The heavy top: a geometric treatment

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Abstract. We consider the steady group motions of a rigid body with a fixed point moving in a gravitational field. For an asymmetric top, rotation about the axis of gravity is the only permissible group motion; for a Lagrange top, simultaneous rotation about the axis of gravity and spin about the axis of symmetry of the top is permissible. Our analysis of the heavy top follows the reduced energy momentum method of Simo *et al* , which is applicable to a wide range of conservative systems with symmetry. Steady group motions are characterized as solutions of a variational problem on the configuration space; local minima of the amended potential correspond to nonlinearly orbitally stable steady motions. The combination of a low-dimensional configuration space and a relatively large number of parameters that produce substantial qualitative changes in the dynamics makes possible a thorough, detailed analysis, which not only reproduces the classical results for this well known system, but leads to some results which we believe are new.

We determine general equilibrium and nonlinear stability conditions for steady group motions of a heavy top with a fixed point. We rederive the classical equilibrium and stability conditions for sleeping tops and precessing Lagrange tops, analyse in detail the stability of a family of steady rotations of tilted tops which bifurcate from the branch of sleeping tops parametrized by angular velocity, and classify the possible stability transitions of an arbitrary top as its angular velocity is increased. We obtain a simple, general expression for the characteristic polynomial of the linearized equations of motion and analyse the linear stability of both sleeping tops and the family of tilted top motions previously mentioned. Finally, we demonstrate the coexistence of stable branches of steadily precessing tops that bifurcate from the branch of sleeping Lagrange tops throughout the range of angular velocities for which the sleeping top is stable.

AMS classification scheme numbers: 70E15, 70K20, 58F05, 58F14, 70H30

1. Introduction

The heavy top is one of the most familiar classical mechanical systems. The system, which consists of a rigid body with a fixed point moving in a gravitational field, is of sufficiently low dimension that a reasonably thorough analysis is tractable, yet rich

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enough to possess many interesting features in common with other, more complicated systems. The literature on the heavy top is substantial; Klein and Sommerfeld (1910) is an exhaustive study. Routh (1882, 1884) contains an elegant and thorough analysis of the stability of simple motions of heavy tops with a fixed point, as well as more complicated rigid body systems. Modern, topological treatments of the heavy top in the spirit of Smale (1970a, b) have been carried out by Iacob (1971), Katok (1972) and Tatarinov (1974); see Arnold (1989) for a summary (in English) of Tatarinov's results.

Our goal in this paper is the derivation of old, as well as new, results by means of energy-momentum techniques. Part of our goal is the demonstration of the effectiveness of these techniques in a classical setting. As shown in Simo *et al* (1990, 1991a, b), Lewis and Simo (1990), and Simo *et al* (1991), these methods are applicable to a wide variety of mechanical systems. We believe that some of our results, including a thorough bifurcation analysis of the linear stability of sleeping tops and a classification of the possible stability transitions of a broad class of steady motions, are new. The focus of our analysis is steady group motion of a heavy top. By a group motion, we mean a motion that is compatible with the symmetries of the top. For an asymmetric top, rotation about the axis of gravity is the only permissible group motion; for a Lagrange top, i.e., a heavy top whose centre of mass lies along its axis of symmetry, simultaneous rotation about the axis of gravity and spin about the axis of symmetry of the top is permissible. For these classes of motions, it is possible to characterize configurations that will maintain steady motion with a specified velocity as solutions of a variational problem. In fact, configurations that are constrained minima of the appropriate functional determine nonlinearly stable (modulo symmetries) motions. There are a number of variational analyses of steady motions of conservative systems in general and the heavy top in particular. Routh (1884) makes frequent use of variational methods; for a more recent treatment, see Maddocks (1989).

The variational technique used in the present work is the reduced energy momentum (REM) method of Simo *et al* (1991). This method is a synthesis of techniques developed by Riemann (1860), Routh (1882, 1884), Smale (1970a, b), Arnold (1966), Poincaré (1892) and the authors. One of the advantages of the REM method over other variational formulations is the conversion of the original constrained variational problem on phase space into an unconstrained variational problem on the configuration space. This reduced problem may be further simplified at the linear level by decomposing variations into 'rigid' and 'internal' modes with respect to which the second variation of the associated functional block diagonalizes. In consequence, the equilibrium and stability conditions can be easily derived as simple, explicit algebraic conditions.

The variational characterization of steady motion is also useful for the detection of bifurcations. When the second variation of the functional to be minimized changes signature, a new branch of configurations in steady motion may bifurcate. These branches can be detected and described in increasing detail by computing increasingly high-order variations of the functional. (See Golubitsky *et al* (1985, 1988) for a detailed discussion of singularity theory approaches to bifurcation theory.) We show here that the REM technique can greatly simplify the search for bifurcations. In particular, it sets a readily verifiable bifurcation condition on degenerate variations.

Given that a possible bifurcation of relative equilibria is signalled by a zero eigenvalue of the second variation and that the second variation is central to the stability analysis, it is natural to consider the relationship between bifurcation and stability. Typically, one expects a 'transfer of stability' at a bifurcation point: the original branch of solutions loses stability and a new stable branch of solutions appears.

However, this need not be the case for Hamiltonian systems with symmetry. In fact, we show that under certain conditions a family of solutions parametrized by precession and spin bifurcates from a branch of sleeping Lagrange tops, while the sleeping solution branch retains stability throughout the entire range in which bifurcations occur. The Lagrange top provides an example of this phenomenon. Associated to each angular velocity for which the *sleeping* Lagrange top is stable, there are two one-parameter families of precessing top motions that limit to the sleeping top with the specified velocity.

1.1. Basic notation: canonical phase space

We consider a heavy top, i.e. a rigid body acted upon by a gravitational force, with a fixed point. Once we have specified the shape of the top and the point of the top to be fixed, the configuration space Q is the set of all permissible orientations of the top. If we choose a reference configuration $\mathscr{B}$, we can identify the current configuration $\mathscr{B}_t$ with the element Λ_t of the rotation group SO(3) taking $\mathscr{B}$ to $\mathscr{B}_t$; Λ_t is uniquely determined by the relation $\Lambda_t \mathscr{B} = \mathscr{B}_t$. Hence we set $Q = \text{SO}(3)$. The tangent bundle TQ consists of pairs $(\Lambda, \delta\Lambda)$, where $\Lambda \in \text{SO}(3)$ and the matrix $\delta\Lambda$ satisfies the linearized orthogonality constraint

$$\delta\Lambda^T \Lambda + \Lambda^T \delta\Lambda = 0 \tag{1.1}$$

i.e. $\Lambda^T \delta\Lambda$ is a skew-symmetric matrix:

$$\text{sym}[\Lambda^T \delta\Lambda] = \text{sym}[\delta\Lambda\Lambda^T] = 0. \tag{1.2}$$

Let $\widehat{}$ denote the map from $\mathbb{R}^3$ to the space of skew-symmetric matrices determined by

$$\widehat{\xi}\mathbf{x} = \xi \times \mathbf{x} \qquad \forall \, \mathbf{x} \in \mathbb{R}^3. \tag{1.3}$$

The condition (1.2) implies that the pair $(\Lambda, \delta\Lambda)$ is an element of TQ if and only if $\Lambda \in \text{SO}(3)$ and

$$\Lambda \widehat{\delta\Theta} = \delta\Lambda = \widehat{\delta\theta}\Lambda \tag{1.4}$$

for some vectors $\delta\Theta$ and $\delta\theta \in \mathbb{R}^3$. Thus there are two distinct identifications of the tangent bundle TQ with the product $\text{SO}(3) \times \mathbb{R}^3$. The *body* variables $(\Lambda, \delta\Theta)$ correspond to the *left* trivialization of TQ; the *spatial* variables $(\Lambda, \delta\theta)$ correspond to the *right* trivialization.

The canonical phase space for the top is the cotangent bundle $T^*Q = T^* \text{SO}(3)$ consisting of pairs (Λ, Π_Λ) of configurations and conjugate momenta. T^*Q also has a body and a spatial representation: $(\Lambda, \Pi_\Lambda) \in T^*Q$ if and only if there exist vectors Π and $\pi \in \mathbb{R}^3$ such that

$$\Lambda \widehat{\Pi} = \Pi_\Lambda = \widehat{\pi}\Lambda. \tag{1.5}$$

The covector–vector pairings in the body and spatial variables are

$$\langle \Pi_\Lambda, \delta\Lambda \rangle = \tfrac{1}{2}\mathrm{tr}\big((\Lambda\widehat{\Pi})^T(\Lambda\widehat{\delta\Theta})\big) = \Pi \cdot \delta\Theta \tag{1.6}$$

and

$$\langle \Pi_\Lambda, \delta\Lambda \rangle = \tfrac{1}{2}\mathrm{tr}\big((\widehat{\pi}\Lambda)^T(\widehat{\delta\theta}\Lambda)\big) = \pi \cdot \delta\theta. \tag{1.7}$$

The identity

$$\widehat{\Lambda\xi} = \Lambda\widehat{\xi}\Lambda^T \tag{1.8}$$

which holds for any vector $\xi \in \mathbb{R}^3$ and any matrix $\Lambda \in$ SO(3), implies that the body and spatial representations are related by the equalities

$$\delta\theta = \Lambda\delta\Theta \qquad \text{and} \qquad \pi = \Lambda\Pi. \tag{1.9}$$

The kinetic energy of a top moving with velocity $\dot\Lambda$ is

$$\tfrac{1}{2}\int_{\mathscr{B}} \rho_{\mathrm{ref}}(X)(\dot\Lambda X) \cdot (\dot\Lambda X)\mathrm{d}^3 X \tag{1.10}$$

where ρ_{ref} is the reference density. If we define the energy inner product $\langle\,,\,\rangle_{\mathbf{I}}$ on TQ by

$$\begin{aligned}
\langle U, V \rangle_{\mathbf{I}} &:= \int_{\mathscr{B}} \rho_{\mathrm{ref}}(X)(UX) \cdot (VX)\mathrm{d}^3 X \\
&= \mathrm{tr}\,(U^T V \mathbf{I})
\end{aligned} \tag{1.11}$$

where

$$\mathbf{I} := \int_{\mathscr{B}} \rho_{\mathrm{ref}}(X) X \otimes X \mathrm{d}^3 X \tag{1.12}$$

then the kinetic energy equals $\tfrac{1}{2}\langle\dot\Lambda, \dot\Lambda\rangle_{\mathbf{I}}$. The gravitational potential energy is $\mathbf{g} \cdot \Lambda\mathbf{M}$, where $\mathbf{g} = g e_3$ denotes the gravity vector and $\mathbf{M} := \int_{\mathscr{B}} \rho_{\mathrm{ref}}(X) X \, \mathrm{d}^3 X$. If $m := \int_{\mathscr{B}} \rho_{\mathrm{ref}}(X)\, \mathrm{d}^3 X$ is the total mass of the body, then the centre of mass of the reference body is given by $\mathbf{M}/m$. Thus $|\mathbf{M}| = m\ell$, where ℓ is the distance from the fixed point of the body to the centre of mass.

The inner product $\langle\,,\,\rangle_{\mathbf{I}}$ given by (1.11) can be expressed in terms of the pairing (1.6) in body representation in the following fashion. Define the *reference inertia tensor* $\mathbb{I}_{\mathrm{ref}}$ by

$$\begin{aligned}
\mathbb{I}_{\mathrm{ref}} &= \int_{\mathscr{B}} \rho_{\mathrm{ref}}(X)\big(|X|^2 \mathbf{1}_3 - X \otimes X\big)\mathrm{d}^3 X \\
&= (\mathrm{tr}\,\mathbf{I})\,\mathbf{1}_3 - \mathbf{I}
\end{aligned} \tag{1.13}$$

where $\mathbf{I}_3$ denotes the three by three identity matrix. Then a straightforward computation shows that

$$\langle \Lambda \widehat{\delta\Theta}_1, \Lambda \widehat{\delta\Theta}_2 \rangle_{\mathbf{I}} = \delta\Theta_1 \cdot \mathbb{I}_{\mathrm{ref}} \delta\Theta_2 \qquad (1.14)$$

for any $\Lambda \in SO(3)$. Note that (1.13) implies that the eigenvalues I_1, I_2, and I_3 of $\mathbb{I}_{\mathrm{ref}}$ satisfy $I_i = \mathrm{tr}\,\mathbf{I} - \lambda_i$ for $i \in \{1,2,3\}$, where the λ_i's are the (positive) eigenvalues of $\mathbf{I}$. Thus $I_j + I_k - I_i = 2\lambda_i > 0$ implies that the inequalities $I_i < I_j + I_k$ hold for all i, j, $k \in \{1,2,3\}$, $j \neq k$.

The analogous expression for the inner product $\langle\ ,\ \rangle_{\mathbf{I}}$ in terms of the spatial variables involves the *current spatial inertia tensor* $\mathbb{I}_\Lambda$ given by

$$\mathbb{I}_\Lambda := \Lambda \mathbb{I}_{\mathrm{ref}} \Lambda^T. \qquad (1.15)$$

Using (1.9) and (1.14), we determine the relationship

$$\langle \widehat{\delta\theta}_1 \Lambda, \widehat{\delta\theta}_2 \Lambda \rangle_{\mathbf{I}} = \langle \Lambda(\widehat{\Lambda^T \delta\theta_1}) \qquad \Lambda(\widehat{\Lambda^T \delta\theta_2}) \rangle_{\mathbf{I}} = (\Lambda^T \delta\theta_1) \cdot \mathbb{I}_{\mathrm{ref}}(\Lambda^T \delta\theta_2) = \delta\theta_1 \cdot \mathbb{I}_\Lambda \delta\theta_2 \qquad (1.16)$$

for any $\Lambda \in SO(3)$. Note that (1.14) shows that the metric $\langle\ ,\ \rangle_{\mathbf{I}}$ on TQ is left invariant, whereas (1.16) shows that the metric is typically not right invariant.

1.2. Lagrangian and Hamiltonian structures

The Lagrangian L for the heavy top is the difference between the kinetic and potential energies, i.e.,

$$L(\Lambda, \dot\Lambda) = \tfrac{1}{2}\langle \dot\Lambda, \dot\Lambda \rangle_{\mathbf{I}} - \mathbf{g} \cdot \Lambda \mathbf{M}. \qquad (1.17)$$

The Legendre transformation FL associated to a Lagrangian L, defined by

$$FL(\Lambda, \dot\Lambda) \cdot (\Lambda, \delta\Lambda) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} L(\Lambda, \dot\Lambda + \epsilon\,\delta\Lambda) \qquad (1.18)$$

for $\dot\Lambda$, $\delta\Lambda \in T_\Lambda Q$, maps velocity fields to momentum fields. In the case of the heavy top, the Legendre transformation $FL : T_\Lambda Q \to T^*_\Lambda Q$ at a configuration Λ is

$$FL(\Lambda, \Lambda\widehat{\delta\Theta}) = (\Lambda, \langle \Lambda\widehat{\delta\Theta}, \cdot \rangle_{\mathbf{I}}) = (\Lambda, \Lambda(\widehat{\mathbb{I}_{\mathrm{ref}}\delta\Theta})) \qquad (1.19)$$

or, equivalently,

$$FL(\Lambda, \widehat{\delta\theta}\Lambda) = (\Lambda, \langle \widehat{\delta\theta}\Lambda, \cdot \rangle_{\mathbf{I}}) = (\Lambda, (\widehat{\mathbb{I}_\Lambda \delta\theta})\Lambda). \qquad (1.20)$$

Formulae (1.19) and (1.20) are a direct consequence of (1.14) and (1.16) and the definition of the pairing between T^*Q and TQ given by (1.6) and (1.7). The total energy associated to the configuration-velocity pair $(\Lambda, \dot\Lambda)$ is

$$E(\Lambda, \dot\Lambda) = FL(\dot\Lambda) \cdot \dot\Lambda - L(\Lambda, \dot\Lambda) = \tfrac{1}{2}\langle \dot\Lambda, \dot\Lambda \rangle_{\mathbf{I}} + \mathbf{g} \cdot \Lambda \mathbf{M}. \qquad (1.21)$$

To express the Lagrangian in terms of the spatial variables, define the *spatial angular velocity* by

$$\widehat{\omega} := \dot{\Lambda}\Lambda^T. \tag{1.22}$$

Then (1.16) gives the expression of the Lagrangian in spatial variables as

$$L(\Lambda, \omega) := L(\Lambda, \widehat{\omega}\Lambda) = \tfrac{1}{2}\langle \widehat{\omega}\Lambda, \widehat{\omega}\Lambda \rangle_{\mathbb{I}} - \mathbf{g} \cdot \Lambda\mathbf{M} = \tfrac{1}{2}\omega \cdot \mathbb{I}_\Lambda \omega - \mathbf{g} \cdot \Lambda\mathbf{M}. \tag{1.23}$$

The Legendre transformation $FL : SO(3) \times \mathbb{R}^3 \to SO(3) \times \mathbb{R}^3$ has the spatial expression

$$FL(\Lambda, \omega) = (\Lambda, \mathbb{I}_\Lambda \omega), \tag{1.24}$$

which follows directly from (1.20). Note that we use the same letter L to denote the Lagrangian with respect to both material and spatial variables; in general, we shall use the same notation to denote a mapping with respect to material, spatial or body coordinates.

Next, we determine the Hamiltonian structure of the heavy top. The canonical phase space is the cotangent bundle T^*Q consisting of pairs (Λ, Π_Λ) of configurations and conjugate momenta. Given the Lagrangian L, the cotangent bundle T^*Q is the image of the tangent bundle TQ under the Legendre transformation. The Hamiltonian on the phase space T^*Q is defined by

$$H(\Lambda, \Pi_\Lambda) := E(\Lambda, FL^{-1}(\Pi_\Lambda)) = \mathbf{g} \cdot \Lambda\mathbf{M} + \tfrac{1}{2}\langle \Pi_\Lambda, \Pi_\Lambda \rangle_{\mathbb{I}^{-1}} \tag{1.25}$$

where $\langle \, , \, \rangle_{\mathbb{I}^{-1}}$ is the metric on covectors induced by FL, i.e.

$$\langle \Pi_\Lambda, \Pi'_\Lambda \rangle_{\mathbb{I}^{-1}} = \langle FL^{-1}(\Pi_\Lambda), FL^{-1}(\Pi'_\Lambda) \rangle_{\mathbb{I}}. \tag{1.26}$$

In spatial representation, (1.17), (1.16) and (1.20) imply that

$$H(\Lambda, \pi) := \mathbf{m} \cdot \mathbf{g} + \tfrac{1}{2}\pi \cdot \mathbb{I}_\Lambda^{-1}\pi \tag{1.27}$$

where $\mathbf{m} := \Lambda\mathbf{M}$ is the spatial representation of the mass vector. In the body representation (1.17), (1.14), and (1.19) imply that

$$H(\Lambda, \Pi) := g\gamma \cdot \mathbf{M} + \tfrac{1}{2}\Pi \cdot \mathbb{I}_{\text{ref}}^{-1}\Pi \tag{1.28}$$

where $\gamma := \Lambda^T \mathbf{e}_3$ is the direction of gravity in the body frame.

1.3. Equations of motion

Recall from section 1.2 that the spatial angular velocity ω defined by (1.22) satisfies $\widehat{\omega} = \dot{\Lambda}\Lambda^T$. Similarly, the body angular velocity Ω is given by

$$\widehat{\Omega} = \Lambda^T \dot{\Lambda}. \tag{1.29}$$

Note that (1.9) implies that

$$\omega = \Lambda\Omega \qquad \text{and} \qquad \pi = \Lambda\Pi \tag{1.30}$$

where the body and spatial momenta

$$\Pi = \mathbb{I}_{ref}\Omega \qquad and \qquad \pi = \mathbb{I}_A\omega \tag{1.31}$$

are given by the Legendre transformation according to (1.19) and (1.20).

The following expressions are the well-known body and spatial forms of the equations of motion; see e.g. section (6.1) or Arnold (1966, 1989) for the derivation of these equations.

The body form of the equations of motion is

$$\dot{\Lambda} = \Lambda\widehat{\Omega} \qquad and \qquad \dot{\Pi} = \Pi \times \Omega + g\gamma \times \mathbf{M}. \tag{1.32}$$

The spatial form of the equations of motion is

$$\dot{\Lambda} = \widehat{\omega}\Lambda \qquad and \qquad \dot{\pi} = \mathbf{g} \times \mathbf{m}. \tag{1.33}$$

Note that the second equation in (1.33) is the Euler equation for a rigid body under the influence of a gravitational potential.

1.4. Symmetries

The symmetries of the Hamiltonian are central to the stability analysis. To obtain optimal results, it is typically necessary to work with the largest available symmetry group; failure to incorporate all symmetries can result in overly restrictive stability conditions.

We first consider the spatial symmetries of the heavy top. Material frame indifference (i.e., left SO(3) invariance) of the free rigid body is broken by the gravitational potential. While the kinetic energy is invariant under the full group of spatial rotations, the potential energy is invariant only under spatial rotations that preserve the gravity vector **g**. The gravitational potential is invariant not only with respect to *spatial* (left) rotations about **g**, but also with respect to *body* (right) rotations about the vector passing through the centre of mass **M**. If the top possesses an axis of symmetry, then the kinetic energy is right invariant under rotations about the axis of symmetry. If the top has a symmetric moment of inertia tensor and its centre of mass lies on the axis of symmetry, i.e. it is a Lagrange top, then the Hamiltonian is invariant under the group $G = S^1 \times S^1$ consisting of spatial (left) rotations about the direction of gravity and body (right) rotations about the axis of symmetry.

We consider three distinct cases of steady motion, each determined by the symmetries of the top and its motion: first, the 'asymmetric' top (i.e., *not* a Lagrange top); second, a 'tilted' Lagrange top, for which the axis of symmetry is *not* parallel to the gravity vector **g**; finally, the sleeping Lagrange top, for which the axis of symmetry and the gravity vector **g** *are* parallel. In the case of an asymmetric top, the appropriate symmetry group is $G = S^1$, which acts freely by spatial rotations about **g**. The appropriate symmetry group in the case of a Lagrange top is $G = S^1 \times S^1$, which acts by rotations about **g** and the axis of symmetry. Tilted configurations have trivial isotropy, i.e. there are no elements of the group G that fix the configuration. On the other hand, a sleeping Lagrange top is fixed by the one-dimensional subgroup of G consisting of a rotation about **g** paired with a reverse rotation of equal magnitude about the axis of symmetry.

2. The asymmetric top

We briefly outline some of the essential constructions of the reduced energy momentum (REM) method here, using the asymmetric top to illustrate the definitions, and refer the reader to Simo *et al* (1991) and Lewis (1991a) for the general statement of the method. A glossary of the essential constructions and a very brief summary of the method are given in the appendix. An essential feature of the analysis is the incorporation of the constant momentum constraint given by Noether's theorem. In the case of the asymmetric top, where the symmetry group is S^1, the momentum in question is simply the e_3 component of the angular momentum. (In the case of the Lagrange top, which possesses a body symmetry, an additional scalar 'material momentum' is conserved.) For the asymmetric top the symmetry group G is

$$G := \left\{ \exp_{SO(3)} \left[s\hat{e}_3 \right] : s \in \mathbb{R} \right\} \approx S^1 \tag{2.1}$$

with Lie algebra $g = \left\{ s\hat{e}_3 : s \in \mathbb{R} \right\} \approx \mathbb{R}$. The dual g^* to g is also isomorphic to $\mathbb{R}$, with pairing given by multiplication. G acts on Q and T^*Q by matrix multiplication on the left.

