THE HARTOGS EXTENSION THEOREM ON $(n-1)$-COMPLETE COMPLEX SPACES

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ABSTRACT. Employing Morse theory and the method of analytic discs but no $\overline{\partial}$ techniques, we establish a version of the Hartogs extension theorem in a singular setting, namely: for every domain Ω of an $(n-1)$-complete normal complex space of pure dimension $n \geqslant 2$, and for every compact set $K \subset \Omega$ such that $\Omega \backslash K$ is connected, holomorphic or meromorphic functions in $\Omega \backslash K$ extend holomorphically or meromorphically to Ω.

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[5 colored illustrations]

§1. INTRODUCTION

The goal of the present article is to perform a generalization of the classical Hartogs extension theorem in certain singular complex spaces which enjoy appropriate convexity conditions, using the method of analytic discs for local extensional steps and Morse-theoretical tools for the global topological control of monodromy.

In its original form, the theorem states that in an arbitrary bounded domain $\Omega \Subset \mathbb{C}^n$ ($n \geqslant 2$), every compact set $K \subset \Omega$ with $\Omega \backslash K$ connected is an illusory singularity for holomorphic functions, namely $\mathcal{O}(\Omega \backslash K) = \mathcal{O}(\Omega)\big|_{\Omega \backslash K}$ (for history, motivations and background, we refer *e.g.* to [12, 21, 22]). By now, the shortest proof, due to Ehrenpreis, follows easily from the simple proposition that $\overline{\partial}$-cohomology with compact support vanishes in bidegre $(0,1)$ (*see* [14]). Along these lines and after results due to Kohn-Rossi, the Hartogs theorem was generalized to $(n-1)$-complete complex manifolds by Andreotti-Hill [2], *i.e.* manifolds exhausted by a $\mathcal{C}^\infty$ function whose Levi-form has at least 2 positive eigenvalues at every point. We also refer to [17] for an approach via the holomorphic Plateau boundary problem.

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To endeavor the theory in general singular complex spaces $\big(X, \mathcal{O}_X\big)$, it is at present advisable to look for methods avoiding global $\overline{\partial}$ techniques, as well as global integral kernels, because such tools are not yet available. The geometric Hartogs theory was attacked long ago by Rothstein, who introduced the notion of q-convexity. On the other hand, within the modern sheaf-theoretic setting, the so-called *Andreotti-Grauert theory* allows to perform extension (of holomorphic functions, of differentials forms, of coherent sheaves, *etc.*) from shell-like regions of the form $\big\{z \in X : a < \rho(z) < b\big\}$ into their inside $\big\{z \in X : \rho(z) < b\big\}$, where ρ is a fixed $(n-1)$-convex exhaustion function for X. Geometrically speaking, one performs holomorphic extension by means of the *Grauert bump method* through the level sets of ρ in the direction of decreasing values, jumping finitely many times across the critical points of ρ.

However, a satisfying, complete generalization of the Hartogs theorem should apply to general excised bounded domains $\Omega \backslash K$ lying in an $(n-1)$-complete complex space $\big(X, \mathcal{O}_X\big)$, not only to shells $\{a < \rho < b\}$ relative to the $(n-1)$-convex exhaustion function. But then, after perturbing and smoothing out $\partial\Omega$, one must unavoidably take account of the critical points of $\rho\big|_{\partial\Omega}$ and also of the possible multi-sheetedness of the intermediate step-wise extensions. This causes considerably more delicate topological problems than in the well known Grauert bump method, in which monodromy of the holomorphic (or meromorphic, or sheaf-theoretic) extensions from $\{a < \rho < b\}$ to $\{a' < \rho < b\}$ with $a' < a$ is almost freely assured[1], even across critical points of ρ. Considering simply a domain $\Omega \Subset \mathbb{C}^n$ $(n \geqslant 2)$, with obvious exhaustion $\rho(z) := \|z\|$, the classical Hartogs theorem based on analytic discs and on Morse theory was worked out in [19], where emphasis was put on rigor in order to provide with firm grounds the subsequent works on the subject. The essence of the present article is to transfer such an approach to $(n-1)$-complete general complex spaces, where $\overline{\partial}$ techniques are still lacking, with some new difficulties due to the singularities.

§2. STATEMENT OF THE RESULTS

Thus, let $\big(X, \mathcal{O}_X\big)$ be a reduced complex analytic space of pure dimension $n \geqslant 2$, equipped with an open cover $X = \bigcup_{j \in J} U_j$ together with holomorphic isomorphisms $\varphi_j : U_j \to \mathsf{A}_j$ onto some closed complex analytic sets A_j contained in balls $\widetilde{\mathsf{B}}_j \subset \mathbb{C}^{N_j}$, some $N_j \geqslant 2$. By definition ([5, 10]), a $\mathcal{C}^\infty$ function $f : X \to \mathbb{C}$ is locally represented as $f|_{U_j} = \widetilde{\mathsf{f}}_j \circ \varphi_j$ for some collection of $\mathcal{C}^\infty$ "ambient" functions $\widetilde{\mathsf{f}}_j : \widetilde{\mathsf{B}}_j \to \mathbb{C}$, $j \in J$. A

[1] The reader in referred to point **2)** of the proof of Prosition 4.1 below and to Figure 3 in Section 4 for an illustration of the concerned univalent extension argument.

real-valued continuous function ρ on X is an *exhaustion function* if sub-level sets $\{z \in X : \rho(z) < c\}$ are relatively compact in X for every $c \in \mathbb{R}$. A $\mathcal{C}^\infty$ function $\rho : X \to \mathbb{R}$ is called *strongly q-convex* if the $\mathcal{C}^\infty$ ambient $\widetilde{\rho}_j : \widetilde{\mathsf{B}}_j \to \mathbb{R}$ can be chosen to be strongly q-convex, *i.e.* their Levi-forms $i\,\partial\overline{\partial}(\widetilde{\rho}_j)$ have at least $N_j - q + 1$ positive eigenvalues at every point, for all $j \in J$. Finally[2], X is called *q-complete* if it possesses a $\mathcal{C}^\infty$ strongly q-convex exhaustion function. Note that the 1-complete spaces are precisely the Stein spaces.

We will mainly work with a *normal* $(n-1)$-complete X, and we recall that a reduced complex space $(X, \mathcal{O}_X)$ is *normal* if the sheaf of *weakly holomorphic functions*, namely functions defined and holomorphic on the regular part $X_{\rm reg} = X \backslash X_{\rm sing}$ which are $L^\infty_{\rm loc}$ on X, coincides with the complete sheaf $\mathcal{O}_X$ of holomorphic functions on X. Then $X_{\rm sing}$ is of codimension $\geqslant 2$ at every point of X ([5, 10]) and for every open set $U \subset X$, both restriction maps

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X\big(U\backslash X_{\rm sing}\big) \quad \text{and} \quad \mathcal{M}_X(U) \longrightarrow \mathcal{M}_X\big(U\backslash X_{\rm sing}\big) \tag{2.1}$$

are bijective[3], where $\mathcal{M}_X$ denotes the meromorphic sheaf. To generalize Hartogs extension, normality of X is an unavoidable assumption, because there are examples of Stein surfaces S having a single singular point $\widehat{p}$ which are *not* normal ([10], vol. II, p. 196), whence $K := \{\widehat{p}\}$ fails to be removable for holomorphic functions defined in a neighborhood of K.

We can now state our main result.

