcal{A}^{(n)}\lambda^{-2n+1}}{4^n}\right)_-,\mathcal{A}^{(n)}\right]=\\ &= \left[\left(\frac{\mathcal{A}^{(n)}\lambda^{-2n+1}}{4^n}\right)_-,\left(\mathcal{A}^{(n)}\right)_+\right],\end{aligned} \tag{29}$$

where the last step is due to the fact that $\left(\frac{\mathcal{A}^{(n)}\lambda^{-2n+1}}{4^n}\right)_-$ commutes with $\left(\mathcal{A}^{(n)}\right)_-$. ☐

Remark 4.3. *The fact that we can neglect the singular part of $\mathcal{A}^{(n)}$ at $\lambda = 0$ is more general than in the above proof. Suppose we want to compute $\mathrm{ad}^*_{X_k}\mathcal{A}^{(n)}$ for $X_k = \left(\frac{\mathcal{A}_t^{(n)}\lambda^{-2k+1}}{4^k}\right)_-$, then for every $Y \in \mathfrak{g}_-$ we have*

$$\begin{aligned}\langle \mathrm{ad}^*_{X_k}\mathcal{A}^{(n)}, Y\rangle &= \left\langle \left[\left(\frac{\mathcal{A}^{(n)}\lambda^{-2k+1}}{4^k}\right)_-, \mathcal{A}^{(n)}\right], Y\right\rangle \\ &= \left\langle \left[\left(\frac{\mathcal{A}^{(n)}\lambda^{-2k+1}}{4^k}\right)_-, \left(\mathcal{A}^{(n)}\right)_+\right], Y\right\rangle,\end{aligned}$$

because the singular part of $\mathcal{A}^{(n)}$ does not contribute to the residue.

Similarly it is easy to see that

$$(\partial_z - \partial_z^w)\,\mathcal{A}^{(n)} = (\partial_z - \partial_z^w)\,A.$$

Resuming, we proved the following Lemma:

Lemma 4.4. *The monodromy preserving deformation equation (17) is the same as*

$$(\partial_z - \partial_z^w)\,A = \mathrm{ad}^*_B A, \tag{30}$$

where $A = \left(\mathcal{A}^{(n)}\right)_+ \in \mathfrak{g}^*_-$ is the dynamical part of $\mathcal{A}^{(n)}$, and $B = \left(\frac{\mathcal{A}^{(n)}\lambda^{1-2n}}{4^n}\right)_- \in \mathfrak{g}_-$.

This Lemma allows us to interpret the evolution along $(\partial_z - \partial_z^w)$ as a vector field on a coadjoint orbit of the twisted loop algebra $\mathfrak{g}_-$.

Let us now recall the Poisson structure on $\mathfrak{g}^*_-$. This is fairly standard (see for example the beautiful book [8]), but we recall some details here in order to fix notations and adapt the computations to our special case.

The Poisson structure on $\mathfrak{g}^*_-$ is given by observing that every $X \in \mathfrak{g}_-$ defines a linear function X_* on $\mathfrak{g}^*_- \ni \Xi$:

$$X_* : \begin{array}{ccc} \mathfrak{g}^*_- & \to & \mathbb{C} \\ \Xi & \to & \langle \Xi, X\rangle. \end{array}$$

This fact allows one to identify ${\mathfrak{g}^*_-}^*$ with $\mathfrak{g}_-$ and to define the Poisson bracket between two linear functions on $\mathfrak{g}^*_-$ as S. Lie did

$$\{X_*, Y_*\}(\Xi) := \langle \Xi, [X, Y]\rangle\,.$$

The Poisson bracket between two functions f and g on $\mathfrak{g}^*_-$ is given by

$$\{f, g\}\,(\Xi) = \langle \Xi, [\mathrm{d}f, \mathrm{d}g]\rangle \tag{31}$$

where the differential $\mathrm{d}f$ of a function f on $\mathfrak{g}^*_-$ is a linear function $\mathrm{d}f \in {\mathfrak{g}^*_-}^* \sim \mathfrak{g}_-$ defined by

$$\langle \mathrm{df}, \delta\Xi\rangle := f\,(\Xi + \delta_X\Xi) - f(\Xi) + \mathcal{O}(\delta_X\Xi)^2, \tag{32}$$

where

$$\delta_X\Xi := \mathrm{ad}^*_X\Xi \in \mathfrak{g}^*_-.$$

In particular one has $\mathrm{d}X_* = X$.

It is well known that this Poisson bracket is degenerate and its symplectic leaves are the coadjoint orbits of its elements. The kernel of this bracket consisting of the *Casimirs,* i.e. functions f such that

$$\mathrm{ad}^*_X\Xi(\mathrm{d}f) = 0, \qquad \forall X \in \mathfrak{g}_-,\ \Xi \in \mathfrak{g}^*_-.$$

Lemma 4.5. *The times $t_1, \ldots, t_{n-1}$ are the Casimirs of the Poisson bracket (31).*

Proof. We show that the Hamiltonian vector fields generated by $t_1, \ldots, t_{n-1}$ are zero.

Denote the eigenvalues of $\mathcal{A}(\lambda)$ by $\pm\mu(\lambda)$. The polynomial part of $\mu(\lambda)$ is a polynomial of order $2n$ and all the coefficients of the odd positive powers of λ are zero. Therefore the polynomial part of $\mu(\lambda)$ has exactly only $n+1$ non zero coefficients, of which the first one is 4^n. Our claim is that all the other coefficients of the positive powers of λ give our times $t_1, \ldots, t_{n-1}$ (we shall see in Section 7 that the coefficient of the -2 power of λ is the Hamiltonian in the variable z).

In fact, as proved in corollary 8.2 below, the times t_i are given by the 'spectral residue formula' [24] as follows.

$$t_l = \frac{1}{4^l}\mathrm{Res}_\infty \lambda^{-2l-1}\mu(\lambda)d\lambda = \langle A, \frac{\lambda^{-2l-1}}{2\,4^l}\Psi\sigma_3\Psi^{-1},\rangle$$

where Ψ is the eigenvector matrix of $\mathcal{A}$ and $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Since $\frac{\lambda^{-2l-1}}{2\,4^l}\Psi\sigma_3\Psi^{-1}$ commutes with $\mathcal{A}(\lambda)$, it produces a trivial vector field and therefore t_l is a Casimir. □

Thanks to the above Lemma, the coadjoint orbits are obtained by fixing the values of $t_1,\dots,t_{n-1}$:

$$\mathcal{O}_A:=\left\{\left(\begin{array}{cc}\sum_{k=0}^n a^{(n)}_{2k+1}\lambda^{2k} & \sum_{k=1}^n b^{(n)}_{2k}\lambda^{2k-1}+\sum_{k=0}^{n-1} b^{(n)}_{2k+1}\lambda^{2k}\\ \sum_{k=1}^n b^{(n)}_{2k}\lambda^{2k-1}-\sum_{k=0}^{n-1} b^{(n)}_{2k+1}\lambda^{2k} & -\sum_{k=0}^n a^{(n)}_{2k+1}\lambda^{2k}\end{array}\right)\Bigg|_{t_1=t_1^0,\dots,t_{n-1}=t^0_{n-1}}\right\}$$

The dimension of the coadjoint orbits is $2n$. It is well known that the Poisson bracket (31) restricted to the coadjoint orbits is non-degenerate, so that it defines the so-called Kostant-Kirillov symplectic structure ω on them. We will write

$$\omega(f,g)=\{f,g\},$$

for every pair of functions on the coadjoint orbit.

To compute the Poisson brackets between the coefficients $a^{(n)}_{2k+1}$, $b^{(n)}_{2k}$ and $b^{(n)}_{2k+1}$, we observe that their differentials are

$$\begin{array}{rcll}\mathrm{d}a^{(n)}_{2k+1} &=& \frac{1}{2}(E_{11}-E_{22})\lambda^{-(2k+1)} & \text{for}\quad 0\le k\le n,\\ \mathrm{d}b^{(n)}_{2k+1} &=& \frac{1}{2}(-E_{12}+E_{21})\lambda^{-(2k+1)} & \text{for}\quad 0\le k\le n-1,\\ \mathrm{d}b^{(n)}_{2k} &=& \frac{1}{2}(E_{12}+E_{21})\lambda^{-2k} & \text{for}\quad 0\le k\le n,\end{array}\tag{33}$$

where with a slight abuse of notation we are calling $a^{(n)}_{2k+1}$, $b^{(n)}_{2k}$ and $b^{(n)}_{2k+1}$ the elements of $\mathfrak{g}_-^{*\,*}\sim\mathfrak{g}_-$ which applied to Ξ produce the coefficients $a^{(n)}_{2k+1}$, $b^{(n)}_{2k}$ and $b^{(n)}_{2k+1}$ respectively.

By using these gradients, we can compute the Poisson brackets between the matrix entries

$$\begin{array}{rcll}\left\{a^{(n)}_{2k+1},b^{(n)}_{2l+1}\right\} &=& -b^{(n)}_{2(k+l+1)}, & \text{for}\quad 0\le k\le n,\quad 0\le l\le n-1,\quad k+l\le n-1,\\ \left\{a^{(n)}_{2k+1},b^{(n)}_{2l}\right\} &=& -b^{(n)}_{2(k+l)+1} & \text{for}\quad 0\le k\le n,\quad 1\le l\le n,\quad k+l\le n-1,\\ \left\{b^{(n)}_{2k},b^{(n)}_{2l+1}\right\} &=& a^{(n)}_{2(k+l)+1} & \text{for}\quad 1\le k\le n,\quad 0\le l\le n-1,\quad k+l\le n,\end{array}\tag{34}$$

while all the other brackets vanish.

5. Canonical coordinates for the isomonodromic deformations

Our first attempt to build the canonical coordinates for the second Painlevé hierarchy is to use the general framework of the algebro–geometric Darboux coordinates (see [13, 35, 4]). In this setting one considers the spectral curve

$$\Gamma(\mu,\lambda)=\left\{\det(\mu-\mathcal{A}^{(n)}(\lambda))=0\right\}=\left\{\mu^2=-\det\left(\mathcal{A}^{(n)}(\lambda)\right)\right\}.\tag{35}$$

The *characteristic equation* $\mu^2=-\det\left(\mathcal{A}^{(n)}(\lambda)\right)$ defines the eigenvalue $\mu(\lambda)$ of $A(\lambda)$ as a function on the corresponding 2-sheeted Riemannian surface of genus g. The *Baker–Akhiezer function* $\psi(\lambda)$ is defined then as the eigenvector of $\mathcal{A}^{(n)}(\lambda)$

$$\mathcal{A}^{(n)}(\lambda)\psi(\lambda)=\mu(\lambda)\psi(\lambda)$$

corresponding to the eigenvalue $\mu(\lambda)$. Generally, ψ has $g+1$ poles.

Following [37], let us briefly illustrate how to construct canonical coordinates $p_1,\dots,p_g$ and $q_1,\dots,q_g$ on the cotangent bundle of the Jacobian T^*J of the curve Γ.

Denote by q the λ–projection of the generic point in the divisor of ψ. We fix the following normalization

$$(c_1,c_2)\cdot\psi(q)=1,$$

for some choice of c_1,c_2. The q_j variables are the roots of

$$c_1^2A_{12}(q_j)-c_1c_2(A_{11}(q_j)-A_{22}(q_j))-c_2^2A_{21}(q_j)=0,\tag{36}$$

while the p_j variables are the eigenvalues

$$p_j=\left(A_{11}(q_j)-\frac{c_1}{c_2}A_{12}(q_j)\right).\tag{37}$$

Choosing the normalization $c_1 = -c_2 = 1$ we get roots $q_1, \dots, q_{2n}$ such that $q_{n+j} = -q_j$, $j = 1, \dots, n$. They are the roots of the following equation:

$$\sum_{k=0}^{n-1} (b^{(n)}_{2k+1} + a^{(n)}_{2k+1})\lambda^{2k} + a^{(n)}_{2n+1}\lambda^{2n} = 0. \tag{38}$$

The corresponding p_j are given by

$$p_j = \sum_{k=0}^{n} b^{(n)}_{2k} q_j^{2k-1}.$$

In the generic case, it is well-known that the coordinates $q_1, \dots, q_{2n}$, $p_1, \dots, p_{2n}$ are canonical with respect to the Konstant-Kirillov Poisson structure as well

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \tag{39}$$

however, a proof of this fact for non–generic cases is still missing.

Generically, the dimension $2g$ of T^*J coincides with the dimension of the symplectic leaves in the coadjoint orbit associated to (30). This allows one to identify these symplectic leaves with T^*J and to treat $p_1, \dots, p_g$ and $q_1, \dots, q_g$ as canonical coordinates on the symplectic leaves themselves.