The infinitesimal generator associated to the G action on Q is

$$\xi_Q(\Lambda) = (\Lambda, \xi\hat{e}_3\Lambda) \tag{2.2}$$

i.e., an infinitesimal rotation about the axis e_3 with angular velocity ξ. The infinitesimal generator in the spatial representation is $\xi_Q(\Lambda) = (\Lambda, \xi e_3)$. The *locked inertia tensor* $\mathbb{I}(\Lambda) : g \to g^*$ measures the momentum associated to the specified group motion; e.g., $\mathbb{I}(\Lambda)\xi$ is the e_3 component of the angular momentum of the top in orientation Λ rotating with angular velocity ξ about the e_3 axis. Specifically, the locked inertia tensor associated to the S^1 action is defined by the relationship

$$\eta \, \mathbb{I}(\Lambda)\xi := \left\langle FL(\xi_Q(\Lambda)), \eta_Q(\Lambda) \right\rangle = (\eta e_3) \cdot \mathbb{I}_\Lambda(\xi e_3) = \eta \, \xi \, e_3 \cdot \mathbb{I}_\Lambda e_3 \tag{2.3}$$

for all ξ and $\eta \in \mathbb{R}$, where the second equality follows from (1.20) and (2.2). Hence $\mathbb{I}(\Lambda) = e_3 \cdot \mathbb{I}_\Lambda e_3$.

There are two functionals that play a role in the variational problem. The first, the *augmented potential*, is constructed by evaluating the negative of the Lagrangian at a specified infinitesimal generator. Given $\xi \in g \approx \mathbb{R}$, the *augmented potential* for the asymmetric heavy top is defined by

$$V_\xi(\Lambda) = -L(\xi_Q(\Lambda)) = \mathbf{m} \cdot \mathbf{g} - \tfrac{1}{2}\xi^2 \mathbb{I}(\Lambda). \tag{2.4}$$

A point Λ_e is a critical point of V_ξ if and only if $FL(\xi_Q(\Lambda_e))$ is a *relative equilibrium*, i.e. if Λ_e can be maintained in steady motion with angular velocity ξ. (This result is proved for general simple mechanical systems in Simo *et al* (1991).) However, the second variation of V_ξ does not provide sharp stability conditions. Hence we introduce the *amended potential*, which is constructed by evaluating the energy at an infinitesimal generator obtained via the locked inertia tensor from a specified value of the momentum. Specifically, given $\mu \in g^* \approx \mathbb{R}$, the *amended potential* for the asymmetric heavy top is

$$V_\mu(\Lambda) = E\left(\left(\mathbb{I}(\Lambda)^{-1}\mu \right)_Q(\Lambda) \right) = \mathbf{m} \cdot \mathbf{g} + \tfrac{1}{2}\mathbb{I}(\Lambda)^{-1}\mu^2. \tag{2.5}$$

Critical points Λ_e of V_μ are also relative equilibria and remain in steady rotation with the specified momentum μ. Definiteness of the second variation $D^2 V_\mu(\Lambda_e)$ on some complement to the tangent space $g \cdot \Lambda_e$ to the group orbit is sufficient to guarantee nonlinear stability of the motion modulo rotations about e_3. The functional V_μ is often less convenient to work with than V_ξ; these inconveniences can be avoided to a great extent by noting that the second variation $D^2 V_\mu(\Lambda_e)$ of the amended potential can be constructed by adding a positive semi-definite bilinear form to the second variation $D^2 V_\xi(\Lambda_e)$ of the augmented potential. Define the *locked momentum map* $\mathbb{I}_\xi : Q \to g^*$ by

$$\mathbb{I}_\xi(\Lambda) := \mathbb{I}(\Lambda)\xi. \tag{2.6}$$

Then the symmetric bilinear form $\mathscr{B}_e := D^2 V_\mu(\Lambda_e)$ satisfies

$$\mathscr{B}_e(\delta\Lambda, \Delta\Lambda) = D^2 V_\xi(\Lambda_e)(\delta\Lambda, \Delta\Lambda) + (D\mathbb{I}_\xi(\Lambda_e) \cdot \delta\Lambda) \cdot \mathbb{I}(\Lambda_e)^{-1}(D\mathbb{I}_\xi(\Lambda_e) \cdot \Delta\Lambda) \tag{2.7}$$

for all variations $\delta\Lambda, \Delta\Lambda \in T_{\Lambda_e} Q$. (See Simo *et al* (1991) for the general statement of this result for simple mechanical systems.)

2.1. Relative equilibria—the first variation

We use the augmented potential V_ξ to determine relative equilibria. Using standard vector product identities, the first variation of the locked inertia tensor is computed as follows:

$$\begin{aligned}
D\mathbb{I}(\Lambda) \cdot \delta\Lambda &= e_3 \cdot \left(\delta\Lambda \mathbb{I}_{ref}\Lambda^T + \Lambda \mathbb{I}_{ref}\delta\Lambda^T\right)e_3 \\
&= e_3 \cdot \left(\widehat{\delta\theta}\mathbb{I}_\Lambda - \mathbb{I}_\Lambda\widehat{\delta\theta}\right)e_3 \\
&= e_3 \cdot \left(\delta\theta \times \mathbb{I}_\Lambda e_3\right) - \mathbb{I}_\Lambda e_3 \cdot \left(\delta\theta \times e_3\right) \\
&= 2\delta\theta \cdot \left(\left(\mathbb{I}_\Lambda e_3\right) \times e_3\right)
\end{aligned} \tag{2.8}$$

where $\widehat{\delta\theta} = \delta\Lambda\Lambda^T$. The first variation of the gravitational potential in the direction of the variation $\delta\Lambda$ is

$$g e_3 \cdot \delta\Lambda M = g\delta\theta \cdot (m \times e_3). \tag{2.9}$$

Thus relative equilibria are determined by the condition

$$0 = DV_\xi(\Lambda_e) \cdot \delta\Lambda = \delta\theta \cdot \left(\left(g m_e - \xi^2 \mathbb{I}_{\Lambda_e} e_3\right) \times e_3\right) \tag{2.10}$$

for all $\delta\Lambda \in T_{\Lambda_e} Q$, i.e.,

$$\left(g m_e - \xi^2 \mathbb{I}_{\Lambda_e} e_3\right) \times e_3 = 0. \tag{2.11}$$

The relative equilibria determined by (2.11) are traditionally referred to as the 'axes of Staude' (see, e.g., Maddocks (1989)). When performing the stability analysis, we shall find it convenient to express the equilibrium condition in the form

$$g m_e - \xi^2 \mathbb{I}_{\Lambda_e} e_3 = \Gamma_e e_3 \tag{2.12}$$

where

$$\Gamma_e := m_e \cdot g - \xi^2 \mathbb{I}(\Lambda_e) = \left(g m_e - \xi^2 \mathbb{I}_{\Lambda_e} e_3\right) \cdot e_3. \tag{2.13}$$

2.2. Stability—the second variation

Let $\delta\Lambda = \widehat{\delta\theta}\Lambda_e$ and $\Delta\Lambda = \widehat{\Delta\theta}\Lambda_e$ denote two elements of $T_{\Lambda_e}Q$. The second variation of the augmented potential is

$$
\begin{aligned}
D^2 V_\xi(\Lambda_e)(\delta\Lambda, \Delta\Lambda) &= (\mathbf{e}_3 \times \delta\theta) \cdot \left(\Delta\theta \times \mathbf{gm}_e - \xi^2\left(\Delta\theta \times \mathbb{II}_{\Lambda_e}\mathbf{e}_3 - \mathbb{II}_{\Lambda_e}(\Delta\theta \times \mathbf{e}_3)\right)\right) \\
&= -\xi^2(\mathbf{e}_3 \times \delta\theta) \cdot \mathbb{II}_{\Lambda_e}(\mathbf{e}_3 \times \Delta\theta) \\
&\quad + (\mathbf{e}_3 \times \delta\theta) \cdot \Delta\theta \times \left(\mathbf{gm}_e - \xi^2\mathbb{II}_{\Lambda_e}\mathbf{e}_3\right) \\
&= -(\mathbf{e}_3 \times \delta\theta) \cdot \left(\xi^2\mathbb{II}_{\Lambda_e} + \Gamma_e\mathbf{1}_3\right)(\mathbf{e}_3 \times \Delta\theta).
\end{aligned}
\tag{2.14}
$$

The momentum correction term in (2.7) is given by

$$
\begin{aligned}
&\left(D\mathbb{II}_\xi(\Lambda_e) \cdot \delta\Lambda\right) \cdot \mathbb{II}(\Lambda_e)^{-1}\left(D\mathbb{II}_\xi(\Lambda_e) \cdot \Delta\Lambda\right) \\
&\quad = \frac{\xi^2}{\mathbb{II}(\Lambda_e)}\left(D\mathbb{II}(\Lambda_e) \cdot \delta\Lambda\right)\left(D\mathbb{II}(\Lambda_e) \cdot \Delta\Lambda\right) \\
&\quad = \frac{4\xi^2}{\mathbb{II}(\Lambda_e)}(\mathbf{e}_3 \times \delta\theta) \cdot \left(\mathbb{II}_{\Lambda_e}\mathbf{e}_3 \otimes \mathbb{II}_{\Lambda_e}\mathbf{e}_3\right)(\mathbf{e}_3 \times \Delta\theta).
\end{aligned}
\tag{2.15}
$$

Substitution of expressions (2.14) and (2.15) into (2.7) yields

$$
\mathscr{B}_e(\delta\Lambda, \Delta\Lambda) = (\mathbf{e}_3 \times \delta\theta) \cdot \left(\frac{4\xi^2}{\mathbb{II}(\Lambda_e)}\mathbb{II}_{\Lambda_e}\mathbf{e}_3 \otimes \mathbb{II}_{\Lambda_e}\mathbf{e}_3 - \xi^2\mathbb{II}_{\Lambda_e} - \Gamma_e\mathbf{1}_3\right)(\mathbf{e}_3 \times \Delta\theta).
\tag{2.16}
$$

The equilibrium is orbitally nonlinearly stable if $\mathscr{B}_e$ is positive semi-definite, with kernel spanned by $\hat{\mathbf{e}}_3\Lambda_e$, i.e., if the restriction $\widetilde{\mathscr{B}}_e$ of the form

$$
\frac{4\xi^2}{\mathbb{II}(\Lambda_e)}\mathbb{II}_{\Lambda_e}\mathbf{e}_3 \otimes \mathbb{II}_{\Lambda_e}\mathbf{e}_3 - \xi^2\mathbb{II}_{\Lambda_e} - \Gamma_e\mathbf{1}_3
\tag{2.17}
$$

to $\mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\}$ is positive definite. The equilibrium condition (2.12) implies that the restriction of (2.17) to $\mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\}$ equals

$$
\frac{4g^2}{\xi^2\mathbb{II}(\Lambda_e)}\mathbf{m}_e \otimes \mathbf{m}_e - \xi^2\mathbb{II}_{\Lambda_e} - \Gamma_e\mathbf{1}_3.
\tag{2.18}
$$

The stability conditions can easily be formulated in terms of the body vectors $\mathbf{M}$ and γ. The matrix (2.17) can be written in the form

$$
\Lambda_e\left(\frac{4\xi^2}{(\gamma_e \cdot \mathbb{II}_{\mathrm{ref}}\gamma_e)}\mathbb{II}_{\mathrm{ref}}\gamma_e \otimes \mathbb{II}_{\mathrm{ref}}\gamma_e - \xi^2\mathbb{II}_{\mathrm{ref}} - \Gamma_e\mathbf{1}_3\right)\Lambda_e^T.
\tag{2.19}
$$

Hence $\widetilde{\mathscr{B}}_e$ is positive definite if and only if the restriction of

$$
\frac{4\xi^2}{I_{\gamma_e}}\mathbb{II}_{\mathrm{ref}}\gamma_e \otimes \mathbb{II}_{\mathrm{ref}}\gamma_e - \xi^2\mathbb{II}_{\mathrm{ref}} - \Gamma_e\mathbf{1}_3
\tag{2.20}
$$

or, equivalently,

$$
\frac{4g^2}{\xi^2 I_{\gamma_e}}\mathbf{M} \otimes \mathbf{M} - \xi^2\mathbb{II}_{\mathrm{ref}} - \Gamma_e\mathbf{1}_3
\tag{2.21}
$$

to the plane orthogonal to γ_e is positive definite.

In many cases, the test for positive definiteness of $\widetilde{\mathscr{B}}_e$ is most easily implemented by testing for positivity of the trace and determinant of $\widetilde{\mathscr{B}}_e$. Hence, we derive below some simple expressions for these invariants. If we define

$$\mathscr{C}_e := \xi^2 \mathbb{I}_{\Lambda_e}|_{e_1 - e_2} + \Gamma_e \mathbf{1}_2 \tag{2.22}$$

then

$$\widetilde{\mathscr{B}}_e = \frac{4\xi^2}{\mathbb{II}(\Lambda_e)} \left(\mathbb{I}_{\Lambda_e} e_3 - \mathbb{II}(\Lambda_e) e_3 \right) \otimes \left(\mathbb{I}_{\Lambda_e} e_3 - \mathbb{II}(\Lambda_e) e_3 \right) - \mathscr{C}_e. \tag{2.23}$$

Using the identities

$$\mathrm{tr}\,(\mathbf{x} \otimes \mathbf{x} - A) = |\mathbf{x}|^2 - \mathrm{tr}\,A \tag{2.24}$$

and

$$\det\,(\mathbf{x} \otimes \mathbf{x} - A) = \det A - (\mathbf{e}_3 \times \mathbf{x}) \cdot A(\mathbf{e}_3 \times \mathbf{x}) \tag{2.25}$$

for all two by two symmetric matrices A and all $\mathbf{x} \in \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, we see that

$$\mathrm{tr}\,\widetilde{\mathscr{B}}_e = \frac{4\xi^2}{\mathbb{II}(\Lambda_e)} |\mathbf{e}_3 \times \mathbb{I}_{\Lambda_e} \mathbf{e}_3|^2 - \mathrm{tr}\,\mathscr{C}_e \tag{2.26}$$

and

$$\det\,\widetilde{\mathscr{B}}_e = \det \mathscr{C}_e - \frac{4\xi^2}{\mathbb{II}(\Lambda_e)} (\mathbf{e}_3 \times \mathbb{I}_{\Lambda_e} \mathbf{e}_3) \cdot \mathscr{C}_e (\mathbf{e}_3 \times \mathbb{I}_{\Lambda_e} \mathbf{e}_3). \tag{2.27}$$

Hence $\widetilde{\mathscr{B}}_e$ is positive definite if and only if

$$\frac{4\xi^2}{\mathbb{II}(\Lambda_e)} |\mathbf{e}_3 \times \mathbb{I}_{\Lambda_e} \mathbf{e}_3|^2 > \mathrm{tr}\,\mathscr{C}_e \tag{2.28}$$

and

$$\det \mathscr{C}_e > \frac{4\xi^2}{\mathbb{II}(\Lambda_e)} (\mathbf{e}_3 \times \mathbb{I}_{\Lambda_e} \mathbf{e}_3) \cdot \mathscr{C}_e (\mathbf{e}_3 \times \mathbb{I}_{\Lambda_e} \mathbf{e}_3). \tag{2.29}$$

Making use of the equilibrium condition (2.11), i.e.

$$\mathbf{e}_3 \times \mathbb{I}_{\Lambda_e} \mathbf{e}_3 = \frac{g}{\xi^2} \mathbf{e}_3 \times \mathbf{m}_e \tag{2.30}$$

we can express the stability conditions in the form

$$\frac{4g^2}{\xi^2 \mathbb{II}(\Lambda_e)} |\mathbf{e}_3 \times \mathbf{m}_e|^2 > \mathrm{tr}\,\mathscr{C}_e \tag{2.31}$$

and

$$\det \mathscr{C}_e > \frac{4g^2}{\xi^2 \mathbb{I}(\Lambda_e)}(\mathbf{e}_3 \times \mathbf{m}_e) \cdot \mathscr{C}_e(\mathbf{e}_3 \times \mathbf{m}_e). \tag{2.32}$$

In summary, we have derived the following equilibrium and stability conditions:

An asymmetric top with spatial inertia tensor $\mathbb{I}_{\Lambda_e}$ and mass vector $\mathbf{m}_e$ can be maintained in steady rotation about the axis of gravity $\mathbf{g} = g\mathbf{e}_3$ with angular velocity ξ if and only if

$$g\mathbf{m}_e - \xi^2 \mathbb{I}_{\Lambda_e}\mathbf{e}_3 = \Gamma_e \mathbf{e}_3$$

equivalently, if and only if

$$g\mathbf{M} - \xi^2 \mathbb{I}_{\mathrm{ref}}\gamma_e = \Gamma_e \gamma_e$$

where $\gamma_e := \Lambda_e^T \mathbf{e}_3$ is the direction of gravity with respect to the body frame and

$$\Gamma_e = \mathbf{m}_e \cdot \mathbf{g} - \xi^2 \mathbb{I}(\Lambda_e) = g\gamma_e \cdot \mathbf{M} - \xi^2 \gamma_e \cdot \mathbb{I}_{\mathrm{ref}}\gamma_e.$$

This steady rotation is nonlinearly orbitally stable if the restriction $\widetilde{\mathscr{B}}_e$ of the symmetric matrix

$$\frac{4\xi^2}{\mathbb{I}(\Lambda_e)}\mathbb{I}_{\Lambda_e}\mathbf{e}_3 \otimes \mathbb{I}_{\Lambda_e}\mathbf{e}_3 - \xi^2 \mathbb{I}_{\Lambda_e} - \Gamma_e \mathbf{1}_3$$

to the $\mathbf{e}_1$–$\mathbf{e}_2$ plane is positive definite.

2.3. Stability of special classes of relative equilibria

In this section, we explicitly parametrize some special classes of steady motions of heavy tops that are characterized by conditions on the position of the axis of gravity relative to the eigenvectors of the inertia tensor and present the equilibrium and stability conditions in terms of these parameters. The Hamiltonian is determined by the eigenvalues of the reference inertia tensor $\mathbb{I}_{\mathrm{ref}}$, the mass of the top, the coordinates of the centre of mass with respect to the principal axes of the reference inertia tensor, and the strength of gravity. The quantities characterizing the steady motion of the top are the angle of the cone traced out by the motion of the vector from the fixed point to the centre of mass, and the angular velocity ξ.

We first apply the stability criteria to a sleeping asymmetric top, i.e., a top for which the gravity vector $\mathbf{g} = g\mathbf{e}_3$ is an eigenvector of the equilibrium inertia tensor $\mathbb{I}_{\Lambda_e}$. Taking the remaining eigenvectors of the inertia tensor as our basis vectors, we may assume that $\mathbb{I}_{\Lambda_e} = \mathrm{diag}\left[I_1, I_2, I_3\right]$ for some positive constants I_i satisfying $I_1 > I_2$. The equilibrium equations (2.12) imply that $\mathbf{m}_e = \pm m\ell\mathbf{e}_3$. A positive sign implies that the top is upright; a negative sign implies that the top is hanging straight down. Conversely, if $\mathbf{m}_e$ and $\mathbf{e}_3$ are proportional, then $\mathbf{e}_3$ must be an eigenvector of $\mathbb{I}_{\Lambda_e}$ in the case of steady motion, i.e., sleeping tops are characterized by the condition that the mass vector $\mathbf{m}_e$ is parallel to the gravity vector $\mathbf{g}$.

A sleeping top Λ_e can be kept in steady rotation at any angular velocity, i.e. $DV_\xi(\Lambda_e) = 0$ for any $\xi \in \mathbb{R}$. The stability matrix $\widetilde{\mathscr{B}}_e$ for an upright top is given by

$$\widetilde{\mathscr{B}}_e = \text{diag}\left[\xi^2(I_3 - I_1) - mg\ell, \, \xi^2(I_3 - I_2) - mg\ell\right]. \tag{2.33}$$

Hence an upright sleeping top, i.e. a top for which ℓ is positive, rotating at angular velocity ξ is nonlinearly stable modulo rotations about e_3 if it is in rotation about the axis of maximal inertia, i.e. if $I_3 > I_1 > I_2$, and

$$\xi^2 > \frac{mg\ell}{I_3 - I_1}. \tag{2.34}$$

A hanging sleeping top has stability matrix $\widetilde{\mathscr{B}}_e = \text{diag}\left[\xi^2(I_3 - I_1) + mg\ell, \, \xi^2(I_3 - I_2) + mg\ell\right]$. Hence a hanging top in rotation about the axis of maximal inertia is nonlinearly stable *at any velocity*. A hanging sleeping top for which either $I_1 > I_3 > I_2$ or $I_1 > I_2 > I_3$ is stable for angular velocities ξ satisfying

$$\xi^2 < \frac{mg\ell}{I_1 - I_3}. \tag{2.35}$$

These stability conditions for sleeping tops can be found in Routh (1884, section 211).

A tilted top is a top for which the gravity vector $g = ge_3$ is *not* an eigenvector of the inertia tensor $\mathbb{I}_{\Lambda_e}$. If it can maintain a steady rotation about the axis of gravity without spin, then it is a relative equilibrium. There is a special family of tilted relative equilibria that generalizes the family of sleeping tops analysed above; specifically, we consider tops for which the axis of gravity lies in a plane determined by two of the principal axes of the inertia tensor. To simplify the general calculations carried out in sections 2.1 and 2.2, we shall construct a coordinate system tailored to the candidate relative equilibrium. By assumption, there is an eigenvector e_1 of $\mathbb{I}_{\Lambda_e}$ such that $e_1 \cdot g = 0$. Hence we can set $e_2 := e_3 \times e_1$. In this coordinate system, the spatial inertia tensor $\mathbb{I}_{\Lambda_e}$ is given by

$$\mathbb{I}_{\Lambda_e} = R_\theta \text{diag}\left[I_1, I_2, I_3\right] R_\theta^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 \cos^2\theta + I_3 \sin^2\theta & (I_2 - I_3)\cos\theta\sin\theta \\ 0 & (I_2 - I_3)\cos\theta\sin\theta & I_2 \sin^2\theta + I_3 \cos^2\theta \end{pmatrix} \tag{2.36}$$

where

$$R_\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{2.37}$$

for some angle $\theta \neq 0, \pi$. Note that the stability of a specific top is determined by the spatial equilibrium values $\mathbb{I}_{\Lambda_e}$ and m_e, as well as the equilibrium velocity ξ; the stability conditions are independent of the specific choice of a reference orientation of the top. Hence we are free to make any convenient choice of principal axes of the body frame. We set $\mathbb{I}_{\text{ref}} = \text{diag}\left[I_1, I_2, I_3\right]$ and $\Lambda_e = R_\theta$.