Theorem 2.2. *Let X be a connected $(n-1)$-complete normal complex space of pure dimension $n \geqslant 2$. Then for every domain $\Omega \subset X$ and every compact set $K \subset \Omega$ with $\Omega\backslash K$ connected, holomorphic or meromorphic functions on $\Omega\backslash K$ extend holomorphically or meromorphically and uniquely to Ω:*

$$\mathcal{O}_X(\Omega\backslash K) = \mathcal{O}_X(\Omega)\big|_{\Omega\backslash K} \qquad \text{or} \qquad \mathcal{M}_X(\Omega\backslash K) = \mathcal{M}_X(\Omega)\big|_{\Omega\backslash K}.$$

Some comments on the hypotheses are in order. Firstly, connectedness of X is not a restriction, since otherwise, Ω would be contained in a single component of X. Secondly, as X is $(n-1)$-complete, $i\,\partial\overline{\partial}\,\big(\rho\big|_{X_{\rm reg}}\big)$ has at least 2 positive eigenvalues at every point $z \in X_{\rm reg}$, and consequently, each super-level set

$$\big\{z \in X : \rho(z) > c\big\},$$

has a pseudoconcave boundary at every smooth point $z \in X_{\rm reg}$ with $d\rho(z) \neq 0$ and in fact, the Levi-form of this boundary has at least one

[2] The previous definitions are known to be independent of the choices — covering, embeddings φ_j, dimensions N_j, extensions $\widetilde{(\bullet)}$, *see* [5, 7, 10].

[3] The first statement yields immediately that every point $z \in X$ has a neighborhood basis $\big(\mathcal{V}_k\big)_{k\in\mathbb{N}}$ such that $X_{\rm reg} \cap \mathcal{V}_k$ is connected; also, $X_{\rm reg}$ itself is connected. The second statement is known as Levi's extension theorem ([8], p. 185).

negative eigenvalue at z. Thirdly, by a theorem of Ohsawa ([20]), every (connected) n-dimensional *noncompact* complex manifold is n-complete, and in fact, easy examples show that Hartogs extension may fail: take the product $X := R \times S$ of two Riemann surfaces, with R compact and S *noncompact*, take a point $s \in S$ and set $K := R \times \{s\}$; by [6], there exists a meromorphic function function having a pole of order 1 at s, whence $\mathcal{O}(X)$ does not extend through K. Consequently, in the category of strong Levi-form assumptions, $(n-1)$-convexity is sharp.

For the theorem, the main strategy of proof consists of performing holomorphic or meromorphic extension entirely within the regular part of X.

Proposition 2.3. *With X, Ω and K as in Theorem 2.2, holomorphic or meromorphic functions on $\big[\Omega\backslash K\big]_{\rm reg}$ extend holomorphically or meromorphically to $\Omega_{\rm reg}$.*

Notice that both $\big[\Omega\backslash K\big]_{\rm reg}$ and $\Omega_{\rm reg}$ are connected (footnote 3). Then by (2.1), extension immediately holds to Ω. This yields Theorem 2.2 if one takes the proposition for granted; Sections 3 and 4 below are devoted to prove this proposition.

For meromorphic extension, one could in principle well avoid the assumption of normality. In the case of meromorphic extension, we get a general result valid for reduced spaces without further local assumptions.

Theorem 2.4. *Let X be a globally irreducible $(n-1)$-complete reduced complex space of pure dimension $n \geqslant 2$. Then for every domain $\Omega \subset X$ and every compact set $K \subset \Omega$ with $[\Omega\backslash K]_{\rm reg}$ connected, meromorphic functions on $\Omega\backslash K$ extend meromorphically and uniquely to Ω:*

$$\mathcal{M}_X(\Omega\backslash K) = \mathcal{M}_X(\Omega)\big|_{\Omega\backslash K}.$$

If moreover the data lie in $\mathcal{O}_X(\Omega\backslash K)$, the extension is weakly holomorphic.

The proof, also relying upon an application of Proposition 2.3, is postponed to Section 5; an example in §5.1 shows that requiring only that $\Omega\backslash K$ is connected does not suffices.

For the proposition, the main difficulty is that $X_{\rm sing}$ can in general cross $\Omega\backslash K$. We will approach $X_{\rm sing}$ from the regular part and fill in progressively $\Omega_{\rm reg}$ by means of the super-level sets of a suitable modification μ of the exhaustion ρ, such that μ is still strongly $(n-1)$-convex but exhausts only $X_{\rm reg}$ in a neighborhood of $\overline{\Omega}$. To verify that the extension procedure devised in [19] can be performed, preliminaries are required.

§3. GEOMETRICAL PREPARATIONS

3.1. Smoothing out the boundary. To launch the filling procedure, we want to view the connected open set $\Omega\backslash K$ as a neighborhood of some convenient connected hypersurface M contained in $\big(\Omega\backslash K\big) \cap X_{\rm reg}$.

Lemma 3.2. *Let X, Ω and K be as in Theorem 2.2. Then there is a domain $D \Subset \Omega$ containing K such that $M := \partial D \cap X_{\rm reg}$ is a $\mathcal{C}^\infty$* connected *hypersurface of $X_{\rm reg}$.*

Proof. Suppose first that $X = \mathbb{C}^n$. Let d be a regularized distance function ([23]) for K, *i.e.* a $\mathcal{C}^\infty$ real-valued function with $K = \{\mathsf{d} = 0\}$ and $\frac{1}{c}\,{\rm dist}\,(x, K) \leqslant \mathsf{d}(x) \leqslant c\,{\rm dist}\,(x, K)$ for some constant $c > 1$, where ${\rm dist}$ is the Euclidean distance in $\mathbb{R}^{2n}$. By Sard's theorem, there are arbitrarily small $\varepsilon > 0$ such that $\widehat{M} := \{\mathsf{d} = \varepsilon\}$ is a $\mathcal{C}^\infty$ hypersurface of $\mathbb{R}^{2n}$ bounding the open set $\widehat{\Omega} := \{\mathsf{d} < \varepsilon\}$ which satisfies $K \subset \widehat{\Omega} \Subset \Omega$. However, since $\widehat{M}$ need not be connected, we must modify it.

To this aim, we pick finitely many disjoint closed simple $\mathcal{C}^\infty$ arcs $\gamma_1, \dots, \gamma_r$ which meet $\widehat{M}$ transversally only at their endpoints such that $\widehat{M} \cup \gamma_1 \cup \cdots \cup \gamma_r$ is connected. Since $\Omega\backslash K$ is connected, we can insure that each γ_k is contained in $\Omega\backslash K$.

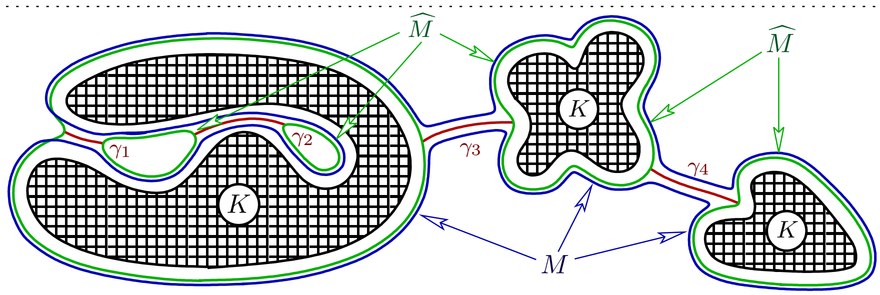

Fig. 1: Connectifying the smoothed out boundary

We can then modify $\widehat{M}$ in the following way: we cut out a very small ball in $\widehat{M}$ around each endpoint of every γ_k, and we link up the connected components of the excised hypersurface with r thin tubes $\simeq \mathbb{R} \times S^{2n-2}$ almost parallel to the γ_k, smoothing out the corners appearing near the endpoints. The resulting hypersurface M is $\mathcal{C}^\infty$ and connected. Since each γ_k is either contained in $\widehat{\Omega} \cup \widehat{M}$ or in $\mathbb{R}^{2n}\big\backslash\widehat{\Omega}$, a new open set D with $\partial D = M$ is obtained by either deleting from $\widehat{\Omega}$ or adding to $\widehat{\Omega}$ the thin tube around each γ_k. All the tubes around the γ_k which are contained in $\mathbb{R}^{2n}\big\backslash\widehat{\Omega}$ constitute thin open tunnels between the components of $\widehat{\Omega}$, whence D is connected.

On a general complex space X, the idea is to embed a neighborhood of $\overline{\Omega}$ smoothly into some Euclidean space $\mathbb{R}^N$ and then to proceed similarly.

