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The Heckscher—Ohlin—Samuelson Trade Theory and the Cambridge Capital Controversies: On the Validity of Factor Price Equalisation Theorem

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The Heckscher–Ohlin–Samuelson Trade Theory and the Cambridge Capital Controversies: On the Validity of Factor Price Equalisation Theorem*

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Abstract

This paper examines the validity of the factor price equalisation theorem (FPET) in relation to capital theory. First, it presents a survey of the literature on Heckscher–Ohlin–Samuelson (HOS) models that treat capital as a primary factor, beginning with Samuelson (1953). In addition, by consulting the Cambridge capital controversies, this paper observes that the validity of the FPET relies crucially on this setting. It does no longer hold whenever capital is assumed to be a bundle of reproducible commodities. This paper also refers to the recent literature on the dynamic HOS trade theory and argues that such studies ignore the difficulties posed by the capital controversies. It thereby concludes that the FPET holds even when capital is modelled as a reproducible factor. In conclusion, the paper suggests the necessity of reconstructing basic theories of international trade without relying on the FPET.

JEL Classification Code: B51, D33, F11.

Keywords: factor price equalisation, global univalence, capital as a bundle of reproducible commodities, reswitching of techniques, capital reversing.

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1 Introduction

'Globalisation' is undoubtedly one of the key words that characterises the 21st Century. Various definitions of 'globalisation' have been proposed, as Wolf (2004) discusses. For our purposes, it is sufficient to define 'globalisation' as the integration of markets for goods, services, capital and labour, formerly segmented by political barriers, into a single 'world market'.

The first driving force behind modern globalisation was the establishment of a free trade system under the International Monetary Fund and the General Agreement on Tariffs and Trade (IMF-GATT), which gradually reduced tariffs after the end of WWII. Thereafter, the collapse of the IMF's fixed exchange rate system and its replacement by the flexible exchange rate system further drove globalisation. Additionally, the relaxation and abolishment of various regulations, the product of the counter-Keynesian revolution, resulted in the free international movement of capital. Finally, the World Trade Organisation (WTO) was founded. It established comprehensive rules for international transactions of goods, finances, information (i.e. communication), intellectual property, and services.[1]

Almost all economic theories have been supportive of globalisation. Indeed, the classical economists of the 19th century, such as Smith, Ricardo, J.S. Mill, and Marx, asserted that open economies were superior to closed ones.[2] Modern neo-classical economics also asserts the superiority of an open economy by establishing the Heckscher–Ohlin–Samuelson (HOS) model. Economists typically support globalisation by referring to the potential gains from trade enjoyed by every economic agent under open economies (Anderson, 2008). Ricardo's theory of 'comparative advantage', an explanation of the potential gains from trade, remains one of the cornerstones of international economics. Neo-classical economics also argues 'comparative advantage' within the HOS framework.

However, there are several differences between classical (including Marxian) and neo-classical models of gains from trade. Classical models assume that each country is endowed with its own techniques, which may vary. Neo-classical models assume that every country is faced with a common set of techniques but differs in terms of factor endowments. Note that in the modern economy, globalisation actively promotes the international movement *not only* of goods, services, capital, and labour, *but also* of information and knowledge. This allows everyone anywhere to access common information and knowledge of production technology, at least in the long run. To capture this feature of the modern economy, we can assume that every country is faced with a common set of techniques. This is formalised by a common production possibility set, as in the HOS model. It should be noted, however, that access to information and knowledge of production technology does not necessarily imply that every country can use them effectively. In order for a country to use a technique, it must have the necessary capital formation and labour force. Given the imperfection of international factor markets, the choice of a technique is dependent on the country's factor endowments.

In order to analyse globalisation with the HOS model, the validity of a set of theorems

[1] See Wolf (2004) concerning the history of the world economy's construction in detail.

[2] However, Malthus and List severely criticise the free trade doctrine. It is well known that Malthus (1815) criticises the free trade system for its effects on food security and the stability of prices. List (1904) also criticises free trade for its failure to protect infant industries.

must be examined; these include the HO theorem, the factor price equalisation theorem, the Stolper-Samuelson theorem, and the Rybczynski theorem. Although the discovery of the Leontief Paradox (Leontief, 1953, 1956) precipitated this type of examination, we focus on the factor price equalisation theorem (FPET), the cornerstone of the HOS model. According to this theorem, the equilibrium international price, as determined by free trade, ensures the equalisation of factor prices, and the other three basic theorems are developed by presuming this consequence. Thus, it is important to determine whether factor prices tend to converge in modern globalisation.

