

THE HARTMAN-WINTNER LAW OF THE ITERATED LOGARITHM FOR MARTINGALES¹

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According to the Hartman-Wintner law of the iterated logarithm [4], $\{Y_i, i \geq 1\}$ independent identically distributed with $EY_1 = 0$ and $EY_1^2 = 1$ implies that $\limsup \sum_{i=1}^n Y_i / (2n \log_2 n)^{\frac{1}{2}} = 1$ almost surely (a.s.). We generalize this result to stationary ergodic martingale difference sequences.

THEOREM. *Let $(Y_i, i \geq 1)$ be a stationary ergodic stochastic sequence with $E[Y_i | Y_1, Y_2, \dots, Y_{i-1}] = 0$ a.s. for all $i \geq 2$ and $EY_1^2 = 1$. Then $\limsup \sum_{i=1}^n Y_i / (2n \log_2 n)^{\frac{1}{2}} = 1$ a.s.*

PROOF. In order to prove the result, we assume a particular representation for the stochastic sequence, namely that the sample space Ω is the Cartesian product $\prod_{i=-\infty}^{\infty} R_i$ of copies of the real line, that the Y_i 's are the coordinate variables of Ω , that \mathcal{F} is the σ -field of events generated by the Y_i 's, and that P is the probability induced on (Ω, \mathcal{F}) by the finite dimensional distributions of the Y_i 's. Letting \mathcal{F}_n be the σ -field generated by $\{Y_m, -\infty < m \leq n\}$, we note that $E[Y_i | \mathcal{F}_{i-1}] = 0$ a.s. for all integers i . Let K_i be a sequence of positive constants to be specified later and let $b_i = K_i(i/\log_2 i)^{\frac{1}{2}}$. Let $Y_i' = Y_i I(|Y_i| \leq b_i) + b_i I(Y_i > b_i) - b_i I(Y_i < -b_i)$ and $Z_i' = Y_i' - E[Y_i' | \mathcal{F}_{i-1}]$. Likewise let $Y_i'' = Y_i I(|Y_i| \leq M) + MI(Y_i > M) - MI(Y_i < -M)$ and $Z_i'' = Y_i'' - E[Y_i'' | \mathcal{F}_{i-1}]$ for $M > 0$. This modification of the usual truncation procedure enables us to conclude that $E[Y_i^2 | \mathcal{F}_{i-1}] \geq E[(Z_i')^2 | \mathcal{F}_{i-1}] \geq E[(Z_i'')^2 | \mathcal{F}_{i-1}]$ a.s. when $b_i \geq M$. (See Corollary 4 of [3] for a proof of this.) By the Birkoff ergodic theorem, $\sum_{i=1}^n E[(Z_i'')^2 | \mathcal{F}_{i-1}] / n \rightarrow E(Z_0'')^2$ a.s. and $\sum_{i=1}^n E[Y_i^2 | \mathcal{F}_{i-1}] / n \rightarrow 1$ a.s. K_i will be chosen such that $K_i \rightarrow 0$ and $b_i \rightarrow \infty$. We then obtain $1 = \limsup \sum_{i=1}^n E[Y_i^2 | \mathcal{F}_{i-1}] / n \geq \limsup \sum_{i=1}^n E[(Z_i')^2 | \mathcal{F}_{i-1}] / n \geq \limsup \sum_{i=1}^n E[(Z_i'')^2 | \mathcal{F}_{i-1}] / n = E(Z_0'')^2$. Since $E(Z_0'')^2 \rightarrow 1$ as $M \rightarrow \infty$, it follows that

$$(1) \quad \sum_{i=1}^n E[(Z_i')^2 | \mathcal{F}_{i-1}] / n \rightarrow 1 \text{ a.s.} \quad \text{and hence that}$$

$$(2) \quad \sum_{i=1}^n E[(Z_i'')^2 | \mathcal{F}_{i-1}] \rightarrow \infty \text{ a.s.}$$

According to [5], if $(Z_i, \mathcal{F}_i, i \geq 1)$ is a martingale difference sequence with $s_n^2 = \sum_{i=1}^n E[Z_i^2 | \mathcal{F}_{i-1}] \rightarrow \infty$ a.s., $u_n = (2 \log_2 s_n^2)^{\frac{1}{2}}$, \mathcal{F}_{i-1} measurable random variables $L_i \rightarrow 0$ a.s., and $|Z_i| \leq L_i s_i / u_i$ a.s. for all $i \geq 1$, then $\limsup \sum_{i=1}^n Z_i / (s_n u_n) = 1$ a.s.

Recalling (1) and (2), Z_i' satisfies the hypotheses of this theorem with $L_i = 2K_i u_i (i/\log_2 i)^{\frac{1}{2}} / s_i$ since $|Z_i'| \leq 2K_i (i/\log_2 i)^{\frac{1}{2}}$ a.s. Thus, using (1), $\limsup \sum_{i=1}^n Z_i' / (2n \log_2 n)^{\frac{1}{2}} = \limsup \sum_{i=1}^n Z_i' / (s_n u_n) = 1$ a.s.