In our case instead, it is not hard to realize that the suitably de-singularised spectral curve Γ is an hyperelliptic curve of genus $g = 2n$, so that $\dim(T^*J) = 4n$, which is twice the dimension of our symplectic leaves. In fact the characteristic equation has the following form

$$\mu^2 = 4^{2n}\lambda^{4n} + Pol_{2n-1}(\lambda^2) + \frac{\alpha_n^2}{\lambda^2},$$

where Pol_{2n-1} is a polynomial of degree $2n-1$. By doing if necessary a small monodromy preserving deformation, we can assume this polynomial to be irreducible. Setting $\tilde{\mu} = \lambda\mu$, we get

$$\tilde{\mu}^2 = Pol_{4n+2}(\lambda),$$

where Pol_{4n+2} is an irreducible polynomial in λ of degree $4n+2$. We see that the genus is $2n$.

Another problem is that the coordinates $q_1, \dots, q_g$ are defined by taking the roots of the polynomial (38), so they may not satisfy the Painlevé property of the isomonodromic deformations equations (see [32, 33]). Therefore we propose a new set of canonical coordinates:

Theorem 5.1. *Consider the following*

$$P_k = \Pi_{2k} = \frac{a^{(n)}_{2(n-k)+1} + b^{(n)}_{2(n-k)+1}}{a^{(n)}_{2n+1}}, \qquad Q_k = \sum_{j=1}^{n} \frac{1}{2j} b^{(n)}_{2j} \frac{\partial S_{2j}}{\partial \Pi_{2k}}, \qquad k = 1, \dots, n,$$

where $S_k = \sum_{j=1}^{2n} q_j^k$ *for* $k = 1, \dots, 2n$ *and* $\Pi_1, \dots, \Pi_{2n}$ *are the symmetric functions of* $q_1, \dots, q_{2n}$*:*

$$\Pi_1 = q_1 + q_2 + \cdots + q_{2n}, \quad \Pi_2 = \sum_{1 \le j < k \le 2n} q_j q_k, \quad \dots, \quad \Pi_{2n} = q_1 q_2 \dots q_{2n}.$$

Then

(1) $P_1, \dots, P_n, Q_1, \dots, Q_n$ *are coordinates in the symplectic leaves.*
(2) $P_1, \dots, P_n, Q_1, \dots, Q_n$ *are canonical, namely*

$$\{P_i, P_j\} = \{Q_i, Q_j\} = 0, \qquad \{P_i, Q_j\} = \delta_{ij}.$$

Remark 5.2. *Observe that the first statement of our theorem could have been guessed by noticing that, as in the case of the Kowalevski top,* Γ *admits one extra symmetry* $\lambda \to -\lambda$ *apart from the hyperelliptic involution* $\mu \to -\mu$*. As a consequence* Γ *is a two-sheeted covering of a genus* n *hyperelliptic curve* C *obtained by setting* $z = \lambda^2$*:*

$$C = \left\{ \mu^2 = 4^{2n} z^{2n} + Pol_{2n-1}(z) + \frac{\alpha_n^2}{z} \right\}.$$

Differently from the case of the Kowalevski top, this cover is branched at 0 *and at* ∞*. Having fixed the normalization* $c_1 = -c_2 = 1$*, we see that the Baker-Akhiezer function* ψ *has a simple pole at* $\lambda = \infty$ *and* $2n$ *simple poles at* $\lambda = q_1, \dots, q_{2n}$*. This pole divisor is clearly invariant w.r.t. the involution* $\lambda \to -\lambda$*. In fact the poles* $\lambda = q_1, \dots, q_{2n}$ *come in pairs* q_j, q_{n+j} *and each pair projects to one pole on* C*. Due to the* *construction by A. Weil (see for example* [34]*), the symmetric functions of the pole divisor on C appear naturally as coordinates when endowing $Jac(C)$ with the structure of algebraic variety.*[4]

Proof. The first statement of the theorem follows as a straightforward corollary of Theorem 6.1 proved in Section 6 below.

Let us prove the second statement of our theorem, i.e. that our coordinates $P_1,\dots,P_n,Q_1,\dots,Q_n$ are canonical.

To compute the Poisson brackets between our coordinates, we observe that $\Pi_{2k+1}=S_{2k+1}=0$ in our case and

$$\Pi_{2k}=\frac{a^{(n)}_{2(n-k)+1}+b^{(n)}_{2(n-k)+1}}{a^{(n)}_{2n+1}},\qquad k=1,\dots,n.$$

First let us compute the bracket $\{P_k,P_l\}$

$$\begin{aligned}\{P_k,P_l\} &= \left\{\frac{a^{(n)}_{2(n-k)+1}+b^{(n)}_{2(n-k)+1}}{a^{(n)}_{2n+1}},\frac{a^{(n)}_{2(n-l)+1}+b^{(n)}_{2(n-l)+1}}{a^{(n)}_{2n+1}}\right\}=\\ &= \left(\frac{1}{a^{(n)}_{2n+1}}\right)^2\left(\left\{b^{(n)}_{2(n-l)+1},a^{(n)}_{2(n-k)+1}\right\}-\left\{b^{(n)}_{2(n-k)+1},a^{(n)}_{2(n-l)+1}\right\}\right)=\\ &= \left(\frac{1}{a^{(n)}_{2n+1}}\right)^2\left(b^{(n)}_{2(2n-l-k+1)}-b^{(n)}_{2(2n-l-k+1)}\right)=0.\end{aligned}$$

Therefore we have

$$\{P_k,P_l\}=\{\Pi_{2k},\Pi_{2l}\}=0,\qquad k,l=1,\dots,n. \tag{40}$$

To compute the brackets that involve the Q_k, we make use of the following formula:

$$\ln\left(\sum_{j=0}^{\infty}\Pi_j\gamma^j\right)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{S_k}{k}\gamma^k, \tag{41}$$

where γ is an auxiliary variable. Since in our case, the roots of the polynomial

$$\sum_{k=0}^{n-1}(b^{(n)}_{2k+1}+a^{(n)}_{2k+1})\lambda^{2k}+a^{(n)}_{2n+1}\lambda^{2n}=0$$

are given by $q_1,\dots,q_n,-q_1,\dots,-q_n$, we have

$$\Pi_{2k+1}=S_{2k+1}=0.$$

Therefore, by differentiating (41) with respect to Π_{2k}, we can express $\frac{\partial S_{2j}}{\partial\Pi_{2k}}$ as follows

$$\frac{\partial S_{2j}}{\partial\Pi_{2k}}=-(2j)\left[\left(\sum_{i=0}^{n}\Pi_{2i}\gamma^{2i}\right)^{-1}\right]_{2j-2k}, \tag{42}$$

where $[X(\gamma)]_{2j-2k}$ is the coefficient of γ^{2j-2k} of $X(\gamma)$ considered as a power series in γ near 0. Note that although the sum in the left hand side of (41) goes from 1 to ∞, only terms where $j\le n$ enter in (42) as $j-k\le n$.

We will now compute the bracket $\{P_k,Q_l\}$

$$\{P_k,Q_l\}=\sum_{j=1}^{n}\{\Pi_{2k},b^{(n)}_{2j}\}\frac{1}{2j}\frac{\partial S_{2j}}{\partial\Pi_{2l}}+\sum_{j=1}^{n}b^{(n)}_{2j}\frac{1}{2j}\left\{\Pi_{2k},\frac{\partial S_{2j}}{\partial\Pi_{2l}}\right\}. \tag{43}$$

[4] We are grateful to M. Talon for pointing out to us the similarity with the Kowalevski case and with the construction by A. Weil.

Since $\frac{\partial S_{2j}}{\partial \Pi_{2l}}$ is a polynomial in Π_{2m} with $m \leq j$, the second term in (43) is zero because of (40). The first term in (43) is

$$
\begin{aligned}
\{P_k, Q_l\} &= \sum_{j=1}^{n} \{\Pi_{2k}, b_{2j}^{(n)}\} \frac{1}{2j} \frac{\partial S_{2j}}{\partial \Pi_{2l}} = \\
&= \sum_{j=1}^{n} \Pi_{2(k-j)} \left[\left(\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1} \right]_{2j-2l} = \\
&= \sum_{j=l}^{n} \Pi_{2(k-j)} \left[\left(\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1} \right]_{2j-2l},
\end{aligned}
$$

where we replaced the sum from 1 to n by a sum from l to n in the last equation because the expression

$$
\left(\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1}
$$

does not contain any negative power. Since the expression

$$
\sum_{j=l}^{n} \Pi_{2(k-j)} \left[\left(\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1} \right]_{2j-2l}
$$

is just the coefficient of γ^{2k-2l} in

$$
\left(\sum_{i=0}^{n} \Pi_{2i} \gamma^{2i} \right) \left(\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1} = 1,
$$

we see that

$$
\{P_k, Q_l\} = \delta_{kl}.
$$

To compute the brackets between Q_k and Q_l, once again, note that since the brackets

$$
\{b_{2j}^{(n)}, b_{2i}^{(n)}\} = \left\{ \frac{\partial S_{2j}}{\partial \Pi_{2k}}, \frac{\partial S_{2i}}{\partial \Pi_{2l}} \right\} = 0,
$$

the only contributions to the bracket $\{Q_k, Q_l\}$ come from the cross terms

$$
\begin{aligned}
\{Q_k, Q_l\} &= \sum_{j,i=1}^{n} \left\{ \frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}, \frac{1}{2i} b_{2i}^{(n)} \frac{\partial S_{2i}}{\partial \Pi_{2l}} \right\} = \\
&= \sum_{j,i=1}^{n} \frac{1}{4ij} \left(b_{2j}^{(n)} \left\{ \frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)} \right\} \frac{\partial S_{2i}}{\partial \Pi_{2l}} - b_{2i}^{(n)} \left\{ \frac{\partial S_{2i}}{\partial \Pi_{2l}}, b_{2j}^{(n)} \right\} \frac{\partial S_{2j}}{\partial \Pi_{2k}} \right) = \qquad (44) \\
&= \sum_{j,i=1}^{n} \frac{1}{4ij} \left(b_{2j}^{(n)} \left\{ \frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)} \right\} \frac{\partial S_{2i}}{\partial \Pi_{2l}} - b_{2j}^{(n)} \left\{ \frac{\partial S_{2j}}{\partial \Pi_{2l}}, b_{2i}^{(n)} \right\} \frac{\partial S_{2i}}{\partial \Pi_{2k}} \right).
\end{aligned}
$$

The bracket between $\frac{\partial S_{2j}}{\partial \Pi_{2k}}$ and $b_{2i}^{(n)}$ can be computed as

$$\begin{aligned}
\left\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\right\} &= 2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n}\left\{\Pi_{2s}, b_{2i}^{(n)}\right\}\gamma^{2s}\right]_{2j-2k} = \\
&= -2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n}\Pi_{2(s-i)}\gamma^{2s}\right]_{2j-2k} = \\
&= -2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n-i}\Pi_{2s}\gamma^{2s+2i}\right]_{2j-2k} = \\
&= -2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n-i}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2i} = \\
&= -2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2i},
\end{aligned}$$

where in the last line, we replaced the upper limit of the second sum by n. This is because $2j-2k-2i \leq 2(n-i)$, so if $p > n-i$, the coefficient of γ^{2p} in $\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}$ will not enter in the final expression.

Therefore we have

$$\sum_{i=1}^{n}\frac{1}{4ij}\left\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\right\}\frac{\partial S_{2i}}{\partial \Pi_{2l}} = \left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-3}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2l}.$$

Similarly, the second term in (44) is given by

$$\sum_{i=1}^{n}\frac{1}{4ij}\left\{\frac{\partial S_{2j}}{\partial \Pi_{2l}}, b_{2i}^{(n)}\right\}\frac{\partial S_{2i}}{\partial \Pi_{2k}} = \left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-3}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2l},$$

therefore the first and second terms in (44) equal each other and we have $\{Q_k, Q_l\} = 0$.

In summary, we have

$$\{P_k, Q_l\} = \delta_{kl}, \quad \{P_k, P_l\} = 0, \quad \{Q_k, Q_l\} = 0,$$

as we wanted to prove. □

Example 5.3. Case $n=1$. In this case $Q_1 = \frac{1}{2}b_2^{(1)}\frac{\partial S_2}{\partial \Pi_2} = -b_2^{(1)}$ and $P_1 = \Pi_2 = \frac{a_1^{(1)}+b_1^{(1)}}{a_3^{(1)}}$, that is

$$Q = 4w, \qquad P = \frac{1}{2}\left(w_z - w^2 - \frac{z}{2}\right).$$

These coincide with Okamoto's canonical coordinates (up to a constant factor) [36].