In their analysis of the US current account imbalance, Obstfeld and Rogoff (2005) reveal that the income return on US-owned assets exceeded that on US liabilities by an average of 1.2% a year from 1983 to 2003. Furthermore, the return on US foreign investments, including capital gains, exceeded that on US liabilities by a remarkable 3.1% during the same period. If the return is regarded as a measurement of factor price, how can this persistent difference be explained? To address the gap between this observation and the lesson from the HOS trade theory, this paper critically reviews the validity of factor price equalisation in international trades.[3]

We pay particular attention to the relationship between the theoretical development in the HOS trade theory and the outcome of the Cambridge capital controversies, which revealed that the neo-classical principle of marginal productivity does not, in general, hold. The neo-classical production function treats capital as a primary factor of production. As a result, its amount is given independently of the price system. Moreover, the rate of profit maintains a one-to-one correspondence with a technique. In this case, there is a monotonically decreasing relationship between capital intensity and the rate of profit (the principle of marginal productivity). However, if capital is treated as a bundle of reproducible commodities, the neo-classical theory does not hold. A technique may correspond to multiple rates of profit, a phenomenon termed *the reswitching of techniques*. In this case, the monotonically decreasing relationship between capital intensity and the rate of profit generally does not hold. In other words, capital intensity may rise as the rate of profit increases, a phenomenon termed *capital reversing*.

As we shall argue later, the outcome of the controversies can be used to re-examine the validity of the HOS model, which assumes the neoclassical production function and treats capital as a primary factor. In particular, we rigorously show that the FPET can be preserved in the HOS model simply because capital is treated as a primary factor. It no longer holds whenever capital consists of heterogeneously reproducible commodities. This kind of analysis was also conducted by neo-Ricardian scholars, influenced by Sraffa (1960), who contributed to the literature of the HOS model in the 1960s and 70s, such as Steedman, Metcalfe, and Mainwaring.[4] Responding to such criticisms, counterarguments were developed by neoclassical scholars such as Burmeister (1978) and Dixit (1981). Moreover, new results on the FPET have been developed since then, such as Wong (1990). Given this sequence of

[3] While we have a special interest in capital, several studies examine the inexplicable relationship between international trade and wage disparities; see, for example, Kurokawa (2014).

[4] See also Depoortère and Ravix (2015) concerning the Neo-Ricardian re-evaluations of international trade theory. They re-evaluate the theory from the viewpoint of not only capital theory but also the concept of comparative advantage, gains from trade, and others.

debates, a main contribution of this paper is to update and even strengthen the neo-Ricardian criticism of the FPET by reviewing these neoclassical works and contrasting them with our examples of production economies with alternative Leontief techniques, in which factor price equalisation cannot be observed.

First, this paper shows that the failure of factor price equalisation easily arises with alternative Leontief techniques, assuming the simplest international economies of two nations with two commodities and two factors (capital and labour). This contrasts with the work of Wong (1990) which, combined with the classical work of Samuelson (1953), verified that the failure of the factor price equalisation is never observed in HOS economies with the same simple international framework. Second, this paper presents a numerical example of the two-integrated-sector Leontief production economy with two tradable final goods and two non-tradable intermediate goods, in which the FPET does not hold despite a lack of sectorial capital intensity reversal. This example refutes Dixit (1981), who argues that the failure of the factor price equalisation in economies with non-tradable intermediate goods comes from the difficulty of preserving no capital intensity reversal in such economies. Indeed, our example exhibits such a failure in the presence of the reswitching of techniques and capital reversing, even though no capital intensity reversal is observed.

Finally, this paper reviews the counterarguments of neo-classical economists against the critiques popularised by the Cambridge capital controversies. In particular, it examines Burmeister (1978), who addressed the neo-Ricardian criticisms of the FPET most seriously by referring to the controversies. Burmeister (1978) presented the most influential form of the HOS model after the controversies, in which capital is modelled as reproducible commodities and the two capital intensity conditions are then provided to preserve the FPET. This paper argues that such an excellent result is possible because the HOS model in Burmeister (1978) is ingeniously constructed to avoid the occurrence of the inconvenient phenomena emphasised by the controversies.