We now determine the equilibrium mass vector $m_e = \Lambda_e M$. Since Λ_e is a rotation about e_1 and hence fixes e_1, (2.36) and the equilibrium condition (2.12) imply that

$$g M \cdot e_1 = g m_e \cdot (\Lambda_e e_1) = g m_e \cdot e_1 = \xi^2 e_1 \cdot \mathbb{I}_{\Lambda_e} e_3 = 0. \tag{2.38}$$

Hence

$$\mathbf{M} = m\ell(0, \sin\phi, \cos\phi) \tag{2.39}$$

and

$$\mathbf{m}_e = \Lambda_e \mathbf{M} = m\ell R_{(\theta-\phi)}\mathbf{e}_3 = (0, m\ell \sin(\phi-\theta), m\ell \cos(\phi-\theta)) \tag{2.40}$$

for some angle ϕ. The $\mathbf{e}_2$ component of (2.12) yields the equilibrium condition

$$mg\ell \sin(\phi-\theta) = \xi^2(I_2 - I_3)\cos\theta\sin\theta. \tag{2.41}$$

Since we assume that the top is not a sleeping top, the right hand side of (2.41) is non-zero. Thus, the equilibrium velocity ξ satisfies

$$\xi^2 = \frac{2mg\ell \sin(\theta-\phi)}{(I_3 - I_2)\sin 2\theta}. \tag{2.42}$$

The stability conditions for relative equilibria tilted about $\mathbf{e}_1$ are obtained by substituting (2.36) and (2.42) into (2.16). The equilibrium value of the locked inertia tensor is

$$\mathbb{I}(\Lambda_e) = \mathbf{e}_3 \cdot \mathbb{I}_{\Lambda_e}\mathbf{e}_3 = I_2 \sin^2\theta + I_3 \cos^2\theta = \tfrac{1}{2}(I_2 + I_3 + (I_3 - I_2)\cos 2\theta). \tag{2.43}$$

Hence, using (2.42), we see that

$$\begin{aligned}
\Gamma_e &= g\mathbf{m}_e \cdot \mathbf{e}_3 - \xi^2 \mathbb{I}(\Lambda_e) \\
&= \tfrac{1}{2}\xi^2\big((I_3 - I_2)\sin 2\theta \cot(\theta-\phi) - (I_2 + I_3 + (I_3 - I_2)\cos 2\theta)\big) \\
&= -\tfrac{1}{2}\xi^2\big(I_2 + I_3 - (I_3 - I_2)\rho(\theta,\phi)\big)
\end{aligned} \tag{2.44}$$

where

$$\rho(\theta,\phi) := \frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}. \tag{2.45}$$

The stability matrix is $\widetilde{\mathscr{B}}_e = \operatorname{diag}\big[b_1, b_2\big]$, where

$$b_1 := -\xi^2 \mathbf{e}_1 \cdot \mathbb{I}_{\Lambda_e}\mathbf{e}_1 - \Gamma_e = \tfrac{1}{2}\xi^2\big(I_2 + I_3 - 2I_1 - (I_3 - I_2)\rho(\theta,\phi)\big) \tag{2.46}$$

and

$$\begin{aligned}
b_2 &:= \frac{4\xi^2}{\mathbf{e}_3 \cdot \mathbb{I}_{\Lambda_e}\mathbf{e}_3}\big(\mathbf{e}_2 \cdot \mathbb{I}_{\Lambda_e}\mathbf{e}_3\big)^2 - \xi^2 \mathbf{e}_2 \cdot \mathbb{I}_{\Lambda_e}\mathbf{e}_2 - \Gamma_e \\
&= \tfrac{1}{2}\xi^2(I_3 - I_2)\left(\frac{4(I_3 - I_2)\sin^2 2\theta}{I_2 + I_3 + (I_3 - I_2)\cos 2\theta} + \cos 2\theta - \rho(\theta,\phi)\right).
\end{aligned} \tag{2.47}$$

The steady rotation of the top is stable if b_1 and b_2 are both positive. Hence, if we define

$$\tau_1 := \frac{I_2 + I_3 - 2I_1}{I_3 - I_2} \qquad \text{and} \qquad \tau_2 := \frac{4(I_3 - I_2)\sin^2 2\theta}{I_2 + I_3 + (I_3 - I_2)\cos 2\theta} + \cos 2\theta \tag{2.48}$$

then the steady rotation is stable if

$$I_3 > I_2 \qquad \text{and} \qquad \min\{\tau_1, \tau_2\} > \rho(\theta, \phi) \tag{2.49}$$

or if

$$I_2 > I_3 \qquad \text{and} \qquad \rho(\theta, \phi) > \max\{\tau_1, \tau_2\}. \tag{2.50}$$

If $\phi = 0$, i.e., if the reference centre of mass $\mathbf{M}$ lies on the principal axis of $\mathbb{I}_{\text{ref}}$ with eigenvalue I_3 (and hence the spatial centre of mass $\mathbf{m}_e$ lies along the principal axis of $\mathbb{I}_{\Lambda_e}$ with eigenvalue $\mathbb{I}(\Lambda_e)$), then the expressions appearing above are substantially simplified. The equation (2.42) for the equilibrium velocity then takes the form

$$\xi^2 = \frac{mg\ell}{(I_3 - I_2)\cos\theta}. \tag{2.51}$$

Hence, if $I_3 > I_2$, then tilted relative equilibria with centre of mass along a principal axis exist only for $\pi/2 > \theta > -\pi/2$, i.e. the centre of mass is above the fixed point of the top. On the other hand, if $I_2 > I_3$, then the centre of mass lies below the fixed point, with $3\pi/2 > \theta > \pi/2$. The diagonal entries of the stability matrix $\widetilde{\mathscr{B}}_e$ are given by $b_1 = \xi^2(I_2 - I_1)$ and

$$b_2 = \frac{\xi^2(I_3 - I_2)}{I_2 + I_3 \cot^2\theta}\left[3(I_3 - I_2)\cos^2\theta - I_2\right]. \tag{2.52}$$

The condition $b_1 > 0$ yields $I_2 > I_1$. To impose the condition $b_2 > 0$, we distinguish two cases. If $I_3 > I_2 > I_1$, then $b_2 > 0$ implies that $3(I_3 - I_2)\cos^2\theta > I_2$, which implies that $I_2 > 3I_3/4$ and

$$\cos^2\theta > \frac{I_2}{3(I_3 - I_2)}. \tag{2.53}$$

If $I_2 > I_1$ and $I_2 > I_3$, then both b_1 and b_2 are positive for any angle θ, i.e. this steady motion of the top is stable.

Remark: *The Kovalevskaya top:* A special class of Kovalevskaya tops is determined by the conditions $I_1 = I_3 = 2I_2$ and $\phi = 0$. If $\theta = 0$, then $\widetilde{\mathscr{B}}_e = \text{diag}\left[-mg\ell, \xi^2 I_2 - mg\ell\right]$; hence the top is formally unstable. If $\theta = \pi$, then $\widetilde{\mathscr{B}}_e = \text{diag}\left[mg\ell, \xi^2 I_2 + mg\ell\right]$ is positive definite; hence the hanging Kovalevskaya top is stable for all velocities. Finally, if $\cos^2\theta \neq 1$, then $b_1 = \xi^2(I_2 - I_1) = -\xi^2 I_2 < 0$ implies that the motion is formally unstable. These results are given in Levi-Civita (1901). We shall show in section 4 that for θ satisfying $0 < \cos^2\theta < 1/3$, these relative equilibria are *linearly stable*, even though they fail to be formally stable.

We summarize our results as follows:

Consider an asymmetric top, whose inertia tensor has eigenvalues I_1, I_2, and I_3, and an eigenvector $\mathbf{e}_1$ (with eigenvalue I_1) of $\mathbb{I}_{\Lambda_e}$ satisfying $\mathbf{e}_1 \cdot \mathbf{g} = \mathbf{e}_1 \cdot \mathbf{m}_e = 0$. Let θ denote the angle in the plane orthogonal to $\mathbf{e}_1$ between the axis of gravity $\mathbf{g}$ and the eigenvector of

$\mathbb{I}_{\mathrm{A}_e}$ *with eigenvalue* I_3 *and let* ϕ *denote the angle in the same plane between* $\mathbf{m}_e$ *and this eigenvector, i.e.* $\mathbf{m}_e \cdot \mathbf{g} = mg\ell \cos(\phi - \theta)$. *The top can be maintained in steady rotation about the axis of gravity* $\mathbf{e}_3$ *with angular velocity* ξ *if either*

$$\theta = 0, \pi \qquad or \qquad \xi^2 = \frac{2mg\ell \sin(\theta - \phi)}{(I_3 - I_2) \sin 2\theta}.$$

The nonlinear orbital stability conditions for steady motion of the asymmetric top are

(i) $\theta = \phi = 0$. *Upright sleeping tops are nonlinearly orbitally stable if* I_3 *is the maximal eigenvalue of* $\mathbb{I}_{\mathrm{ref}}$ *and*

$$\xi^2 > \frac{mg\ell}{I_3 - I_m}$$

where $I_m := \max \{I_1, I_2\}$.

(ii) $\theta = \pi$ *and* $\phi = 0$. *Hanging sleeping tops are stable if either* I_3 *is the maximal axis of inertia, or*

$$\xi^2 < \frac{mg\ell}{I_m - I_3}.$$

(iii) $\theta \neq 0, \pi$ *and* $\phi = 0$. *Tilted tops for which the centre of mass lies on the principal axis of inertia with eigenvalue* I_3 *are stable if either* I_2 *is the maximal axis of inertia, or*

$$I_3 > I_2 > I_1 \qquad and \qquad 1 > \cos^2 \theta > \frac{I_2}{3(I_3 - I_2)}.$$

(iv) $\phi \neq 0$. *Tilted tops for which the position vector of the centre of mass is orthogonal to the principal axis* $\mathbf{e}_1$ *are stable if*

$$I_3 > I_2 \qquad and \qquad \min \{\tau_1, \tau_2\} > \rho(\theta, \phi)$$

or if

$$I_2 > I_3 \qquad and \qquad \rho(\theta, \phi) > \max \{\tau_1, \tau_2\}$$

where

$$\rho(\theta, \phi) := \frac{\sin(\theta + \phi)}{\sin(\theta - \phi)} \qquad \tau_1 := \frac{I_2 + I_3 - 2I_1}{I_3 - I_2}$$

and

$$\tau_2 := \frac{4(I_3 - I_2) \sin^2 2\theta}{I_2 + I_3 + \cos 2\theta} + \cos 2\theta.$$

2.4. Stability of general relative equilibria

An arbitrary relative equilibrium can be viewed as a point on a curve of relative equilibria parametrized by angular velocity. We shall characterize below all possible sequences of stability transitions along such curves. We make use of equations (2.27) and (2.26), which can be expressed in the form

$$\operatorname{tr} \widetilde{\mathscr{B}}_e = \frac{4(gm\ell \sin \theta)^2}{\xi^2 \mathbb{I}(\Lambda_e)} - \xi^2 (\operatorname{tr} \mathbb{I}_{\Lambda_e} - 3\mathbb{I}(\Lambda_e)) - 2gm\ell \cos \theta$$

$$= -\frac{(gm\ell)^2}{\kappa} \left(\kappa^2 \sigma_1 + 2\kappa \cos \theta - 4\sin^2 \theta \right) \tag{2.54}$$

and

$$\det \widetilde{\mathscr{B}}_e = \frac{\xi^4}{\mathbb{I}(\Lambda_e)} \left(4\det \mathbb{I}_{\Lambda_e} - 3\mathbb{I}(\Lambda_e) \operatorname{si} \mathbb{I}_{\Lambda_e} + 2\mathbb{I}(\Lambda_e)^2 \operatorname{tr} \mathbb{I}_{\Lambda_e} - \mathbb{I}(\Lambda_e)^3 \right)$$

$$+ \xi^2 gm\ell \left(\operatorname{tr} \mathbb{I}_{\Lambda_e} - 3\mathbb{I}(\Lambda_e) \right) \cos \theta + (gm\ell \sin \theta)^2 \left(1 - 4\frac{gm\ell \cos \theta}{\xi^2 \mathbb{I}(\Lambda_e)} \right)$$

$$= \frac{(gm\ell)^2}{\kappa} \left(\kappa^3 (2\sigma_1 - 3\sigma_2 + 4\sigma_3) + \kappa^2 \sigma_1 \cos \theta + \kappa - 4\sin^2 \theta \, \cos \theta \right) \tag{2.55}$$

where $\operatorname{si} A := \frac{1}{2}\left((\operatorname{tr} A)^2 - \operatorname{tr} A^2 \right)$ for all three by three matrices M and

(i) the σ_i's are the scaled invariants of the inertia tensor

$$\sigma_1 := \frac{\operatorname{tr} \mathbb{I}_{\text{ref}}}{\mathbb{I}(\Lambda_e)} - 3 \quad = \frac{I_1 + I_2 + I_3}{\mathbb{I}(\Lambda_e)} - 3$$

$$\sigma_2 := \frac{\operatorname{si} \mathbb{I}_{\text{ref}}}{\mathbb{I}(\Lambda_e)^3} - 3 = \frac{I_1 I_2 + I_1 I_3 + I_2 I_3}{\mathbb{I}(\Lambda_e)^2} - 3 \tag{2.56}$$

$$\sigma_3 := \frac{\det \mathbb{I}_{\text{ref}}}{\mathbb{I}(\Lambda_e)^3} - 1 = \frac{I_1 I_2 I_3}{\mathbb{I}(\Lambda_e)^3} - 1$$

where the I_i's are the eigenvalues of $\mathbb{I}_{\text{ref}}$ (and hence of $\mathbb{I}_\Lambda$ for all Λ)

(ii) $\kappa := \frac{\xi^2 \mathbb{I}(\Lambda_e)}{mg\ell} = \frac{\xi^2 \gamma_e \cdot \mathbb{I}_{\text{ref}} \gamma_e}{mg\ell}$ is the scaled kinetic energy

(iii) θ is the angle between $\mathbf{m}_e$ and $\mathbf{e}_3$, i.e. $\mathbf{m}_e \cdot \mathbf{e}_3 = m\ell \cos \theta = \gamma_e \cdot \mathbf{M}$.

Expressions (2.55) and (2.54) are obtained by means of the identities

$$\operatorname{tr} S|_{\mathbf{v}^\perp} = \operatorname{tr} S - (\mathbf{v} \cdot S\mathbf{v}) \tag{2.57}$$

$$\det S|_{\mathbf{v}^\perp} = \operatorname{si} S - (\mathbf{v} \cdot S\mathbf{v}) \operatorname{tr} S + |S\mathbf{v}|^2 \tag{2.58}$$

and

$$(\mathbf{v} \times S\mathbf{v}) \cdot S(\mathbf{v} \times S\mathbf{v}) = (\mathbf{v} \cdot S\mathbf{v})\det S|_{\mathbf{v}^\perp} - \det S$$

$$= (\mathbf{v} \cdot S\mathbf{v})\left(\operatorname{si} S - (\mathbf{v} \cdot S\mathbf{v}) \operatorname{tr} S + |S\mathbf{v}|^2 \right) - \det S \tag{2.59}$$

for any symmetric matrix S and unit vector $\mathbf{v}$. Applying (2.58) and (2.57) to $\mathbb{II}_{\Lambda_e}|_{\mathbf{e}_1-\mathbf{e}_2}$ and making use of the equilibrium identity (2.30), we see that

$$
\begin{aligned}
\det \mathscr{C}_e &= \xi^4 \det\left(\mathbb{II}_{\Lambda_e}|_{\mathbf{e}_1-\mathbf{e}_2}\right) + \xi^2 \Gamma_e \operatorname{tr}\left(\mathbb{II}_{\Lambda_e}|_{\mathbf{e}_1-\mathbf{e}_2}\right) + \Gamma_e^2 \\
&= \xi^4\left(\operatorname{si}\mathbb{II}_{\Lambda_e} - \mathbb{I}(\Lambda_e)\operatorname{tr}\mathbb{II}_{\Lambda_e} + \mathbb{I}(\Lambda_e)^2\right) + (gm\ell\sin\theta)^2 \\
&\quad + \xi^2\Gamma_e\left(\operatorname{tr}\mathbb{II}_{\Lambda_e} - \mathbb{I}(\Lambda_e)\right) + \Gamma_e^2
\end{aligned}
\tag{2.60}
$$

$$
\begin{aligned}
&= \xi^4\left(\operatorname{si}\mathbb{II}_{\Lambda_e} - 2\mathbb{I}(\Lambda_e)\operatorname{tr}\mathbb{II}_{\Lambda_e} + 3\mathbb{I}(\Lambda_e)^2\right) \\
&\quad + \xi^2 g\mathbf{e}_3\cdot\mathbf{m}_e\left(\operatorname{tr}\mathbb{II}_{\Lambda_e} + \mathbb{I}(\Lambda_e)\right) + (gm\ell)^2.
\end{aligned}
\tag{2.61}
$$

The identity (2.59) implies that

$$
\begin{aligned}
(\mathbf{e}_3 \times \mathbb{II}_{\Lambda_e}\mathbf{e}_3)&\cdot\mathscr{C}_e(\mathbf{e}_3 \times \mathbb{II}_{\Lambda_e}\mathbf{e}_3) \\
&= \xi^2\left(\mathbb{I}(\Lambda_e)\operatorname{si}\mathbb{II}_{\Lambda_e} - \mathbb{I}(\Lambda_e)^2\operatorname{tr}\mathbb{II}_{\Lambda_e} + \mathbb{I}(\Lambda_e)|\mathbb{II}_{\Lambda_e}\mathbf{e}_3|^2 - \det\mathbb{II}_{\Lambda_e}\right) \\
&\quad + \left(gm_e\cdot\mathbf{e}_3 - \xi^2\mathbb{I}(\Lambda_e)\right)|\mathbb{II}_{\Lambda_e}\mathbf{e}_3|^2
\end{aligned}
\tag{2.62}
$$

$$
\begin{aligned}
&= \xi^2\left(\mathbb{I}(\Lambda_e)\operatorname{si}\mathbb{II}_{\Lambda_e} - \mathbb{I}(\Lambda_e)^2\operatorname{tr}\mathbb{II}_{\Lambda_e} - \det\mathbb{II}_{\Lambda_e} + \mathbb{I}(\Lambda_e)^3\right) \\
&\quad + \frac{(gm\ell)^3\cos\theta\sin^2\theta}{\xi^4}.
\end{aligned}
$$

Equation (2.55) follows directly from (2.60) and (2.62).

We now use the expressions (2.54) and (2.55) to obtain a rough general classification of the possible behaviours of the system as the scaled kinetic energy κ is increased. The classifications are determined by applying the Routh–Hurwitz criteria to the polynomials

$$
P_t(\kappa) := -\sigma_1\kappa^2 - 2\cos\theta\,\kappa + 4\sin^2\theta
\tag{2.63}
$$

and

$$
P_d(\kappa) := \beta\kappa^3 + \sigma_1\cos\theta\,\kappa^2 + \kappa - 4\sin^2\theta\cos\theta
\tag{2.64}
$$

where $\beta := 2\sigma_1 - 3\sigma_2 + 4\sigma_3$.

The Routh–Hurwitz criterion states that the number of roots of P_t and P_d in the right half-plane are given by the number of sign changes in the sequences

$$
\sigma_1,\ \cos\theta,\ -1 \qquad \text{and} \qquad \beta,\ \sigma_1\cos\theta,\ \frac{\sigma_1 + 4\beta\sin^2\theta}{\sigma_1},\ -\cos\theta
\tag{2.65}
$$

respectively. (For a discussion of the Routh–Hurwitz criterion, see, for example, Gantmacher (1989).)

For $\beta\cos\theta > 0$, the Routh–Hurwitz criterion implies that P_d has either one or three positive real roots. For $\beta\cos\theta < 0$, P_d has either zero or two positive real roots. P_d is positive (negative) for sufficiently large κ if β is positive (negative). For sufficiently small κ, P_d is positive (negative) for $\cos\theta$ negative (positive). We next note that the condition that the determinant have a local extremum, namely, that

$$
3\beta\kappa^2 + 2\sigma_1\kappa\cos\theta + 1 = 0
\tag{2.66}
$$

Table 1. Possible stability transitions for relative equilibria of general tops.

	$\cos\theta > 0$	$\cos\theta < 0$
$\beta < 0$	I, IPI or INI	PI
$\beta > 0$ and $\sigma_1 < 0$	IP, IPIP or INIP	P or PIP
$\beta > 0$ and $\sigma_1 > 0$	IN	PIN

cannot be satisfied by tops for which β and $\sigma_1 \cos\theta$ are both positive, or

$$\beta > \tfrac{1}{3}\sigma_1^2 \cos^2\theta. \tag{2.67}$$

For such tops, the determinant can change sign at most once.

Ignoring the special cases associated to simultaneous roots of P_t and P_d, or their derivatives, we summarize the possible stability transitions in table 1. The entries of the table are interpreted as follows: P (respectively N) indicates positive (respectively negative) definiteness of $\widetilde{\mathscr{B}}_e$, while I indicates indefiniteness of $\widetilde{\mathscr{B}}_e$. The transition sequences read from left to right with increasing κ.

(i) Negative definiteness of the second variation is possible only if either

$$\sigma_1 > 0 \qquad \text{or} \qquad \cos^2\theta > \frac{4\sigma_1}{4\sigma_1 - 1} > 0. \tag{2.68}$$

(ii) Multiple changes of signature are possible only if

$$\cos^2\theta > \frac{3\beta}{\sigma_1^2} \quad \text{and either} \quad \sigma_1 > 0 \quad \text{or} \quad \cos^2\theta > \frac{4\beta + \sigma_1}{4\beta}. \tag{2.69}$$

For example, for $\beta > \sigma_1^2/3$, $\sigma_1 < 0$, and $\cos\theta > 0$, the only possible sequence is IP. The second inequality in equation (2.68) is obtained by requiring that the roots of P_t be real. Condition (2.69) implies that the second sequence in (2.65) has three sign changes and the polynomial (2.66) has real roots, i.e., that the polynomial P_d has two positive critical points.

Some fairly detailed information regarding the linear stability of the tops can be obtained from table 1, making use of the following observations. Positive definiteness of $\widetilde{\mathscr{B}}_e$ implies nonlinear, and hence linear, stability of the relative equilibrium. The characteristic polynomial (4.18) for the linearized dynamics, which is derived in section 4, has both real and imaginary roots if det $\widetilde{\mathscr{B}}_e$ is negative; hence indefiniteness of $\widetilde{\mathscr{B}}_e$ implies linear instability. Negative definiteness of $\widetilde{\mathscr{B}}_e$ can be associated with either stability or instability. Some general conclusions regarding linear stability can be summarized as follows:

(i) $\cos\theta > 0$ implies that the top is unstable at sufficiently low velocities
(ii) $\cos\theta < 0$ implies that the top is stable at sufficiently low velocities
(iii) $\beta < 0$ implies that the top is unstable at sufficiently high velocities
(iv) $\beta > 0$ and $\sigma_1 < 0$ implies that the top is stable at sufficiently high velocities.