Example 5.4. Case $n=2$. In this case $P_1 = \Pi_2 = \frac{a_3^{(2)}+b_3^{(2)}}{a_5^{(2)}}$, $P_2 = \Pi_4 = \frac{a_1^{(2)}+b_1^{(2)}}{a_5^{(2)}}$, $Q_1 = \frac{1}{2}b_2^{(2)}\frac{\partial S_2}{\partial \Pi_2} + \frac{1}{4}b_4^{(2)}\frac{\partial S_4}{\partial \Pi_2} = -b_2^{(2)} + b_4^{(2)}\Pi_2$, and $Q_2 = \frac{1}{4}b_4^{(2)}\frac{\partial S_4}{\partial \Pi_4} = -b_4^{(2)}$ so, finally

$$\begin{aligned}
P_1 &= -\frac{1}{2}(w^2 - w_z - \frac{t_1}{2}),\\
P_2 &= \frac{1}{16}(-z + 6w^4 - 12w^2w_z + 2w_z^2 - 4ww_{zz} + 2w_{zzz} + 2t_1(w_z - w^2)),\\
Q_1 &= -8ww_z + 4w_{zz},\\
Q_2 &= 16w.
\end{aligned}$$

6. The coefficients of $\mathcal{A}^{(n)}$ as polynomials in the canonical coordinates

This Section is completely devoted to the proof of the theorem below which expresses the matrix $\mathcal{A}^{(n)}$ as a polynomial in $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$ and $z, t_1, \ldots, t_{n-1}$.

Theorem 6.1. *Let $\mathcal{A}$, $\mathcal{B}_{odd}$ and $\mathcal{B}_{even}$ be the following polynomials in λ:*

$$\mathcal{A}=\sum_{i=0}^{n} a_{2i+1}^{(n)}\lambda^{2i+1},\quad \mathcal{B}_{odd}=\sum_{i=0}^{n} b_{2i+1}^{(n)}\lambda^{2i+1},\quad \mathcal{B}_{even}=\sum_{i=1}^{n} b_{2i}^{(n)}\lambda^{2i},\tag{45}$$

and $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{T}$ the following truncated series:

$$\mathcal{Q}=\sum_{i=1}^{n} Q_i\lambda^{2i},\quad \mathcal{P}=\sum_{i=1}^{n} P_i\lambda^{-2i},\quad \mathcal{T}\;=\;\sum_{i=1}^{n-1} t_i(2\lambda)^{2i-2n}-z(2\lambda)^{-2n}.\tag{46}$$

Then the following relations hold true:

$$\mathcal{A}\;=\;\left(\frac{1}{4}(2\lambda)^{2n+1}\left(1+\mathcal{P}-\frac{(1+\mathcal{T})^2}{1+\mathcal{P}}\right)-(2\lambda)^{-2n-1}\mathcal{Q}^2(1+\mathcal{P})\right)_+,\tag{47a}$$

$$\mathcal{B}_{odd}\;=\;\left(\frac{1}{4}(2\lambda)^{2n+1}\left(1+\mathcal{P}-\frac{(1+\mathcal{T})^2}{1+\mathcal{P}}\right)+(2\lambda)^{-2n-1}\mathcal{Q}^2(1+\mathcal{P})\right)_+,\tag{47b}$$

$$\mathcal{B}_{even}\;=\;-\lambda^2\left(\lambda^{-2}\mathcal{Q}(1+\mathcal{P})\right)_+,\tag{47c}$$

where in the above, the inverse $(1+\mathcal{P})^{-1}$ is to be interpreted as:

$$(1+\mathcal{P})^{-1}=\sum_{i=0}^{\infty}(-\mathcal{P})^i\tag{48}$$

and X_+ indicates the polynomial part of the Laurent series X.

In particular, by comparing the coefficients of λ in (47), we can express the matrix entries $a_{2i+1}^{(n)}$, $b_{2i+1}^{(n)}$ and $b_{2i}^{(n)}$ as polynomials in the canonical coordinates and the times.

Proof. Throughout the proof of this theorem we shall use the following facts: let X, X' and Y, Y' be Laurent series in λ with no positive part, such that

$$\begin{aligned}(\lambda^{2n-1}X)_+ &= (\lambda^{2n-1}X')_+\\ (\lambda^{2n-1}Y)_+ &= (\lambda^{2n-1}Y')_+\end{aligned}$$

then

$$(\lambda^{2n-1}XY)_+\;=\;(\lambda^{2n-1}X'Y')_+,\quad\text{and}\quad(\lambda^{2n-1}X^{-1})_+=(\lambda^{2n-1}(X')^{-1})_+.\tag{49}$$

Let us prove (47c) first. Using the definition of Q_k given in Theorem 5.1, we have

$$\begin{aligned}b_{2k}^{(n)} &= \frac{-1}{2k}b_{2k}^{(n)}\frac{\partial S_{2k}}{\partial\Pi_{2k}}=\\ &= -Q_k+\frac{1}{2k+2}b_{2k+2}^{(n)}\frac{\partial S_{2k+2}}{\partial\Pi_{2k}}+\frac{1}{2k+4}b_{2k+4}^{(n)}\frac{\partial S_{2k+4}}{\partial\Pi_{2k}}+\cdots\end{aligned}$$

now apply the above repeatedly to $b_{2k+2l}^{(n)}$ to obtain the following

$$\begin{aligned}b_{2k}^{(n)} &= -Q_k-Q_{k+1}\left(\frac{1}{2k+2}\frac{\partial S_{2k+2}}{\partial\Pi_{2k}}\right)-\\ &- Q_{k+2}\left(\frac{1}{2k+4}\frac{\partial S_{2k+4}}{\partial\Pi_{2k}}+\frac{1}{2k+4}\frac{1}{2k+2}\frac{\partial S_{2k+4}}{\partial\Pi_{2k+2}}\frac{\partial S_{2k+2}}{\partial\Pi_{2k}}\right)-\\ &- \cdots-Q_{k+l}\left(\frac{1}{2k+2l}\frac{\partial S_{2k+2l}}{\partial\Pi_{2k}}+\sum_{m=1}^{\infty}U_{2k+2l}^m\right)+\cdots\end{aligned}\tag{50}$$

where

$$\begin{aligned}U_{2k+2l}^m &= \sum_{j_1=m}^{l-1}\frac{1}{2k+2l}\frac{\partial S_{2k+2l}}{\partial\Pi_{2k+2j_1}}\sum_{j_2=m-1}^{j_1-1}\frac{1}{2k+2j_1}\frac{\partial S_{2k+2j_1}}{\partial\Pi_{2k+2j_2}}\\ &\cdots\sum_{j_m=1}^{j_{m-1}-1}\frac{1}{2k+2j_{m-1}}\frac{\partial S_{2k+2j_{m-1}}}{\partial\Pi_{2k+2j_m}}\frac{1}{2k+2j_m}\frac{\partial S_{2k+2j_m}}{\partial\Pi_{2k}}.\end{aligned}\tag{51}$$

Note that, by (42), we have

$$\frac{\partial S_{2p}}{\partial \Pi_{2q}} = -(2p)\left[\left(\sum_{j=0}^{n}\Pi_{2j}(\gamma)^{2j}\right)^{-1}\right]_{2p-2q}. \tag{52}$$

This allows us to interpret the U^m_{2k+2l} as coefficients of an infinite series, which is a product of the series in the right hand side of (52). However, we need to be careful since the sums in (51) begin with terms that correspond to the coefficient of γ^2 in (52). This means that

$$U^m_{2k+2l} = \text{ coefficient of } \gamma^{2l} \text{ in } \left(1+\left(-\sum_{j=0}^{n}\Pi_{2j}\gamma^{2j}\right)^{-1}\right)^{m+1}.$$

We can now identify Q_k as coefficients of λ in the polynomial $\mathcal{Q}$, $b^{(n)}_{2k}$ as coefficients of λ in the polynomial $\mathcal{B}_{even}$. If we now substitute $\gamma=\lambda^{-1}$ and interpret U^m_{2k+2l} also as coefficients of λ in the product series, we can rewrite (51) as follows

$$\begin{aligned}
\mathcal{B}_{even} &= -\lambda^2\left(\lambda^{-2}\mathcal{Q}\sum_{m=0}^{\infty}\left(1+\left(-\sum_{j=0}^{n}\Pi_{2j}\lambda^{-2j}\right)^{-1}\right)^{m}\right)_+ =\\
&= -\lambda^2\left(\lambda^{-2}\mathcal{Q}\left(\left(\sum_{j=0}^{n}\Pi_{2j}\lambda^{-2j}\right)^{-1}\right)^{-1}\right)_+ =\\
&= -\lambda^2\left(\lambda^{-2}\mathcal{Q}\left(\sum_{j=0}^{n}\Pi_{2j}\lambda^{-2j}\right)\right)_+ =\\
&= -\lambda^2\left(\lambda^{-2}\mathcal{Q}(1+\mathcal{P})\right)_+,
\end{aligned}$$

which proves (47c).

Let us now prove (47b). Notice that in the expressions (15c) for $b^{(n)}_{2k+1}$ the Lenard operators appear together with their first and second derivatives. The canonical coordinates instead involve only the Lenard operators and their first derivatives. In fact, the $P_1,\dots,P_n$ are expressed in terms of sums $a^{(n)}_{2j+1}+b^{(n)}_{2j+1}$ which depend only on the Lenard operators without derivatives, and the $Q_1,\dots,Q_n$ depend on the same sums and on the even coefficients $b^{(n)}_{2k}$ which are only expressed in terms of the Lenard operators and their first derivatives.

Therefore, to express the odd coefficients $b^{(n)}_{2k+1}$ in terms of the canonical coordinates, we will need to express the second derivatives of the Lenard operators in terms of the Lenard operators themselves and their first derivatives. In terms of the generating function, this is given by

Proposition 6.2. *Let $\mathcal{L}$ be the generating function of the Lenard recursion operator:*

$$\mathcal{L} \ := \ \sum_{i=1}^{\infty}\mathcal{L}_i\xi^i,$$

where ξ is an auxiliary variable. Then the following relation holds true:

$$\begin{aligned}
\partial_z^2\mathcal{L} &= \frac{1}{2}(\xi^{-1}\mathcal{L}-\mathcal{L}_1)-2\mathcal{L}\mathcal{L}_1\\
&+ (1+2\mathcal{L})^{-1}\left(\frac{1}{2}(\xi^{-1}\mathcal{L}-\mathcal{L}_1)-\mathcal{L}\mathcal{L}_1+(\partial_z\mathcal{L})^2\right).
\end{aligned} \tag{53}$$

In particular, the above equation expresses the second derivatives of the Lenard operators as a polynomial in the Lenard operators and their first derivatives.

Proof. We prove (53) at each order in ξ. At order ξ, the equation (53) is trivially satisfied. At order ξ^n, $n>0$, by multiplying (53) by $(1+2\mathcal{L})$, we get:

$$\mathcal{L}_{n+1} = \partial_z^2\mathcal{L}_n+3\mathcal{L}_n\mathcal{L}_1+\sum_{j=1}^{n-1}\left(4\mathcal{L}_1\mathcal{L}_j\mathcal{L}_{n-j}-\partial_z\mathcal{L}_j\partial_z\mathcal{L}_{n-j}-\mathcal{L}_{j+1}\mathcal{L}_{n-j}-2\mathcal{L}_{n-j}\partial_z^2\mathcal{L}_j\right). \tag{54}$$

To prove this, we integrate relation (13) by parts iteratively. At the first step, we get:

$$\mathcal{L}_{n+1} = \left(\partial_z^2 + 2\mathcal{L}_1\right)\mathcal{L}_n + 2\int \mathcal{L}_1\partial_z\mathcal{L}_n \mathrm{d}z, \tag{55}$$

where we have replaced $w_z - w^2$ by $\mathcal{L}_1$. It is therefore sufficient to compute $\int \mathcal{L}_1\partial_z\mathcal{L}_n \mathrm{d}z$. To achieve this, we will now compute a more general term $\int \mathcal{L}_i\partial_z\mathcal{L}_k \mathrm{d}z$ for any k and i. First let us replace the term $\partial_z\mathcal{L}_k$ by using (13):

$$\int \mathcal{L}_i\partial_z\mathcal{L}_k \mathrm{d}z = \int \mathcal{L}_i\left(\partial_z^3 + 4\mathcal{L}_1\partial_z + 2(\mathcal{L}_1)_z\right)\mathcal{L}_{k-1}\mathrm{d}z.$$

We now integrate the first and last terms by parts, where the integration by parts in the last term is performed as follows, let

$$\begin{aligned} F(z) &= 2\int \mathcal{L}_i(\mathcal{L}_1)_z \mathrm{d}z \\ &= \partial_z^2\mathcal{L}_i + 4\mathcal{L}_1\mathcal{L}_i - \mathcal{L}_{i+1}, \end{aligned}$$

where we used the Lenard recursion relation (13) to obtain the above equality. We then have:

$$2\int \mathcal{L}_i(\mathcal{L}_1)_z\mathcal{L}_{k-1}\mathrm{d}z = F(z)\mathcal{L}_{k-1} - \int F(z)\partial_z\mathcal{L}_{k-1}\mathrm{d}z.$$

¿From this we have

$$\begin{aligned} \int \mathcal{L}_i\partial_z\mathcal{L}_k \mathrm{d}z &= \mathcal{L}_i\partial_z^2\mathcal{L}_{k-1} - \partial_z\mathcal{L}_i\partial_z\mathcal{L}_{k-1} + \mathcal{L}_{k-1}\partial_z^2\mathcal{L}_i + 4\mathcal{L}_1\mathcal{L}_i\mathcal{L}_{k-1} \\ &- \mathcal{L}_{i+1}\mathcal{L}_{k-1} + \int \mathcal{L}_{i+1}\partial_z\mathcal{L}_{k-1}\mathrm{d}z \end{aligned}$$

We can replace the term $\int \mathcal{L}_{i+1}\partial_z\mathcal{L}_{k-1}\mathrm{d}z$ on the right hand side by similar formula and express $\int \mathcal{L}_i\partial_z\mathcal{L}_k \mathrm{d}z$ as a polynomial in $\mathcal{L}_j$, $\partial_z\mathcal{L}_j$ and $\partial_z^2\mathcal{L}_j$ for $j < k$. In particular, we have

$$\begin{aligned} 2\int \mathcal{L}_1\partial_z\mathcal{L}_n \mathrm{d}z &= \mathcal{L}_1\mathcal{L}_n + \sum_{j=1}^{n-1}\Bigg(4\mathcal{L}_1\mathcal{L}_j\mathcal{L}_{n-j} - \partial_z\mathcal{L}_j\partial_z\mathcal{L}_{n-j} - \\ &- \mathcal{L}_{j+1}\mathcal{L}_{n-j} + 2\mathcal{L}_j\partial_z^2\mathcal{L}_{n-j}\Bigg). \end{aligned}$$

By substituting this into (55), we finally get (54). □

We can now express the odd coefficients $b_{2k+1}^{(n)}$ in terms of the canonical coordinates.