The general impossibility of factor price equalisation presented in this paper may provide a rational basis for the persistent differences between the returns earned by the US and those of the rest of the world. It is clear, in the globalised economy, that capital is not a primary factor, but rather is composed of a bundle of reproducible commodities. Even if the globalised economy were perfectly competitive, as the theory assumes, there could still be differences in the returns.

The paper is organised as follows: Section 2 presents a survey of the representative literatures on the traditional HOS model, in which capital is treated as a primary factor. Section 3 shows the impossibility of factor price equalisation in simple production economies with alternative Leontief techniques by providing two numerical examples. Section 4 reviews Burmeister (1978). Section 5 presents our concluding remarks.

Throughout the paper, we assume the perfect mobility of factors within each domestic market and the perfect flexibility of prices; that international trade does not incur any costs (e.g. transportation costs and tariffs) except for the direct cost of production; that there is no perfect specialisation, and thus every country produces all commodities; and that there is no joint production unless otherwise stated.

2 The HOS Model with Capital as a Primary Factor of Production

In this section, we examine the traditional HOS model in which capital is treated as a primary factor of production. Heckscher (1919) and Ohlin (1933) defined the structure of comparative advantage as the difference in countries' factor endowments.

Samuelson (1953) formulates the basic model by using the general equilibrium model and conjectures the sufficient condition for the FPET: *no factor intensity reversal*. In the HOS model, no factor intensity reversal has been assumed. As far as the two-commodity case with two factors is concerned, however, Wong (1990) shows that no factor intensity reversal occurs.

2.1 FPET in the basic model with two commodities and two factors

Let us introduce the common basic structure of the models used here. The structure is developed by Samuelson (1953). Simplifying the model, assume that there are n commodities and m productive factors. For each commodity $j = 1, \dots, n$, there is a production function $f_j : \mathbb{R}^m_+ \to \mathbb{R}_+$ such that f_j is (i) increasing; (ii) homogeneous of degree one; (iii) strictly quasi-concave; (iv) differentiable on $\mathbb{R}^m_+$; and it permits free disposal. Burmeister and Dobell (1970, pp. 8-12) calls such a production function a *neo-classical production function*. The intrinsic feature of this type of production function is that *all productive factors are primary*. There is no structure of the production of commodities by commodities. Let us denote $X_j = f_j(V_{1j}, \cdots, V_{mj})$, where $\sum_{j=1}^{n} V_{ij} = V_i$. As the function is homogeneous of degree one, it can be rewritten as follows:

$$1 = f_j(a_{1j}, \cdots, a_{mj}), \text{ where } a_{ij} \equiv \frac{V_{ij}}{X_j}. \tag{1}$$

For each commodity j, there is the set of input vectors which produce one unit of commodity j by using the technology f_j, which is denoted by $\mathcal{A}_j$.[5] The set $\mathcal{A}_j$ represents the *set of alternative techniques* for the production of commodity j, over which each producer can choose. It is easy to see that $\mathcal{A}_j$ is a closed and convex set due to the continuity and the quasi-concavity of f_j.

Let $\boldsymbol{w} \equiv (w_i)_{i=1,\dots,m} \in \mathbb{R}^m_+$ be a profile of factor prices. Then, given a factor price vector $\boldsymbol{w} \in \mathbb{R}^m_+$, for each commodity j, let

$$c_j(\boldsymbol{w}) \equiv \min_{\boldsymbol{a}_j \in \mathcal{A}_j} \boldsymbol{w} \cdot \boldsymbol{a}_j.$$

Thus, we can define the (*indirect*) *cost function* $c_j : \mathbb{R}^m_+ \to \mathbb{R}_+$ for each commodity j.