Remark: Many of the cases determined by the Routh–Hurwitz criterion can be ruled out as possible sequences of behaviour by taking into account the fact that the trace cannot change sign in a region where the determinant is positive; hence the transitions

PN and NP cannot occur unless the trace and determinant are simultaneously equal to zero. We let the letter Z denote the state $\widetilde{\mathscr{B}}_e = 0$. A further condition on the existence of the transitions PN and NP is that the relevant root of P_d also be a critical point of P_d. Degenerate cases, in which, for example, a critical point of P_d is simultaneously a root of P_d, may result in the following sequences: IZI, for $\beta < 0$ and $\cos\theta > 0$; IZIP, IPZP, or INZP, for $\beta > 0$, $\sigma_1 < 0$, and $\cos\theta > 0$; and PZN, for $\beta < 0$, $\sigma_1 > 0$, and $\cos\theta < 0$.

3. Lagrange tops

We now consider a Lagrange top, i.e. a top satisfying

$$R\mathbb{I}_{\text{ref}}R^T = \mathbb{I}_{\text{ref}} \tag{3.1}$$

for all rotations R about the axis $\mathbf{M}$. Set $R = \exp(\epsilon\widehat{\mathbf{M}})$, where exp denotes the exponential map $\exp : so(3) \rightarrow SO(3)$ given explicitly by Rodrigues' formula

$$\exp\widehat{\omega} = 1_3 + \frac{\sin(|\omega|)}{|\omega|}\widehat{\omega} + 2\frac{\sin^2(|\omega|/2)}{|\omega|^2}\widehat{\omega}^2. \tag{3.2}$$

Differentiating with respect to ϵ, we see that (3.1) holds if and only if

$$\text{sym}\left[\mathbb{I}_{\text{ref}}\widehat{\mathbf{M}}\right] = \mathbf{0}. \tag{3.3}$$

The symmetry condition (3.1) also implies that $\mathbf{M}$ *is an eigenvector of* $\mathbb{I}_{\text{ref}}$. Indeed, since $\mathbf{M}$ is the axis of rotation of R, we have $R\mathbf{M} = \mathbf{M}$; thus (3.1) implies that $R\mathbb{I}_{\text{ref}}\mathbf{M} = \mathbb{I}_{\text{ref}}\mathbf{M}$, i.e. $\mathbb{I}_{\text{ref}}\mathbf{M}$ is also the axis of the rotation R. Hence $\mathbb{I}_{\text{ref}}\mathbf{M} = \lambda_M\mathbf{M}$ for some $\lambda_M \in \mathbb{R}$. Moreover, *the remaining two eigenvalues of* $\mathbb{I}_{\text{ref}}$ *are necessarily equal*, which can be shown as follows. If $\mathbb{I}_{\text{ref}}$ has three equal eigenvalues, then there is nothing to prove. If not, let $\lambda \neq \lambda_M$ be an eigenvalue of $\mathbb{I}_{\text{ref}}$ with eigenvector $\mathbf{x}$. The vector $\mathbf{x}$ is not parallel to $\mathbf{M}$; hence $\mathbf{x}$ and $R\mathbf{x}$ are linearly independent. The symmetry condition (3.1) implies that $\lambda R\mathbf{x} = R\mathbb{I}_{\text{ref}}\mathbf{x} = \mathbb{I}_{\text{ref}}R\mathbf{x}$; hence $R\mathbf{x}$ is a *distinct eigenvector* with the same eigenvalue λ. The reference symmetry condition (3.3) leads to the spatial symmetry condition $\text{sym}\left[\mathbb{I}_\Lambda\widehat{\mathbf{m}}\right] = \mathbf{0}$. An analogous argument to that used for $\mathbf{M}$ and $\mathbb{I}_{\text{ref}}$ shows that $\mathbf{m}$ is an eigenvector of the current spatial inertia tensor $\mathbb{I}_\Lambda$.

The Hamiltonian (1.27) associated to such a top is invariant not only under *left*, i.e. spatial, rotations about the axis $\mathbf{e}_3$, but also under the *right* S^1 action of body rotation through the angle θ about the axis $\mathbf{M}$, given by

$$\theta \cdot \Lambda := \Lambda(\exp(\theta\widehat{\Sigma}))^T \tag{3.4}$$

where $\Sigma := \frac{1}{|\mathbf{M}|}\mathbf{M}$. Note that the induced action on the spatial representation (Λ, π) is $\theta \cdot (\Lambda, \pi) = (\theta \cdot \Lambda, \pi)$ and the invariance condition for the Hamiltonian reads $H(\theta \cdot \Lambda, \pi) = H(\Lambda, \pi)$ for any $\theta \in S^1$. In this case, the full symmetry group of the Hamiltonian is $G = S^1 \times S^1$, with Lie algebra $g = \mathbb{R}^2$ and infinitesimal generator

$$(\xi, \omega)_Q(\Lambda) = \xi\widehat{\mathbf{e}}_3\Lambda - \omega\Lambda\widehat{\Sigma}. \tag{3.5}$$

The locked inertia tensor $\mathbb{I}(\Lambda) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the relationship

$$
\begin{aligned}
(\eta, \zeta) \cdot \mathbb{I}(\Lambda)(\xi, \omega) &= \left\langle FL((\xi, \omega)_Q(\Lambda)), (\eta, \zeta)_Q(\Lambda) \right\rangle \\
&= \left\langle \Lambda(\xi \widehat{\Lambda^T \mathbf{e}_3} - \omega \widehat{\Sigma}), \Lambda(\eta \widehat{\Lambda^T \mathbf{e}_3} - \zeta \widehat{\Sigma}) \right\rangle_{\mathrm{I}} \\
&= (\xi \Lambda^T \mathbf{e}_3 - \omega \Sigma) \cdot \mathbb{I}_{\mathrm{ref}}(\eta \Lambda^T \mathbf{e}_3 - \zeta \Sigma)
\end{aligned}
\tag{3.6}
$$

using the left SO(3) invariance of the inner product $\langle \, , \, \rangle_{\mathrm{I}}$, the formula $\Lambda \hat{\mathbf{v}} \Lambda^T = \widehat{\Lambda \mathbf{v}}$, (1.14), and (1.19). Hence

$$
\mathbb{I}(\Lambda) = \begin{pmatrix} \mathbf{e}_3 \cdot \Lambda \mathbb{I}_{\mathrm{ref}} \Lambda^T \mathbf{e}_3 & -\mathbf{e}_3 \cdot \Lambda \mathbb{I}_{\mathrm{ref}} \Sigma \\ -\mathbf{e}_3 \cdot \Lambda \mathbb{I}_{\mathrm{ref}} \Sigma & \Sigma \cdot \mathbb{I}_{\mathrm{ref}} \Sigma \end{pmatrix} = \begin{pmatrix} \mathbf{e}_3 \cdot \mathbb{I}_\Lambda \mathbf{e}_3 & -\mathbf{e}_3 \cdot \mathbb{I}_\Lambda \mathbf{s} \\ -\mathbf{e}_3 \cdot \mathbb{I}_\Lambda \mathbf{s} & \mathbf{s} \cdot \mathbb{I}_\Lambda \mathbf{s} \end{pmatrix}
\tag{3.7}
$$

where $\mathbf{s} := \Lambda \Sigma$.

We are now ready to determine the conditions of relative equilibrium for the Lagrange top. The augmented potential takes the form

$$
V_{(\xi, \omega)}(\Lambda) = \Lambda \mathbf{M} \cdot \mathbf{g} - \tfrac{1}{2}(\xi, \omega) \cdot \mathbb{I}(\Lambda)(\xi, \omega)
\tag{3.8}
$$

with first variation

$$
DV_{(\xi, \omega)}(\Lambda_e) \cdot \widehat{\delta\theta} \Lambda_e = (\mathbf{e}_3 \times \delta\theta) \cdot \left(g\mathbf{m}_e - \xi \mathbb{I}_{\Lambda_e}(\xi \mathbf{e}_3 - \omega \mathbf{s}_e) \right)
\tag{3.9}
$$

where $\mathbf{s}_e := \Lambda_e \Sigma$. Hence the configuration Λ_e is a relative equilibrium if and only if

$$
g\mathbf{m}_e - \xi \mathbb{I}_{\Lambda_e}(\xi \mathbf{e}_3 - \omega \mathbf{s}_e) = \Gamma_e \mathbf{e}_3,
\tag{3.10}
$$

where

$$
\begin{aligned}
\Gamma_e &:= \left(g\mathbf{m}_e - \xi \mathbb{I}_{\Lambda_e}(\xi \mathbf{e}_3 - \omega \mathbf{s}_e) \right) \cdot \mathbf{e}_3 \\
&= mg\ell \mathbf{s}_e \cdot \mathbf{e}_3 - (\xi, 0) \cdot \mathbb{I}(\Lambda_e)(\xi, \omega).
\end{aligned}
\tag{3.11}
$$

While the equilibrium conditions (3.10) can be expressed as two scalar equations (namely, that the $\mathbf{e}_1$ and $\mathbf{e}_2$ components of the left hand side equal zero), we shall see that, due to the symmetry of the Lagrange top, relative equilibria are determined by a single non-trivial scalar equation.

We now discuss two distinct classes of relative equilibria for Lagrange tops: precessing tops and sleeping tops. The motion of the spatial axis of symmetry of a precessing top traces out a cone about the axis of gravity; the spatial axis of symmetry $\mathbf{s}_e$ of a sleeping Lagrange top coincides with the axis of gravity $\mathbf{e}_3$. It is convenient to distinguish precessing and sleeping tops when considering the equilibrium conditions; it is *essential* to distinguish them when performing the stability analysis. The crucial distinction is the additional symmetry of the sleeping top: spatial rotations about the axis of gravity are indistinguishable from body rotations about the axis of symmetry. For example, a spatial rotation combined with a reverse body rotation through the same angle fixes the sleeping top. A stability analysis that neglects this additional symmetry may yield overly restrictive stability conditions. We first treat the case of a precessing top, using the standard stability algorithm, and then consider the sleeping Lagrange top.

3.1. *The precessing Lagrange top*

If the axis of gravity and the axis of spatial symmetry are distinct, i.e., $\mathbf{s}_e \neq \pm \mathbf{e}_3$, then the group G acts freely in a neighbourhood of Λ_e. The two-dimensional tangent space $g \cdot \Lambda_e$ to the group orbit is given by (3.5);

$$g \cdot \Lambda_e = \mathrm{span}\{\widehat{\mathbf{e}}_3 \Lambda_e, \widehat{\mathbf{s}}_e \Lambda_e\}. \tag{3.12}$$

Since the group G is abelian, the augmented and amended potentials are G invariant and $g \cdot \Lambda_e$ lies in the kernel of the first and second variations of the modified potentials. Thus the spatial representation of the space of non-trivial variations is spanned by $\delta\Lambda_v := \widehat{\delta\theta}_v \Lambda_e$, where $\delta\theta_v := (\mathbf{e}_3 \times \mathbf{s}_e)/|\mathbf{e}_3 \times \mathbf{s}_e|$, and Λ_e is a relative equilibrium if and only if the single scalar equation

$$DV_{(\xi,\omega)}(\Lambda_e) \cdot \delta\Lambda_v = 0 \tag{3.13}$$

is satisfied.

We now show that any steadily precessing Lagrange top is orbitally nonlinearly stable. As we did for the tilted asymmetric relative equilibria, we choose a coordinate system that simplifies the necessary calculations. Set

$$\mathbf{e}_1 := \delta\theta_v = \frac{\mathbf{e}_3 \times \mathbf{s}_e}{|\mathbf{e}_3 \times \mathbf{s}_e|} \qquad \text{and} \qquad \mathbf{e}_2 := \mathbf{e}_3 \times \mathbf{e}_1. \tag{3.14}$$

Thus $\mathbf{s}_e = R_\theta \mathbf{e}_3$ for some $\theta \neq 0, \pi$, where R_θ is given by (2.37). The symmetry assumptions imply that the reference inertia matrix $\mathbb{I}_{\mathrm{ref}}$ has a double eigenvalue, say I_1, and a distinct eigenvalue, say I_3, and that $\mathrm{sym}[\mathbb{I}_{\Lambda_e}\widehat{\mathbf{s}}_e] = \mathbf{0}$. Hence we set $\mathbb{I}_{\mathrm{ref}} = \mathrm{diag}[I_1, I_1, I_3]$ and $\Lambda_e = R_\theta$.

Equation (3.7) implies that the equilibrium locked inertia tensor $\mathbb{I}(\Lambda_e)$ is given by

$$\mathbb{I}(\Lambda_e) = \begin{pmatrix} I_1 \sin^2\theta + I_3 \cos^2\theta & -I_3 \cos\theta \\ -I_3 \cos\theta & I_3 \end{pmatrix}. \tag{3.15}$$

The non-trivial equilibrium condition is given by

$$DV_{(\xi,\omega)}(\Lambda_e) \cdot \delta\Lambda_v = \sin\theta\big(\xi^2(I_3 - I_1)\cos\theta - \xi\omega I_3 - mg\ell\big) = 0 \tag{3.16}$$

which is obtained by substituting $\mathbf{m}_e = m\ell \mathbf{s}_e = m\ell R_\theta \mathbf{e}_3$ into (3.9). Neglecting for the moment the solutions $\theta = 0, \pi$ corresponding to the sleeping Lagrange top, the equilibrium condition (3.16) may be solved for ω to yield

$$\omega = \frac{\xi^2(I_3 - I_1)\cos\theta - mg\ell}{\xi I_3}. \tag{3.17}$$

Hence, $FL((\xi,\omega)_Q(\Lambda_e))$ is a precessing relative equilibrium if and only if ω satisfies (3.17). The equilibrium condition (3.16) and the following stability result can be found in Routh (1884, section 207), with the change of variables $\mu = \xi$ and $n = \xi\cos\theta - \omega$.

The relative equilibrium with configuration Λ_e is orbitally stable if

$$\begin{aligned} \mathscr{B}_e(\delta\Lambda_v, \delta\Lambda_v) &= D^2 V_{(\xi,\omega)}(\Lambda_e)(\delta\Lambda_v, \delta\Lambda_v) \\ &\quad + (D\mathbb{I}_{(\xi,\omega)}(\Lambda_e) \cdot \delta\Lambda_v) \cdot \mathbb{I}(\Lambda_e)^{-1}(D\mathbb{I}_{(\xi,\omega)}(\Lambda_e) \cdot \delta\Lambda_v) > 0 \end{aligned} \tag{3.18}$$

where $\mathbb{I}_{(\xi,\omega)}(\Lambda) := \mathbb{I}(\Lambda)(\xi,\omega)$ is the locked momentum map for the G action. The second variation of the augmented potential (3.8) takes the form

$$D^2 V_{(\xi,\omega)}(\Lambda_e)(\widehat{\delta\theta}\Lambda_e, \widehat{\Delta\theta}\Lambda_e) = -(\mathbf{e}_3 \times \delta\theta) \cdot \left(\xi^2 \mathbb{I}_{\Lambda_e} + \Gamma_e \mathbf{1}_3\right)(\mathbf{e}_3 \times \Delta\theta) \tag{3.19}$$

where $\Gamma_e = -\xi^2 I_1$ is determined using (3.11) and (3.17). Hence (3.19) implies that the second variation of the augmented potential in the direction of the non-trivial mode $\delta\Lambda_v = \widehat{\mathbf{e}}_1 \Lambda_e$ is given by

$$D^2 V_{(\xi,\omega)}(\Lambda_e)(\delta\Lambda_v, \delta\Lambda_v)) = -\mathbf{e}_2 \cdot \left(\xi^2 \mathbb{I}_{\Lambda_e} + \Gamma_e \mathbf{1}_3\right)\mathbf{e}_2 = \xi^2(I_1 - I_3)\sin^2\theta. \tag{3.20}$$

The locked momentum map $\mathbb{I}_{(\xi,\omega)}$ has first variation

$$D\mathbb{I}_{(\xi,\omega)}(\Lambda_e) \cdot \widehat{\delta\theta}\Lambda_e = \begin{pmatrix} 2\xi(\mathbf{e}_3 \times \delta\theta) \cdot \mathbb{I}_{\Lambda_e}\mathbf{e}_3 - \omega(\mathbf{e}_3 \times \delta\theta) \cdot \mathbb{I}_{\Lambda_e}\mathbf{s}_e \\ -\xi(\mathbf{e}_3 \times \delta\theta) \cdot \mathbb{I}_{\Lambda_e}\mathbf{s}_e \end{pmatrix} \tag{3.21}$$

In particular, for ω given by (3.17),

$$D\mathbb{I}_{(\xi,\omega)}(\Lambda_e) \cdot \delta\Lambda_v = \xi\sin\theta \begin{pmatrix} (I_1 - I_3)\cos\theta - \frac{mg\ell}{\xi^2} \\ I_3 \end{pmatrix} \tag{3.22}$$

and hence

$$(D\mathbb{I}_{(\xi,\omega)}(\Lambda_e) \cdot \delta\Lambda_v) \cdot \mathbb{I}(\Lambda_e)^{-1}(D\mathbb{I}_{(\xi,\omega)}(\Lambda_e) \cdot \delta\Lambda_v)$$
$$= \xi^2(I_1\cos^2\theta + I_3\sin^2\theta) - 2mg\ell\cos\theta + \frac{(mg\ell)^2}{\xi^2 I_1}. \tag{3.23}$$

Thus, using (3.18), we obtain

$$\mathscr{B}_e(\delta\Lambda_v, \delta\Lambda_v) = \xi^2 I_1 - 2mg\ell\cos\theta + \frac{(mg\ell)^2}{\xi^2 I_1}$$
$$= \frac{1}{\xi^2 I_1}\left(\left(\xi^2 I_1 - mg\ell\cos\theta\right)^2 + \left(mg\ell\sin\theta\right)^2\right) > 0 \tag{3.24}$$

for all $\theta \neq 0, \pi$. It follows that *all tilted Lagrange tops in steady precession are nonlinearly stable modulo rotations about the axis of gravity and the axis of symmetry.*

3.2. The sleeping Lagrange top

We consider a steady motion in which the axis of symmetry, the centre of mass vector, and the axis of gravity all coincide. The spatial symmetry condition $\text{sym}\left[\mathbb{I}_{\Lambda_e}\widehat{\mathbf{s}}_e\right] = \mathbf{0}$ implies that $\mathbf{e}_3$ is an eigenvector of $\mathbb{I}_{\Lambda_e}$. Hence we can assume without loss of generality that $\mathbb{I}_{\Lambda_e}$ is diagonal; the symmetry assumption then implies that $\mathbb{I}_{\Lambda_e} = \text{diag}\left[I_1, I_1, I_3\right]$ for some constants $I_1, I_3 \in \mathbb{R}$.

As was previously discussed, the group $G = S^1 \times S^1$ fails to act freely when the axis of gravity and the spatial axis of symmetry coincide. In this case, $\mathbf{s}_e = \pm\mathbf{e}_3$ and hence $\mathbf{m}_e = \pm m\ell\mathbf{e}_3$. We shall analyse in detail the case $\mathbf{s}_e = \mathbf{e}_3$ and simply present the results for the case $\mathbf{s}_e = -\mathbf{e}_3$, which are derived analogously, at the end of the section. If $\mathbf{s}_e = \mathbf{e}_3$, then

$$(\xi, \xi)_Q(\Lambda_e) = \xi(\widehat{\mathbf{e}}_3 - \widehat{\mathbf{s}}_e)\Lambda_e = \mathbf{0}; \tag{3.25}$$

hence the tangent space $g \cdot \Lambda_e$ to the group orbit is only one-dimensional. This has the crucial consequence that the inertia tensor is no longer invertible;

$$
\begin{aligned}
\mathbb{I}(\Lambda_e) &= \begin{pmatrix} \mathbf{e}_3 \cdot \mathbb{I}_{\Lambda_e} \mathbf{e}_3 & -\mathbf{e}_3 \cdot \mathbb{I}_{\Lambda_e} \mathbf{s}_e \\ -\mathbf{e}_3 \cdot \Lambda_e \mathbb{I}_{\Lambda_e} \mathbf{s}_e & \mathbf{s}_e \cdot \mathbb{I}_{\Lambda_e} \mathbf{s}_e \end{pmatrix} \\
&= (\mathbf{e}_3 \cdot \mathbb{I}_{\Lambda_e} \mathbf{e}_3) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\end{aligned}
\tag{3.26}
$$

has a non-trivial kernel equal to $\mathrm{span}\{(1,1)\}$. Hence we no longer have a one to one correspondence between angular velocities and angular momenta. Given $\lambda \in \mathbb{R}$, define $\mathbf{v}_\lambda := (\lambda, \lambda - 1)$; it follows from (3.25) that the motion of the sleeping Lagrange top determined by the infinitesimal generator $\xi \mathbf{v}_\lambda$ is the same as that generated by a pure spatial rotation with angular velocity $\xi \mathbf{e}_3$, regardless of the choice of λ. Hence, when considering a sleeping Lagrange top rotating with spatial angular velocity ξ, we are free to choose an arbitrary value of λ and take $\xi \mathbf{v}_\lambda$ as the generator of the relative equilibrium.

The reduced energy momentum method can still be applied to systems for which the group action fails to be free; however, some slight modifications to the method are required. The treatment of symmetric equilibria is described in Lewis (1991a); while the definitions there are given for Lagrangian systems, they are readily translated into the Hamiltonian context. For symmetric equilibria of a system with Hamiltonian of the form 'kinetic plus potential', relative equilibria still correspond to critical points of the augmented potential; the only change in the expression for the stability form $\mathscr{B}_e$ is in the momentum correction term. In the case of the sleeping Lagrange top, we shall show that this change is, in fact, transparent.

We consider the one-parameter family of augmented potentials $V_{\xi \mathbf{v}_\lambda}$. Any sleeping top is a relative equilibrium, since (3.21) implies that $D\mathbb{I}_{(\xi,\omega)}(\Lambda_e) \equiv \mathbf{0}$ for any $(\xi, \omega) \in \mathbb{R}^2$ and $\Lambda_e \mathbf{e}_3 = \mathbf{e}_3$. Thus, using (3.9), we obtain

$$
DV_{\xi \mathbf{v}_\lambda}(\Lambda_e) = mg\ell(\Lambda_e \mathbf{e}_3) \times \mathbf{e}_3 - \xi^2 \mathbf{v}_\lambda \cdot D\mathbb{I}(\Lambda_e)\mathbf{v}_\lambda = 0.
\tag{3.27}
$$

We now determine the stability conditions. Since the tangent to the group orbit $g \cdot \Lambda_e$ is one-dimensional if Λ_e corresponds to a sleeping configuration of a Lagrange top, even though g is two-dimensional, the space of non-trivial variations of V_ξ is two-dimensional for such equilibria. We now briefly discuss the definition of the stability form $\mathscr{B}_e$ at a symmetric equilibrium. Since $\mathbb{I}(\Lambda_e)$ is not invertible if the configuration is symmetric, the inverse $\mathbb{I}(\Lambda_e)^{-1}$ appearing in (2.7) must be replaced by a generalized inverse $\widetilde{\mathbb{I}(\Lambda_e)}^{-1}$: range $\mathbb{I}(\Lambda_e) \to g/(\ker \mathbb{I}(\Lambda_e))$. However, since $D\mathbb{I}_{(\xi \mathbf{v}_\lambda)}(\Lambda_e) \equiv \mathbf{0}$, the stability form $\mathscr{B}_e$ associated to a sleeping Lagrange top is equal to the second variation $D^2 V_{\xi \mathbf{v}_\lambda}(\Lambda_e)$. Using (3.19) and (3.11), we see that

$$
\Gamma_e = mg\ell - \xi^2 \lambda I_3
\tag{3.28}
$$

and hence

$$
\begin{aligned}
D^2 V_{\xi \mathbf{v}_\lambda}(\Lambda_e)(\widehat{\delta\theta \Lambda_e}, \widehat{\Delta\theta \Lambda_e}) &= -(\mathbf{e}_3 \times \delta\theta) \cdot \left[\xi^2 \lambda^2 \mathbb{I}_{\Lambda_e} + \Gamma_e \mathbf{1}_3 \right] (\mathbf{e}_3 \times \Delta\theta) \\
&= \left[\xi^2 \lambda(I_3 - \lambda I_1) - mg\ell \right] (\mathbf{e}_3 \times \delta\theta) \cdot (\mathbf{e}_3 \times \Delta\theta).
\end{aligned}
\tag{3.29}
$$

Hence $D^2 V_{\xi v_\lambda}(\Lambda_e)$ is positive semi-definite, with kernel equal to $\mathrm{span}\{\hat{e}_3 \Lambda_e\}$, for $\lambda_\xi^- < \lambda < \lambda_\xi^+$, where

$$\lambda_\xi^\pm := \frac{I_3 \pm \sqrt{I_3^2 - 4mg\ell I_1/\xi^2}}{2I_1}. \tag{3.30}$$

The relative equilibrium is nonlinearly stable if we can find *any* real number λ such that $D^2 V_{\xi v_\lambda}(\Lambda_e)$ is positive definite; this is clearly possible if $\lambda_\xi^\mp$ are real valued and distinct, i.e., if

$$\xi^2 > \frac{4mg\ell I_1}{I_3^2}. \tag{3.31}$$

This is the well-known linear stability condition for an upright sleeping Lagrange top. Condition (3.31) can also be derived by the energy-Casimir method, as in Holm *et al* (1985), or by a linear stability analysis, as in section 4.