We can assume that the holomorphic isomorphisms $\phi_j : U_j \to \mathsf{A}_j \subset \widetilde{\mathsf{B}}_j \subset \mathbb{C}^{N_j}$ are defined in slightly larger open sets $U_j' \Supset U_j$, for all $j \in J$. Pick $\mathcal{C}^\infty$ functions λ_j having compact support in U_j' and satisfying $\lambda_j = 1$ on $\overline{U}_j$; prolong them to be 0 on X outside U_j'. By compactness, there is a finite open cover:

$$\overline{\Omega} \subset U_{j_1} \cup \cdots \cup U_{j_m}.$$

Consider the $\mathcal{C}^\infty$ map, valued in $\mathbb{R}^N$ with $N := 2(N_{j_1} + \cdots + N_{j_m}) + m$, which is defined by:

$$\Psi := \big(\lambda_{j_1} \cdot \phi_{j_1}, \dots, \lambda_{j_m} \cdot \phi_{j_m},\ \ \lambda_{j_1}, \dots, \lambda_{j_m}\big).$$

It is an immersion at every point x of $U_{j_1} \cup \cdots \cup U_{j_m}$, because x belongs to some U_{j_k}, whence the j_k-th component $\lambda_{j_k} \cdot \phi_{j_k} \equiv \phi_{j_k}$ of Ψ is even an embedding of $U_k \ni x$. Furthermore, we claim that Ψ separates points. Indeed, if we set:

$$W_{j_k} := \big\{z \in X :\ \lambda_{j_k}(z) = 1\big\},$$

then clearly $U_{j_k} \subset W_{j_k} \subset U_{j_k}'$. Pick two distinct points $x, y \in U_{j_1} \cup \cdots \cup U_{j_m}$. Then x belongs to some U_{j_k}, so $\lambda_{j_k}(x) = 1$. If $\lambda_{j_k}(y) \neq 1$, then $\Psi(y) \neq \Psi(x)$ and we are done. If $\lambda_{j_k}(y) = 1$, *i.e.* if $y \in W_{j_k}$, then the j_k-th component of Ψ distinguishes x from y, since $\lambda_{j_k} \cdot \phi_{j_k}(y) = \phi_{j_k}(y)$ differs from $\phi_{j_k}(x)$ because ϕ_{j_k} embeds U_{j_k}' into $\mathbb{R}^{2N_{j_k}}$. So Ψ embeds into $\mathbb{R}^N$ the neighborhood $U_{j_1} \cup \cdots \cup U_{j_m}$ of $\overline{\Omega}$.

We choose a regularized distance function $\mathsf{d}_{\Psi(K)}$ for $\Psi(K)$ in $\mathbb{R}^N$. We stratify X so that $X_{\rm reg}$ is the single largest stratum (remind it is connected) and then stratify $X_{\rm sing}$ by listing all connected components of $\big[X_{\rm sing}\big]_{\rm reg}$, then continuing with $\big[X_{\rm sing}\big]_{\rm sing}$, and so on inductively. By Sard's theorem and the stratified transversality theorem ([13]), for almost every $\varepsilon > 0$, the intersection

$$\big\{x \in \mathbb{R}^N :\ \mathsf{d}_{\Psi(K)}(x) = \varepsilon\big\} \cap \Psi\big(\Omega_{\rm reg}\big)$$

is a $\mathcal{C}^\infty$ real hypersurface of $\Psi(\Omega_{\rm reg})$ having finitely many connected components which are contained in $\Psi\big([\Omega \backslash K]_{\rm reg}\big)$. Importantly, we can construct the thin connecting tubes so that they *lie all entirely inside* $\Psi\big(\big[\Omega \backslash K\big]_{\rm reg}\big)$, thanks to the fact that $\Psi\big(\Omega_{\rm reg}\big)$ is locally (arcwise) connected, also near points of $\Psi\big(\Omega_{\rm sing}\big)$. Then the remaining arguments are the same and we put everything back to X *via* Ψ^{-1}, getting a connected $\mathcal{C}^\infty$ hypersurface $M \subset [\Omega \backslash K]_{\rm reg}$ and a domain D with $K \subset D \Subset \Omega$. (We remark that normality of X was crucially used.) $\square$

As we said, we will perform the filling procedure entirely inside $X_{\rm reg}$. This is possible thanks to an idea of Demailly which consists of modifying the initial exhaustion ρ so that $X_{\rm sing}$ is put at $-\infty$. A recent application of this idea also appears in [4].

3.3. Putting $X_{\rm sing}$ into a well. By Lemma 5 in [3], there exists an *almost plurisubharmonic function*[4] v on X which is $\mathcal{C}^\infty$ on $X_{\rm reg}$ and has poles along $X_{\rm sing}$:

$$X_{\rm sing} = \big\{v = -\infty\big\}.$$

As in Section 2, if $\mathsf{A}_j = \varphi_j(U_j)$ is represented in a local ball $\widetilde{\mathsf{B}}_j \subset \mathbb{C}^{N_j}$ of radius $r_j > 0$ centered at $z_j \in \mathbb{C}^{N_j}$ as the zero-set $\{g_{j,\nu} = 0\}$ of finitely many functions $g_{j,\nu}$ holomorphic in a neighborhood of the closure of $\widetilde{\mathsf{B}}_j$, the local ambient $\widetilde{v}_j : \widetilde{\mathsf{B}}_j \to \{-\infty\} \cup \mathbb{R}$ is essentially of the form[5]:

$$\widetilde{v}_j = \log\Big(\sum_\nu |g_{j,\nu}|^2\Big) - \frac{1}{r_j^2 - |z-z_j|^2}.$$

Thus, locally on each $\widetilde{\mathsf{B}}_j$, the function v we pick from [3] is of the form:

$$\widetilde{v}_j = \widetilde{u}_j + \widetilde{\mathsf{r}}_j,$$

with $\widetilde{u}_j$ strictly psh, $\mathcal{C}^\infty$ on $\widetilde{\mathsf{B}}_j \backslash \big[\mathsf{A}_j\big]_{\rm sing}$, equal to $\{-\infty\}$ on $\big[\mathsf{A}_j\big]_{\rm sing}$ and with a remainder $\widetilde{\mathsf{r}}_j$ which is $\mathcal{C}^\infty$ on the whole of $\widetilde{\mathsf{B}}_j$. Notice that each $\widetilde{v}_j$ is $L^\infty_{\rm loc}$.

3.4. Modified strongly $(n-1)$-convex exhaustion function μ. Pick a constant $C > 0$ such that $\max_{\overline{D}}(\rho) < C$.

Lemma 3.5. *There exists $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leqslant \varepsilon_0$, the function*:

$$\mu := \rho + \varepsilon\, v$$

is $\mathcal{C}^\infty$ on $X_{\rm reg}$ and satisfies:

- **(a)** $\max_{\overline{D}}(\mu) < C$;
- **(b)** $X_{\rm sing} = \{\mu = -\infty\}$;
- **(c)** *μ is strongly $(n-1)$-convex in a neighborhood of $\{\rho \leqslant C\}$.*

Proof. Property **(b)** holds provided only that $\varepsilon < \frac{C - \max_{\overline{D}}(\rho)}{\max_{\overline{D}}(v)}$. Furthermore, **(a)** is clear since ρ is $\mathcal{C}^\infty$ and since $X_{\rm sing} = \{v = -\infty\}$. To check **(c)**, we compute Levi-forms as $(1,1)$-forms:

$$\begin{aligned} i\,\partial\overline{\partial}\,\widetilde{\mu}_j &= i\,\partial\overline{\partial}\,\widetilde{\rho}_j + \varepsilon\, i\,\partial\overline{\partial}\,\widetilde{v}_j \\ &= i\,\partial\overline{\partial}\,\widetilde{\rho}_j + \varepsilon\, i\,\partial\overline{\partial}\,\widetilde{u}_j + \varepsilon\, i\,\partial\overline{\partial}\,\widetilde{\mathsf{r}}_j. \end{aligned} \tag{3.6}$$

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[4] *i.e.* by definition, a function which is locally the sum of a psh function and of a $\mathcal{C}^\infty$ function, or equivalently, a function v whose complex Hessian $i\,\partial\overline{\partial}\,v$ has bounded negative part.