[5] That is:

$$\mathcal{A}_j \equiv \left\{ (a_{1j}, \dots, a_{mj}) \in \mathbb{R}^m_+ \mid f_j(a_{1j}, \dots, a_{mj}) \geqq 1 \right\}.$$

Let $\boldsymbol{a}_j(\boldsymbol{w}) = \arg\min_{\boldsymbol{a}_j \in \mathcal{A}_j} \boldsymbol{w} \cdot \boldsymbol{a}_j$ for each commodity j. Since $\mathcal{A}_j$ is closed and strictly convex, note that $\boldsymbol{a}_j(\boldsymbol{w})$ is uniquely determined for each $\boldsymbol{w} \in \mathbb{R}^m_+$. Therefore, $c_j(\boldsymbol{w}) = \boldsymbol{w} \cdot \boldsymbol{a}_j(\boldsymbol{w})$.[6]

For each profile of factor prices $\boldsymbol{w} \in \mathbb{R}^m_+$, let us define

$$\boldsymbol{A}(\boldsymbol{w}) \equiv [\boldsymbol{a}_j(\boldsymbol{w})]_{j=1,\ldots,n}$$

which is a non-negative $m \times n$ matrix. Denote a vector of commodity prices by $\boldsymbol{p} \in \mathbb{R}^n_+$. Then, a vector of perfectly competitive equilibrium prices $(\boldsymbol{p}, \boldsymbol{w}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$ should satisfy the following conditions:

$$\boldsymbol{p} \leqq \boldsymbol{w}\boldsymbol{A}(\boldsymbol{w}), \tag{2}$$

$$[\boldsymbol{p} - \boldsymbol{w}\boldsymbol{A}(\boldsymbol{w})]\,\boldsymbol{X} = 0, \tag{3}$$

$$\boldsymbol{A}(\boldsymbol{w})\,\boldsymbol{X} \leqq \boldsymbol{V}, \tag{4}$$

where $\boldsymbol{X} \equiv [X_j] \in \mathbb{R}^n_+$ and $\boldsymbol{V} \equiv [V_i] \in \mathbb{R}^m_+$ respectively denote the vectors of commodity outputs and factor endowments. Given the definition of $\boldsymbol{A}(\boldsymbol{w})$, $\boldsymbol{w}\boldsymbol{A}(\boldsymbol{w})$ denotes the vector of each sector's cost function. Therefore, (2) is the zero-profit condition for competitive equilibrium prices; (3) is the condition for profit maximizing production plans; and (4) is the condition of factor constraints for feasible production plans. If all commodities are produced and all factors are fully utilised in equilibrium, then the equality associated with $(\boldsymbol{p}, \boldsymbol{w}) > \boldsymbol{0}$ holds in both (2) and (4).

The basic idea to prove the FPET was put forward by Samuelson (1953). Let $c(\boldsymbol{w}) \equiv \boldsymbol{w}\boldsymbol{A}(\boldsymbol{w})$. Denote the range of the cost functions by $\mathcal{P}$, which consists of commodity price vectors $\boldsymbol{p} \in \mathbb{R}^n_+$ whose values are equal to the costs $c(\boldsymbol{w})$ for some factor price vector $\boldsymbol{w} \in \mathbb{R}^m_+$.[7] Then, $c : \boldsymbol{w} \mapsto \boldsymbol{p}$ is **global univalent** if it is bijective on $\mathcal{P}$. Given that the commodity prices that are determined by international trade satisfies $\boldsymbol{p} = c(\boldsymbol{w})$ in equilibrium with $\boldsymbol{X} > \boldsymbol{0}$, the global univalence of c ensures the FPET. Under what conditions is the global univalence of c verified? Simplifying the argument, let $m = n$. As $c(\boldsymbol{w}) = \boldsymbol{w}\boldsymbol{A}(\boldsymbol{w})$ holds by definition, c is global univalent whenever $\boldsymbol{A}(\boldsymbol{w})$ is non-singular for any $\boldsymbol{w} \geq \boldsymbol{0}$. Indeed, if $\boldsymbol{A}(\boldsymbol{w}')$ is singular for some $\boldsymbol{w}' \geq \boldsymbol{0}$, then it does not have its inverse matrix, thus an infinite number of factor price vectors would satisfy the condition (2) with equality. Therefore, the FPET does not hold.

[6] It is well-known that the cost function has the following properties:

Property 1: $c_j(\boldsymbol{w})$ is differentiable with respect to $\boldsymbol{w}$.

Property 2: $c_j(\boldsymbol{w})$ is a homogeneous of degree one.