Remark: Note that if one ignores the symmetry of the equilibrium and considers only the case $\lambda = 1$ corresponding to a pure left rotation, then the apparent condition of stability, associated to positive definiteness of (3.29), is

$$\xi^2 > \frac{mg\ell}{I_3 - I_1}. \tag{3.32}$$

Clearly, failure to utilize all symmetries of the equilibrium can result in an overly restrictive stability condition. (As the top approaches a planar disc ($I_3 = 2I_1$), the ξ value for loss of stability coincides with the ξ value for a pure left rotation.)

The stability form for a hanging sleeping top is obtained by replacing ℓ with $-\ell$ in (3.30). Setting $\lambda = 0$, we see that the hanging Lagrange top is always stable.

A Lagrange top with reference inertia tensor $\mathrm{diag}\left[I_1, I_1, I_3\right]$ *tilted by an angle* $\theta \neq 0, \pi$ *about the* $\mathbf{e}_1$ *axis with spatial (left) angular velocity* ξ *and spin (right) angular velocity* ω *remains in nonlinearly orbitally stable steady motion if*

$$\omega = \frac{\xi^2 (I_3 - I_1) \cos \theta - mg\ell}{\xi I_3}.$$

A sleeping top $(\theta = 0, \pi)$ *can be maintained in steady rotation at any angular velocity* ξ. *The upright sleeping top, with* $\theta = 0$, *is orbitally stable if*

$$\xi^2 > \frac{4mg\ell I_1}{I_3^2}.$$

The hanging sleeping top, with $\theta = \pi$, *is always stable.*

---

4. Linearized stability

The energy analysis provides sufficient conditions for orbital nonlinear stability of relative equilibria. However, these conditions need not be necessary. If the energy analysis fails, i.e., if the stability form $\mathcal{B}_e$ fails to be positive definite on some complement to the tangent space $g \cdot \Lambda_e$ to the group orbit, then it is necessary to consider other criteria to determine the stability or instability of the relative equilibrium. In this section, we consider the linear stability of relative equilibria. As was shown in section 3.2, in the case of a sleeping Lagrange top, the nonlinear stability conditions given by the energy analysis coincide with the classical linear stability criteria. However, indefiniteness of $\mathcal{B}_e$ is not always a reliable indicator of linear instability. The non-canonical structure of the reduced manifold can introduce 'magnetic' terms that may shift the spectrum.

We do not carry out the linear stability analysis on the canonical equations of motion (1.33) or (1.32); instead, we construct a reduced system of equations on a manifold $\mathscr{P}$ that is isomorphic to the quotient manifold T^*Q/S^1 and linearize this reduced set of equations. In the process of this reduction, we perform a momentum shift; the resulting linearized equations can be directly related to the stability matrix $\widetilde{\mathcal{B}}_e$ that appears in the energy stability analysis. The reduced variables are defined with respect to the body frame of reference *within* a moving frame with total momentum μ. These variables are the unit direction of gravity γ and the shifted momentum $\widetilde{\Pi}$, which are related to the canonical variables by the mapping

$$\Sigma_\mu : (\Lambda, \Lambda\widehat{\Pi}) \mapsto (\gamma, \widetilde{\Pi}) = \left(\Lambda^T \mathbf{e}_3, \Pi - \mu\zeta(\Lambda^T \mathbf{e}_3)\right) \tag{4.1}$$

where the vector field $\zeta : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\zeta(\gamma) := (\gamma \cdot \mathbb{I}_{\mathrm{ref}}\gamma)^{-1}\mathbb{I}_{\mathrm{ref}}\gamma = I_\gamma^{-1}\mathbb{I}_{\mathrm{ref}}\gamma \tag{4.2}$$

for $I_\gamma := \gamma \cdot \mathbb{I}_{\mathrm{ref}}\gamma$, satisfies

$$\left(\Lambda, \mu\Lambda\widehat{\zeta}(\Lambda^T\mathbf{e}_3)\right) = FL\left(\left(\mathbb{I}(\Lambda)^{-1}\mu\right)_Q(\Lambda)\right). \tag{4.3}$$

(Note that I_γ is the body expression of the inertia tensor $\mathbb{I}(\Lambda)$.) The motivation for this change of variables is discussed in section 6.2. The mapping (4.1) is invariant under spatial rotations about the axis of gravity; in fact,

$$\Sigma_\mu(\Lambda_1, \Lambda_1\widehat{\Pi}_1) = \Sigma_\mu(\Lambda_2, \Lambda_2\widehat{\Pi}_2) \quad \Longleftrightarrow \quad (\Lambda_2, \Pi_2) = R_\theta \cdot (\Lambda_1, \Pi_1) = (R_\theta\Lambda_1, \Pi_1) \tag{4.4}$$

for some rotation R_θ about $\mathbf{e}_3$. The set of all pairs $(\gamma, \widetilde{\Pi})$ makes up the Poisson reduced manifold

$$\mathscr{P} := \left\{(\gamma, \widetilde{\Pi}) : |\gamma| = 1, \ \widetilde{\Pi} \in \mathbb{R}^3\right\} \approx S^2 \times \mathbb{R}^3. \tag{4.5}$$

It follows from (4.4) that the space $\mathscr{P}$ can be identified with the quotient manifold T^*Q/S^1.

The dynamics on $\mathscr{P}$ induced by the dynamics (1.32) on T^*Q are given by

$$\dot{\gamma} = \gamma \times \widetilde{\Omega} \qquad \text{and} \qquad \dot{\widetilde{\Pi}} = \widetilde{\Pi} \times \widetilde{\Omega} + \gamma \times \left(g\mathbf{M} - \xi^2\mathbb{I}_{\mathrm{ref}}\gamma - \xi r_\gamma\widetilde{\Omega}\right) \tag{4.6}$$

where

(i) $\widetilde{\Omega} := \mathbb{I}_{\text{ref}}^{-1}\widetilde{\Pi}$ is the shifted angular velocity in the body frame
(ii) $\xi := I_\gamma^{-1}\mu$ is the 'background velocity' associated to the momentum shift, i.e.

$$\dot{\Lambda} = \Lambda\left(\widehat{\widetilde{\Omega}} + \xi\widehat{\gamma}\right) \tag{4.7}$$

(iii) $r_\gamma := \operatorname{tr}\mathbb{I}_{\text{ref}} - 2I_\gamma^{-1}\,|\mathbb{I}_{\text{ref}}\gamma|^2$.

These equations are geometrically derived and motivated in section 6.2; this derivation is not central to the remainder of the paper and may be skipped at the reader's discretion.

The linearized equations of motion associated to a relative equilibrium with angular velocity ξ are obtained by linearizing (4.6) at the pair $(\gamma_e, 0)$ associated to a relative equilibrium with spatial coordinates $(\Lambda_e, \xi\mathbb{I}_{\Lambda_e}\mathbf{e}_3)$ and momentum $\mu = I_{\gamma_e}\xi$, yielding

$$\begin{pmatrix} \delta\gamma \\ \widetilde{\delta\Pi} \end{pmatrix} = \begin{pmatrix} \gamma_e \times \mathbb{I}_{\text{ref}}^{-1}\widetilde{\delta\Pi} \\ \gamma_e \times \left(B_e\delta\gamma - \xi r_{\gamma_e}\mathbb{I}_{\text{ref}}^{-1}\widetilde{\delta\Pi}\right) \end{pmatrix}. \tag{4.8}$$

Ignoring the null vector $(0, \mathbb{I}_{\text{ref}}\gamma_e)$ associated to variations of the 'background velocity', we see that the linearized dynamics are determined by the four by four matrix

$$L_H = \begin{pmatrix} \chi_{\gamma_e} & 0 \\ 0 & \chi_{\gamma_e} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I}_\perp^{-1} \\ B_e & -\xi r_{\gamma_e}\mathbb{I}_\perp^{-1} \end{pmatrix}. \tag{4.9}$$

Here $\mathbb{I}_\perp^{-1}$, χ_{γ_e}, and B_e denote the restrictions to the orthogonal complement to γ_e of $\mathbb{I}_{\text{ref}}^{-1}$, $\widehat{\gamma}_e$, and the symmetric matrix

$$\xi^2\left(4I_{\gamma_e}^{-1}(\mathbb{I}_{\text{ref}}\gamma_e) \otimes (\mathbb{I}_{\text{ref}}\gamma_e) - \mathbb{I}_{\text{ref}} + I_{\gamma_e}\mathbb{1}_3\right) - (g\mathbf{M}\cdot\gamma_e)\mathbb{1}_3 = \xi^2\left(\mathbb{I}_{\text{ref}} - 2I_{\gamma_e}\nabla\zeta(\gamma_e)\right) + \Gamma_e\mathbb{1}_3 \tag{4.10}$$

where the symmetric matrix

$$\nabla\zeta(\gamma) = I_\gamma^{-1}\mathbb{I}_{\text{ref}} - 2\zeta(\gamma) \otimes \zeta(\gamma) \tag{4.11}$$

is the matrix representation of the first variation of the map ζ and Γ_e is given by (2.13). The characteristic polynomial of the matrix L_H is

$$\chi(\lambda) = \frac{I_{\gamma_e}}{\det\mathbb{I}_{\text{ref}}}\left(\lambda^4 I_{\gamma_e}^{-1}\det\mathbb{I}_{\text{ref}} + \lambda^2\left(\xi^2 r_{\gamma_e}^2 + \langle B_e, S_e\rangle\right) + \det B_e\right) \tag{4.12}$$

where S_e is the restriction of the linear mapping

$$\mathbb{I}_{\text{ref}} - I_{\gamma_e}^{-1}(\mathbb{I}_{\text{ref}}\gamma_e) \otimes (\mathbb{I}_{\text{ref}}\gamma_e) \tag{4.13}$$

to the orthogonal complement to γ_e. The polynomial χ has purely imaginary roots if and only if B_e has positive determinant and

$$\xi^2 r_{\gamma_e}^2 + \langle B_e, S_e\rangle > 2\sqrt{I_{\gamma_e}^{-1}\det\mathbb{I}_{\text{ref}}\det B_e}. \tag{4.14}$$

To relate the linear stability conditions as directly as possible to the energy stability conditions derived in section 2.2, we can rewrite the characteristic polynomial (4.12) in terms of the spatial variables as follows. Given $\Lambda_e \in Q$, let $\mathscr{S}_e$ denote the restriction of the mapping

$$\mathbb{II}_{\Lambda_e} - \mathbb{II}(\Lambda_e)^{-1}(\mathbb{II}_{\Lambda_e}\mathbf{e}_3) \otimes (\mathbb{II}_{\Lambda_e}\mathbf{e}_3) \tag{4.15}$$

to the $\mathbf{e}_1$–$\mathbf{e}_2$ plane. Positive definiteness of $\mathbb{II}_{\Lambda_e}$ implies that

$$\delta\pi \cdot \mathscr{S}_e \delta\pi = \mathbb{II}(\Lambda_e)^{-1}\det \mathbb{II}_{\Lambda_e}(\mathbf{e}_3 \times \delta\pi) \cdot \mathbb{II}_{\Lambda_e}^{-1}(\mathbf{e}_3 \times \delta\pi) > 0 \tag{4.16}$$

for all $\delta\pi$ in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane; hence $\mathscr{S}_e$ is positive definite. Define

$$\rho_e := \operatorname{tr} \mathbb{II}_{\Lambda_e} - 2\mathbb{II}(\Lambda_e)^{-1} |\mathbb{II}_{\Lambda_e}\mathbf{e}_3|^2. \tag{4.17}$$

If $\gamma_e = \Lambda_e \mathbf{e}_3$, then $\mathbb{II}(\Lambda_e) = I_{\gamma_e}$, $\rho_e = r_{\gamma_e}$, and $\mathscr{S}_e = \Lambda_e S_e \Lambda_e^T$. From (1.15) and (2.17) we obtain $\mathbb{II}_{\Lambda_e} = \Lambda_e \mathbb{II}_{\mathrm{ref}} \Lambda_e^T$ and $\widetilde{\mathscr{B}}_e = \Lambda_e B_e \Lambda_e^T$. Hence the characteristic polynomial (4.12) can be expressed in the form

$$\chi(\lambda) = \frac{\mathbb{II}(\Lambda_e)}{\det \mathbb{II}_{\Lambda_e}} \left(\lambda^4 \, \mathbb{II}(\Lambda_e)^{-1}\det \mathbb{II}_{\Lambda_e} + \lambda^2 \left(\xi^2 \rho_e^2 + \langle \widetilde{\mathscr{B}}_e, \mathscr{S}_e \rangle \right) + \det \widetilde{\mathscr{B}}_e \right). \tag{4.18}$$

The energy analysis implies that positive definiteness of $\widetilde{\mathscr{B}}_e$ (equivalently, positive semi-definiteness of $\mathscr{B}_e$) is sufficient to guarantee linear stability of the relative equilibrium, while (4.18) implies that indefiniteness is sufficient to guarantee linear instability. However, *negative* definiteness of $\widetilde{\mathscr{B}}_e$ need *not* imply instability. For certain classes of tops, linear stability does imply positive semi-definiteness of $\mathscr{B}_e$, and hence nonlinear stability. However, there are some classes of tops for which there exists a range of angular velocities at which the top is linearly stable, while $\mathscr{B}_e$ is *negative* semi-definite.

4.1. Linear stability of sleeping tops

If the equilibrium configuration Λ_e is a sleeping top, i.e., if $\mathbf{e}_3$ is an eigenvector of $\mathbb{II}_{\Lambda_e}$, then the linear stability analysis is substantially simplified. If Λ_e determines a sleeping top, then $\mathscr{S}_e = \mathbb{II}_2$, where $\mathbb{II}_2$ denotes the restriction of $\mathbb{II}_{\Lambda_e}$ to the $\mathbf{e}_1$–$\mathbf{e}_2$ plane,

$$\det \mathbb{II}_{\Lambda_e} = \mathbb{II}(\Lambda_e) \det \mathbb{II}_2, \qquad \text{and} \qquad \rho_e = \operatorname{tr} \mathbb{II}_2 - \mathbb{II}(\Lambda_e). \tag{4.19}$$

In this case, the spatial form (4.18) of the characteristic polynomial simplifies to

$$\chi(\lambda) = \frac{1}{\det \mathbb{II}_2} \left(\lambda^4 \det \mathbb{II}_2 + \lambda^2 \left(\xi^2 \rho_e^2 + \langle \widetilde{\mathscr{B}}_e, \mathbb{II}_2 \rangle \right) + \det \widetilde{\mathscr{B}}_e \right) \tag{4.20}$$

where $\widetilde{\mathscr{B}}_e$ is given by (2.17). If $\mathbb{II}_{\Lambda_e} = \operatorname{diag}\left[I_1, I_2, I_3\right]$ and $\mathbf{m}_e = m\ell \mathbf{e}_3$, then (4.20) can be written as

$$\chi(\lambda) = \frac{1}{I_1 I_2} \Big((I_1 + I_2 - I_3)^2 \xi^2 \lambda^2$$
$$+ \left(mg\ell - I_1 \lambda^2 + (I_2 - I_3)\xi^2 \right) \left(mg\ell - I_2 \lambda^2 + (I_1 - I_3)\xi^2 \right) \Big). \tag{4.21}$$

The characteristic polynomial (4.21) for the linearized equations of motion about an upright sleeping top appears, with slightly different sign conventions, in Routh (1884, section 212).

To simplify the analysis, we introduce the parameters

$$\delta := I_1 + I_2 \qquad \psi := \frac{2mg\ell}{\delta}$$

$$\alpha := \frac{I_1 - I_2}{\delta} \qquad \varphi := \frac{\delta - I_3}{\delta}. \tag{4.22}$$

Note that the conditions $0 < I_i < I_j + I_k$ for all i, j, and k imply $\varphi > 0$ and $1 - \varphi > |\alpha|$. We first assume that ψ is positive, i.e. that the top is upright; a negative value of ψ indicates that the top is hanging. The characteristic polynomial (4.21) of the linearized equations takes the form

$$\chi(\lambda) = \frac{4}{1 - \alpha^2} \Big[\varphi^2 \xi^2 \lambda^2 + \tfrac{1}{4} \big(\psi - (1 + \alpha)\lambda^2 + (2\varphi - (1 + \alpha))\xi^2 \big)$$

$$\times \big(\psi - (1 - \alpha)\lambda^2 + (2\varphi - (1 - \alpha))\xi^2 \big) \Big] \tag{4.23}$$

with respect to these parameters. The eigenvalue transitions are determined by the parameters α and φ that characterize the equilibrium inertia tensor. The regions associated to distinct sequences of eigenvalue transitions are indicated in figure 1.
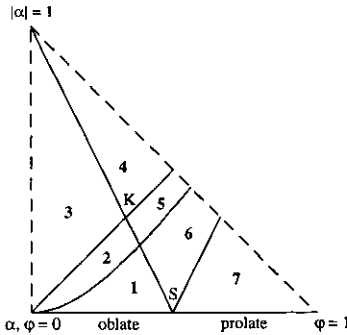

Figure 1. Regions in 'top space' for sleeping tops: the φ–α plane.

The broken lines correspond to the limiting configurations in which one of the principal axes has length zero. The vertical broken line represents discs rotating in the plane; the diagonal broken line represents discs rotating about an out-of-plane axis. The points $(1, 0)$ and $(0, 1)$ correspond to configurations with *two* principal axes of zero length. The line $\alpha = 0$ consists of Lagrange tops; the point $S = (1/2, 0)$ corresponds to a spherical top and the point $K = (1/3, 1/3)$ is the Kovalevskaya top. Regions **1–3** correspond to tops in rotation about the shortest principal axis (i.e., the axis of greatest moment of inertia). Regions **4–6** correspond to tops rotating about the middle axis and region **7** contains all tops in rotation about the longest axis. Note that only tops in regions 1–3 can ever be stable according to the energy analysis.

Given a top in one of the regions described above, we can consider the possible transitions of the eigenvalues of the linearized system as the angular velocity is

decreased. The relevant values of ξ^2 are defined as follows:

$$\tau_{\text{sf-f}}^+ := \frac{\psi}{1 - 2\varphi + \alpha} = \frac{mg\ell}{I_3 - I_2}$$

$$\tau_{\text{sf-f}}^- := \frac{\psi}{1 - 2\varphi - \alpha} = \frac{mg\ell}{I_3 - I_1}$$

$$\tau_{\text{f-s}}^\pm := \frac{\psi\left(\varphi - \alpha^2 \pm \sqrt{\left(1 - \alpha^2\right)\left(\varphi^2 - \alpha^2\right)}\right)}{\varphi(1 - \varphi)^2}$$

$$= \frac{mg\ell\left(4I_1 I_2 - (I_1 + I_2)I_3 \pm 2\sqrt{I_1 I_2 (2I_1 - I_3)(2I_2 - I_3)}\right)}{I_3^2(I_1 + I_2 - I_3)}. \tag{4.24}$$

The subscripts sf–f and f–s stand for 'superfast–fast' and 'fast–slow'; the terminology 'fast tops' and 'slow tops' is traditional. The values $\tau_{\text{sf-f}}^\pm$ correspond to values of ξ^2 at which zero double eigenvalues occur. We shall show in section 5 that bifurcations to tilted relative equilibria (moving without spin) occur when $\xi^2 = \tau_{\text{sf-f}}^\pm$; tops for which $\tau_{\text{sf-f}}^+$ and $\tau_{\text{sf-f}}^-$ take on distinct real values undergo a loss of stability when $\xi^2 = \tau_{\text{sf-f}}^+$ and regain *linear* stability when $\xi^2 = \tau_{\text{sf-f}}^-$. The values $\tau_{\text{f-s}}^\pm$ correspond to values of ξ^2 at which non-zero eigenvalues (necessarily double) occur. All upright sleeping tops are linearly unstable for $\xi^2 < \tau_{\text{f-s}}^+$.

The values $\tau_{\text{sf-f}}^\pm$ and $\tau_{\text{f-s}}^\pm$ are determined as follows: The eigenvalues of the linearized system satisfy

$$\lambda^2 = \frac{\varpi \pm 2\sqrt{\sigma}}{1 - \alpha^2} \tag{4.25}$$

where

$$\varpi = \psi - \left[(1 - \varphi)^2 + \varphi^2 - \alpha^2\right]\xi^2 \tag{4.26}$$

and

$$\sigma = \tfrac{1}{4}\alpha^2\psi^2 + \varphi(\alpha^2 - \varphi)\psi\xi^2 + \varphi^2(1 - \varphi)^2\xi^4$$
$$= \left(\varphi(1 - \varphi)\xi^2 + \tfrac{1}{2}\alpha\psi\right)^2 - (1 - \alpha)(\varphi + \alpha)\varphi\psi\xi^2.$$

Non-zero double eigenvalues occur when $\sigma = 0$, $\varpi \neq 0$; zero eigenvalues occur when $\varpi^2 = 4\sigma$, equivalently when

$$\det \mathcal{B}_e = \delta^2\left((1 - 2\varphi - \alpha)\xi^2 - \psi\right)\left((1 - 2\varphi + \alpha)\xi^2 - \psi\right) = 0 \tag{4.27}$$

i.e., when the stability form $\mathcal{B}_e$ is singular. The values $\tau_{\text{f-s}}^\pm$ are the roots of the equation $\sigma = 0$; $\tau_{\text{sf-f}}^\pm$ are the roots of the equation $\varpi^2 = 4\sigma$.