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To complete the proof it suffices to show that

$$(3) \quad \sum_{i=1}^n E[Y_i' | \mathcal{F}_{i-1}] / (2n \log_2 n)^{\frac{1}{2}} \rightarrow 0 \quad \text{a.s.} \quad \text{and that}$$

$$(4) \quad \sum_{i=1}^n |Y_i - Y_i'| / (2n \log_2 n)^{\frac{1}{2}} \rightarrow 0 \quad \text{a.s.}$$

Noting that $E[Y_i' | \mathcal{F}_{i-1}] = -E[Y_i - Y_i' | \mathcal{F}_{i-1}]$, it suffices by the Kronecker Lemma to establish $\sum_{i=16}^{\infty} E|Y_i - Y_i'| / (2i \log_2 i)^{\frac{1}{2}} < \infty$ in order to prove (3) and (4).

To this end (following the approach of [2]) the sequence K_i is chosen to depend on the distribution of Y_0 in a rather involved manner. Let $c_i = i^2 P[i-1 < |Y_0| \leq i]$, noting that $\sum_{i=1}^{\infty} c_i < \infty$ follows from $EY_0^2 < \infty$. Let $n_k \geq 2$ be an increasing sequence of integers such that $n_{k+1} > 2^{n_k}$ and $\sum_{i=n_k}^{\infty} c_i < 2^{-k}$ for all $k \geq 1$. For each $i \geq n_1$ letting $K_i = (k)^{-\frac{1}{2}}$ when $n_{k-1} \leq i < n_k$ it follows that

$$(5) \quad \sum_{i=n_1}^{\infty} c_i / K_i = \sum_{k=2}^{\infty} k^{\frac{1}{2}} \sum_{i=n_{k-1}}^{n_k-1} c_i \leq \sum_{k=2}^{\infty} k^{\frac{1}{2}} 2^{-k+1} < \infty.$$

Note that the above choice of K_i is consistent with prior requirements that $K_i \rightarrow 0$ and $b_i \rightarrow \infty$. With $b_i = K_i(i/\log_2 i)^{\frac{1}{2}}$ let $N(m)$ be the largest integer n such that $[b_n] \leq m$ where $[\cdot]$ is the greatest integer function, noting that $b_i \rightarrow \infty$. Since $K_i \downarrow$ and $K_i/K_{i^3} \rightarrow 1$, it follows that there exists an integer m_0 such that $m \geq m_0$ implies

$$\begin{aligned} & \{(4m^2 \log_2 m / K_m^2) K_{[4m^2 \log_2 m / K_m^2]}^2 / \log_2 (4m^2 \log_2 m / K_m^2)\}^{\frac{1}{2}} \\ & \geq \{(4m^2 \log_2 m / K_m^2) K_{m^3}^2 / \log_2 m^3\}^{\frac{1}{2}} > m. \end{aligned}$$

Thus for $m \geq m_0$

$$(6) \quad N(m) \leq 4m^2 \log_2 m / K_m^2.$$

$$\begin{aligned} & \sum_{i=16}^{\infty} E|Y_i - Y_i'| / (2i \log_2 i)^{\frac{1}{2}} \\ & \leq \sum_{i=16}^{\infty} E|Y_i| I(|Y_i| > b_i) / (2i \log_2 i)^{\frac{1}{2}} \\ & \leq \sum_{i=16}^{\infty} \sum_{m=[b_i]}^{\infty} (m+1) P[m < |Y_0| \leq m+1] / (2i \log_2 i)^{\frac{1}{2}} \\ & = \sum_{m=[b_{16}]}^{\infty} \sum_{i=16}^{N(m)} (m+1) P[m < |Y_0| \leq m+1] / (2i \log_2 i)^{\frac{1}{2}}. \end{aligned}$$

By elementary integration and (6),

$$\begin{aligned} & \sum_{i=16}^{N(m)} (2i \log_2 i)^{-\frac{1}{2}} < c(N(m) / \log_2 N(m))^{\frac{1}{2}} \\ & \leq c((4m^2 \log_2 m / K_m^2) / \log_2 (4m^2 \log_2 m / K_m^2))^{\frac{1}{2}} \approx 2cm / K_m. \end{aligned}$$

Using (5) and combining, it follows that $\sum_{i=16}^{\infty} E|Y_i - Y_i'| / (2i \log_2 i)^{\frac{1}{2}} < \infty$, thus completing the proof.

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REFERENCES

[1] BILLINGSLEY, PATRICK (1961). The Lindeberg-Lévy theorem for martingales. *Proc. Amer. Math. Soc.* **12** 788-792.
 [2] CHOW Y. S. Unpublished lecture notes on the Hartman-Wintner law of the iterated logarithm.

- [3] CHOW, Y. S. and STUDDEN, W. J. (1969). Monotonicity of the variance under truncation and variations of Jensen's inequality. *Ann. Math. Statist.* **40** 1106–1108.
- [4] HARTMAN, P. and WINTNER, A. (1941). On the law of the iterated logarithm. *Amer. J. Math.* **63** 169–176.
- [5] STOUT, W. F. (1970). A martingale analogue of Kolmogorov's law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **15** 279–290.