If we make the substitution $\xi = \frac{1}{4}\lambda^{-2}$ in the generating function $\mathcal{L}$, then by (15c), we have

$$\mathcal{B}_{odd} = \left((2\lambda)^{2n-1}\left(\frac{1}{2}\partial_z^2\mathcal{L} + w\partial_z\mathcal{L} + w_z(\mathcal{L}+\mathcal{L}_0)\right)(1+\mathcal{T})\right)_+, \tag{56}$$

$$\mathcal{B}_{even} = (2\lambda)^2\left((2\lambda)^{2n-2}\left(-\partial_z\mathcal{L} - 2w(\mathcal{L}+\mathcal{L}_0)\right)(1+\mathcal{T})\right)_+. \tag{57}$$

(58)

Note that, since

$$\left((2\lambda)^{2n-1}\partial_z\mathcal{L}(1+\mathcal{T})\right)_+$$

is a series in odd powers of λ only, while

$$\left((2\lambda)^{2n-2}\partial_z\mathcal{L}(1+\mathcal{T})\right)_+$$

is a series in even powers of λ only, we have

$$2\lambda\left((2\lambda)^{2n-2}\partial_z\mathcal{L}(1+\mathcal{T})\right)_+ = \left((2\lambda)^{2n-1}\partial_z\mathcal{L}(1+\mathcal{T})\right)_+.$$

¿From this, and the second equation in (56), we can express the second term in $\mathcal{B}_{odd}$ in terms of $\mathcal{B}_{even}$

$$\left((2\lambda)^{2n-1}\partial_z\mathcal{L}(1+\mathcal{T})\right)_+ = (2\lambda)^{-1}\mathcal{B}_{even} + \left((2\lambda)^{2n-1}(2\mathcal{L}+1)w(1+\mathcal{T})\right)_+.$$

By substituting this into the right hand side of $\mathcal{B}_{odd}$, we have

$$\begin{aligned} \mathcal{B}_{odd} &= \left((2\lambda)^{2n-1}\left(\frac{1}{2}\partial_z^2\mathcal{L} - w^2(\mathcal{L}+\mathcal{L}_0) + \mathcal{L}_1(\mathcal{L}+\mathcal{L}_0)\right)(1+\mathcal{T}) - (2\lambda)^{-1}\mathcal{B}_{even}\right)_+, \\ \mathcal{B}_{even} &= (2\lambda)^2\left((2\lambda)^{2n-2}\left(-\partial_z\mathcal{L} - 2w\mathcal{L} - w+\right)(1+\mathcal{T})\right)_+. \end{aligned}$$

By substituting this into (53), we obtain

$$\begin{aligned} \mathcal{B}_{odd} &= \frac{1}{2}\Bigg((2\lambda)^{2n-1}\Bigg[-2w^2\mathcal{L} - w^2 + \frac{1}{2}((2\lambda)^2\mathcal{L}+\mathcal{L}_1) + \\ &+ (1+2\mathcal{L})^{-1}\Bigg(\frac{1}{2}\left((2\lambda)^2\mathcal{L}-\mathcal{L}_1\right) - \mathcal{L}\mathcal{L}_1 + \\ &+ (\partial_z\mathcal{L})^2\Bigg)\Bigg](1+\mathcal{T}) - 2w(2\lambda)^{-1}\mathcal{B}_{even}\Bigg)_+. \end{aligned} \tag{59}$$

To complete the calculation, we need to express $\mathcal{L}$ in terms of $\mathcal{P}$. Since we have

$$P_k = \frac{a^{(n)}_{2(n-k)+1} + b^{(n)}_{2(n-k)+1}}{a^{(n)}_{2n+1}} = \frac{2}{4^k}\sum_{l=n-k}^{n} t_l\mathcal{L}_{k+(l-n)}, \qquad k=1,\dots,n,$$

where $t_n = 1$, we see that

$$\begin{aligned} \mathcal{L}_k &= 2^{2k-1}P_k - t_{n-1}\mathcal{L}_{k-1} + \dots - t_{n-i}\mathcal{L}_{k-i} + \cdots, \\ \mathcal{L}_k &= 2^{2k-1}P_k + 2^{2k-3}P_{k-1}(-t_{n-1}) + \\ &+ 2^{2(k-i)-1}P_{k-i}\left(-t_{n-i} + \dots + \sum_{j_1=1}^{n-i-1}(-t_{n-i-j_1})\sum_{j_2=1}(-t_{j_1-j_2})\cdots\sum_{j_l=1}^{j_{l-1}-1}(-t_{j_l}) + \dots\right) + \\ &+ \mathcal{L}_0\left(t_{n-k} + \dots + \sum_{j_1=1}^{n-k-1}(-t_{n-k-j_1})\sum_{j_2=1}(-t_{j_1-j_2})\cdots\sum_{j_l=1}^{j_{l-1}-1}(-t_{j_l}) + \dots\right). \end{aligned}$$

Finally, by using similar argument as before, we see that

$$\left((2\lambda)^{2n-1}(\mathcal{L}+\mathcal{L}_0)\right)_+ = \left((2\lambda)^{2n-1}\left(\mathcal{L}_0 + \frac{1}{2}\mathcal{P}\right)(1+\mathcal{T})^{-1}\right)_+, \tag{60}$$

where $(1+\mathcal{T})^{-1}$ in the above is to be interpreted as in (48). Thanks to (49), this implies the following

$$\left((2\lambda)^{2n-1}(\mathcal{L}+\mathcal{L}_0)^{-1}\right)_+ = \left((2\lambda)^{2n-1}\left(\mathcal{L}_0 + \frac{1}{2}\mathcal{P}\right)^{-1}(1+\mathcal{T})\right)_+ \tag{61}$$

By substituting (60) and (61) into (59), we get(47b).

To finish the proof of the theorem, note that by the definition of the coordinates $P_1,\dots,P_n$ in Theorem 5.1, we have

$$\lambda(2\lambda)^{2n}(1+\mathcal{P}) = (\mathcal{A}+\mathcal{B}_{odd}) \quad \Rightarrow$$

$$\mathcal{A} = \left(-\mathcal{B}_{odd} + \lambda(2\lambda)^{2n}(1+\mathcal{P})\right) \tag{62}$$

The proves the theorem. □

Example 6.3. Let us illustrate how formulae (47a), (47b) and (47c) work in a concrete but non trivial example. In the case $n = 3$ the definitions (45) and (46) read:

$$\begin{aligned}
\mathcal{A} &= a_1^{(3)}\lambda + a_3^{(3)}\lambda^3 + a_5^{(3)}\lambda^5 + a_7^{(3)}\lambda^7,\\
\mathcal{B}_{odd} &= b_1^{(3)}\lambda + b_3^{(3)}\lambda^3 + b_5^{(3)}\lambda^5,\\
\mathcal{B}_{even} &= b_2^{(3)}\lambda^2 + b_4^{(3)}\lambda^4 + b_6^{(3)}\lambda^6,\\
\mathcal{Q} &= Q_1\lambda^2 + Q_2\lambda^4 + Q_3\lambda^6,\\
\mathcal{P} &= \frac{P_1}{\lambda^2} + \frac{P_2}{\lambda^4} + \frac{P_3}{\lambda^6},\\
\mathcal{T} &= \frac{t_2}{(2\lambda)^2} + \frac{t_1}{(2\lambda)^4} - \frac{z}{(2\lambda)^6}.
\end{aligned} \tag{63}$$

Theorem 6.1 allows to express the coefficients $a_1^{(3)}, a_3^{(3)}, a_5^{(3)}, a_7^{(3)}$ and the coefficients $b_1^{(3)}, \dots, b_6^{(3)}$ in terms of $P_1, P_2, P_3, Q_1, Q_2, Q_3$ and z, t_1, t_2. Let us use (47c) first:

$$\mathcal{Q}(1+\mathcal{P}) = \frac{P_3Q_1}{\lambda^4} + \frac{P_2Q_1 + P_3Q_2}{\lambda^4} + P_1Q_1 + P_2Q_2 + P_3Q_3 + \lambda^2(Q_1 + P_1Q_2 + P_2Q_3) + \lambda^4(Q_2 + P_1Q_3) + \lambda^6 Q_3,$$

and after dividing by λ^2 and throwing away all negative powers, we get

$$\left(\lambda^{-2}\mathcal{Q}(1+\mathcal{P})\right)_+ = Q_1 + P_1Q_2 + P_2Q_3 + \lambda^2(Q_2 + P_1Q_3) + \lambda^4 Q_3.$$

Now, we need to multiply by λ^2 again and to compare with $\mathcal{B}_{even}$. We get

$$b_2^{(3)} = -Q_1 - P_1Q_2 - P_2Q_3, \qquad b_4^{(3)} = -Q_2 - P_1Q_3, \qquad b_6^{(3)} = -Q_3.$$

Let us briefly illustrate how to obtain the other coefficients. The procedure is the same as above with the only complication of the term

$$\left(\frac{1}{4}(2\lambda)^7\frac{1+\mathcal{T}}{1+\mathcal{P}}\right)_+.$$

Let us compute this term explicitly. Since we have a power 7 in λ in the numerator, we need to only the terms up to order $\frac{1}{\lambda^7}$ in the expansion of $(1+\mathcal{P})^{-1}$ at ∞:

$$(1+\mathcal{P})^{-1} = 1 - \frac{P_1}{\lambda^2} + \frac{P_1^2 - P_2}{\lambda^4} + \frac{2P_1P_2 - P_3 - P_1^3}{\lambda^6} + \mathcal{O}(\lambda^8).$$

In this way we see that

$$\begin{aligned}
\left(\frac{1}{4}(2\lambda)^7\frac{1+\mathcal{T}}{1+\mathcal{P}}\right)_+ &= (64P_1 - 16t_2)\lambda^5 + (64P_2 - 32P_1^2 - 4t_1 + 16t_2P_1 - 2t_2^2)\lambda^3 + \\
&+\lambda(64P_3 + z + 32P_1^3 - 64P_1P_2 + 4t_1P_1 - 16t_2P_1^2 + 16t_2P_2 - t_1t_2 + 2t_2^2P_1).
\end{aligned}$$

Analogously, one can compute the other terms in (47a) and (47b) to obtain:

$$\begin{aligned}
b_1^{(3)} &= z + 32P_1^3 - 64P_1P_2 + 64P_3 + \frac{Q_2^2}{128} + \frac{Q_1Q_3}{64} + \frac{P_1Q_2Q_3}{64} + \\
&\quad + \frac{P_2Q_3^2}{128} + 4t_1P_1 - 16t_2P_1^2 + 16t_2P_2 - t_1t_2 + 2t_2^2P_1,\\
b_3^{(3)} &= 64P_2 - 32P_1^2 + \frac{Q_2Q_3}{64} + \frac{P_1Q_3^2}{128} - 4t_1 + 16t_2P_1 - 2t_2^2,\\
b_5^{(3)} &= 64P_1 + \frac{Q_3^2}{128} - 16t_2,\\
a_1^{(3)} &= z + 32P_1^3 - 64P_1P_2 + \frac{Q_2^2}{128} + \frac{Q_1Q_3}{64} + \frac{P_1Q_2Q_3}{64} + \\
&\quad + \frac{P_2Q_3^2}{128} + 4t_1P_1 - 16t_2P_1^2 + 16t_2P_2 - t_1t_2 + 2t_2^2P_1,\\
a_3^{(3)} &= -32P_1^2 + \frac{Q_2Q_3}{64} + \frac{P_1Q_3^2}{128} - 4t_1 + 16t_2P_1 - 2t_2^2,\\
a_5^{(3)} &= +\frac{Q_3^2}{128} - 16t_2, \qquad a_7^{(3)} = 64.
\end{aligned}$$

It is clear that we have only used linear algebra to obtain these coefficients, instead of using differentiation, integration and recursion.