[5] In addition, a regularized maximum function ([3]) is used to smoothly glue these different definitions on all finite intersections $A_{j_1} \cap \cdots \cap A_{j_m}$ and the formula given here is exact on a sub-ball $\widetilde{\mathsf{C}}_j \subset \widetilde{\mathsf{B}}_j$.

Here, $\varepsilon \, i \, \partial\overline{\partial} \, \widetilde{u}_j$ adds positivity to $i \, \partial\overline{\partial} \, \widetilde{\rho}_j$ (since $\widetilde{u}_j$ is psh), whereas the negative contribution due to $i \, \partial\overline{\partial} \, \widetilde{\mathsf{r}}_j$ is bounded from below on $\{\rho \leqslant 2C\}$, and consequently, $\varepsilon > 0$ can be chosen small enough so that $i \, \partial\overline{\partial} \, \widetilde{\mu}_j$ still has 2 eigenvalues > 0 at every point. □

In the next section, while applying the holomorphic extension procedure of [19], we shall have to insure that the extensional domains attached to M from either the outside or the inside *cannot go beyond* $\{\rho \leqslant C\}$. So we have to prepare in advance the curvature of the limit hypersurface $\{\rho = C\} \cap X_{\text{reg}}$.

Enlarging C of an arbitrarily small increment if necessary, we can assume (thanks to Sard's theorem) that C is a regular value of $\rho\big|_{X_{\text{reg}}}$, so that

$$\Lambda := \{\rho = C\} \cap X_{\text{reg}}$$

is a $\mathcal{C}^\infty$ real hypersurface of X_{reg}.

Lemma 3.7. *Lowering again $\varepsilon > 0$ if necessary, the following holds*:

- **(d)** *At every point q of the $\mathcal{C}^\infty$ real hypersurface $\Lambda = \{\rho = C\} \cap X_{\text{reg}}$, one can find a complex line $E_q \subset T_q^c \Lambda$ on which the Levi-forms of both ρ and μ are positive.*

Here, $q \mapsto E_q$ might well be discontinuous, but this shall not cause any trouble in the sequel.

Proof. Each $p \in \{\rho = C\}$ is contained in some $U_{j(p)}$, whence ρ is represented by an ambient function $\widetilde{\rho}_{j(p)} : \widetilde{\mathsf{B}}_{j(p)} \to \mathbb{R}$ whose Levi-form has at least $N_{j(p)} - n + 2$ eigenvalues > 0. By diagonalizing the Levi matrix $i \, \partial\overline{\partial}\widetilde{\rho}_{j(p)}$ at the central point of $\widetilde{\mathsf{B}}_{j(p)}$, we may easily define, in some small open sub-ball $\widetilde{\mathsf{C}}_{j(p)} \subset \widetilde{\mathsf{B}}_{j(p)}$ having the same center, a $\mathcal{C}^\infty$ family $\widetilde{q} \mapsto \widetilde{F}_{\widetilde{q}}$ of complex $(N_{j(p)} - n + 2)$-dimensional affine subspaces such that the Levi-form of $\widetilde{\rho}_{j(p)}$ is positive definite on every $\widetilde{F}_{\widetilde{q}}$, for every $\widetilde{q} \in \widetilde{\mathsf{C}}_{j(p)}$.

Next, if we set $V_{j(p)} := \varphi_{j(p)}^{-1}\big(\widetilde{\mathsf{C}}_{j(p)}\big)$, which is an open subset of $U_{j(p)}$, we can cover the compact set $\{\rho = C\}$ by finitely many $V_{j(p)}$, hence there is a finite number of points p_a, $a = 1, \dots, A$, such that

$$\{\rho = C\} \subset V_{j(p_1)} \cup \cdots \cup V_{j(p_A)}.$$

According to (3.6), on each $\widetilde{\mathsf{C}}_{j(p_a)}$, $a = 1, \dots A$, we have:

$$i \, \partial\overline{\partial} \, \widetilde{\mu}_{j(p_a)} = i \, \partial\overline{\partial} \, \widetilde{\rho}_{j(p_a)} + \varepsilon \, i \, \partial\overline{\partial} \, \widetilde{u}_{j(p_a)} + \varepsilon \, i \, \partial\overline{\partial} \widetilde{\mathsf{r}}_{j(p_a)}.$$

We choose $\varepsilon > 0$ so small that the remainder $\varepsilon \, i \, \partial\overline{\partial} \widetilde{\mathsf{r}}_{j(p_a)}$ does not perturb positivity on $\widetilde{\mathsf{C}}_{j(p_a)}$ for every $a = 1, \dots A$, and we get that $i \, \partial\overline{\partial} \, \widetilde{\mu}_{j(p_a)}$ is still positive on $\widetilde{F}_{\widetilde{q}}$ for every $\widetilde{q} \in \widetilde{\mathsf{C}}_{j(p_a)}$, and every $a = 1, \dots A$.

Let $q \in \{\rho = C\} \cap X_{\rm reg}$. Then $q \in V_{j(p_a)}$ for some a. We set $\widetilde{q} := \varphi_{j(p_a)}(q) \in \widetilde{\mathsf{C}}_{j(p_a)}$ and we define:

$$F_q := \big(d\,\varphi_{j(p_a)}\big)^{-1}\Big(\widetilde{F}_{\widetilde{q}} \cap T_{\widetilde{q}}\,\mathsf{A}_{j(p_a)}\Big).$$

Then the complex linear spaces $\widetilde{F}_{\widetilde{q}}$ and F_q are at least of dimension 2 and the Levi-form of μ is positive on any 1-dimensional subspace $E_q \subset F_q \cap T_q^c\,\Lambda$.
□

Next, applying Morse transversality theory, we may perturb μ in $X_{\rm reg} \cap \{\rho < 2\,C\}$ in an arbitrarily small way, so that[6]:

- **(e)** *μ is a Morse function on $X_{\rm reg} \cap \{\rho < 2\,C\}$ having finitely many or at most countably many critical points; moreover, different critical points of μ are located in different level sets $\{\mu = c\}$.*

Of course, if they are infinite in number, critical values can only accumulate at $-\infty$. Similarly, we may perturb ρ very slightly near $\{\rho = C\}$ so that:

- **(f)** the $\mathcal{C}^\infty$ hypersurface $\{\rho = C\} \cap X_{\rm reg}$ does not contain any critical point of μ.

Finally, again thanks to Morse transversality theory, we may perturb the connected $\mathcal{C}^\infty$ hypersurface $M \subset \partial D$ of Lemma 3.2 in an arbitrarily small way so that[7]:

- **(g)** *M does not contain critical points of μ, and $\mu\big|_M$ is a Morse function on M having finitely many or at most countably many critical points; moreover, any two different critical points of μ or of $\mu\big|_M$ have different critical values.*

We draw a diagram, where $X_{\rm sing}$ is symbolically represented as a continuous broken line having spikes, with a level-set $\{\mu = \widehat{c}\}$ which is critical for $\mu\big|_M$ and a single critical point $\widehat{p} \in M \cap \{\mu = \widehat{c}\}$.

[6] The previous four properties being preserved, especially **(d)** on $\{\rho = C\}$.

[7] The perturbed M being still contained in $\{\rho < C\}$ and in the original connected corona $\Omega \backslash K$.