Property 3: $c_j(\boldsymbol{w})$ is concave.

Property 4: $c_j(\boldsymbol{w})$ is monotonically increasing.

See, for example, Mas-Colell et al. (1995, p. 141) with respect to the proof.

[7] That is, $\mathcal{P} \equiv \left\{\boldsymbol{p} \in \mathbb{R}^n_+ \mid \exists \boldsymbol{w} \in \mathbb{R}^m_+ : \boldsymbol{p} = c(\boldsymbol{w})\right\}$.

It is well-known by the Shephard's lemma that

$$\frac{\partial p_j}{\partial w_i} = \frac{\partial c_j(\boldsymbol{w})}{\partial w_i} = a_{ij}(\boldsymbol{w}), \quad i, j = 1, \cdots, n. \tag{5}$$

Therefore, the matrix $\boldsymbol{A}(\boldsymbol{w})$ is the Jacobian of the cost function c. Samuelson (1953) asserts that the non-vanishing Jacobian determinant of the unit cost functions is the sufficient condition for the validity of FPET in the case of $n = 2$:[8]

$$\frac{a_{11}(\boldsymbol{w})}{a_{21}(\boldsymbol{w})} > \frac{a_{12}(\boldsymbol{w})}{a_{22}(\boldsymbol{w})} \text{ for any } \boldsymbol{w} \geq \boldsymbol{0}, \text{ or } \frac{a_{11}(\boldsymbol{w})}{a_{21}(\boldsymbol{w})} < \frac{a_{12}(\boldsymbol{w})}{a_{22}(\boldsymbol{w})} < 0 \text{ for any } \boldsymbol{w} \geq \boldsymbol{0}. \tag{6}$$

In (6), Sector 1 is relatively intensive in factor 1, compared with Sector 2, in the former and is relatively intensive in factor 2 in the latter. Samuelson's (1953) use of the Jacobian matrix of the unit cost functions to characterise the condition for the validity of the FPET had a decisive impact on the direction of later research.[9]

2.1.1 No Factor Intensity Reversal in Two-commodity Case

Consider a two-commodity model in more detail. Wong (1990) considers economies with two commodities ($n = 2$) and m primary factors of production ($m \geqq 2$). For any pair of factors, say h and k ($h, k = 1, \cdots, m, h \neq k$), we define the factor intensity of Sector 1 (Υ_{hk}) and Sector 2 (Φ_{hk}), respectively, as follows:

Definition 1 (Factor Intensity): The factor intensity of Sector 1 and 2 with respect to factor h and k is respectively defined as follows:

$$\begin{cases} \Upsilon_{hk}(w_1, \cdots, w_m) = \frac{a_{h1}(w_1, \cdots, w_m)}{a_{k1}(w_1, \cdots, w_m)}, \\ \Phi_{hk}(w_1, \cdots, w_m) = \frac{a_{h2}(w_1, \cdots, w_m)}{a_{k2}(w_1, \cdots, w_m)}. \end{cases}$$

Here, no factor intensity reversal means that Sector 1 is relatively more intensive in factor h than in factor k compared with Sector 2 if and only if $\Upsilon_{hk}(\boldsymbol{w}) > \Phi_{hk}(\boldsymbol{w})$ for any $\boldsymbol{w} \geq \boldsymbol{0}$. Wong (1990) assumes that a cost function from which $\Upsilon_{hk}(\boldsymbol{w})$ and $\Phi_{hk}(\boldsymbol{w})$ are derived satisfies Properties 1–4 of footnote 6. Note that such a cost function does not necessarily mean that the underlying production function is neoclassical. The properties are valid even though the production function is of the Leontief type.