7. Hamiltonians

In Section 4 we proved that the equation (16) is the same as (30):

$$(\partial_z - \partial_z^w)\, A = \mathrm{ad}^*_B A,$$

where $A = \left(\mathcal{A}^{(n)}\right)_+ \in \mathfrak{g}^*_-$ is the dynamical part of $\mathcal{A}^{(n)}$, and $B = \left(\frac{\mathcal{A}^{(n)}\lambda^{1-2n}}{4^n}\right)_- \in \mathfrak{g}_-$. This allows us to interpret the evolution along $(\partial_z - \partial_z^w)$ as a vector field on a coadjoint orbit of the twisted loop algebra $\mathfrak{g}_-$. We are now going to show that this vector field is Hamiltonian and that the isomonodromic deformation Hamiltonian for the n-th equation in the PII hierarchy is given by

$$H^{(n)} := -\frac{1}{2\,4^n}\mathrm{TrRes}\left(\lambda^{1-2n}\left(\mathcal{A}^{(n)}\right)^2\right). \tag{64}$$

This fact is actually a consequence of a more general result:

Proposition 7.1. *The vector field*

$$\mathcal{X}_k(A) := -\left[\left(\mathcal{A}^{(n)}\lambda^{1-2k}\right)_-, A\right],$$

is Hamiltonian with the Hamiltonian function

$$h_k^{(n)} := \frac{1}{2}\mathrm{TrRes}\left(\lambda^{1-2k}\left(\mathcal{A}^{(n)}\right)^2\right). \tag{65}$$

Proof. Let us denote

$$\hat{L}_k = -\left(\mathcal{A}^{(n)}\lambda^{1-2k}\right)_- \in \mathfrak{g}_-.$$

We are interested in the vector field

$$\mathcal{X}_k(A) = \left[\hat{L}_k, A\right].$$

To show that it is Hamiltonian and to compute the Hamiltonian function f, we use the following definition:

$$\omega(\mathcal{X}_k, Y)(\Xi) := -\langle [Y,\Xi], \mathrm{d}f\rangle, \quad Y \in \mathfrak{g}_-,\ \Xi \in \mathfrak{g}^*_-,$$

so that

$$\omega(\mathcal{X}_k, Y)(A) = -\left\langle \left[\hat{L}_k, Y\right], A\right\rangle = \left\langle [Y,A], -\hat{L}_k\right\rangle =< [Y,A], \mathrm{d}f > .$$

This shows that if we can prove that there exist f such that $\mathrm{d}f = -\hat{L}_k$, then $[\hat{L}_k, A]$ defines a Hamiltonian vector field of Hamiltonian f.

We are now going to show that the Hamiltonian (65) is such that $\mathrm{d}h_k^{(n)} = -\hat{L}_k$. For every $X \in \mathfrak{g}_-$ and $\Xi \in \mathfrak{g}^*_-$, we can identify $[X,\Xi] = \mathrm{ad}^*_X \Xi$ with a vector tangent to the coadjoint orbit. Denote this vector by $\delta_X \Xi$. Let $\delta_X A \in T_A\mathcal{O}_A$ then using the definition 32, we get

$$h_k^{(n)}(A + \delta_X A) - h_k^{(n)}(A) + \mathcal{O}(\delta_X A)^2 = \frac{1}{2}\left\langle \lambda^{1-2k}A, (2\delta_X \mathcal{A}^{(n)})\right\rangle = \left\langle \delta_X A, \mathrm{d}h_k^{(n)}\right\rangle,$$

which is the contraction between $\delta_X A$ and $\mathrm{d}h_k^{(n)}$, as we wanted to prove. □

We now compute the Hamiltonian $\mathcal{H}^{(n)}$ in terms of the canonical coordinates.

Theorem 7.2. *Define*

$$\mathcal{H}^{(n)}(P_1,\dots,P_n,Q_1,\dots,Q_n,z) := -\frac{1}{4^n}\left(\sum_{l=0}^{n-1} a^{(n)}_{2l+1}a^{(n)}_{2(n-l)-1} - \sum_{l=0}^{n-1} b^{(n)}_{2l+1}b^{(n)}_{2(n-l)-1} + \sum_{l=0}^{n} b^{(n)}_{2l}b^{(n)}_{2(n-l)}\right) + \frac{Q_n}{4^n}, \tag{66}$$

in which we are thinking of $a^{(n)}_{2l+1}, b^{(n)}_{2l+1}, b^{(n)}_{2l}$ as the functions of $P_1,\dots,P_n,Q_1,\dots,Q_n,t_1,\dots,t_{n-1},z$ given by (47). Then the n-th member of the second Painlevé hierarchy is given by the equations

$$\frac{\mathrm{d}P_k}{\mathrm{d}z} = -\frac{\partial\mathcal{H}^{(n)}}{\partial Q_k}, \qquad \frac{\mathrm{d}Q_k}{\mathrm{d}z} = \frac{\partial\mathcal{H}^{(n)}}{\partial P_k}. \tag{67}$$

Proof. Observe that equation (64) gives

$$H^{(n)} = -\frac{1}{4^n}\left(\sum_{l=0}^{n-1} a^{(n)}_{2l+1}a^{(n)}_{2(n-l)-1} - \sum_{l=0}^{n-1} b^{(n)}_{2l+1}b^{(n)}_{2(n-l)-1} + \sum_{l=0}^{n} b^{(n)}_{2l}b^{(n)}_{2(n-l)}\right),$$

where we are treating $a_j^{(n)}, b_j^{(n)}$ as coordinates on the coadjoint orbit. When expressing this Hamiltonian in our canonical coordinates, we need to take into account a shift h due to the explicit z dependence in the variable P_n. All other canonical coordinates depend on $a_j^{(n)}, b_j^{(n)}$ only. To compute this shift we use the following well-known result:

Lemma 7.3. *Let*

$$\frac{\mathrm{d}y_i}{\mathrm{d}z} = \{y_i, H(y,z)\}, \tag{68}$$

be a Hamiltonian system on a Poisson manifold with Poisson brackets $\{\cdot,\cdot\}$ *and*

$$y = \phi(x,z),$$

be a local diffeomorphism depending explicitly on z*. Let the vector field* $\partial_z\phi$ *be a Hamiltonian vector field with Hamiltonian* δH*. Then (68) is a Hamiltonian system also in the* x*–coordinates*

$$\frac{\mathrm{d}x_i}{\mathrm{d}z} = \{x_i, \hat{H}(x,z)\},$$

where

$$\hat{H}(x,z) = H(\phi(x,z),z) - \delta H(\phi(x,z),z).$$

Let us compute this shift in our case. The only coordinate depending explicitly on z is $P_n = \frac{-z}{4^n} + f(a_1^{(n)},\dots,a_{2n+1}^{(n)}, b_0^{(n)}, b_1^{(n)},\dots,b_{2n}^{(n)})$. So for $y = (P_1,\dots,P_n,Q_1,\dots,Q_n)$, we have $\delta H^{(n)} = \frac{Q_n}{4^n}$ which gives (66). □

Remark 7.4. *It is clear that the Hamiltonian equations (67) satisfy the Painlevé property. In fact (17) satisfies the Painlevé property* [32, 33]*, and since* $\mathcal{A}^{(n)}$ *is a polynomial in* $P_1,\dots,P_n,Q_1,\dots,Q_n$*, no movable critical points are introduced.*

Example 7.5. In the case $n=1$, we get the $\mathrm{P_{II}}$ Hamiltonian [36, 29]

$$\mathcal{H}^{(1)} = 4P^2 + \frac{1}{4}Q + \frac{1}{4}PQ^2 + 2Pz - \frac{1}{2}Q\alpha_1.$$

The Hamilton's equations

$$\frac{\mathrm{d}^2Q}{\mathrm{d}z^2} = \frac{1}{8}Q^3 + Qz + 4\alpha_1,$$

give the second Painlevé equation for $w = \frac{1}{4}Q$:

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + wz + \alpha_1.$$

8. Higher order flows as time-dependent Hamiltonian systems

As illustrated in Section 2, the time t_k dependence is described by the rescaled mKdV equation (14), which is equivalent to the compatibility equation (18). In the Appendix A, we proved that all equations (17), (18) and (19) are consistent, so that they indeed define isomonodromic deformations. In this Section, we shall deduce the Hamiltonian functions $\mathcal{H}_k^{(n)}$ such that equation (14) is equivalent to

$$\frac{\partial Q_i}{\partial t_k} = \frac{\partial \mathcal{H}_k^{(n)}}{\partial P_i}, \qquad \frac{\partial P_i}{\partial t_k} = -\frac{\partial \mathcal{H}_k^{(n)}}{\partial Q_i}, \quad i = 1,\dots,n. \tag{69}$$

Since we are dealing with isomonodromic deformations, the correct way to deduce these Hamiltonians is to express equation (19) as a time-dependent Hamiltonian system.

8.1. **Spectral invariants.** The spectral curve Γ of the matrix $\mathcal{A}^{(n)}$ defined by

$$\det(\mu I - \mathcal{A}^{(n)}) = 0$$

plays an important role in defining the Hamiltonians of the isomonodromic flows and of the Painlevé hierarchy. We now express Γ in terms of the generating functions of the canonical coordinates and the time:

$$\begin{aligned}
\mu^2 &= \lambda^{-2}\left(\mathcal{A}^2 - \mathcal{B}_{odd}^2 + (\mathcal{B}_{even} - \alpha_n)^2\right)\\
&= \left((2\lambda)^{4n}(1+\mathcal{P})^2 - 4(2\lambda)^{2n-1}(1+\mathcal{P})\mathcal{B}_{odd} + \lambda^{-2}(\mathcal{B}_{even} - \alpha_n)^2\right)
\end{aligned}$$

By substituting (47) into the above, we obtain the spectral curve in terms of the canonical coordinates

$$\begin{aligned}
\mu^2 &= (2\lambda)^{4n}(1+\mathcal{P})^2 - 4(2\lambda)^{2n-1}(1+\mathcal{P})\mathcal{B}_{odd} + \lambda^{-2}(\mathcal{B}_{even} - \alpha_n)^2 = \\
&= (2\lambda)^{4n}\left(1+\mathcal{P}\right)^2 - (2\lambda)^{2n-1}(1+\mathcal{P})\Bigg[(2\lambda)^{2n+1}\left(1+\mathcal{P} - \frac{(1+\mathcal{T})^2}{1+\mathcal{P}}\right) + \\
&+ 4(2\lambda)^{-2n-1}\mathcal{Q}^2(1+\mathcal{P})\Bigg]_+ + \lambda^{-2}\left(\lambda^2\left[\lambda^{-2}\mathcal{Q}(1+\mathcal{P})\right]_+ + \alpha_n\right)^2 = \\
&= (2\lambda)^{2n-1}(1+\mathcal{P})\left[(2\lambda)^{2n+1}\frac{(1+\mathcal{T})^2}{1+\mathcal{P}} - 4(2\lambda)^{-2n-1}\mathcal{Q}^2(1+\mathcal{P})\right]_+ + \\
&+ \lambda^{-2}\left(\lambda^2\left[\lambda^{-2}\mathcal{Q}(1+\mathcal{P})\right]_+ + \alpha_n\right)^2
\end{aligned}$$

where $\frac{1}{1+\mathcal{P}} = \sum_{i=0}^{\infty}(-1)^i\mathcal{P}^i$. The last equality follows be $\lambda^{2n}(1+\mathcal{P})$ contains positive powers only.