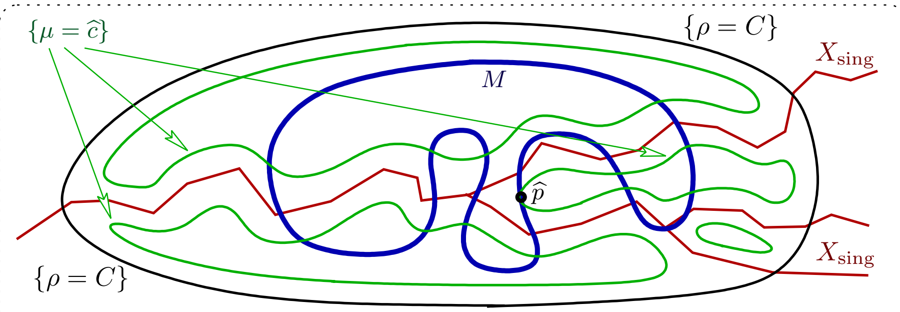

Fig. 2: The smooth boundary M, a level-set of μ and $X_{\rm sing}$

§4. Holomorphic extension to $D_{\rm reg}$

For $c \in \mathbb{R}$, we introduce

$$X_{\mu>c} := \{z \in X : \mu(z) > c\}.$$

This open set is contained in $X_{\rm reg}$, since $X_{\rm sing} = \{\mu = -\infty\}$. For every connected component $M'_{\mu>c}$ of

$$M_{\mu>c} := M \cap X_{\mu>c} = M \cap \{\mu > c\},$$

we want to fill in (by means of a finite number of families of analytic discs) a certain domain $Q'_{\mu>c}$ which is enclosed by $M'_{\mu>c}$ inside $\{\mu > c\}$. Similarly as in Proposition 5.3 of [19], we must consider *all* the connected components $M'_{\mu>c}$ and analyze the combinatorics of how they merge or disappear.

Let $\mathcal{V}(M)$ be a thin tubular neighborhood of M, whose thinness shrinks to zero while approaching $X_{\rm sing}$. For every connected component $M'_{\mu>c}$ of $M_{\mu>c}$, we denote by $\mathcal{V}\big(M'_{\mu>c}\big)_{\mu>c}$ the part of $\mathcal{V}(M)$ around $M'_{\mu>c}$ again intersected with $\{\mu > c\}$. It is a connected tubular neighborhood of $M'_{\mu>c}$ inside $\{\mu > c\}$.

Proposition 4.1. *Let $c \in \mathbb{R}$ with $c < \max_M(\mu) < C$ be any regular value of μ and of $\mu\big|_M$. Let $M'_{\mu>c}$ be any nonempty connected component of $M \cap X_{\mu>c}$. Then there is a unique connected component $Q'_{\mu>c}$ of $X_{\mu>c}\big\backslash M'_{\mu>c}$ which is relatively compact in $X_{\rm reg}$ and contained in $\{\rho < C\}$ with the property that two different domains $Q'_{\mu>c}$ and $Q''_{\mu>c}$ are either disjoint or one is contained in the other. Furthermore, for every holomorphic or meromorphic function f defined in the thin tubular neighborhood $\mathcal{V}(M)$ of M, there exists a unique holomorphic or meromorphic extension F, constructed by means of a finite number of $(n-1)$-concave Levi-Hartogs figures and defined in*

$$Q'_{\mu>c} \bigcup \mathcal{V}\big(M'_{\mu>c}\big)_{\mu>c},$$

such that $F = f$ when both functions are restricted to $\mathcal{V}\big(M'_{\mu>c}\big)_{\mu>c}$.

Proof. We only describe the modifications one must bring to the arguments of [19].

1) The Levi-form of the compact $\mathcal{C}^\infty$ boundary $\{\mu = c\}$ of the super-level set $\{\mu > c\}$ (contained in $X_{\rm reg}$) has 1 negative eigenvalue, so that the Levi extension theorem with analytic discs (*cf.* the survey [18]) applies at each point of $\{\mu = c\}$. In Section 3 of [19], we defined $(n-a)$-concave Hartogs figures for $1 \leqslant a \leqslant n-1$, but we used only 1-concave ones, because the Levi-form of exterior of spheres $\{\|z\| < r\}$ in $\mathbb{C}^n$ had $(n-1)$ negative eigenvalues. Here, we start from $(n-1)$-concave Hartogs figures, we modify them similarly as in Section 3 of [19] (details are skipped) and we call them $(n-1)$*-concave Levi-Hartogs figures*.

Next, we use a finite number of these figures, *via* some local charts of $X_{\rm reg}$, to cover $\{\mu = c\}$ and to show that holomorphic[8] (or meromorphic) functions in $\{\mu > c\}$ extend to a slightly deeper super-level set $\{\mu > c - \eta\}$ (provided no critical point of μ or of $\mu\big|_M$ is encountered in the shell $\{c \geqslant \mu > c - \eta\}$), for some $\eta > 0$ which depends on X, on n, on μ, but not on c.

2) Contrary to the $\mathbb{C}^n$ case treated in [19], μ may have critical points on $X_{\rm reg}$. Grauert's theory shows how to jump across them with $\overline{\partial}$ techniques, and we summarize how we can proceed here[9], using only analytic discs in Levi-Hartogs figures.

Consider a point $\widehat{p} \in X_{\rm reg}$ which is critical: $d\mu(\widehat{p}) = 0$, and set $\widehat{c} := \mu(\widehat{p})$. The Morse lemma provides local real coordinates centered at $\widehat{p}$ in which $\mu = x_1^2 + \cdots + x_k^2 - y_1^2 - \cdots - y_{2n-k}^2$, for some k. Since $i\,\partial\overline{\partial}\,\mu$ has at least 2 positive eigenvalues everywhere, k is $\geqslant 2$. This is a crucial fact, because this implies that super-level sets $\{\mu > \widehat{c} + \delta\}$ are all connected[10] in a neighborhood of $\widehat{p}$, for every $\delta \in \mathbb{R}$ close to 0, and moreover, that these domains grow regularly and continuously as δ decreases from positive values to negative values.

[8] Since the configuration is always local and biholomorphic to $\mathbb{C}^n$ ($n = \dim X_{\rm reg}$) and since holomorphic envelopes coincide with meromorphic envelopes in $\mathbb{C}^n$, meromorphic functions enjoy exactly the same extension properties. Thus, in [19], results stated for holomorphic functions are immediately true for meromorphic functions too.

[9] We emphasize that, from the point of view of holomorphic extension, jumping across critical points of μ on $X_{\rm reg}$ is much simpler than jumping across critical points of $\mu\big|_M$, *cf.* the $\mathbb{C}^n$ case [19].

[10] In $\mathbb{R}^3$ already, this is true for the "exterior" $x^2 + y^2 - z^2 > \delta$ of the standard cone.

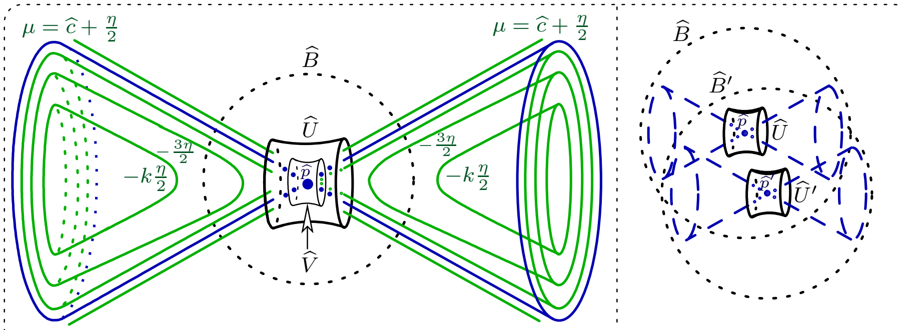

Fig. 3: Filling outside a neighborhood of $\widehat{p}$ and shifting $\widehat{p}$

Next, we fix a ball $\widehat{B}$ centered at $\widehat{p}$ and we cut out a small neighborhood $\widehat{U} \subset \widehat{B}$ of $\widehat{p}$. If $\widehat{V} \subset \widehat{U}$ is a small neighborhood, we consider the $\mathcal{C}^\infty$ hypersurface:

$$\big\{\mu > \widehat{c} + \tfrac{\eta}{2}\big\} \setminus \widehat{V}.$$

Placing finitely many $(n-1)$-concave Levi-Hartogs figures at points of this hypersurface, we get holomorphic or meromorphic extension to $\big\{\mu > \widehat{c} - \frac{\eta}{2}\big\} \setminus \widehat{V}_1$, where $\widehat{V}_1 \subset \widehat{V}$ is slightly bigger than $\widehat{V}$. Repeating the filling process finitely many times until $\big\{\mu = \widehat{c} - \frac{k\eta}{2}\big\}$ does not intersect $\widehat{B}$, where k is an odd integer, we fill in $\widehat{B} \backslash \widehat{U}$. At each step, monodromy of the extension is assured thanks to connectedness of $\big\{\mu > \widehat{c} + \delta\big\} \setminus \widehat{U}$, for every small $\delta \in \mathbb{R}$. However, we cannot fill in $\widehat{U}$ directly this way.