For tops lying in region **1** and **2**, all of the above values are positive, and are hence attained by ξ^2. For tops in regions **3** and **4**, the values $\tau_{\text{f-s}}^\pm$ are imaginary. We note that $\sigma = (\alpha\psi/2)^2 \geqslant 0$ when $\xi = 0$ and that

$$|\varphi - \alpha^2| \geqslant \sqrt{\left(1 - \alpha^2\right)\left(\varphi^2 - \alpha^2\right)} \tag{4.28}$$

implies that both roots of $\sigma = 0$ must have the same sign. Hence σ is negative, if ever, only for ξ^2 between the two positive roots. In regions **4**, **5** and **6**, where I_3 is neither

the minimum nor the maximum eigenvalue of $\mathbb{I}_{\Lambda_e}$, $\tau_{\text{sf}-\text{f}}^- < 0$; in these regions ξ^2 cannot attain the value $\tau_{\text{sf}-\text{f}}^-$, hence the eigenvalues pass through zero only once, at $\tau_{\text{sf}-\text{f}}^+$. In region 7 ($I_3 < I_2 < I_1$), both $\tau_{\text{sf}-\text{f}}^+$ and $\tau_{\text{sf}-\text{f}}^-$ are negative, hence the eigenvalues never pass through the origin. Along the straight line boundaries of the regions 1–7, the behaviour resembles that of the region to the right of the boundary. Along the curve

$$\alpha^2 = \tfrac{1}{2}\left(1 + 4\varphi^2 - \sqrt{8\varphi^2 + 1}\right) \tag{4.29}$$

where ϖ and σ vanish simultaneously for $\xi^2 = \tau_{\text{f}-\text{s}}^- = \tau_{\text{sf}-\text{f}}^+$, an entire window of eigenvalue behaviour shown in figure 2 collapses to a quadruple zero eigenvalue. Note that this curve separates the region of parameter space for which the energy stability criteria and the linear stability criteria agree from the region for which 'linear restabilization' can occur. In regions 1, 6, and 7, the magnetic terms may dominate the second variation $\mathscr{B}_e$ for a range of angular velocities for which $\mathscr{B}_e$ is negative definite, i.e. it is possible that

$$\xi^2 \rho_e^2 > -\langle \widetilde{\mathscr{B}}_e, \mathbb{I}_2 \rangle > 0. \tag{4.30}$$

Looking at figure 1, we see that this is possible for relatively oblate or nearly symmetric tops, i.e., tops for which φ is sufficiently greater than $|\alpha|$.

The Lagrange tops, which lie along the line $\alpha = 0$, are distinct from the asymmetric tops in the following respects: first, $\tau_{\text{sf}-\text{f}}^+ = \tau_{\text{sf}-\text{f}}^-$, hence the 'window of instability' disappears and the eigenvalues of Lagrange tops in region 1 pass through zero only once, remaining on the imaginary axis; second, the transition values $\tau_{\text{f}-\text{s}}^\pm$ simplify to $\tau_{\text{f}-\text{s}}^+ = \psi/(1-\varphi)^2$ and $\tau_{\text{f}-\text{s}}^- = 0$, hence the collapse of the four distinct imaginary eigenvalues to two double real eigenvalues takes place at $\xi^2 = 0$ and the split to four distinct real eigenvalues does not occur. The difference in behaviour between Lagrange and asymmetric tops is an example of *system symmetry breaking*.

The eigenvalue transitions associated to each region in figure 1 are given in figures 2–4. The eigenvalue portraits are displayed from left to right in order of decreasing angular velocity. We assume here that $I_1 > I_2$ and, hence, $\tau_{\text{sf}-\text{f}}^+ > \tau_{\text{sf}-\text{f}}^-$; if $I_2 > I_1$, then the appropriate portraits can be obtained by exchanging the labels $\tau_{\text{sf}-\text{f}}^+$ and $\tau_{\text{sf}-\text{f}}^-$. The numbers of the portrait sequences are associated to the numbers of the regions in figure 1; the sequences labeled L 1 and L 7 are associated to the Lagrange tops lying on the intersection of the line $\alpha = 0$ with regions 1 and 7. The portrait sequence labelled 1–2, respectively 5–6, describes the eigenvalue behaviour of tops lying on the boundary of regions 1 and 2, respectively 5 and 6.

The eigenvalue behaviour along the boundaries of the regions is characterized as follows: On the boundaries between regions 1 and 2 and regions 5 and 6, the lower 'superfast-fast' and upper 'fast-slow' transitions collide, as indicated in the eigenvalue transition diagrams. Along the boundary of the regions 2 and 3, and regions 4 and 5, i.e. where $|\alpha| = \varphi$, the eigenvalues have the form

$$\lambda^2 = \frac{\psi + \xi^2(3\varphi - 1)}{1 + \varphi} \qquad \text{and} \qquad \lambda^2 = \frac{\psi}{1 - \varphi} - \xi^2. \tag{4.31}$$

Hence no 'fast–slow' transition occurs along these boundaries and the eigenvalues portraits along the 2–3 boundary (respectively 4–5 boundary) resemble the portraits for region 3 (respectively 4). Along the 1–6, 2–5, and 3–4 boundaries, the upper
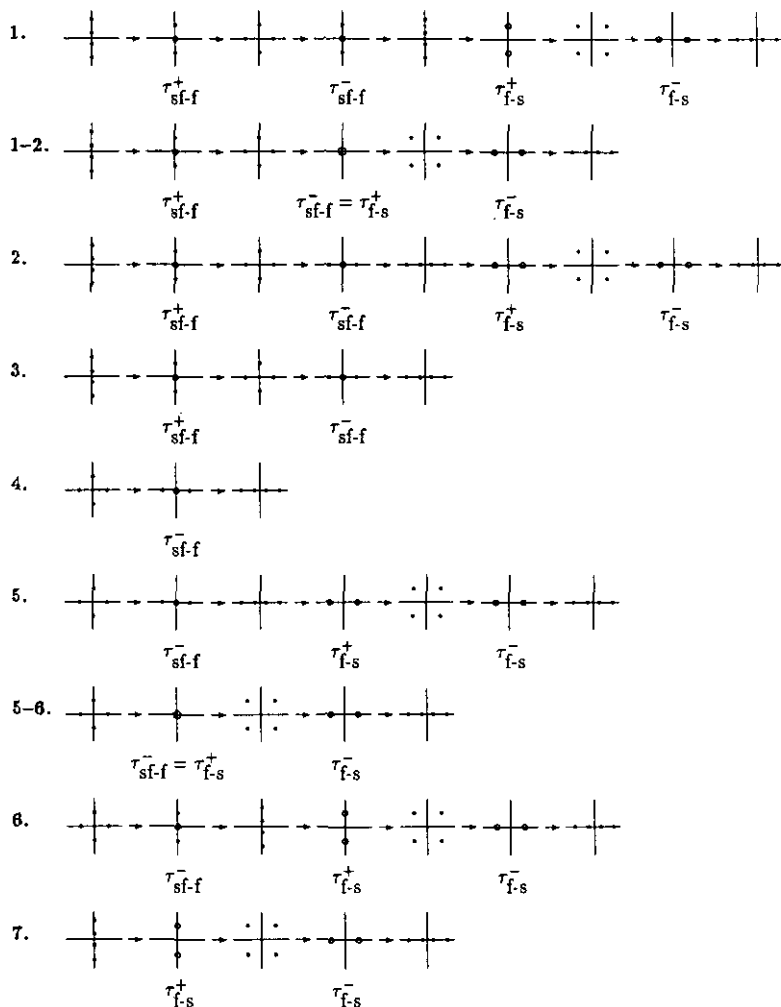
Figure 2. Eigenvalue transitions for upright sleeping tops.

'superfast-fast' transition point τ_{sf-f}^+ occurs at infinite angular velocity; along the **6–7** boundary, the lower 'superfast–fast' transition point τ_{sf-f}^- occurs at infinite velocity.

Remark: A Hamiltonian–Hopf bifurcation occurs at the τ_{f-s}^+ transition for tops in regions **1, 6** and **7** and for all Lagrange tops. The transition of double conjugate purely imaginary eigenvalues to a conjugate quadruple of eigenvalues with non-zero real part is a characteristic feature of Hamiltonian–Hopf bifurcation, however, some non-degeneracy conditions must be checked to establish the actual occurence of a bifurcation. (See Van der Meer (1985, 1990).) We do not carry out that analysis here.

The eigenvalue behaviour of the hanging sleeping top is far simpler than that of the upright top. If ψ is negative, σ is positive for all values of α, φ and ξ. Hence the 'fast–slow' transitions never occur for hanging tops; the eigenvalues always consist of purely real or purely imaginary pairs. Hence the detailed structure of the 'top space' given in figure 1 is no longer relevant; the sequences of eigenvalue transitions are
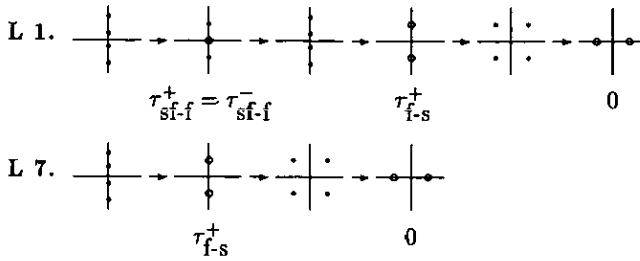
Figure 3. Eigenvalue transitions for upright sleeping Lagrange tops.

determined solely by the relative magnitudes of the principal axes of the inertia tensor. As indicated by the energy analysis of section 2, hanging sleeping tops for which the axis of rotation e_3 is the axis of maximal inertia, i.e., for which $I_3 > I_1$ and $I_3 > I_2$, are always stable, with purely imaginary eigenvalues. Tops for which I_3 is the middle axis, i.e. for which $I_2 > I_3 > I_1$ or $I_1 > I_3 > I_2$, are stable only for sufficiently small angular velocities; the eigenvalues undergo a single 'superfast–fast' transition. Finally, if I_3 is the smallest axis of inertia, then the top is stable for sufficiently large or sufficiently small angular velocities; both of the 'superfast–fast' transitions occur. The eigenvalue transitions are given below; the eigenvalue portraits are displayed from left to right in order of decreasing angular velocity.
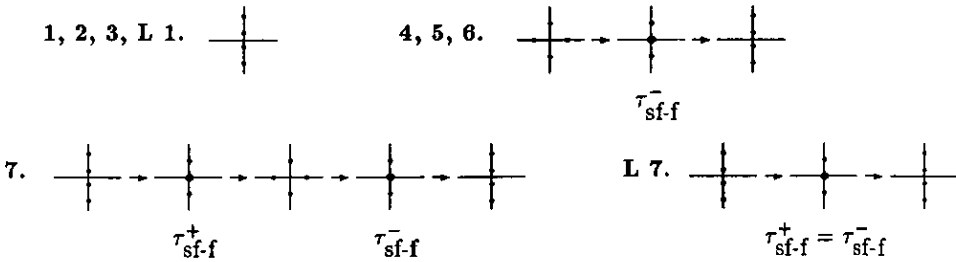

Figure 4. Eigenvalue transitions for hanging sleeping tops.

The possible sequences of eigenvalue behaviour for sleeping tops are summarized in tables 2 and 3.. The characteristics of the eigenvalues are indicated by the following conventions, where κ and v are distinct non-zero real numbers:

$$
\begin{array}{lll}
\text{II}:\pm\kappa i, \pm vi & \text{DI}:\pm\kappa i \text{ (double)} & \text{IZ}:\pm\kappa i, 0 \text{ (double)} \\
\text{RR}:\pm\kappa, \pm v & \text{DR}:\pm\kappa \text{ (double)} & \text{ZR}:\pm\kappa, 0 \text{ (double)} \\
\text{IR}:\pm\kappa i, \pm v & \text{Q}: \pm\kappa \pm vi & \text{ZZ}:0 \text{ (quadruple)}.
\end{array}
$$

For example, II indicates two distinct pairs of conjugate imaginary eigenvalues, while DI indicates a conjugate pair of double imaginary eigenvalues, Q indicates a quartet.

As in the preceding tables, the information is displayed from left to right in order of decreasing velocity; the notation '$>>$' denotes the range of angular velocities between the specific left and right hand values; e.g., the third column describes the eigenvalue behaviour of the top for angular velocities ξ satisfying $\tau^{+}_{sf-f} > \xi^2 > \tau^{-}_{sf-f}$. The first column describes the behaviour of tops for which $\xi^2 > \tau^{+}_{sf-f}$, while the last column decribes tops for which $\tau^{-}_{f-s} > \xi^2$. A dash indicates that the eigenvalue behaviour in that velocity range is the same as that in the ajoining ranges.

Table 2. Eigenvalue sequences for an upright sleeping top.

Region \ ξ^2	$>$	τ^{+}_{sf-f}	$>>$	τ^{-}_{sf-f}	$>>$	τ^{+}_{f-s}	$>>$	τ^{-}_{f-s}	$>$
1	II	IZ	IR	IZ	II	DI	Q	DR	RR
1–2	II	IZ	IR	ZZ	—	ZZ	Q	DR	RR
2	II	IZ	IR	ZR	RR	DR	Q	DR	RR
3	II	IZ	IR	ZR	RR	—	—	—	—
4	—	—	IR	ZR	RR	—	—	—	—
5	—	—	IR	ZR	RR	DR	Q	DR	RR
5–6	—	—	IR	ZZ	—	ZZ	Q	DR	RR
6	—	—	IR	IZ	II	DI	Q	DR	RR
7	—	—	—	—	II	DI	Q	DR	RR
L 1	II	IZ	—	IZ	II	DI	Q	DR	—
L 7	—	—	—	—	II	DI	Q	DR	—

Table 3. Eigenvalue sequences for a hanging sleeping top.

Region \ ξ^2	$>$	τ^{+}_{sf-f}	$>>$	τ^{-}_{sf-f}	$>$
1, 2, 3, L 1	—	—	—	—	II
4, 5, 6	—	—	IR	IZ	II
7	II	IZ	IR	IZ	II
L 7	II	IZ	—	IZ	II

4.2. Linear stability of a special class of tilted equilibria

We can express the characteristic polynomial (4.18) for the simple family of tilted relative equilibria introduced in section 2.3 in terms of the parametrization given in that section. Recall that these are relative equilibria for which the axis of gravity lies in the plane determined by two of the principal axes of the inertia tensor; we restrict our attention to the relative equilibria that satisfy the additional condition that the centre of mass lies along one of these axes, specifically, that the angle ϕ determined by (2.39) equals zero. If $\mathbb{I}_{\Lambda_e}$ is given by (2.36), then $\mathscr{S}_e = \mathrm{diag}\left[I_1, \tau_s\right]$, where

$$\tau_s := \frac{I_2 I_3}{\mathbb{I}(\Lambda_e)} = \frac{I_2 I_3}{I_2 \sin^2\theta + I_3 \cos^2\theta} \tag{4.32}$$

and $\rho_e = I_1 - I_2 - I_3 + 2\tau_s$. Since $\widetilde{\mathscr{B}}_e = \mathrm{diag}\left[b_1, b_2\right]$, where b_1 and b_2 are given by (2.46) and (2.47), the characteristic polynomial $\chi(\lambda)$ satisfies

$$\chi(\lambda) = \frac{1}{I_1 \tau_s}\left(\lambda^4 I_1 \tau_s + \lambda^2\left(\xi^2(I_1 - I_2 - I_3 + 2\tau_s)^2 + I_1 b_1 + \tau_s b_2\right) + b_1 b_2\right)$$

$$= \frac{\xi^4}{I_1 \tau_s} \left(\left(\tfrac{\lambda}{\xi} \right)^4 I_1 \tau_s + \left(\tfrac{\lambda}{\xi} \right)^2 \left((I_1 - I_2 - I_3 + 2\tau_s)^2 + I_1 c_1 + \tau_s c_2 \right) + c_1 c_2 \right)$$

$$(4.33)$$

where $c_i := b_i / \xi^2$. Hence the relative equilibrium is linearly stable if $c_1 c_2 > 0$ and

$$(I_1 - I_2 - I_3 + 2\tau_s)^2 + I_1 c_1 + \tau_s c_2 > 2 \sqrt{\tau_s I_1 c_1 c_2}. \tag{4.34}$$
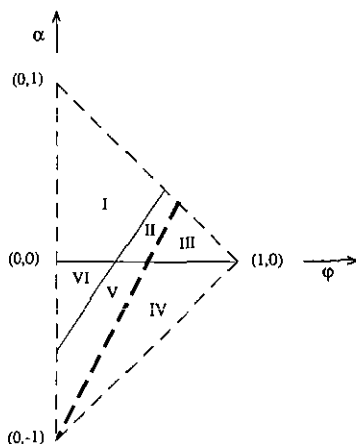

Figure 5. Regions in 'top space' for tilted relative equilibria: the φ–α plane.

The bifurcation diagram for these tops is given in figure 5. The boundaries of the regions are determined by the lines $I_1 = I_2$, $I_2 = 3I_3/4$, and $I_2 = I_3$. Note that relative equilibria of this class do *not* exist if $I_2 = I_3$. The centre of mass of a relative equilibrium satisfying $I_3 > I_2$ sits above the fixed point of the top, i.e. satisfies $\cos \theta > 0$, while the centre of mass of a relative equilibrium satisfying $I_2 > I_3$ sits below the fixed point. Unlike sleeping tops, the stability behaviour of the tilted tops depends on the sign of α. As before, the eigenvalue diagrams in figure 6 are displayed form left to right in order of *decreasing* angular velocity.
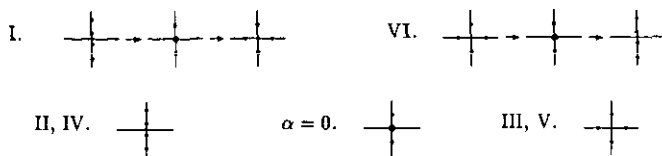

Figure 6. Eigenvalue transitions for tilted tops.

The relationship between formal and linear stability is much simpler for these tilted tops than for the sleeping tops. We shall show that linear stability corresponds to definiteness (either positive or negative) of the stability matrix $\widetilde{\mathscr{B}}_e$, while linear instability corresponds to indefiniteness of $\widetilde{\mathscr{B}}_e$. In regions III–VI, $\widetilde{\mathscr{B}}_e$ is either positive definite or indefinite; hence our earlier discussion of the characteristic polynomial (4.12) shows that formal stability and linear stability coincide in these regions. In region I, $\widetilde{\mathscr{B}}_e$ is negative definite for sufficiently high velocities, as determined by (2.51) and (2.52), and indefinite for lower velocities. The stability form $\widetilde{\mathscr{B}}_e$ is negative definite for all relative equilibria with top shape parameters in region II.

In general, negative definiteness of $\widetilde{\mathscr{B}}_e$ can correspond to either linear stability or instability. However, we shall show that for this family of relative equilibria, negative definiteness implies linear stability. If we define

$$\tau_t := (I_1 - I_2 - I_3 + 2\tau_s)^2 + I_1 c_1 + \tau_s c_2 \qquad \text{and} \qquad \tau_d := \tau_t^2 - 4 I_1 \tau_s c_1 c_2 \qquad (4.35)$$

then the linear stability condition is equivalent to the condition that both τ_t and τ_d be positive. We shall derive parametrizations of $\cos^2 \theta$ and I_i, $i = 1, 2, 3$, which hold whenever $\widetilde{\mathscr{B}}_e$ is negative definite and use these parametrizations to show that τ_t and τ_d are positive.

The equations $c_1 = I_2 - I_1$ and (2.52) imply that negative definiteness of $\widetilde{\mathscr{B}}_e$ holds for relative equilibria with 'top shape parameters' in region I of figure 5 and sufficiently high velocity, i.e. sufficiently small value of $\cos^2 \theta$, and for all relative equilibria with 'top shape parameters' in region II of figure 5. We shall derive a parametrization of $\cos^2 \theta$ that holds for tops having negative definite second variation; we shall derive distinct parametrizations of the eigenvalues I_i of $\mathbb{I}_{A_e}$ for such relative equilibria in regions I and II. Using (2.52), we see that $I_3 > I_2$ and $0 > b_2 = \xi^2 c_2$ imply that $I_3 > I_2 > 3(I_3 - I_2) \cos^2 \theta$. Hence there exists $\varepsilon_c > 0$ such that

$$\cos^2 \theta < \frac{I_2}{3(I_3 - I_2)(1 + \varepsilon_c)}. \qquad (4.36)$$

In region I, the eigenvalues I_2 and I_3 satisfy the relationship $3I_3/4 > I_2$; in terms of the parameters φ and α introduced in section 4.1, this inequality takes the form $\alpha > (3\varphi - 1)/2$. Combining this with the general constraint $1 - \varphi > \alpha$ derived in section 4.1, we see that

$$\alpha = \frac{1}{2}\left(3\varphi - 1 + \frac{3 - 5\varphi}{1 + \varepsilon_\alpha}\right) \qquad (4.37)$$

for some $\varepsilon_\alpha > 0$. The parameter φ attains its maximum value in region I when $1 - \varphi = (3\varphi - 1)/2$, i.e. when $\varphi = 3/5$. Since φ is always positive, we see that $(\varphi, \alpha) \in$ region I implies that $\varphi = 3/(5(1 + \varepsilon_\varphi))$ for some $\varepsilon_\varphi > 0$. Substituting these values into the formulas for τ_t and τ_d yields rational functions with positive coefficients; hence, since δ, ε_c, ε_α, and ε_φ are all positive, τ_t and τ_d are positive. These substitutions are too unwieldy to be readily carried out by hand; however, they can easily be performed using a symbolic computation language, such as *Mathematica*™.

An analogous argument can be applied to region II. The constraint $I_3 > I_2 > 3I_3/4$ implies that $(3\varphi - 1)/2 > \alpha > 2\varphi - 1$. The parameter φ satisfies $2/3 > \varphi > 1/3$ in region II. Hence $(\varphi, \alpha) \in$ region II implies that

$$\varphi = \frac{1}{3}\left(1 + \frac{1}{(1 + \varepsilon_\varphi)}\right) \qquad \text{and} \qquad \alpha = 2\varphi - 1 + \frac{1 - \varphi}{2(1 + \varepsilon_\alpha)} \qquad (4.38)$$

for some $\varepsilon_\varphi > 0$, $\varepsilon_\alpha > 0$. Once again, substituting these values into the expressions for τ_t and τ_d yields rational functions in δ, ε_c, ε_α, and ε_φ with positive coefficients.

Remark: *The Kovalevskaya top:* Note that the special class of tilted Kovalevskaya tops discussed in section 2.3, which satisfy $I_1 = I_3 = 2I_2$ and $\mathbb{I}_{ref}M = I_3M$, have shape parameters $(\varphi, \alpha) = (1/3, 1/3)$ and hence lie in region I. The analysis given in section 2.3 shows that the stability matrix $\widetilde{\mathscr{B}}_e$ is indefinite, and hence the motion is unstable, for $\xi^2 < 2\sqrt{3}mg\ell/I_3$. For $\xi^2 > 2\sqrt{3}mg\ell/I_3$, $\widetilde{\mathscr{B}}_e$ is negative definite and thus the motion is linearly stable. The linear stability conditions can be stated in terms of the angle θ as follows: the members of this class of tilted Kovalevskaya tops are linearly stable if $\cos^2\theta < 1/3$ and unstable if $\cos^2\theta > 1/3$. This result supplements those of Levi-Civita (1901).