In particular, we have proved the following

Proposition 8.1. *The coefficients of the spectral curve μ^2 of the matrix $\mathcal{A}^{(n)}$ can be expressed as polynomials of the canonical coordinates and the times as follows*

$$\begin{aligned}
\mu^2 &= (2\lambda)^{2n-1}(1+\mathcal{P})\left[(2\lambda)^{2n+1}\frac{(1+\mathcal{T})^2}{1+\mathcal{P}} - 4(2\lambda)^{-2n-1}\mathcal{Q}^2(1+\mathcal{P})\right]_+ \\
&+ \lambda^{-2}\left(\lambda^2\left[\lambda^{-2}\mathcal{Q}(1+\mathcal{P})\right]_+ - \alpha_n\right)^2. \qquad (70)
\end{aligned}$$

Corollary 8.2. *The constant α_n, the times $t_1, \dots t_{n-1}$ and the Hamiltonian $H^{(n)}$ are spectral invariants. In particular the spectral curve can be written as*

$$\mu^2 = (2\lambda)^{2n}\left((2\lambda)^{2n}(1+\mathcal{T})^2\right)_+ - 4^n H^{(n)}\lambda^{2n-2} + \sum_{k=1}^{n-1} h_k^{(n)}\lambda^{2k-2} + \frac{\alpha_n^2}{\lambda^2},$$

where

$$h_k^{(n)} = \frac{1}{2}\mathrm{ResTr}\left(\lambda^{1-2k}\left(\mathcal{A}^{(n)}\right)^2\right), \qquad k = 1, \dots, n.$$

Proof. By definition of $h_k^{(n)}$, it is clear that $h_k^{(n)}$ is the coefficient of the $2k-2$ power in λ. Analogously for $H^{(n)}$. So we only need to prove that the coefficients of the powers $4n, 4n-2, \dots, 2n$ are given by $(2\lambda)^{2n}\left((2\lambda)^{2n}(1+\mathcal{T})^2\right)_+$. In particular, this implies that the polynomial part of μ is given by the following

$$\mu = 4^n t_n\lambda^{2n} + 4^{n-1}t_{n-1}\lambda^{2n-2} + \dots - z.$$

In fact, let the expansion of the spectrum μ at $\lambda = \infty$ be the following

$$\mu = \sum_{i=-\infty}^{2n} \mu_i\lambda^i$$

Let us denote the coefficients of μ^2 by D_i, that is

$$\mu^2 = \sum_{i=-2}^{2n} D_i\lambda^i$$

We can express D_i as quadratic polynomials in the μ_k and if $k<0$, then μ_k will only appear in the coefficient D_i when $i<2n$. Therefore, to compute the polynomial part of μ, we only need to consider the coefficients D_i with $i>2n-1$. These coefficients are given by the coefficients of

$$((2\lambda)^{-2n}\mu^2)_+ = \left((2\lambda)^{2n}\left((2\lambda)^{-4n}\mu^2\right)\right)_+$$

By using the relations (49) in (70), we see that

$$\begin{aligned} ((2\lambda)^{-2n}\mu^2)_+ &= \left[(2\lambda)^{2n}(1+\mathcal{T})^2 - 4(2\lambda)^{-2n-2}\mathcal{Q}^2(1+\mathcal{P})^2\right]_+ \\ &+ \left[4(2\lambda)^{-2n-2}\mathcal{Q}^2(1+\mathcal{P})^2\right]_+ \\ ((2\lambda)^{-2n}\mu^2)_+ &= \left[(2\lambda)^{2n}(1+\mathcal{T})^2\right]_+ \end{aligned}$$

This implies the corollary. □

8.2. **Time flows Hamiltonians.** We want to adapt the construction of Section 4 to express equation (19) as a time-dependent Hamiltonian system and the computations of Section 7 to find the Hamiltonians.

The main difficulty we encounter is that now

$$\partial_\lambda \hat{M}^{(k)} \neq (2k+1)\partial^w_{t_k}\mathcal{A}^{(n)}.$$

The main idea to handle this problem is the following: suppose there exists a set of coordinates $u_1, u_2, \ldots, u_{2n}$ in our coadjoint orbit $\mathcal{O}_A$ such that

$$\partial_\lambda \hat{M}^{(k)} = (2k+1)\partial^u_{t_k}\mathcal{A}^{(n)}, \tag{71}$$

where $\partial^u_{t_k}$ denotes the t_k–derivative with the u coordinates fixed (in the sense explained at the beginning of Section 4). Then (19) becomes

$$(2k+1)\left(\partial_{t_k} - \partial^u_{t_k}\right)\mathcal{A}^{(n)} = [\hat{M}^{(k)}, \mathcal{A}^{(n)}] \tag{72}$$

after cancelation. This allows us to interpret the evolution along $\partial_{t_k} - \partial^u_{t_k}$ as a Hamiltonian vector field on the coadjoint orbit, as explained in Section 4, provided that the coordinates $u_1, \ldots, u_{2n}$ exist (which is a non–trivial fact because equation (71) is overdetermined).

Our strategy is as follows: since in Proposition 7.1 we computed the Hamiltonians $h^{(n)}_k$ corresponding to the matrices $\hat{L}_k$ which give the same flow as

$$L_k = \left[\lambda^{1-2k}\mathcal{A}^{(n)}\right]_+, \qquad k=1,\ldots,n, \tag{73}$$

we introduce some new times $s_1, \ldots, s_n$ corresponding to the flows along $L_1, \ldots . L_n$:

$$\partial_{s_k}\mathcal{A}^{(n)} = [L_k, \mathcal{A}^{(n)}] + \partial_\lambda L_k, \quad k=1,\ldots,n. \tag{74}$$

Observe that $s_n = -\frac{z}{4^n}$. Since the matrices L_k are related to the matrices $\hat{M}^{(k)}$ by

$$L_k = 4^k \sum_{i=k+1}^{n} t_i \hat{M}^{(i-k)} - 4^k t_k \mathcal{B}, \qquad k=1,\ldots,n, \tag{75}$$

the times $t_1, \ldots, t_{n-1}$ and the $s_1, \ldots, s_{n-1}$ must be related by:

$$\partial_{s_k} = 4^k \sum_{i=k+1}^{n} (2(i-k)+1)t_i\partial_{t_{i-k}} - 4^k t_k \partial_z, \qquad k=1,\ldots,n. \tag{76}$$

This relation is the key to our procedure: on one side, it will allow us to prove that there exist coordinates $u_1, \ldots, u_{2n}$ such that

$$\partial^u_{s_k}\mathcal{A}^{(n)} = \partial_\lambda L_k, \tag{77}$$

on the other side relation (75), will allow us to compute the Hamiltonians $H^{(n)}_k$ in terms of the Hamiltonians $h^{(n)}_k$.

The following proposition shows that (76)guarantees the compatibility of the over–determined system (71).

Proposition 8.3. *Let s_k be the times corresponding to the equations*

$$\partial_{s_k}\mathcal{A}^{(n)} - \partial_\lambda L_k = [L_k, \mathcal{A}^{(n)}], \quad k = 1, \dots, n$$

where L_k are given by (73). Then the system of differential equations

$$\partial^u_{s_k}\mathcal{A}^{(n)} = \partial_\lambda L_k$$

is compatible only if t_k and s_k satisfy the following relations

$$\partial_{s_k} t_j = 4^k(2j+1)t_{j+k}, \quad \partial_{s_k} z = -4^k t_k, \tag{78}$$

where $t_l = 0$ for $l > n$.

Observe that equation (78) is equivalent to (76).

Proof. Let us consider the explicit s_k derivative given by

$$\partial^u_{s_k}\mathcal{A}^{(n)} = \partial_\lambda L_k.$$

This is an over–determined system of equations. We are going to prove that it admits a solution. In terms of the entries of $\mathcal{A}^{(n)}$, this is equivalent to

$$\begin{array}{rcl} \partial^u_{s_k} a^{(n)}_{2j+1} & = & (2j+1)a^{(n)}_{2j+2k+1}, \quad j = 0, \dots, n, \\ \partial^u_{s_k} b^{(n)}_j & = & j b_{j+2k}, \quad j = 1, \dots, 2n, \\ a^{(n)}_j & = & 0, \quad j > 2n+1, \quad b^{(n)}_j = 0, \quad j > 2n. \end{array}$$

This means that

$$\begin{array}{rcl} \lambda^{2n}\partial^u_{s_k}(1+\mathcal{P}) & = & \left[\partial_\lambda\left(\lambda^{2n-2k+1}(1+\mathcal{P})\right)\right]_+, \\ \lambda^{-1}\partial^u_{s_k}\mathcal{B}_{even} & = & \left[\partial_\lambda\left(\lambda^{-2k}\mathcal{B}_{even}\right)\right]_+, \\ \lambda^{-1}\partial^u_{s_k}\mathcal{B}_{odd} & = & \left[\partial_\lambda\left(\lambda^{-2k}\mathcal{B}_{odd}\right)\right]_+, \end{array} \tag{79}$$

where $\mathcal{P}$, $\mathcal{B}_{even}$ and $\mathcal{B}_{odd}$ are given by (46) and (47). We will now rewrite the expression of $\mathcal{B}_{odd}$ in (47) as

$$\mathcal{B}_{odd} = \frac{1}{4}\left[(2\lambda)^{2n+1}(1+\mathcal{P}) - \frac{((2\lambda)^{2n+1}(1+\mathcal{T}))^2}{(2\lambda)^{2n+1}(1+\mathcal{P})} + \right. \\ \left. + \frac{4\mathcal{B}^2_{even}}{(2\lambda)^{2n+1}(1+\mathcal{P})}\right)\Bigg]_+ . \tag{80}$$

Then, by computing

$$\lambda^{-1}\partial^u_{s_k}\mathcal{B}_{odd} \quad = \quad \left[\partial_\lambda\left(\lambda^{-2k}\mathcal{B}_{odd}\right)\right]_+, \tag{81}$$

and applying (79) and (80), we see that, in order that the over–determined system (77) is compatible, the derivative of $\mathcal{T}$ must satisfy

$$\lambda^{2n}\partial^u_{s_k}(1+\mathcal{T}) \quad = \quad \left[\partial_\lambda\left(\lambda^{2n-2k+1}(1+\mathcal{T})\right)\right]_+.$$

This gives

$$\partial^u_{s_k} t_j = 4^k(2j+1)t_{j+k}, \quad \partial^u_{s_k} z = -4^k t_k,$$

where $t_l = 0$ if $l > n$ in the above equation.

If we chose another set of coordinates,

$$\{y_1(u,t), \dots, y_{2n}(u,t), t_1, \dots, t_n\}$$

Then the vector field $\partial^y_{s_k}$ is given by

$$\partial^u_{s_k} = \partial^y_{s_k} + \sum_{j=1}^n \partial^u_{s_k} y_j \partial_{y_j}$$

Therefore $\partial^u_{s_k} t_j = \partial^y_{s_k} t_j$ for $j = 1, \dots, n$ and the same for z. Hence we can drop the superscript u in (82) to obtain (78). □

As proved in Proposition 7.1, the Hamiltonians $h_k^{(n)}$ corresponding to the vector field

$$\mathcal{X}_k := [L_k, A],$$

are given by equation (65). Due to our derivation, they must be thought of as written in the coordinates u. Since the canonical coordinates $P_1, \dots, Q_n$ depends explicitly on the s_k when expressed in terms of the coordinates u, the actual Hamiltonian differs by a shift given by Lemma 7.3.

We first compute the explicit s_k derivatives of the canonical coordinates. From the definition of the coordinates $Q_1, \dots, Q_n$ we have

$$\lambda^{-2}\mathcal{Q} = -\left[\lambda^{-2}\mathcal{B}_{even}(1+\mathcal{P})^{-1}\right]_+.$$

We can the apply (79) to compute the explicit s_i derivatives of $\mathcal{Q}$. This gives

$$\partial^u_{s_k}\mathcal{Q} = \left[\lambda^{1-2k}\partial_\lambda\mathcal{Q} - (2n+1)\lambda^{-2k}\mathcal{Q}\right]_+.$$

Hence the explicit s_j derivatives of the canonical coordinates are given by

$$\begin{array}{rcl} \partial^u_{s_k}Q_j &=& (2(j+k)-2n-1)Q_{j+k}, \\ \partial^u_{s_k}P_j &=& (2(n-j)+1)P_{j-k}, \quad j\neq k \\ \partial^u_{s_k}P_k &=& (2(n-k)+1), \end{array} \tag{82}$$

where if $l > n$ or $l < 1$, then $P_l = 0$, $Q_l = 0$.