Wong (1990) demonstrates the impossibility of factor intensity reversal in two-commodity HOS models. This means that (6) is always satisfied for any $\boldsymbol{w} \geq \boldsymbol{0}$. From (4), we obtain:

[8] Note that the condition (6) can be written as:

$$\begin{aligned} \det \boldsymbol{A}(\boldsymbol{w}) &> 0 \text{ for any } \boldsymbol{w} \geq \boldsymbol{0}, \text{ or } \det \boldsymbol{A}(\boldsymbol{w}) < 0 \text{ for any } \boldsymbol{w} \geq \boldsymbol{0}, \\ \text{where } \det \boldsymbol{A}(\boldsymbol{w}) &= \det \begin{bmatrix} a_{11}(\boldsymbol{w}) & a_{12}(\boldsymbol{w}) \\ a_{21}(\boldsymbol{w}) & a_{22}(\boldsymbol{w}) \end{bmatrix} = \det \begin{bmatrix} \frac{\partial c_1(\boldsymbol{w})}{\partial w_1} & \frac{\partial c_2(\boldsymbol{w})}{\partial w_1} \\ \frac{\partial c_1(\boldsymbol{w})}{\partial w_2} & \frac{\partial c_2(\boldsymbol{w})}{\partial w_2} \end{bmatrix}. \end{aligned}$$

See the Appendix A7.1 for the rigourous proof of the FPET in the case of $n = 2$.

[9] See Chipman (1966) for a description of other HOS models based on the general equilibrium theory.

$$V_i = a_{i1}(\boldsymbol{w}) X_1 + a_{i2}(\boldsymbol{w}) X_2 \text{ for } i = 1, \cdots m.$$

For any pair of factors, h and k, therefore, we obtain:

$$\begin{aligned} \frac{V_h}{V_k} &= \frac{a_{h1}(\boldsymbol{w}) X_1}{V_k} + \frac{a_{h2}(\boldsymbol{w}) X_2}{V_k} = \frac{a_{h1}(\boldsymbol{w})}{a_{k1}(\boldsymbol{w})} \frac{a_{k1}(\boldsymbol{w}) X_1}{V_k} + \frac{a_{h2}(\boldsymbol{w})}{a_{k2}(\boldsymbol{w})} \frac{a_{k2}(\boldsymbol{w}) X_2}{V_k} \\ &= \rho(\boldsymbol{w}) \Upsilon_{hk}(\boldsymbol{w}) + (1 - \rho(\boldsymbol{w})) \Phi_{hk}(\boldsymbol{w}), \text{ where } \rho(\boldsymbol{w}) \equiv \frac{a_{k1}(\boldsymbol{w}) X_1}{V_k}. \end{aligned} \quad (7)$$

ρ denotes the share of factor k utilised by Sector 1. The incomplete specialisation equilibrium implies $\rho(\boldsymbol{w}) \in (0,1)$. The incomplete specialisation equilibrium is obtained under a condition such as the following:

Condition 1: At any given factor prices $\boldsymbol{w} \geq \mathbf{0}$, full utilisation of factors h and k with incomplete specialisation implies that one and only one of the following cases must occur:

(a) $\Upsilon_{hk}(\boldsymbol{w}) = \frac{V_h}{V_k} = \Phi_{hk}(\boldsymbol{w})$;

(b) $\Upsilon_{hk}(\boldsymbol{w}) > \frac{V_h}{V_k} > \Phi_{hk}(\boldsymbol{w})$;

(c) $\Upsilon_{hk}(\boldsymbol{w}) < \frac{V_h}{V_k} < \Phi_{hk}(\boldsymbol{w})$.

In other words, the incomplete specialisation equilibrium requires that the ratio of factor endowments $\frac{V_h}{V_k}$ in the whole economy just equal those of both sectors, $\Upsilon_{hk}(\boldsymbol{w})$ and $\Phi_{hk}(\boldsymbol{w})$, or be intermediate between the two.

From Condition 1 and Definition 1, the factor intensity reversal between factors h and k means that case (b) occurs at some factor prices and case (c) at other factor prices. The switch from case (b) to case (c), or vice versa, necessarily passes through case (a), since $\Upsilon_{hk}(\boldsymbol{w})$ and $\Phi_{hk}(\boldsymbol{w})$ are continuous functions. Note that $\Upsilon_{hk}(\boldsymbol{w}) > \Phi_{hk}(\boldsymbol{w})$ can be rewritten as $a_{h1}(\boldsymbol{w}) a_{k2}(\boldsymbol{w}) - a_{k1}(\boldsymbol{w}) a_{h2}(\boldsymbol{w}) > 0$, which satisfies (6).