The trick is to shift $\widehat{p}$. One introduces a $\mathcal{C}^\infty$ perturbation μ' of μ localized near $\widehat{p}$ (namely $\mu' = \mu$ elsewhere) such that μ' has another critical point $\widehat{p}'$ (having the same Morse index of course), with corresponding neighborhoods disjoint: $\widehat{U} \cap \widehat{U}' = \emptyset$ and both contained in $\widehat{B} \cap \widehat{B}'$. We repeat the Levi-Hartogs filling with μ', getting holomorphic or meromorphic extension $\big\{\mu' > \widehat{c} - k'\frac{\eta}{2}\big\} \setminus \widehat{U}'$, a domain which contains $\widehat{B}' \backslash \widehat{U}'$, hence contains $\widehat{U}$. Monodromy is again well controlled, just because topologically, $\widehat{B} \backslash \widehat{U}$ and $\widehat{B}' \backslash \widehat{U}'$ are complete shells.

3) We prove the proposition by decreasing c. Provided c does not cross critical values of $\mu\big|_M$, the domains $Q'_{\mu > c}$ do grow regularly and continuously, even when c crosses critical values of μ, according to what has been said just above. At a critical value $\widehat{c}$ of $\mu\big|_M$, for a domain $Q_{\mu > \widehat{c}}$ whose closure contains the corresponding unique critical point $\widehat{p} \in M$, similarly as in [19], three cases may occur:

- **(i)** the domain $Q'_{\mu > \widehat{c} + \delta}$ grows regularly and continuously as δ decreases in a neighborhood of 0;

(ii) precisely when δ becomes negative, the domain $Q'_{\mu>\widehat{c}+\delta}$ is merged with a second domain $Q''_{\mu>\widehat{c}+\delta}$ whose closure also contains $\widehat{p}$ for $\delta = 0$ (the case of three domains or more never occurs);

(iii) the domain $Q'_{\mu>\widehat{c}+\delta}$ is contained in a bigger domain $Q''_{\mu>\widehat{c}+\delta}$ for all small $\delta > 0$, and exactly at $\delta = 0$, the closure of the domain $Q'_{\mu>\widehat{c}}$ is subtracted from $Q''_{\mu>\widehat{c}}$, yielding a new domain $Q'''_{\mu>\widehat{c}}$ which starts to grow regularly and continuously as $Q'''_{\mu>\widehat{c}+\delta}$ for small $\delta < 0$.

We then check by decreasing induction on c that such domains are relatively compact and are either disjoint or one is contained in the other, and we achieve extension by means of $(n-1)$-concave Levi-Hartogs figures similarly as in [19]. But here, a single fact remains to be established, namely that the domains $Q'_{\mu>c}$ remain all contained inside the relatively compact region $\{\rho < C\}$.

This is true at the beginning of the filling process, namely for c slightly smaller than $\max_M(\mu)$, because $M_{\mu>c}$ is then diffeomorphic to a small spherical cap (hence connected) and the relatively compact domain enclosed by $M_{\mu>c}$ in $X_{\mu>c}\backslash M_{\mu>c}$ is just the piece $D_{\mu>c}$ of D, which is diffeomorphic to a thin cut out piece of ball close to M and clearly contained in $\{\rho < C\}$, since $D \cup M \subset \{\rho < C\}$ by **(a)**.

To prove that all $Q'_{\mu>c}$ are contained in $\{\rho < C\}$, we proceed by contradiction. Let c^* be first c (as c decreases) for which some $Q'_{\mu>c}$ is not contained in $\{\rho < C\}$. In the process described above of constructing the domains $Q'_{\mu>c}$, the only discontinuity occurs in **(iii)** and it consists of a suppression. Consequently, the domains $Q'_{\mu>c}$ cannot jump discontinuously across $\{\rho = C\}$, hence at $c = c^*$ (which might be either critical or noncritical), all $Q'_{\mu>c^*}$ are still contained in $\{\rho \leqslant C\}$ and the boundary of at least one domain, say $Q^*_{\mu>c^*}$, touches the $\mathcal{C}^\infty$ border hypersurface $\{\rho = C\} \cap X_{\mathrm{reg}}$.

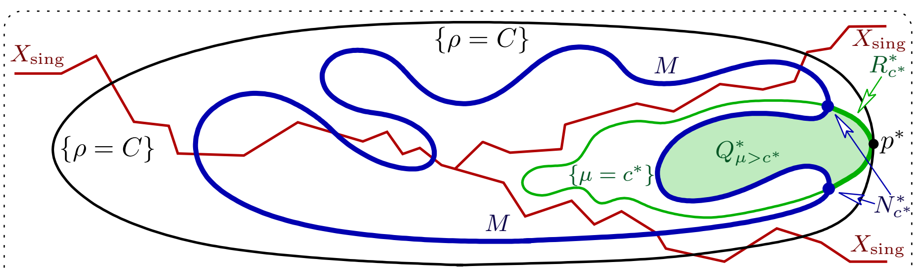

Fig. 4: Tangent contact of the boundary of $Q^*_{\mu>c^*}$ with $\{\rho = C\}$

On the other hand, by definition and by construction, for each c, the boundary of each $Q'_{\mu>c}$ consists of two parts: $M'_{\mu>c}$, which is contained in M, hence remains always in $\{\rho < C\}$, together with a certain closed region $R'_{\mu=c} \cup N'_{\mu=c}$ contained in $\{\mu = c\}$, with $R'_{\mu=c}$ open and $N'_{\mu=c}$ being the boundary in $\{\mu = c\}$ of $R'_{\mu=c}$. In fact, similarly as in Section 5 of [19],

$R'_{\mu=c}$ is always contained in $\{\mu = c\}\backslash M$ and $N'_{\mu=c}$, always contained in $M \cap \{\mu = c\}$ is a $\mathcal{C}^\infty$ real submanifold of X_{reg} of codimension 2 provided c is noncritical for $\mu\big|_M$, while $N'_{\mu=c}$ may have as a single singular (corner) point $\widehat{p}$ for $c = \widehat{p}$ critical. But since $N'_{\mu=c}$ is a subset of $M \cap \{\mu = c\}$, it is always contained in $\{\rho < C\}$.

Consequently, the boundary of $Q^*_{\mu>c^*}$ can touch $\{\rho = C\}$ only at some point $p^* \in R^*_{\mu=c^*}$. So we have $\mu(p^*) = c^*$ and $\rho(p^*) = C$, namely p^* lies in $\{\mu = c^*\}$ and on the $\mathcal{C}^\infty$ hypersurface $\{\rho = C\}$.

By **(f)** above, $p^* \in \{\rho = C\}$ cannot be a critical point of μ, whence $\{\mu = c^*\}$ and $\{\rho = C\}$ are both $\mathcal{C}^\infty$ real hypersurfaces passing through p^*. Furthermore, $\{\mu \geqslant c^*\}$ is still contained in $\{\rho \leqslant C\}$, by definition of c^*, whence $T_{p^*}\{\rho = C\} = T_{p^*}\{\mu = c^*\}$.