5. Bifurcations

A thorough analysis of the possible steady motions of a heavy top is presented in Routh (1884). The heavy top is a sufficiently simple system that it is possible to hypothesize the existence of a particular form of steady motion and then directly determine when, if ever, such motions are possible. For more complicated systems, it can be difficult to 'guess' a priori all possible relative equilibria—abstract results that guarantee the presence (or absence) of particular features can greatly reduce the amount of effort required to locate and characterize families of relative equilibria. Bifurcation theory makes possible the systematic identification of families of relative equilibria; given some initial family of relative equilibria, one can test for the presence of additional families branching from the initial family. An obvious disadvantage of such an approach is its failure to detect isolated families of equilibria; the success of the method depends on the identification of a sufficiently fertile initial branch. (The topological methods of Smale (1970a, b), including the application of Morse theory to the amended potential V_μ, are appropriate for the derivation of global information and location of isolated families.) In the presence of symmetries, families of relative equilibria are often organized about some symmetric branch of equilibria: highly symmetric states are likely candidates for stable motions; once such a 'trivial' solution branch has been found, branches of asymmetric relative equilibria can be detected relatively easily by the local bifurcation analysis. In the present treatment, we make use of only the continuous symmetries of the top; this appears to be sufficient to easily obtain the results we seek. However, recent work by Harnad *et al* (1991) for Hamiltonian systems with both continuous and discrete symmetries, as well the general treatment of bifurcations of symmetric systems described by Golubitsky *et al* (1988), strongly suggest that the explicit consideration of discrete symmetries can substantially simplify the bifurcation analysis of more complicated systems. Near relative equilibria having positive definite $\widetilde{\mathscr{B}}_e$, the Moser–Weinstein theorem guarantees the existence of relative periodic orbits. In fact, one can use discrete symmetries to obtain information on the nature of these periodic orbits using the results of Montaldi *et al* (1990a, b), as was done for planar coupled rigid bodies by Oh *et al* (1989). For details of this sort of analysis, see Chong (1991). In the following sections, we show that the tilted steady motions discussed in section 2.3 and section 3 can be found by a simple bifurcation analysis. In particular, we note that the steady precessing motions of the tilted Lagrange top, which combine overall rotation and spin, bifurcate from the steadily rotating sleeping Lagrange top.

5.1. Bifurcations from asymmetric sleeping tops

We first consider bifurcations from the 'sleeping' configurations, i.e. tops for which the axis of gravity is an eigenvector of the equilibrium inertia dyadic, and then show that no bifurcations from the branches of tilted equilibria to relative equilibria occur. In this case, the axis of gravity e_3 is an eigenvector of the inertia dyadic; we choose the remaining basis vectors so that the inertia dyadic is diagonalized, i.e. $\mathbb{I}_{\Lambda_e} = \text{diag}\left[I_1, I_2, I_3\right]$.

Recall the family of tilted tops for which the centre of mass lies on the principal axis of inertia with eigenvalue I_3 that was discussed in section 2.3. If we fix all of the parameters but the angle θ and the angular velocity ξ, then these tops form branches of relative equilibria parameterized by the angle θ between m_e and e_3. As θ tends towards zero (respectively, π), the family of tops with inertia dyadic (2.36) limit to the sleeping top with inertia dyadic diag $\left[I_1, I_2, I_3\right]$ and angular velocity $\xi^2 = mg\ell/\left(I_3 - I_2\right)$ (respectively $\xi^2 = mg\ell/\left(I_2 - I_3\right)$). Analogously, there is a one-parameter family of relative equilibria tilted through an angle θ about the e_2 axis with angular velocity

$$\xi^2 = \frac{mg\ell}{\cos\theta\left(I_3 - I_1\right)} \tag{5.1}$$

that limits to the sleeping top with angular velocity $\xi^2 = \pm mg\ell/\left(I_3 - I_1\right)$ as θ approaches 0 or π.

The sleeping tops with angular velocity $\xi^2 = \pm mg\ell/\left(I_3 - I_i\right)$, where $i = 1$ or 2, lie at the intersections of the curve of sleeping tops, parameterized by ξ with $\theta = 0$, with the curves of tilted tops, parameterized by θ, with ξ determined by (2.51). Thus these tops are points of bifurcation. Given the branch of sleeping tops, the existence of the branches of tilted relative equilibria is predicted by the singularity theory techniques described in Golubitsky *et al* (1985). The underlying principle is the identification of a simple normal form for the function or vector field in question. Possible directions of bifurcation are determined by the Lyapunov-Schmidt reduction procedure; this procedure determines a mapping from the kernel of the linearization of the original mapping into the complement of the image of the linearized mapping. The solutions along the new branch are tangent to the kernel; higher-order correction terms are determined by computing higher-order derivatives of the reduced mapping.

The starting point for the bifurcation analysis in this case is the equation for a relative equilibrium: $DV_\xi(\Lambda) = 0$. Since the equilibrium equation is automatically satisfied on the tangent $g \cdot \Lambda$ to the group orbit, we can restrict our attention to a complement of $g \cdot \Lambda$. Considering (2.33), we see that the second variation $D^2 V_\xi(\Lambda_e)$ changes signature as ξ^2 passes through the values $mg\ell/(I_3 - I_2)$ and $mg\ell/(I_3 - I_1)$, with null vectors in the direction of the e_1 and e_2 axes. In the case of a sleeping top with critical angular velocity ξ_c satisfying

$$\xi_c^2 = \frac{mg\ell}{I_3 - I_2} \tag{5.2}$$

the reduced mapping takes $\ker D^2 V_\xi(\Lambda_e)/(g \cdot \Lambda_e) \approx \text{span}\{e_1\}$ into itself; hence the reduced bifurcation problem is scalar. As before, let R_θ denote a rotation about e_1. Restricting the augmented potential to rotations about e_1, we see that

$$\begin{aligned} V_\xi(R_\theta) &= \tfrac{1}{4}\xi^2(I_2 + I_3 + (I_3 - I_2)\cos 2\theta) - mg\ell\cos\theta \\ &= T_\xi(\theta) + \text{higher-order terms} \end{aligned} \tag{5.3}$$

where

$$T_\xi(\theta) := (2I_1 - I_3)\xi_c^2 + \frac{mg\ell}{2}\left(\frac{3\theta^2}{2} - \frac{\xi^2 - \xi_c^2}{\xi_c^2}\right)\theta^2. \tag{5.4}$$

Clearly, non-zero critical points of T_ξ exist if and only if $\xi^2 > \xi_c^2$. The general theory guarantees that the behaviour of the full bifurcation problem in a neighbourhood of the possible bifurcation point is described qualitatively by the behaviour of the truncated, reduced system; in particular, the existence of tilted relative equilibria for $\xi^2 > \xi_c^2$ follows from the existence of non-zero critical points of T_ξ. The analysis at the critical velocity $\xi^2 = mg\ell/(I_3 - I_1)$ is analogous.

We now show that no secondary bifurcations, i.e., bifurcations from the branches of tilted relative equilibria, occur. As was previously discussed, relative equilibria can be characterized as critical points of the augmented potential V_ξ; new branches of equilibria must be tangent to a null vector of the second variation of V_ξ at the point of bifurcation. On the other hand, relative equilibria are also critical points of the amended potential V_μ and, by the same reasoning, the second variation of V_μ in the direction of the new branch must equal zero. Hence new branches must be tangent to ker $D\mathbb{I}_\xi(\Lambda_e)$. Thus, ignoring the neutral modes associated to the action of the symmetry group, the space of possible directions of bifurcation from a relative equilibrium rotated from the vertical about the $\mathbf{e}_1$ axis is given by

$$(\ker D\mathbb{I}_\xi(\Lambda_e))/(g_\mu \cdot \Lambda_e) = \mathrm{span}\left\{\mathbb{I}_{\Lambda_e}\mathbf{e}_3, \mathbf{e}_3\right\}/\mathrm{span}\left\{\mathbf{e}_3\right\} = \mathrm{span}\left\{\mathbf{e}_2\right\} \tag{5.5}$$

since (2.36) implies that

$$\mathbb{I}_{\Lambda_e}\mathbf{e}_3 = \left(0, (I_2 - I_3)\cos\theta\sin\theta, I_2\sin^2\theta + I_3\cos^2\theta\right) \tag{5.6}$$

and hence span $\left\{\mathbb{I}_{\Lambda_e}\mathbf{e}_3, \mathbf{e}_3\right\} = \mathrm{span}\left\{\mathbf{e}_2, \mathbf{e}_3\right\}$. However, (2.47) states that

$$D^2 V_\xi(\Lambda_e)(\mathbf{e}_2, \mathbf{e}_2) = D^2 V_\mu(\Lambda_e)(\mathbf{e}_2, \mathbf{e}_2) = \xi^2(I_2 - I_1) > 0 \tag{5.7}$$

so no bifurcation can occur.

We summarize our observations as follows:

Branches of tilted, steadily rotating relative equilibria with angular velocity

$$\xi^2 = \frac{mg\ell}{(I_3 - I_i)\cos\theta} \qquad i = 1, 2 \tag{5.8}$$

where θ denotes the angle between the axis of gravity and the centre of mass, bifurcate from the branch of sleeping top motions of a heavy top at the angular velocities $\xi^2 = mg\ell/(I_3 - I_i)$. The tilted equilibria are stable for sufficiently small angles θ and no secondary bifurcations from these branches occur.

5.2. *Bifurcations from the sleeping Lagrange top*

Associated to each angular velocity for which the sleeping Lagrange top is stable, there are two one-parameter families of tilted precessing motions that limit to the steadily rotating sleeping top with the specified velocity. These branches, like their asymmetric counterparts, were identified by Routh (1884, sections 102 and 201). The branches that bifurcate from the sleeping Lagrange top are characterized as follows: Assume that the top has rotated out of the vertical by an angle θ; i.e., that the equilibrium inertia dyadic is given by $\mathbb{I}_{\Lambda_e} := R_\theta \operatorname{diag}\left[I_1, I_1, I_3\right] R_\theta^T$, where R_θ is given by (2.37). Replacing ξ with $\xi\lambda$ and ω with $\xi(\lambda - 1)$ in (3.16) and solving for λ, we find that the top is a relative equilibrium if and only if $\lambda = \lambda(\theta, \xi)$, where

$$\lambda(\theta, \xi) := \frac{I_3 \pm \sqrt{I_3^2 - 4mg\ell I_\theta/\xi^2}}{2I_\theta} \tag{5.9}$$

for $I_\theta := I_3 + (I_1 - I_3)\cos\theta$. The results of section 3 show that the top tipped at an angle θ with left and right velocities $\pm\xi(\lambda(\theta, \xi), \lambda(\theta, \xi) - 1)$ is a stable relative equilibrium. Note that the traditional concept of 'transfer of stability' need not apply to bifurcations of symmetric relative equilibria; the sleeping top is unstable only for angular velocities below those at which bifurcations occur. This is directly related to the non-uniqueness of the variational characterization of relative equilibria, as is discussed below. Since the second variation of the amended potential is always definite along these branches, no secondary bifurcations can occur.

The bifurcations from the sleeping Lagrange top can be detected by singularity theory methods, as was the case for asymmetric tops. The key to the analysis of bifurcations of a symmetric top from the upright 'sleeping' state is the exploitation of the fact that, while left rotations about the axis of gravity cannot be distinguished from right rotations about the axis of symmetry as long as the top is in the upright state, these motions are distinct as soon as the top is tipped out of the vertical. Hence, for a tilted, precessing top, there is a unique pair $(\xi, \omega) \in g$ that yields the specified motion, while for a sleeping Lagrange top, the curve $\xi v_\lambda = \xi(\lambda, \lambda - 1)$ in g yields the specified angular velocity ξ for any value of λ. In section 3.2, we sought conditions under which some value of λ could be found for which $V_{\xi v_\lambda}$ had a local minimum at the upright top. The same conditions guarantee that two values of λ exist for which $D^2 V_{\xi v_\lambda}(\Lambda_e)$ is indefinite and thus indicate the possibility of a bifurcation.

This analysis follows a general technique for analysing bifurcations from symmetric relative equilibria, i.e. steady motions for which both the configuration and velocity are fixed by some non-trivial group element, that is described in Lewis (1991b). The following proposition provides sufficient conditions for the bifurcation of an entire surface of relative equilibria from a single symmetric relative equilibrium.

Proposition 1. Assume there is an element $q_e \in Q$, algebra elements ζ and $\alpha \in g$, a differentiable function $f : Q \times \operatorname{span}\{\zeta, \alpha\} \to \mathbb{R}$, a differentiable one form $\mathbf{h} : Q \to T^*Q$, and a variation $\delta q_s \in T_{q_e}Q$ satisfying

(i) $DV_\eta(q) = f(q, \eta)\mathbf{h}(q)$ for all $q \in Q$ and $\eta \in \operatorname{span}\{\zeta, \alpha\}$
(ii) $\mathbf{h}(q_e) = 0$
(iii) $D(\mathbf{h} \cdot \delta q_s)(q_e) \cdot \delta q_s \neq 0$.
    Let I_b denote the open set given by

$$I_b := \left\{\sigma \in \mathbb{R} : \sigma^2\left[\left(\zeta \cdot \delta^2\mathbb{I}_\alpha\right)^2 - \left(\alpha \cdot \delta^2\mathbb{I}_\alpha\right)\left(\zeta \cdot \delta^2\mathbb{I}_\zeta\right)\right] > -2\left(\alpha \cdot \delta^2\mathbb{I}_\alpha\right)\delta^2 V\right\} \tag{5.10}$$

where

$$\delta^2 V := D^2 V(q_e)(\delta q_s, \delta q_s) \qquad \delta^2 \mathbb{I}_\alpha := D^2 \mathbb{I}_\alpha(q_e)(\delta q_s, \delta q_s), \text{ etc} \tag{5.11}$$

Then there is a tubular neighbourhood $\mathcal{W} \subset Q \times \mathbb{R}$ of $\{q_e\} \times I_b$ and a function $\lambda : \mathcal{W} \to \mathbb{R}$ such that $FL(\sigma(\zeta + \lambda(q,\sigma)\alpha)_Q(q))$ is a relative equilibrium for all $(q,\sigma) \in \mathcal{W}$.

Condition (iii) is to be interpreted as follows: Let $\mathbf{v}_s$ be a vector field satisfying $\mathbf{v}_s(q_e) = \delta q_s$; the condition that $\mathbf{h}(q_e) = 0$ implies that the first variation at q_e of the scalar function $h_s(q) := \mathbf{h}(q) \cdot \mathbf{v}_s(q)$ does not depend on the choice of $\mathbf{v}_s$. We denote the variation $Dh_s(q_e) \cdot \delta q_s$ by $D(\mathbf{h} \cdot \delta q_s)(q_e) \cdot \delta q_s$. Proposition 1 is a special case of a more general result in which it is not assumed that q_e is a relative equilibrium for all values of ζ; in that case, ζ is viewed as a fixed parameter and λ can be solved for as a function of q alone. (See Lewis (1991b) for the proofs of these results.)

We now apply this result to sleeping Lagrange tops. Let $\mathbb{I}_{\Lambda_e} = \text{diag}[I_1, I_1, I_3]$. Using the G invariance of $V_{(\xi,\omega)}$ and the spatial representation

$$g \cdot \Lambda = \{\mathbf{e}_3, \Lambda\mathbf{e}_3\} \tag{5.12}$$

of the tangent space to the group orbit, we see that

$$DV_{(\xi,\omega)}(\Lambda) \cdot \mathbf{e}_3 \equiv DV_{(\xi,\omega)}(\Lambda) \cdot \Lambda\mathbf{e}_3 \equiv 0. \tag{5.13}$$

Hence the one form $\mathbf{h}$ given in the spatial representation by

$$\mathbf{h}(\Lambda) := \mathbf{e}_3 \times \Lambda\mathbf{e}_3 \tag{5.14}$$

satisfies $DV_{(\xi,\omega)}(\Lambda) = f(\Lambda, (\xi,\omega))\mathbf{h}(\Lambda)$ for some function $f : SO(3) \times \mathbb{R}^2 \to \mathbb{R}$. Clearly, $\mathbf{h}(\Lambda_e) = 0$ if and only if Λ_e is a rotation about $\mathbf{e}_3$. In this case,

$$D(\mathbf{h} \cdot \delta\theta)(\Lambda_e) \cdot \delta\theta = |\mathbf{e}_3 \times \delta\theta|^2. \tag{5.15}$$

Hence $D(\mathbf{h} \cdot \delta\theta)(\Lambda_e) \cdot \delta\theta \neq 0$ for $\delta\theta \notin \text{span}\{\mathbf{e}_3\}$.

Let $\delta\theta_s$ be an arbitrary unit vector in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane. The second variation of the inertia tensor in the direction of $\delta\theta_s$ is

$$D\mathbb{I}(\Lambda_e)(\delta\theta_s, \delta\theta_s) = (\delta\theta_s \times \mathbf{e}_3) \cdot (\delta\theta_s \times \mathbb{I}_{\Lambda_e}\mathbf{e}_3) \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$
$$- (\delta\theta_s \times \mathbf{e}_3) \cdot \mathbb{I}_{\Lambda_e}(\delta\theta_s \times \mathbf{e}_3) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2(I_3 - I_1) & -I_3 \\ -I_3 & 0 \end{pmatrix}. \tag{5.16}$$

We recall the algebra element $\mathbf{v}_\lambda = (\lambda, \lambda - 1)$ introduced in section 3.2; $\mathbf{v}_\lambda = \zeta(\zeta + \lambda\alpha)$ for

$$\zeta = (0, -1) \qquad \text{and} \qquad \alpha = (1, 1). \tag{5.17}$$

The relevant variations of the inertia components are

$$\alpha \cdot \delta^2\mathbb{I}_\alpha = -2I_1 \qquad \zeta \cdot \delta^2\mathbb{I}_\alpha = I_3 \qquad \zeta \cdot \delta^2\mathbb{I}_\zeta = 0. \tag{5.18}$$

The second variation of the potential is $\delta^2 V = mg\ell$. Hence, using (5.10), we see that

$$I_b = \left\{ \xi : \xi^2 I_3^2 > 4 I_1 mg\ell \right\} ; \tag{5.19}$$

i.e., bifurcations occur from all stable sleeping tops.

In summary:

Tilted spinning, precessing steady motions bifurcate from the branch of sleeping Lagrange tops throughout the range of angular velocities for which the sleeping tops are stable. These tilted motions are always stable.

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6. Geometric derivation of the equations of motion

In this section, we derive the unreduced (body and spatial) and reduced equations of motions. While these derivations may be of interest to the geometrically oriented reader, a careful reading of this section is by no means necessary for a thorough understanding of the earlier sections of the paper. The derivations of the spatial and body equations (1.32) and (1.33) are included both for the sake of completeness and to introduce the structures necessary for the derivation of the reduced equations. In section 6.2, we attempt to motivate the construction of the shift map (4.1) and prove that the reduced manifold $\mathscr{P}$ defined in (4.5) is, in fact, a Poisson manifold.

6.1. Derivation of the body and spatial equations of motion

The symplectic forms on $SO(3) \times \mathbb{R}^3$, thought of as the trivialization of $T^* SO(3)$ via left and right reduction, are

$$\omega_B(\Lambda, \Pi)((\Lambda\widehat{\delta\Theta_1}, \delta\Pi_1), (\Lambda\widehat{\delta\Theta_2}, \delta\Pi_2)) = \delta\Pi_2 \cdot \delta\Theta_1 - \delta\Pi_1 \cdot \delta\Theta_2 + \Pi \cdot (\delta\Theta_1 \times \delta\Theta_2) \tag{6.1}$$

and

$$\omega_S(\Lambda, \pi)((\widehat{\delta\theta_1}\Lambda, \delta\pi_1), (\widehat{\delta\theta_2}\Lambda, \delta\pi_2)) = \delta\pi_2 \cdot \delta\theta_1 - \delta\pi_1 \cdot \delta\theta_2 - \pi \cdot (\delta\theta_1 \times \delta\theta_2). \tag{6.2}$$

(See Abraham and Marsden (1978), section 4.4 for the derivation of these forms.)

Denoting the Hamiltonian vector field X_H defined by H and ω_B by

$$X_H(\Lambda, \Pi) = (\Lambda\widehat{X_\Theta}(\Lambda, \Pi), X_\Pi(\Lambda, \Pi)) \tag{6.3}$$

we see that (6.1) implies that

$$\begin{aligned}
\omega_B(\Lambda, \Pi)((\Lambda\widehat{X_\Theta}(\Lambda, \Pi), X_\Pi(\Lambda, \Pi)), (\Lambda\widehat{\delta\Theta}, \delta\Pi)) \\
= \delta\Pi \cdot X_\Theta(\Lambda, \Pi) - X_\Pi(\Lambda, \Pi) \cdot \delta\Theta + \Pi \cdot (X_\Theta(\Lambda, \Pi) \times \delta\Theta) \\
= (\Pi \times X_\Theta(\Lambda, \Pi) - X_\Pi(\Lambda, \Pi)) \cdot \delta\Theta + X_\Theta(\Lambda, \Pi) \cdot \delta\Pi.
\end{aligned} \tag{6.4}$$

If we define the functional derivatives H_Θ and $H_\Pi \in \mathbb{R}^3$ of a function $H : T^*Q \to \mathbb{R}$ by

$$DH(\Lambda, \Pi)(\Lambda\widehat{\delta\Theta}, \delta\Pi) = H_\Theta \cdot \delta\Theta + H_\Pi \cdot \delta\Pi \tag{6.5}$$

for all $\delta\Theta$, $\delta\Pi \in \mathbb{R}^3$, then the Hamiltonian vector field X_H is determined by requiring that the righthand sides of (6.5) and (6.4) be identical for all $\delta\Theta$ and $\delta\Pi \in \mathbb{R}^3$, which yields

$$X_H(\Lambda, \Pi) = (\Lambda\widehat{H}_\Pi, \Pi \times H_\Pi - H_\Theta). \tag{6.6}$$

The differential of the Hamiltonian (1.28) equals

$$
\begin{aligned}
DH(\Lambda, \Pi) \cdot (\Lambda\widehat{\delta\Theta}, \delta\Pi) \\
= g(\Lambda\widehat{\delta\Theta})^T \mathbf{e}_3 \cdot \mathbf{M} + \delta\Pi \cdot \mathbb{I}_{\text{ref}}^{-1}\Pi \\
= -g(\gamma \times \mathbf{M}) \cdot \delta\Theta + \Omega \cdot \delta\Pi.
\end{aligned}
\tag{6.7}
$$

Hence (6.6) yields the Hamiltonian vector field

$$X_\Theta(\Lambda, \Pi) = \Omega \qquad \text{and} \qquad X_\Pi(\Lambda, \Pi) = g\gamma \times \mathbf{M} + \Pi \times \Omega. \tag{6.8}$$

The body form (1.32) of the equations of motion follows from the relationship

$$(\dot{\Lambda}, \dot{\Pi}) = X_H(\Lambda, \Pi). \tag{6.9}$$

The derivation of the spatial form (1.33) of the equations of motion proceeds in a similar manner. If the Hamiltonian vector field associated to the spatial form of the Hamiltonian, given in (1.27), is denoted by $(\widehat{Y}_\theta(\Lambda, \pi)\Lambda, Y_\pi(\Lambda, \pi))$, then

$$
\begin{aligned}
\omega_S(\Lambda, \pi)((\widehat{Y}_\theta(\Lambda, \pi)\Lambda, Y_\pi(\Lambda, \pi)), (\widehat{\delta\theta}\Lambda, \delta\pi)) \\
= -(\pi \times Y_\theta(\Lambda, \pi) + Y_\pi(\Lambda, \pi)) \cdot \delta\theta + Y_\theta(\Lambda, \pi) \cdot \delta\pi.
\end{aligned}
\tag{6.10}
$$

On the other hand, the first variation of the spatial form (1.27) of the Hamiltonian is

$$
\begin{aligned}
DH(\Lambda, \pi) \cdot (\widehat{\delta\theta}\Lambda, \delta\pi) \\
= \widehat{\delta\theta}\mathbf{m} \cdot \mathbf{g} + \delta\pi \cdot \mathbb{I}_\Lambda^{-1}\pi - \tfrac{1}{2}\pi \cdot \mathbb{I}_\Lambda^{-1}(\widehat{\delta\theta}\mathbb{I}_\Lambda - \mathbb{I}_\Lambda\widehat{\delta\theta})\mathbb{I}_\Lambda^{-1}\pi \\
= (\mathbf{m} \times \mathbf{g} - \pi \times \omega) \cdot \delta\theta + \omega \cdot \delta\pi.
\end{aligned}
\tag{6.11}
$$

Hence the right hand sides of (6.11) and (6.10) coincide for all variations $\delta\theta$ and $\delta\pi \in \mathbb{R}^3$ if and only if the spatial equations of motion (1.33) hold.