The method to prove the impossibility of factor intensity reversal is thus to show the non-existence of case (a) or to show that if case (a) exists it must hold at all feasible factor prices under $\rho(\boldsymbol{w}) \in (0,1)$. To show such an impossibility, let us define that Υ_{hk} (resp. Φ_{hk}) is *invariant with respect to any change of factor prices other than h and k* if and only if $\Upsilon_{hk}(\boldsymbol{w}) = \Upsilon_{hk}(\boldsymbol{w}')$ (resp. $\Phi_{hk}(\boldsymbol{w}) = \Phi_{hk}(\boldsymbol{w}')$) holds for any $\boldsymbol{w}, \boldsymbol{w}' \geq \mathbf{0}$ with $(w_h, w_k) = (w'_h, w'_k)$. Then, we have the following.

Theorem 1 (Wong, 1990)**:** If both Υ_{hk} and Φ_{hk} are invariant with respect to any change of factor prices other than that of h and k, then factor intensity reversal is impossible for any $\boldsymbol{w} \geq \mathbf{0}$.

Proof: See the Appendix A. ■

Theorem 1 implies that if the factor intensities of the pair of factors are a function of their factor prices *alone*, and are independent of other prices, then factor intensity reversal is impossible. This implies the following:

Corollary: If there are only two factors in the whole economy with two commodities, no factor intensity reversal occurs.

A special case where no factor intensity reversal occurs is an overall economy with constant elasticity of substitution (CES) production functions, as the following theorem shows.

Theorem 2 (Wong, 1990)**:** If the technologies of both sectors are of the CES type, factor intensity reversal in any pair of factors is impossible.

Proof: See the Appendix A. ■

The theorems are the sufficient conditions for no factor intensity reversal in the HOS model with two commodities. While Theorem 1 implies that no factor intensity reversal occurs whenever the number of productive factors is $m = 2$, Theorem 2 implies that, in the case of economies with the CES technologies where $m \geqq 3$, factor intensity reversal cannot occur for any pair of factors even when $m \geqq 3$.

The CES technologies ensure that the factor intensities with respect to two factors are functions of their factor prices and independent of others, irrespective of the number of factors.[10] From Theorem 2, therefore, we can conclude that for any number of productive factors, no factor intensity reversal does not need to be assumed in the HOS model with two commodities and technologies of the CES type. As is well known, the CES production function includes a broad class of technologies: linear, the Cobb-Douglas, and the Leontief types. Therefore, we can conclude that no factor intensity reversal occurs in a broad class of technologies, as far as two-commodity models are concerned.

The proof of no factor intensity reversal in the case of $n = 2$ depends crucially on Condition 1, which characterises the equilibrium condition for incomplete specialisation. Although this equilibrium condition can be easily characterised in the case of $n = 2$, as Condition 1 shows, the characterisation in the case of $n \geqq 3$ is not as simple as in the case of $n = 2$. Therefore, Theorems 1 and 2 are applicable only to the case of $n = 2$. In contrast, in the case of $n \geqq 3$, *not only* does no factor intensity reversal not automatically hold in general, *but also* it is no longer sufficient for the validity of FPET. Indeed, an even more stringent condition should be assumed in order to ensure the FPE, as carefully discussed in sections 2.2, 2.3, 2.4, and 2.5 of Kurose and Yoshihara (2016) or Appendix B of this paper.

[10] Wong (1990) presents an example in which factor intensity reversal can occur. Suppose the following unit cost functions:

$$\begin{aligned} c_1 &= \left(\alpha_1 w_1^\beta + \alpha_2 w_2^\beta + \alpha_3 w_3^\beta\right)^{\frac{1}{\beta}}, \\ c_2 &= \theta w_1^\sigma \left(w_2^{1-\sigma} + w_3^{1-\sigma}\right), \end{aligned}$$

where $\alpha_i, \theta > 0, \beta < 1, \sigma \in (0, 1)$. The unit cost function of Sector 2 is not of the CES type function. In this case, the factor intensity between factor 1 and 2 of Sector 2 is the function of not only w_1 and w_2 but also w_3.

3 Examination of FPE in economies with reproducible capital

In the previous section, capital is treated as a primary factor of production. As the classical economists, Marx, and Sraffa emphasised, however, an essential feature of the capitalist economic system is that capital is neither homogeneous nor a primary factor of production; it consists of a bundle of heterogeneously reproducible commodities.