Thanks to **(d)**, there is a complex line

$$E_{p^*} \subset T^c_{p^*}\{\rho = C\} = T^c_{p^*}\{\mu = c^*\}$$

on which the Levi-forms of both ρ and μ are positive definite. On the other hand, since $\{-\mu < -c^*\}$ is contained in $\{\rho < C\}$, the Levi-form of $-\mu$ in the direction of E_{p^*} should then be $\geqslant$ the Levi-form of ρ in the same direction. This is a contradiction, and the proof that all $Q'_{\mu>c}$ remain in $\{\rho < C\}$ is completed. This finishes our proof of Proposition 4.1. □

4.2. End of proof of Proposition 2.3. As in Section 2 of [19], one checks that extension holds from $\big[\Omega\backslash K\big]_{\text{reg}}$ to Ω_{reg} provided holomorphic or meromorphic functions defined in the thin tubular neighborhood $\mathcal{V}(M)$ of $M \subset X_{\text{reg}}$ do extend uniquely to $D_{\text{reg}} \bigcup \mathcal{V}(M)$. So we work with M, $\mathcal{V}(M)$ and D_{reg}, and since everything is exhausted as $c \to -\infty$, the conclusion of the proof of Proposition 2.3 is an immediate consequence of the following.

Proposition 4.3. *For every regular value $c > -\infty$ of $\mu\big|_M$, holomorphic or meromorphic functions defined in $\mathcal{V}(M)$ do extend holomorphically or meromorphically and uniquely to*

$$D_{\mu>c} \bigcup \mathcal{V}\big(M_{\mu>c}\big)_{\mu>c}.$$

Proof. We set $c_1 := \max_M(\mu) = \max_{\overline{D}}(\mu) < C$. There is a unique "$\mu$-farthest point" $p_1 \in M$ with $\mu(p_1) = c_1$ and this point is obviously a critical point of Morse index equal to $-(2n-1)$ for $\mu\big|_M$, by virtue of **(g)**. Consequently, for all $c < c_1$ close to c_1, there is a single connected component in $M_{\mu>c}$, namely $M_{\mu>c}$ itself, which is diffeomorphic to a small spherical cap and encloses the domain $D_{\mu>c}$, diffeomorphic to a thin cut out piece of ball. For such $c < c_1$ close to c_1, the proposition is thus a direct consequence of the previous Proposition 4.1.

For arbitrary noncritical c, there is a well defined connected component $M^1_{\mu>c}$ of $M_{\mu>c}$ with $p_1 \in M^1_{\mu>c}$, and we denote by $M^2_{\mu>c}, \dots, M^k_{\mu>c}$ the other connected components of $M_{\mu>c}$. Also, each connected component $D^{\sim}_{\mu>c}$ of $D_{\mu>c}$ is bounded by some of the $M^j_{\mu>c}$, inside $\{\mu > c\}$. The problem is that the various extensions provided by Proposition 4.1 need not stick together, but fortunately, we can go to deeper super-level sets $\{\mu > c'\}$.

Lemma 4.4. *For every c' with $-\infty < c' \leqslant c$ which is noncritical for $\mu\big|_M$, the μ-farthest point p_1 belongs to a unique connected component $M'_{\mu>c'}$ of $M \cap \{\mu > c'\}$ and the enclosed domain $Q'_{\mu>c'}$ constructed by Proposition 4.1 contains D in a neighborhood of p_1.*

Proof. Indeed, if this were not true, there would exist the first $c' = c^*$ (as $c' \leqslant c$ decreases) for which $Q'_{\mu>c'}$ switches to the other side of M near p_1. According to the topological combinatorial processus **(i)**, **(ii)**, **(iii)** above, this could only occur in case **(iii)** with c^* critical, where a component is suppressed from a bigger one $Q''_{\mu>c^*}$ bounded by some $M''_{\mu>c^*}$, the suppressed component necessarily being $Q'_{\mu>c^*}$ itself. Then the bigger component $Q''_{\mu>c^*}$ would contain the side of M which is exterior to D near $\widehat{p}_1$, whence

$$c''_1 := \max\left\{\mu(q) :\ q \in M''_{\mu>c^*}\right\}$$

would necessarily be $> c_1$, which contradicts $c_1 = \max_M(\mu)$. □

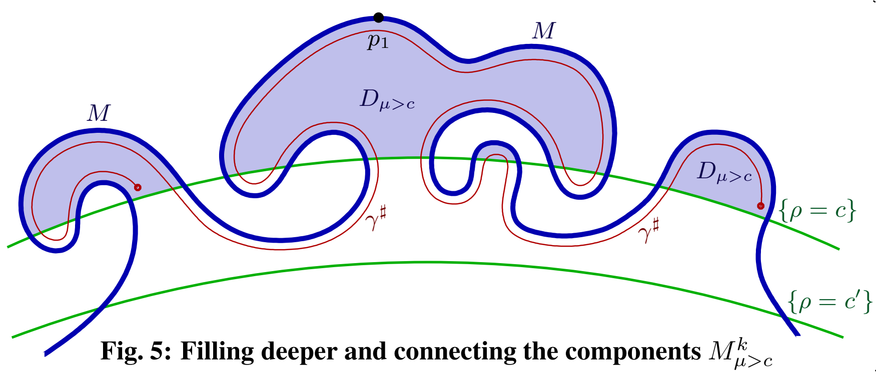

Fig. 5: Filling deeper and connecting the components $M^k_{\mu>c}$

Next, since M is connected (according to Lemma 3.2), we can pick a $\mathcal{C}^\infty$ Jordan arc γ running in M which starts at p_1 and visits every other connected component $M^2_{\mu>c}, \dots, M^k_{\mu>c}$ of $M_{\mu>c}$. Since γ is compact, there is a noncritical $c' > -\infty$ such that $\gamma \subset \{\mu > c'\}$. Fix such a c' and denote by $M'_{\mu>c'}$ the connected component of $M \cap \{\mu > c'\}$ to which p_1 belongs. Then let $Q'_{\mu>c'}$ be as in Lemma 4.4.

Lemma 4.5. *The domain $Q'_{\mu>c'}$ contains $D_{\mu>c}$.*

Proof. Near p_1, this domain already contains a piece of D thanks to Lemma 4.4. From the beginning, M is oriented, since it bounds the domain D. Thus, we can push γ slightly inside D, getting a curve $\gamma^\sharp$ almost parallel to γ which is entirely contained in D, and also contained in $\{\mu > c'\}$

if the push is sufficiently small. Furthermore, $\gamma^\sharp$ is also entirely contained in $Q'_{\mu>c'}$, because the extensional domain $Q'_{\mu>c'}$ is, at least near p_1, located on the same side (with respect to M) as D.

Let $D^\sim_{\mu>c}$ be any connected component of $D_{\mu>c}$. By construction, $\gamma^\sharp$ visits $D^\sim_{\mu>c}$. Thus, every point of $D^\sim_{\mu>c}$ may be joined to some point of $\gamma^\sharp$ by means of some auxiliary $\mathcal{C}^\infty$ curve running in $D^\sim_{\mu>c}$. All such auxiliary curves do not meet M, hence they do not meet $M'_{\mu>c'}$, whence they all run in $Q'_{\mu>c'}$. Consequently, by means of $\gamma^\sharp$ and of the auxiliary curves in each $D^\sim_{\mu>c}$, we may connect, without crossing M even once, every point of $D_{\mu>c}$ with the starting point of $\gamma^\sharp$, contained in $Q'_{\mu>c'}$ near p_1. Thus $D_{\mu>c}$ is effectively contained in $Q'_{\mu>c'}$. □

To conclude, an application of Proposition 4.1 yields unique extension to $Q'_{\mu>c'} \bigcup \mathcal{V}\big(M'_{\mu>c'}\big)_{\mu>c'}$, and by plain restriction, we get unique extension to $D_{\mu>c} \bigcup \mathcal{V}\big(M_{\mu>c}\big)_{\mu>c}$.