For a different derivation of the equations of motion (1.32) and (1.33), obtained by means of the semi-direct product theory, see Marsden, Ratiu and Weinstein (1984).

6.2. Derivation of the reduced equations

The shifted reduced equations of motion (4.6) are constructed by Poisson reduction as follows. We first express the symplectic structure on T^*Q in terms of the left (body) trivialization, i.e. we identify a configuration-momentum pair $(\Lambda, \Pi_\Lambda) \in T^*Q = T^*$ SO(3) with the pair $(\Lambda, \Pi) \in \text{SO}(3) \times \text{so}(3)^* \approx \text{SO}(3) \times \mathbb{R}^3$ determined by the relationship

$$\widehat{\Pi} := \Lambda^T\Pi_\Lambda. \tag{6.12}$$

Recall from section 1.3 that in the body representation the Hamiltonian vector field associated to a function F is given by (6.6). Therefore the *Poisson bracket* $\{F, G\} := \omega_B(X_F, X_G)$ of two functions $F, G : T^*Q \to \mathbb{R}$ is given by

$$\{F, G\}(\Lambda, \Pi) = G_\Pi \cdot F_\Theta - F_\Pi \cdot G_\Theta - \Pi \cdot (F_\Pi \times G_\Pi). \tag{6.13}$$

We now determine a one-parameter family of induced Poisson brackets on the reduced manifold $\mathscr{P}$. The brackets are parametrized by a momentum $\mu \in g^* \approx \mathbb{R}$ that specifies a 'momentum shift' as follows. Associated to the action of a Lie group G on a manifold Q is a *momentum map* $\mathbf{J} : T^*Q \to g^*$, where g^* denotes the dual of the Lie algebra of G; Noether's theorem guarantees that $\mathbf{J}$ is conserved by the Hamiltonian dynamics associated to a G invariant Hamiltonian H on T^*Q. The momentum map $\mathbf{J} : T^*Q \to \mathbb{R}$ associated to the G action on T^*Q is defined by

$$\mathbf{J}(\Lambda, \Lambda\widehat{\Pi}) \cdot \xi := \left\langle \Lambda\widehat{\Pi}, \xi_Q(\Lambda) \right\rangle = \xi\, \Pi \cdot \gamma. \tag{6.14}$$

where $\gamma = \Lambda^T \mathbf{e}_3$. The vector field $\zeta : \mathbb{R}^3 \to \mathbb{R}^3$ given by (4.2) satisfies $\mathbf{J}(\Lambda, \Lambda\widehat{\zeta(\gamma)}) = 1$. Hence

$$(\Lambda, \Lambda\widehat{\Pi}) \in \mathbf{J}^{-1}(\mu) \quad \Longleftrightarrow \quad (\Lambda, \Lambda(\widehat{\Pi} - \mu\widehat{\zeta(\gamma)})) \in \mathbf{J}^{-1}(0). \tag{6.15}$$

Thus Σ_μ maps the momentum level set $\mathbf{J}^{-1}(\mu)$ to the level set $\mathbf{J}^{-1}(0)$. In particular, since (2.2) implies that the body expression for a relative equilibrium in the level $\mathbf{J}^{-1}(\mu)$ is

$$FL(\xi_Q(\Lambda)) = FL(\Lambda, \xi\Lambda\widehat{\gamma}) = (\Lambda, \xi\Lambda(\widehat{\mathbb{I}_{\mathrm{ref}}\gamma})) = (\Lambda, \mu\Lambda\widehat{\zeta(\gamma)}) \tag{6.16}$$

it follows that relative equilibria are mapped to true equilibria by the appropriate shift map. (The momentum shift is an essential ingredient in the cotangent bundle reduction theorem.)

The induced Poisson bracket of functions on $\mathscr{P} = S^2 \times \mathbb{R}^3$ is defined as follows: A function $f : \mathscr{P} \to \mathbb{R}$ induces a function $F := f \circ \Sigma_\mu$ on T^*Q with functional derivatives

$$F_\Theta = f_\gamma \times \gamma - \mu(\nabla\zeta(\gamma)\,\widehat{\gamma})^T f_{\widetilde{\Pi}} \qquad \text{and} \qquad F_\Pi = f_{\widetilde{\Pi}} \tag{6.17}$$

where the matrix $\nabla\zeta(\gamma)$ is given by (4.11). Note that f_γ is defined modulo γ, since $\gamma \in S^2$ implies that all variations $\delta\gamma$ satisfy $\delta\gamma \cdot \gamma = 0$. However, the cross product $f_\gamma \times \gamma$ is well-defined, so F_Θ is well defined. The Poisson bracket of two such induced functions has the form

$$
\begin{aligned}
\{f &\circ \Sigma_\mu, g \circ \Sigma_\mu\}(\Lambda, \Pi) \\
&= g_{\widetilde{\Pi}} \cdot \left(f_\gamma \times \gamma - \mu(\nabla\zeta\,\widehat{\gamma})^T f_{\widetilde{\Pi}}\right) - f_{\widetilde{\Pi}} \cdot \left(g_\gamma \times \gamma - \mu(\nabla\zeta\,\widehat{\gamma})^T g_{\widetilde{\Pi}}\right) \\
&\quad - (\widetilde{\Pi} + \mu\zeta) \cdot \left(f_{\widetilde{\Pi}} \times g_{\widetilde{\Pi}}\right) \\
&= \gamma \cdot (g_{\widetilde{\Pi}} \times f_\gamma - f_{\widetilde{\Pi}} \times g_\gamma) - 2\mu f_{\widetilde{\Pi}} \cdot \mathrm{skew}\left[\nabla\zeta\,\widehat{\gamma}\right] g_{\widetilde{\Pi}} - (\widetilde{\Pi} + \mu\zeta) \cdot \left(f_{\widetilde{\Pi}} \times g_{\widetilde{\Pi}}\right)
\end{aligned}
\tag{6.18}
$$

where $(\gamma, \widetilde{\Pi}) = \Sigma_\mu(\Lambda, \Pi)$. Using the identities

$$2\,\mathrm{skew}\left[S\widehat{\mathbf{x}}\right] = (\mathrm{tr}\,S)\,\widehat{\mathbf{x}} - \widehat{S\mathbf{x}} \tag{6.19}$$

for all symmetric matrices S and all $\mathbf{x} \in \mathbb{R}^3$ and

$$(\nabla\zeta(\gamma))\gamma = \zeta(\gamma) - 2(\zeta(\gamma) \cdot \gamma)\zeta(\gamma) = -\zeta(\gamma) \tag{6.20}$$

we obtain

$$2 \operatorname{skew} \left[\nabla \zeta (\gamma) \, \widehat{\gamma} \right] = (\operatorname{tr} \nabla \zeta (\gamma)) \widehat{\gamma} + \widehat{\zeta} (\gamma) = (\operatorname{div} \zeta (\gamma)) \, \widehat{\gamma} + \widehat{\zeta} (\gamma) \tag{6.21}$$

where

$$\operatorname{div} \zeta (\gamma) = I_\gamma^{-1} \operatorname{tr} \mathbb{I}_{\text{ref}} - 2 |\zeta (\gamma)|^2 = I_\gamma^{-1} r_\gamma \tag{6.22}$$

is the usual divergence of a vector field on $\mathbb{R}^3$. Hence the Poisson bracket $\{ \, , \, \}_\mu$ on $\mathscr{P}$ associated to the momentum μ is given by

$$\{ f , g \}_\mu (\gamma , \widetilde{\Pi}) = \gamma \cdot (g_{\widetilde{\Pi}} \times f_\gamma - f_{\widetilde{\Pi}} \times g_\gamma) + (\mu (\operatorname{div} \zeta) \gamma - \widetilde{\Pi}) \cdot (f_{\widetilde{\Pi}} \times g_{\widetilde{\Pi}}). \tag{6.23}$$

Note that the bracket does not depend on the undetermined γ component of the partial derivatives f_γ and g_γ.

The dynamics on the reduced manifold $\mathscr{P}$ are determined by means of the Poisson bracket $\{ \, , \, \}_\mu$ and the induced Hamiltonian $\widetilde{H}_\mu$ associated to a specified momentum μ. The total energy $\widetilde{H}_\mu$ of the system with 'background momentum' μ, in terms of the reduced variables $(\gamma , \widetilde{\Pi})$, is given by

$$\widetilde{H}_\mu (\gamma , \widetilde{\Pi}) := \tfrac{1}{2} \widetilde{\Pi} \cdot \mathbb{I}_{\text{ref}}^{-1} \widetilde{\Pi} + \frac{\mu^2}{2 I_\gamma} + g \gamma \cdot \mathbf{M}. \tag{6.24}$$

The reduced Hamiltonian satisfies the condition $\widetilde{H}_\mu \circ \Sigma_\mu = H$ and has partial derivatives

$$(\widetilde{H}_\mu)_\gamma = g \mathbf{M} - I_\gamma^{-2} \mu^2 \mathbb{I}_{\text{ref}} \gamma \qquad \text{(modulo } \gamma \text{)} \tag{6.25}$$

and

$$(\widetilde{H}_\mu)_{\widetilde{\Pi}} = \mathbb{I}_{\text{ref}}^{-1} \widetilde{\Pi} = \widetilde{\Omega}. \tag{6.26}$$

Thus

$$\begin{aligned} \dot{f} = \{ f , H_\mu \}_\mu &= \gamma \cdot (\widetilde{\Omega} \times f_\gamma - f_{\widetilde{\Pi}} \times (g \mathbf{M} - 2 I_\gamma^{-2} \mu^2 \mathbb{I}_{\text{ref}} \gamma)) + (\mu (\operatorname{div} \zeta) \gamma - \widetilde{\Pi}) \cdot (f_{\widetilde{\Pi}} \times \widetilde{\Omega}) \\ &= f_\gamma \cdot (\gamma \times \widetilde{\Omega}) + f_{\widetilde{\Pi}} \cdot (\gamma \times (g \mathbf{M} - I_\gamma^{-2} \mu^2 \mathbb{I}_{\text{ref}} \gamma - \mu (\operatorname{div} \zeta) \widetilde{\Omega}) + \widetilde{\Pi} \times \widetilde{\Omega}). \end{aligned} \tag{6.27}$$

Setting $\xi = I_\gamma^{-1} \mu$ in (6.27), we obtain the reduced equations of motion (4.6).

Appendix. The reduced energy-momentum method

A1. Glossary: Simple mechanical systems with symmetry

Q	Configuration space, with elements denoted by $q \in Q$
$P = T^*Q$	Phase space. Points $z = (q, p) \in P$ are configurations and momenta
G	Lie group acting freely on Q on the left; the action of G on P is symplectic, obtained by cotangent lifts
g	Lie algebra of G, with elements denoted by $\eta \in g$
exp	Exponential map $\exp : g \to G$ satisfying $\frac{d}{d\epsilon}\big|_{\epsilon=0} \exp(\epsilon \zeta) = \xi$
g^*	Dual of g, with elements denoted by $\mu \in g^*$
Ad_g	Adjoint action of G on g; $\mathrm{Ad}_g \eta = \frac{d}{d\epsilon}\big|_{\epsilon=0} g(\exp(\epsilon \eta))g^{-1} = [v, \eta]$
Ad_g^*	Coadjoint action of G on g^*; $(\mathrm{Ad}_g^*\mu) \cdot \eta = \mu \cdot \mathrm{Ad}_g \eta$
ad_v	Infinitesimal adjoint action of g on g; $\mathrm{ad}_v \eta = \frac{d}{d\epsilon}\big|_{\epsilon=0} \mathrm{Ad}_{(\exp(\epsilon v))}\eta$
ad_v^*	Infinitesimal coadjoint action of g on g^*; $(\mathrm{ad}_v^*\mu) \cdot \eta = \mu \cdot \mathrm{ad}_v \eta$
G_μ	Isotropy subgroup of $\mu \in g^*$ under the coadjoint action $G_\mu := \{h \in G : \mathrm{Ad}_h^*\mu = \mu\}$
g_μ	Isotropy subalgebra of $\mu \in g^*$ under the infinitesimal coadjoint action $g_\mu := \{\eta \in g : \mathrm{ad}_\eta^*\mu = 0\}$
η_Q	Infinitesimal generator $\eta_Q(q) = \frac{d}{d\epsilon}\big|_{\epsilon=0} \exp(\epsilon \eta) \cdot q$
$g_\mu \cdot q_e$	Tangent to the orbit of the isotropy subgroup G_μ $g_\mu \cdot q_e = \{\eta_Q(q_e) : \eta \in g_\mu\} \subset T_{q_e}Q$
$\mathbf{J}$	Momentum map $\mathbf{J} : P \to g^*$; $\mathbf{J}(q, p) \cdot \eta = \langle p, \eta_Q(q) \rangle$
$\langle\!\langle \cdot, \cdot \rangle\!\rangle$	G invariant Riemannian metric on Q
$\mathbb{I}$	Locked inertia tensor $\mathbb{I} : Q \to L(g, g^*)$; $(\mathbb{I}(q)\zeta) \cdot \eta = \langle\!\langle \eta_Q(q), \zeta_Q(q) \rangle\!\rangle$
$\mathbb{I}_\xi$	Locked momentum map $\mathbb{I}_\xi : Q \to g^*$; $\mathbb{I}_\xi(q) = \mathbb{I}(q)\xi$
FL	Legendre transformation $FL : TQ \to T^*Q$; $\langle FL(v_q), w_q \rangle = \langle\!\langle v_q, w_q \rangle\!\rangle$
V	G–invariant potential energy $V : Q \to \mathbb{R}$
H	Hamiltonian function $H : P \to \mathbb{R}$; $H(q, p) = \frac{1}{2}\left\|FL^{-1}(p)\right\|^2 + V(q)$
V_ξ	Augmented potential $V_\xi : Q \to \mathbb{R}$; $V_\xi(q) = V(q) - \frac{1}{2}\xi \cdot \mathbb{I}(q)\xi$
V_μ	Amended potential $V_\mu : Q \to \mathbb{R}$; $V_\mu(q) = V(q) + \frac{1}{2}\mu \cdot \mathbb{I}^{-1}(q)\mu$
$\mathscr{B}_e$	Stability two form $\mathscr{B}_e : T_{q_e}Q \times T_{q_e}Q \to \mathbb{R}$ $\mathscr{B}_e(\delta q, \Delta q) = D^2 V_\xi(q_e)(\delta q, \Delta q) + (D\mathbb{I}_\xi(q_e) \cdot \delta q) \cdot \mathbb{I}(q_e)^{-1}(D\mathbb{I}_\xi(q_e) \cdot \Delta q)$
$\mathscr{A}$	Generalized Arnold form $\mathscr{A} : g \times g \to \mathbb{R}$ $\mathscr{A}(\zeta, \eta) = \mathrm{ad}_\zeta^*\mu \cdot \left(\mathbb{I}(q_e)^{-1}\mathrm{ad}_\eta^*\mu + \mathrm{ad}_\eta(\mathbb{I}(q_e)^{-1}\mu)\right)$
$\mathscr{Q}_{\mathrm{int}}$	Space of 'internal' variations $\mathscr{Q}_{\mathrm{int}} = \{\delta q \in T_{q_e}Q : \mathbb{I}(q_e)^{-1}(D\mathbb{I}_\xi(q_e) \cdot \delta q) \in g_\mu\}$

A2. Summary of the method

An element $z_e \in P$ is a *relative equilibrium* if the trajectory z_t passing through z_e determined by the Hamiltonian dynamics satisfies

$$z_t = \exp(t \xi) \cdot z_e$$

for some $\xi \in g$. Since the G action is free, $z_e \in T^*_{q_e}Q$ satisfies the above equation if and only if $z_e := FL(\xi_Q(q_e))$ and q_e is a critical point of V_ξ. The equilibrium configuration q_e is also a critical point of V_μ for $\mu := \mathbb{I}_\xi(q_e)$.

The relative equilibrium z_e is (formally) stable modulo G_μ if $D^2 V_\mu(q_e) = \mathscr{B}_e$ is positive semi-definite, with kernel $g_\mu \cdot q_e$. (If the configuration space Q is isomorphic to the symmetry group G, e.g., in the free rigid body, then the relative equilibrium is (formally) stable if $\mathscr{B}_e$ is either positive or negative definite.) The stability conditions can be decomposed into 'rigid' and 'internal' conditions as follows. The stability form $\mathscr{B}_e$ is positive semi-definite with kernel $g_\mu \cdot q_e$ if and only if the restriction of $\mathscr{B}_e$ to the space $\mathscr{Q}_{\text{int}}$ of 'internal' variations is positive semi-definite with kernel $g_\mu \cdot q_e$ and the generalized Arnold form $\mathscr{A}$ is positive semi-definite with kernel g_μ. If the symmetry group G is abelian, as in the case of the heavy top, then $g = g_\mu$ for all values of μ; hence, in this case, the 'rigid' stability conditions are trivially satisfied and $\mathscr{Q}_{\text{int}} = T_{q_e} Q$.

References

Abraham R and Marsden J E 1978 *Foundations of Mechanics* second edn (Reading, MA: Benjamin)

Arnold V I 1966 Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits *Ann. Inst. Fourier* XVI 319–61

—— 1969 The Hamiltonian nature of the Euler equations in the dynamics of a rigid body and an ideal fluid *Usp. Mat. Nauk*, **24** 225–6

—— 1988 *Encyclopaedia of Mathematical Sciences Volume 3; Dynamical Systems III* (New York: Springer)

—— *Mathematical Methods of Classical Mechanics* 2nd edn (New York: Springer)

Chong W D 1991 Quasi-periodic and periodic motions of a heavy rigid body about a fixed point *PhD Thesis* SUNY, Buffalo

Gantmacher F R 1989 *Matrix Theory* vol II (New York: Chelsea)

Golubitsky M and Schaeffer D 1985 *Singularities and Groups in Bifurcation Theory* vol I (New York: Springer)

Golubitsky M, Schaeffer D and Stewart I 1988 *Singularities and Groups in Bifurcation Theory* vol II (New York: Springer)

Harnad J, Hurtubise J and Marsden J E 1991 Reduction of Hamiltonian systems with discrete symmetries (preprint)

Holm D, Marsden J, Ratiu T and Weinstein A 1985 Nonlinear stability of fluid and plasma equilibria *Physics Reports* **123** (1 & 2) 1–116; See also 1983 *Physics Lett.* A **98** 15–21

Iacob A 1971 Invariant manifolds in the motion of a rigid body about a fixed point *Rev. Roum. Math. Pures Appl.* **16** 1497–521

Katok S 1972 Bifurcation sets and integral manifolds in the problem of the motion of a heavy rigid body *Usp. Mat. Nauk* **27** (2) (Russian)

Klein F and Sommerfeld A 1910 *Theorie des Kriesels* (Leipzig: Teubner)

Levi-Civita T 1901 Sui moti stazionari di un corpo rigido nel caso Kowalesky *Atti Accad. Lincei, Rendiconti, Classe di scienze fisiche, matematiche e naturali* **10** I 338–46; II 429–34; III 461–6

Lewis D 1991a Lagrangian block diagonalization *J. Dynam. Diff. Eg.* **3** 619-59

—— 1991b Bifurcations of liquid drops (preprint)

Lewis D and Simo J C 1990 Nonlinear stability of rotating pseudo-rigid bodies *Proc. R. Soc.* A **427** 281–319

Maddocks J H 1989 On the stability of relative equilibria (preprint)

Marsden J, Ratiu T and Weinstein A 1984 Reduction and Hamiltonian structures on duals of semidirect product Lie algebras *Cont. Math.* **28** 55–100

Montaldi J, Roberts M and Stewart I 1990a Existence of nonlinear normal modes of symmetric Hamiltonian systems *Nonlinearity* **3** 695–730

—— 1990b Stability of nonlinear normal mods of symmetric Hamiltonian systems *Nonlinearity* **3** 731–77

Oh Y-G, Sreenath N, Krishnaprasad P S and Marsden J E 1989 The dynamics of coupled planar rigid bodies. II Bifurcations, periodic solutions, and chaos *J. Dyn. Diff. Eqn.* **3** No 1 269–98

Riemann B 1860 Untersuchungen über die Bewegung eines flüssigen gleichartigen Ellipsoides *Abh. d. Königl. Gesell. der Wis. zu Göttingen* **9** 3–36

Routh E J 1882 *Elementary rigid dynamics* (London: McMillan)

—— 1884 *Advanced rigid dynamics* (London: McMillan)

Simo J C, Lewis D and Marsden J E 1991a The stability of relative equilibria. Part I: The reduced energy-momentum method *The Archive for Rational Mechanics and Analysis* **115** 15–59

Simo J C, Posbergh T A and Marsden J E 1990 Stability of coupled rigid body and geometrically exact rods: Block diagonalization and the Energy–Momentum method *Phys. Rep.* **193** 280–360
—— 1991b The stability of relative equilibria. Part II: Application to nonlinear elasticity *The Archive for Rational Mechanics and Analysis* **115** 61–100
Smale S 1970a Topology and Mechanics I *Inventiones mathematicae* **10** 305–31
—— 1970b Topology and Mechanics II *Inventiones mathematicae* **11** 45–64
Tatarinov Ya V 1974 Portraits of the classical integrals of the problem of rotation of a rigid body about a fixed point *Vestn. Mosk. Univ., Series I* **29** No. 6 99–105 (Russian)
Van der Meer J C 1985 The Hamiltonian–Hopf bifurcation *PhD Thesis* Utrecht
—— 1990 Hamiltonian–Hopf bifurcation with symmetry *Nonlinearity* **3** 1041–56