Using Lemma 7.3 and equation (82), we can compute the shifts :

$$\delta h_k^{(n)} = \sum_{j=1}^{n-k}(2(j+k)-2n-1)P_jQ_{j+k} + (2k-2n-1)Q_k. \tag{83}$$

Finally, we use the formulae above to show that the coordinates u actually exist. In fact we construct them recursively as follows using (82). For $j = 1$ this gives

$$\partial^u_{s_k}P_1 = (2n-1)\delta_{k1},$$

so that we choose $u_1 = P_1 - (2n-1)s_1$. Then for $j = 2$ equation (82) gives:

$$\begin{array}{rcl} \partial^u_{s_1}P_2 &=& (2(n-2)+1)P_1, \\ \partial^u_{s_2}P_2 &=& (2n-1) \end{array}$$

and all other derivatives of P_2 are zero, so we put

$$u_2 = P_2 - (2n-1)s_2 - (2(n-2)+1)P_1s_1 + (2(n-2)+1)(2n-1)\frac{s_1^2}{2},$$

so that $p^u_{s_k}u_2 = 0$. By repeating this procedure iteratively we get:

$$\begin{array}{rcl} u_i &=& P_i + \displaystyle\sum_{j=1}^{i-1} W_{ij}P_{i-j} + W_{ii}, \quad i = 1,\dots,n, \\ u_{i+n} &=& Q_i + \displaystyle\sum_{j=1}^{n-i} V_{ij}Q_{i+j}, \quad i = 1,\dots,n, \end{array}$$

where W_{ij} and V_{ij} are given by

$$\begin{array}{rcl} W_{ij} &=& \displaystyle\sum_{m=1}^{j}\sum(-2(n-i+j-l_1)-1)^{k_1}\left(\frac{s_{l_1}}{k_1}\right)^{k_1}\cdots(-2(n-i+j-l_c)-1)^{k_c}\left(\frac{s_{l_c}}{k_c}\right)^{k_c}, \\ V_{ij} &=& \displaystyle\sum_{m=1}^{j}\sum(-2(i+j-l_1)+2n+1)^{k_1}\left(\frac{s_{l_1}}{k_1}\right)^{k_1}\cdots(-2(i+j-l_c)+2n+1)^{k_c}\left(\frac{s_{l_c}}{k_c}\right)^{k_c}, \end{array}$$

where the second summations in the above are taken over all possible combinations of integers

$$\begin{array}{rcl} l_1k_1 &+& l_2k_2 + \cdots + l_ck_c = j \\ k_1 &+& \cdots + k_c = m, \quad 0 < k_i < m, \quad 0 < l_i < j. \end{array}$$

Resuming, we proved the following:

Theorem 8.4. *The Hamiltonians for the equations*

$$\partial_{s_k}\mathcal{A}^{(n)} - \partial_\lambda L_k = [L_k, \mathcal{A}^{(n)}] \tag{84}$$

where L_k are defined by (73), are

$$h_k^{(n)} = \frac{1}{2}\mathrm{ResTr}\left(\lambda^{1-2k}\left(\mathcal{A}^{(n)}\right)^2\right) \tag{85}$$

and their shifts $\delta h_k^{(n)}$ corresponding to the canonical coordinates P_k, Q_k defined in Theorem 5.1 are given by

$$\delta h_k^{(n)} = \sum_{j=1}^{n-k}(2(j+k)-2n-1)P_jQ_{j+k} + (2k-2n-1)Q_k,$$

that is, the equations (84) can be expressed as the following time-dependent Hamiltonian equations

$$\partial_{s_k}\mathcal{A}^{(n)} = \left\{h_k^{(n)} + \delta h_k^{(n)}, \mathcal{A}^{(n)}\right\} + \partial_{s_k}^{can}\mathcal{A}^{(n)}$$

with the Poisson bracket $\{,\}$ defined by

$$\{f,g\} = \sum_{i=1}^{n}\frac{\partial f}{\partial P_i}\frac{\partial g}{\partial Q_i} - \frac{\partial g}{\partial P_i}\frac{\partial f}{\partial Q_i}, \tag{86}$$

and the derivative $\partial_{s_k}^{can}$ is the explicit time derivative of s_k when the canonical coordinates P_i, Q_i are fixed.

Now let us deal with the t_k–flows. Thanks to the fact that the coordinates $u_1,\dots,u_{2n}$ exist and are such that (77) is satisfied, equations (75) and (76) we see that the matrices $\hat{M}^{(i)}$ correspond to the times $(2i+1)t_i$. That is, the equations

$$\begin{aligned}(2k+1)\partial_{t_k}^{u}\mathcal{A}^{(n)} &= \partial_\lambda \hat{M}^{(k)}\\ (2k+1)\partial_{t_k}\mathcal{A}^{(n)} &= [\hat{M}^{(k)}, \mathcal{A}^{(n)}] + \partial_\lambda\hat{M}^{(k)}, \quad k=1,\dots,n\end{aligned}$$

are compatible and hence we can express the higher order isomonodromic flows (14) as time-dependent Hamiltonian flows.

We can now compute the Hamiltonians $H_k^{(n)}$ for the times t_k by using Theorem 8.4 and (76).

Corollary 8.5. *Let $\mathcal{K}_s$ and $\mathcal{H}_{\hat{t}}$ be the following polynomials in λ and $\frac{1}{\lambda}$ respectively:*

$$\mathcal{K}_s = \sum_{k=1}^{n}(h_k^{(n)} + \delta h_k^{(n)})\lambda^{2k}, \qquad \mathcal{H}_t = \sum_{k=1}^{n-1}(H_k^{(n)} + \delta H_k^{(n)})(2\lambda)^{-2k-1} - \frac{1}{2\lambda}(H^{(n)} + \delta H^{(n)}),$$

where $h_k^{(n)}$, $\delta h_k^{(n)}$ are given by (85) and (83) respectively and the coefficients $H_k^{(n)} + \delta H_k^{(n)}$ are given by:

$$\partial_\lambda\mathcal{H}_t = -2(2\lambda)^{-2n}\left((2\lambda)^{-2}\mathcal{K}_s(1+\mathcal{T})^{-1}\right)_+, \tag{87}$$

with $\mathcal{T}$ is defined by (46). Then the equations

$$(2k+1)\partial_{t_k}\mathcal{A}^{(n)} - \partial_\lambda\hat{M}^{(k)} = [\hat{M}^{(k)}, \mathcal{A}^{(n)}]$$

can be expressed as time-dependent Hamiltonian equations with the Poisson bracket given by (86) as follows:

$$\partial_{t_k}\mathcal{A}^{(n)} = \left\{H_k^{(n)} + \delta H_k^{(n)}, \mathcal{A}^{(n)}\right\} + \partial_{t_k}^{can}\mathcal{A}^{(n)},$$

where the derivative $\partial_{t_k}^{can}$ is the explicit time t_k derivative when the canonical coordinates P_i, Q_i are fixed. In particular, the equation (14) is Hamiltonian and it is equivalent to

$$\frac{\partial Q_i}{\partial t_k} = \frac{\partial\mathcal{H}_k^{(n)}}{\partial P_i}, \quad \frac{\partial P_i}{\partial t_k} = -\frac{\partial\mathcal{H}_k^{(n)}}{\partial Q_i}, \qquad k=1,\dots,n-1, \quad i=1,\dots,n,$$

where

$$\mathcal{H}_k^{(n)} = H_k^{(n)} + \delta H_k^{(n)}, \qquad k=1,\dots,n-1.$$

Proof. From (76), we see that the Hamiltonians are related by

$$h_k^{(n)} + \delta h_k^{(n)} = 4^k\sum_{i=k+1}^{n}(2(i-k)+1)t_i(H_{i-k}^{(n)} + \delta H_{i-k}^{(n)}) - 4^k t_k(H^{(n)} + \delta H^{(n)}).$$

This implies the following relation

$$\begin{aligned} 2\mathcal{K}_s &= -(2\lambda)^2\left((2\lambda)^{2n}\partial_\lambda\mathcal{H}_t(1+\mathcal{T})\right), \\ \left((2\lambda)^{2n}(2(2\lambda)^{-2n-2}\mathcal{K}_s)\right)_+ &= -\left((2\lambda)^{2n}\partial_\lambda\mathcal{H}_t(1+\mathcal{T})\right). \end{aligned}$$

Then, by using (49), we obtain (87). □

Example 8.6. Let us demonstrate the results of this Section in the first non-trivial example: $n = 2$. We want to represent the t_1 flow

$$3\partial_{t_1}\mathcal{A}^{(2)} - \partial_\lambda\hat{M}^{(1)} = [\hat{M}^{(1)}, \mathcal{A}^{(2)}] \tag{88}$$

where

$$\hat{M}^{(1)} = \begin{pmatrix} 4\lambda^3 - 2w^2\lambda & -4w\lambda^2 + 2w_z\lambda - w_{zz} + 2w^3 \\ -4w\lambda^2 - 2w_z\lambda - w_{zz} + 2w^3 & -4\lambda^3 + 2w^2\lambda \end{pmatrix},$$

$$\mathcal{A}^{(2)} = \begin{pmatrix} 16\lambda^4 + 4\lambda^2(t_1 - 2w^2) - z - 2t_1w^2 + 6w^4 + 2w_z^2 - 4ww_{zz} & -16\lambda^3 w + 8\lambda^2 w_z + 4\lambda(2w^3 - wt_1 - w_{zz}) + 2t_1w_z - 12w^2w_z - \frac{\alpha_2}{\lambda} \\ -16\lambda^3 w - 8\lambda^2 w_z + 4\lambda(2w^3 - wt_1 - w_{zz}) - 2t_1w_z + 12w^2w_z - \frac{\alpha_2}{\lambda} & -16\lambda^4 - 4\lambda^2(t_1 - 2w^2) + z + 2t_1w^2 - 6w^4 - 2w_z^2 + 4ww_{zz} \end{pmatrix}$$

as a time dependent Hamiltonian equation.

We are now going to show that there exist coordinates $u_1, \ldots, u_4$ such that

$$\partial_\lambda\hat{M}^{(1)} = 3\partial^u_{t_1}\mathcal{A}^{(2)}.$$

This gives the following equations

$$3\partial^u_{t_1}a_3^{(2)} = 3\partial^u_{t_1}\left(-8w^2 + 4t_1\right) = 12, \tag{89a}$$

$$3\partial^u_{t_1}a_1^{(2)} = 3\partial^u_{t_1}\left(2(w_z^2 + 3w^4 - 2ww_{zz}) - 2t_1w^2 - z\right) = -2w^2, \tag{89b}$$

$$3\partial^u_{t_1}b_4^{(2)} = -48\partial^u_{t_1}w = 0, \tag{89c}$$

$$3\partial^u_{t_1}b_3^{(2)} = 24\partial^u_{t_1}w_z = 0, \tag{89d}$$

$$3\partial^u_{t_1}b_2^{(2)} = 3\partial^u_{t_1}\left(-4w_{zz} + 8w^3 - 4t_1w\right) = -8w, \tag{89e}$$

$$3\partial^u_{t_1}b_1^{(2)} = 3\partial^u_{t_1}\left(2(w_{zzz} - 6w^2w_z) + 2t_1w_z\right) = 2w_z. \tag{89f}$$

We also need $\partial^u_{t_1}t_1 = 1$ and $\partial^u_{t_1}z = 0$. Of course, after imposing these two constraints, we have more equations then independent variables. (6 equations and 4 independent variables w, w_z, w_{zz} and w_{zzz}) We need to check that equations (89a)-(89f) are consistent. To see this, first note that equations (89c) and (89d) imply

$$\partial^u_{t_1}w = \partial^u_{t_1}w_z = 0.$$

By substituting this into (89a), we see that

$$3\partial^u_{t_1}a_3^{(2)} = 3\partial^u_{t_1}\left(-8w^2 + 4t_1\right) = 12,$$

which is tautologically true, so (89a) is consistent with (89d) and (89c). By substituting (89d) and (89c) into (89b), we see that

$$\begin{aligned} 3\partial^u_{t_1}a_1^{(2)} &= 3\partial^u_{t_1}\left(2(w_z^2 + 3w^4 - 2ww_{zz}) - 2t_1w^2 - z\right) = -2w^2 \\ \Rightarrow\ \partial^u_{t_1}w_{zz} &= -\frac{1}{3}w. \end{aligned}$$

Then by substituting these into (89e), we find that

$$\begin{aligned} 3\partial^u_{t_1}b_2^{(2)} &= 3\partial^u_{t_1}\left(-4w_{zz} + 8w^3 - 4t_1w\right) \\ &= -12\partial^u_{t_1}w_{zz} - 12w = -8w, \end{aligned}$$

therefore (89e) is also consistent with the other equations. Now the last equation (89f) does not have any consistency issue and it would give us the following

$$3\partial^u_{t_1}w_{zzz} = -2w_z.$$

Therefore these equations are all consistent and $u_1, \ldots, u_4$ exist.