This completes the proofs of Propositions 4.3 and 2.3. □

§5. Meromorphic extension on nonnormal complex spaces

5.1. An example. To see that the weaker assumption that $\Omega \backslash K$ is connected does not suffice, we consider $X = \mathbb{C}^2 / \big((-1,0) \sim (+1,0)\big)$, the euclidean $\mathbb{C}^2$ with two points identified. If we define the structure sheaf by $\mathcal{O}_{\mathbb{C}^2,z}$ at all single points and by $\mathcal{O}_{\mathbb{C}^2,\pm} = \big\{(f,g) \in \mathcal{O}_{\mathbb{C}^2,-1} \times \mathcal{O}_{\mathbb{C}^2,1} : f(-1,0) = g(+1,0)\big\}$ at the double point $(\pm 1, 0)$, the space $\big(X, \mathcal{O}_X\big)$ is reduced and modelled near $(\pm 1,0)$ on $\{(z,w) \in \mathbb{C}^2 \times \mathbb{C}^2 : z = w\}$. This makes it easy to check that the function $|z_1+1|^2 + |z_1-1|^2 + |z_2|^2$ descends to a 1-convex exhaustion of X via the quotient projection $\pi : \mathbb{C}^2 \to X$. Letting $\Omega := X$ and $K := \pi\big(\{|z_1+1|^2+|z_2|^2 = 1\}\big)$, we see that $\Omega \backslash K$ is connected. Furthermore, $\mathcal{O}(\Omega \backslash K)$ consists of all functions holomorphic in $\mathbb{C}^2 \backslash \big\{|z_1 + 1|^2 + |z_2|^2 = 1\big\}$ which satisfy $f(-1,0) = f(+1,0)$. Then obviously, the conclusion of Theorem 2.4 does not hold.

5.2. Proof of Theorem 2.4. To begin with, we observe that Proposition 2.3 carries over without change to the more general setting of Theorem 2.4: indeed, thanks to the connectedness of $[\Omega \backslash K]_{\rm reg}$, we may construct M and D as in Lemma 3.2; the construction of an almost psh function v with $X_{\rm sing} = \{v = -\infty\}$ holds without assumption of normality ([3]), and then Propositions 4.1 and 4.3 do go through (notice that both $\Omega \backslash K$ and $\Omega_{\rm reg}$ are connected). Thus $\mathcal{M}_X(\Omega \backslash K)$ extends uniquely as $\mathcal{M}_X\big(\Omega_{\rm reg} \cup [\Omega \backslash K]\,\big)$, holomorphicity being preserved.

Extension across $\Omega_{\rm sing} \cap K$ is slightly more complicated than in the normal case due to the fact that $\Omega_{\rm sing}$ may have components of codimension one.

Let $\pi : \widehat{X} \to X$ be the normalization of X. Let X_{norm} be the set of the normal points of X. Recall that π restricts to a biholomorphism on $\pi^{-1}(X_{\text{norm}})$. Topologically, π is a local homeomorphism over irreducible points of X and separates the irreducible local components at reducible points. For every open $U \subset X$, setting $\widehat{U} = \pi^{-1}(U)$, we have a canonical isomorphism $\pi^* : \mathcal{M}_X(U) \to \mathcal{M}_{\widehat{X}}(\widehat{U})$ ([8], p. 155). Hence it is enough to extend from $\mathcal{M}_{\widehat{X}}\big(\widehat{\Omega}\backslash L\big)$ to $\mathcal{M}_{\widehat{X}}(\widehat{\Omega})$, where $\widehat{\Omega} := \pi^{-1}(\Omega)$ and $L := \pi^{-1}\big(\Omega_{\text{sing}} \cap K\big)$.

By the Levi extension theorem, we can extend through all points of $z \in L$ with $\dim_z \pi^{-1}(\Omega_{\text{sing}}) \leqslant n-2$. Let H be an irreducible component of Ω_{sing} of codimension one. Since $\dim \widehat{\Omega}_{\text{sing}} \leqslant n-2$, it follows that $\widehat{H}' := \pi^{-1}(H) \cap \widehat{\Omega}_{\text{reg}}$ is dense, open and connected in $\widehat{H} = \pi^{-1}(H)$. Because X is $(n-1)$-convex, it cannot contain any compact analytic hypersurface according to Lemma 5.3 just below, and H has to intersect $\Omega\backslash K$. For dimensional reasons, $\widehat{H}'$ intersects $\big[\pi^{-1}(\Omega\backslash K)\big]_{\text{reg}}$, and we can apply the following version of the Levi extension theorem for complex manifolds ([9]): *Let Y be an analytic subset of a complex manifold of M of codimension at least one. If $U \subset M$ is a domain containing $M\backslash Y$ and intersecting each irreducible one-codimensional component of Y, then holo-/meromorphic functions on U extend holo-/meromorphically to M.*

The remaining part of the singularity lies in $\widehat{\Omega}_{\text{sing}}$ and can be removed by the Levi extension theorem. If the original function on $\Omega\backslash K$ is holomorphic, the extension on $\widehat{\Omega}$ is so too, and its push-forward to Ω is weakly holomorphic. The proof of Theorem 2.4 is complete. □

Lemma 5.3. *An $(n-1)$-convex complex space X of pure dimension n cannot contain any analytic hypersurface Y which is compact.*

Proof. Let ρ be an $(n-1)$-convex exhaustion function. Let $\big(U_j\big)_{j\in J}$ be a locally finite covering of X by open subsets which can be embedded onto analytic subsets A_j of euclidean domains $\widetilde{\mathsf{B}}_j \subset \mathbb{C}^{N_j}$ such that the push-forward of ρ extends as an $(n-1)$-convex function $\widetilde{\rho}_j \in \mathcal{C}^\infty\big(\widetilde{\mathsf{B}}_j\big)$. By an inductive deformation of ρ, we may arrange that all $\widetilde{\rho}_j$ can be chosen to be Morse functions without critical points on A_j.

If there is a compact analytic hypersurface $Y \subset X$, then $\rho|_Y$ attains a global maximum at some point $z_0 \in Y$. We can assume that z_0 lies in some ball $\widetilde{\mathsf{B}}_j$, we denote by $\mathsf{E}_j \subset \mathsf{A}_j \subset \widetilde{\mathsf{B}}_j \subset \mathbb{C}^{N_j}$ the local representative of Y and we drop the index j, because the rest of the argument is local. By construction $\big\{z : \widetilde{\rho}(z) = \widetilde{\rho}(z_0)\big\}$ is a smooth $(n-1)$-convex real hypersurface such that $\mathsf{E} \subset \{\widetilde{\rho} \leqslant \widetilde{\rho}(z_0)\}$. Bending this hypersurface a little, we can arrange that E is in fact contained in $\{\widetilde{\rho} < \widetilde{\rho}(z_0)\} \cup \{z_0\}$ near z_0. By $(n-1)$-convexity of $\widetilde{\rho}$, there is a piece Λ of a small $(N-n+1)$-dimensional complex plane passing through z_0 and contained in the complex tangent plane

$T_{z_0}^c\{\widetilde{\rho} = \widetilde{\rho}(z_0)\}$ on which the Levi-form $i\,\partial\overline{\partial}\widetilde{\rho}$ is positive. Thus Λ is contained in $\{\widetilde{\rho} > \widetilde{\rho}(z_0)\} \cup \{z_0\}$ and has a contact of order exactly two with $\{\widetilde{\rho} = \widetilde{\rho}(z_0)\}$ at z_0. Furthermore, if we pick a nonzero vector $v \in T_{z_0}\mathbb{C}^N$ which points into $\{\rho > \rho(z_0)\}$, the translates $\Lambda_\epsilon := \Lambda + \varepsilon\, v$ do all lie in $\{\rho > \rho(z_0)\}$ for every small $\varepsilon > 0$, whence $\Lambda_\varepsilon \cap \mathsf{E}$ is empty. But given that $\Lambda_0 \cap Y = \{z_0\} \neq \emptyset$, this contradicts the persistence, under perturbation, of the intersection of two complex analytic sets of complementary dimensions in $\mathbb{C}^N$. The lemma is proved. $\square$

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