Using the results of Section 6, we can express the matrix $\mathcal{A}^{(n)}$ in terms of the canonical coordinates P_1, P_2, Q_1, Q_2 and using formula (85) we get

$$h_1^{(2)} = -32zP_2 + 256P_1^2P_2 - 256P_2^2 - 2P_2Q_1Q_2 - P_1P_2Q_2^2 - 2\alpha_2(P_1Q_2 - Q_1) - t_1P_2(128P_1 - 16t_1)$$

$$h_2^{(2)} = -32zP_1 + 256P_1^3 - 512P_1P_2 - P_2Q_2^2 + Q_1^2 + 2\alpha_2Q_2 - t_1\left(128P_1^2 - 128P_2 - 16P_1t_1\right),$$

and using (83) we get for the shifts:

$$\delta h_1^{(2)} = -P_1Q_2 - Q_1, \qquad \delta h_2^{(2)} = -Q_2,$$

so that, using (87), the Hamiltonian of the equation ${\mathrm{P}_{\mathrm{II}}}^{(2)}$ is

$$\mathcal{H}^{(2)} = \frac{1}{16}(-h_1^{(2)} + Q_2).$$

and the Hamiltonian of (14) with $k = 1$ is

$$\mathcal{H}_1^{(2)} = \frac{1}{3}\left(\frac{1}{4}(h_1^{(2)} + \delta h_1^{(2)}) - \frac{1}{16}t_1(h_2^{(2)} + \delta h_2^{(2)})\right).$$

For the sake of completeness, we can compute the shift in $\mathcal{H}_1^{(2)}$ directly from the explicit dependence of P_1, P_2, Q_1, Q_2 on s_1. From (89a)-(89f), we have

$$\begin{aligned}
\partial_{t_1}^u P_1 &= \frac{1}{16}\partial_{t_1}^u(a_3^{(2)} + b_3^{(2)}) = \frac{1}{4},\\
\partial_{t_1}^u P_2 &= \frac{1}{16}\partial_{t_1}^u(a_1^{(2)} + b_1^{(2)}) = \frac{1}{24}\mathcal{L}_1 = \frac{P_1}{12} - \frac{t_1}{48},\\
\partial_{t_1}^u Q_2 &= 0,\\
\partial_{t_1}^u Q_1 &= -\partial_{t_1}^u b_2^{(2)} - Q_2\partial_{t_1}^u P_1 = -\frac{Q_2}{12}.
\end{aligned}$$

Therefore the we can find $\delta H_1^{(2)}$ from the following partial differential equations

$$\begin{aligned}
\frac{\partial \delta H_1^{(2)}}{\partial Q_1} &= -\frac{1}{4},\\
\frac{\partial \delta H_1^{(2)}}{\partial Q_2} &= -\frac{P_1}{12} + \frac{t_1}{48},\\
\frac{\partial \delta H_1^{(2)}}{\partial P_1} &= -\frac{Q_2}{12},\\
\frac{\partial \delta H_1^{(2)}}{\partial P_2} &= 0.
\end{aligned}$$

This gives the shift

$$\delta H_1^{(2)} = -\frac{Q_1}{4} - \frac{P_1Q_2}{12} + \frac{t_1Q_2}{48},$$

which in fact agrees with what we obtained above.

Appendix A. From the mKdV Lax pair to the isomonodromic problem

Let us consider the Lax pair of the mKdV hierarchy:

$$\frac{\partial \Phi}{\partial x} = N\Phi, \qquad \frac{\partial \Phi}{\partial t_{k+1}} = N_k\Phi,$$

where

$$N = \begin{pmatrix} -\zeta & v \\ v & \zeta \end{pmatrix}, \qquad N_k = \begin{pmatrix} \sum_{j=1}^{2k+1} A_j^{(k)}\zeta^j & \sum_{j=0}^{2l} B_j^{(k)}\zeta^j \\ \sum_{j=0}^{2l} C_j^{(k)}\zeta^j & -\sum_{j=1}^{2l+1} A_j^{(k)}\zeta^j \end{pmatrix},$$

where the coefficients $A_j^{(k)}, B_j^{(k)}, C_j^{(k)}$ are given by the formulae (16) with $\mathcal{L}_l$ replaced by $\mathcal{R}_l$, z replaced by x and w replaced by v. Observe that in N_k the terms $B_0^{(k)}$ are nonzero. They are

$$B_0^{(k)} = -(\partial_x + 2v)\mathcal{R}_k.$$

Details can be found in [10].

By the self–similarity reduction (11), and defining

$$\lambda = [(2n+1)T_{n+1}]^{\frac{1}{2n+1}}\zeta,$$

we obtain:

$$(2n+1)T_{n+1}\frac{\partial\Psi}{\partial T_{n+1}} = -z\frac{\partial\Psi}{\partial z} - \sum_{1}^{n-1}(2k+1)t_k\frac{\partial\Psi}{\partial t_k} + \lambda\frac{\partial\Psi}{\partial\lambda},$$
$$T_{k+1}\frac{\partial\Psi}{\partial T_{k+1}} = t_k\frac{\partial\Psi}{\partial t_k},$$
$$x\frac{\partial\Psi}{\partial x} = z\frac{\partial\Psi}{\partial z}.$$

Now, defining

$$\mathcal{B} = [(2n+1)T_{n+1}]^{\frac{1}{2n+1}}N, \qquad \hat{M}^{(k)} = [(2n+1)T_{n+1}]^{\frac{2k+1}{2n+1}}N_k, \quad k = 1,\dots,n,$$

we get

$$\frac{\partial\Psi}{\partial z} = \mathcal{B}\Psi, \tag{90a}$$

$$\frac{\partial\Psi}{\partial\lambda} = \frac{1}{\lambda}\left[\hat{M}^{(n)} + \sum_{l=1}^{n-1}t_l\hat{M}^{(l)} + z\,\mathcal{B}\right]\Psi, \tag{90b}$$

$$(2k+1)\frac{\partial\Psi}{\partial t_k} = \hat{M}^{(k)}\Psi. \tag{90c}$$

To show that (90) coincide with (15) first observe that

$$\hat{M}_k = M^{(k)} - \begin{pmatrix} 0 & (\partial_z + 2w)\mathcal{L}_k \\ (\partial_z + 2w)\mathcal{L}_k & 0 \end{pmatrix}, \tag{91}$$

as it follows from the definition of $\hat{M}_k$ and N_k. Moreover

$$\mathcal{A}^{(n)} = \hat{M}^{(n)} + \sum_{l=1}^{n-1}t_l\hat{M}^{(l)} + z\,\mathcal{B}, \tag{92}$$

because

$$-(\partial_z + 2w)\mathcal{L}_n - \sum_{l}^{n-1}t_l(\partial_z + 2w)\mathcal{L}_l + zw = -\alpha_n$$

thanks to (12).

It is now clear how to prove that the equations (17), (18) and (19) are indeed consistent. This follows from the fact that the mKdV flows commute, so that

$$\frac{\partial N}{\partial T_{k+1}} - \frac{\partial N_k}{\partial x}N_k = [N_k, N],$$

and

$$\frac{\partial N_l}{\partial T_{k+1}} - \frac{\partial N_k}{\partial T_{l+1}} = [N_k, N_l].$$

The equations (17), (18) and (19) then follow automatically from the relations (92), (91).

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, The inverse scattering transform — Fourier analysis for nonlinear problems, *Stud. Appl. Math.*, **53** (1974) 249–315.

[2] M.J. Ablowitz and H. Segur, Exact linearization of a Painlevé transcendent, *Phys. Rev. Lett.*, **38** (1977) 103–1106.

[3] M.J. Ablowitz and H. Segur, "*Solitons and the Inverse Scattering Transform*", SIAM, Philadelphia (1981).

[4] M.R. Adams and J. Harnad and J. Hurtubise, Darboux coordinates and Liouville–Arnold integration in loop algebras, *Comm. math. Phys.*, **155** (1993) 385–413.

[5] H. Airault, Rational solutions of Painlevé equations, *Stud. Appl. Math.*, **61** (1979) 31–53.

[6] M. Audin, Lectures on gauge theory and integrable systems, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, **488** (1997) 1–48.

[7] P. Boalch, Symplectic manifolds and isomonodromic deformations, *Adv. Math.*, **163** (2001) no. 2:137–205.

[8] O. Babelon, D. Bernard and M. Talon, "*Introduction to Classical Integrable Systems*", Cambridge Monographs on Mathematical Physics, CUP (2003).

[9] P.A. Clarkson, N. Joshi and A. Pickering, Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach, *Inverse Problems*, **15** (1999) 175–187.

[10] P.A. Clarkson, N. Joshi and M. Mazzocco, The Lax pair for the mKdV Hierarchy., *Proceedings...*, **volume** (2005) pages.
[11] P.A. Clarkson and E.L. Mansfield, The second Painlevé equation, its hierarchy and associated special polynomials, *Nonlinearity*, **16** (2003) no. 3:R1–R26.
[12] T. Claeys, A.B.J. Kuijlaars and M. Vanlessen, "arXiv: math-ph/0508062" Multi-critical unitary random matrix ensembles and the general Painlevé II equation (2005).
[13] H. Flaschka and D.W. McLaughlin, Canonically conjugate variables for the Korteweg–de Vries equation and the Toda lattice with periodic boundary conditions, *Progr. Theoret. Phys.*, **55** (1976) 438–456.
[14] H. Flaschka and A.C. Newell, Monodromy- and spectrum preserving deformations. I, *Commun. Math. Phys.*, **76** (1980) 65–116.
[15] H. Flaschka, A.C. Newell and T. Ratiu, 300–323, *Phys. D*, **9** (Kac-Moody Lie algebras and soliton equations. II. Lax equations associated with $A_1^{(1)}$) no. 3.1983
[16] H. Flaschka, A.C. Newell and T. Ratiu, 324–332, *Phys. D*, **9** (Kac-Moody Lie algebras and soliton equations. III. Stationary equations associated with $A_1^{(1)}$) no. 3.1983
[17] A.S. Fokas and S. Tanveer, A Hele-Shaw problem and the second Painlevé transcendent, *Math. Proc. Cambridge Philos. Soc.*, **124** (1998) 169–191.
[18] B. Gambier, Sur les équations différentiellles du second ordre et du premier degre dont l'intégrale générale est á points critiques fixés, *Acta Math.*, **33** (1909) 1–55.
[19] J.A. Giannini and R.I. Joseph, The role of the second Painlevé transcendent in nonlinear optics, *Phys. Lett. A*, **141** (1989) 417–419.
[20] P.R. Gordoa, N. Joshi and A. Pickering, Second and fourth Painlevé hierarchies and Jimbo-Miwa linear problems, *J. Math. Phys.*, **47** (2006) 073504.
[21] V.I. Gromak, On the theory of the second Painlevé equation, *Vestnik Beloruss. Gos. Univ. Ser.I* , **3** (1974) 73–75.
[22] J. Harnad, Dual isomonodromic deformations and moment maps to loop algebras, *Comm. Math. Phys*, **166** (1994) no.2: 337–365.
[23] J. Harnad and M. Routhier, R-matrix construction of electromagnetic models for the Painlevé transcendents, *J. Math. Phys.*, **36** (1995) 9:4864–4881.
[24] J. Harnad, "Talk at the Colloque international en l' honneur de Pierre van Moerbeke, Poitiers, France. Unpublished" The Haminltonian structure of the general rational isomonodromic deformations (2005).
[25] N. Hitchin, Frobenius Manifolds (with notes by David Calderbank), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, **488** (1997) 69–112.
[26] S. Kakei, Talk at I. Newton Institute, Cambridge, *Surikaisekikenkyusho Kokyuroku*, **1422** (2005) 192–203.
[27] I. Krichever, Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations. Dedicated to Yuri I. Manin on the occasion of his 65th birthday, *Mosc. Math. J.*, **2** (2002) no. 4:717–752.
[28] N.A. Kudryashov, One generalization of the second Painlevé hierarchy, *J. Phys. A: Math. Gen.*, **35** (2002) 93–99.
[29] M. Jimbo and T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II, *Physica 2D*, **2** (1981) 407–448.
[30] P.D. Lax, Almost periodic solutions of the KdV equation, *SIAM Rev.*, **18** (1976) 351–375.
[31] N.A. Lukashevich, The second Painlevé equation, *Diff. Eqns.*, **7** (1971) 853–854.
[32] B. Malgrange, Sur les déformations isomonodromiques II. Singularités irréguilières, *Progr. Math.*, **37** (1983) 427–438.
[33] T. Miwa, Painlevé property of monodromy preserving deformations and the analyticity of the τ–functions, *Publ. RIMS*, **17** (1981) 703–721.
[34] D. Mumford, "*Tata lectures on theta. II. Jacobian theta functions and differential equations*", Progress in Mathematics (43).Birkhuser Boston, Inc., Boston, MA1984
[35] S.P. Novikov and A.P. Veselov, Poisson brackets and complex tori, *Trudy Mat. Inst. Steklov*, **165** (1984) 49–61.
[36] K. Okamoto, Studies on the Painlevé equations. III. Second and fourth Painlevé equations, *Math. Ann.*, **275** (1986) 2:221–255.
[37] E.K. Sklyanin, Separation of variables. New trends, *preprint arXiv:solv-int*, **9504001** (1995) .
[38] C.A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, *Comm. Math. Phys.*, **159** (1994) no. 1: 151–174.
[39] H. Umemura and H. Watanabe, Solutions of the second and fourth Painlevé equations, *Nagoya Math. J.*, **148** (1997) 151–198.
[40] A.P. Vorob'ev, On rational solutions of the second Painlevé equation, *Diff. Eqns*, **1** (1965) 58–59.
[41] A.I. Yablonskii, On rational solutions of the second Painlevé equation, *Vesti Akad. Navuk. BSSR Ser. Fiz. Tkh. Nauk.*, **3** (1965) 30-35.
[42] N.M.J. Woodhouse, The symplectic and twistor geometry of the general isomonodromic deformation problem, *J. Geom. Phys.*, **39** (2001) no. 2: 97–128.

School of Mathematics, The University of Manchester, Manchester M60 1QD, United Kingdom
E-mail address: Marta.Mazzocco@manchester.ac.uk

Department of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom
E-mail address: M.Mo@bristol.ac.uk