It was the Cambridge capital controversies that highlighted how the difference in the treatment of capital produces significant problems in economic theories. The controversies arose in the 1960's and 1970's between neo-classical economists, such as Samuelson, Solow, Modigliani, Burmeister, Meade, and Hahn, and neo-Ricardian economists in Cambridge, UK, such as J. Robinson, Pasinetti, Garegnani, Kaldor, and Sraffa. The primary sources of the controversies were the concept of capital, the logical validity of the neo-classical production function and the principle of marginal productivity.

The controversies brought to light several problematic issues with the HOS models examined in Section 2. As is shown in detail below, the issues relate to the relationship between factor intensity and factor income distribution. It is revealed that the principle of marginal productivity is not necessarily valid when capital consists of a bundle of heterogeneously reproducible commodities; 'reswitching of techniques' and 'capital reversing' could occur. The former is a phenomenon in which one technique could correspond to some rate of profit. In the latter case the decreasing monotonicity between the rate of profit and capital intensity would not necessarily hold. These phenomena imply that the properties of the neo-classical production function may not hold.

As Sraffa (1960) argues, when capital consists of a bundle of reproducible commodities, the prices of capital must be determined simultaneously with commodity prices and the rate of profit. As such, the capital endowment cannot be formulated independently of commodity prices and factor income distribution. This is in contrast with the models in Section 2, where the capital endowment is given exogenously, independently of other variables. When countries are allowed to choose their optimal techniques, one technique might correspond to multiple rates of profit. If the rate of profit is regarded as the factor price, then it would imply that there is a possibility that the commodity prices and the factor prices are not global univalent. Therefore, when capital consists of a bundle of heterogeneously reproducible commodities, it is not self-evident that factor price equalisation can still be characterised by the theorems described in Section 2. Moreover, in this case, it is not a truism that of no factor intensity reversal is the sufficient condition for global univalence (i.e. the FPET).

3.1 The Cambridge Capital Controversies

First, let us briefly review the most relevant issues highlighted by the Cambridge controversies.[11] In the neo-classical production function, which treats capital as a primary factor of production, the principle of the marginal productivity of capital is at work. The factor price

[11] The controversies covered a wide range of issues; see Blaug (1975), Cohen and Harcourt (2003), Harcourt (1972), and Pasinetti (2000) for further details on their scope.

frontier is convex toward the origin, there is a one-to-one correspondence between the rate of profit and a technique, and capital intensity is monotonically decreasing with respect to the rate of profit.

Samuelson (1962) attempts to apply these properties of the neo-classical production function to the case where capital consists of a bundle of reproducible commodities. He constructs a simple model in which one kind of consumption commodity is produced using labour and *reproducible* capital. This technique is characterised by fixed coefficients. Moreover, this model assumes capital to be heterogeneous, indexed as capital α, capital β, capital $\gamma \cdots$. Capital cannot be produced by using other types of capital. In other words, each type can only be produced using itself as the capital input (i.e. producing capital α requires capital α and labour as inputs). Furthermore, it is assumed that the capital-labour ratio used to produce each type of capital is technologically given and is the same as the capital-labour ratio for the consumption commodity when it is produced using that type of capital. This assumption is imposed on the reproduction of all capital. However, the capital-labour ratio to reproduce capital differs across types; for example, the ratio to reproduce capital β is different from the ratio to reproduce capital α.

In this case, the wage-profit curves that correspond to each type of capital become linear. Because of the assumptions that the techniques are represented by fixed coefficients and that both the consumption commodity and capital require the same capital-labour ratio, the price structure remains unaffected by changes in the factor income distribution. Thus, we obtain the outer envelope of all the straight wage-profit lines, and the outer envelope is convex to the origin.

Samuelson concludes that this outer envelope, obtained from heterogeneous capital, could sufficiently approximate the factor price frontier obtained when capital is a primary factor of production. The approximation of the production function is termed the 'surrogate production function', and the approximated capital is termed 'surrogate capital'.

If Samuelson were correct, then the principle of marginal productivity of capital could be utilised even if capital were heterogeneously reproducible. This is because there would be a one-to-one correspondence between the rate of profit and the technique. The 'non-switching' theorem proven by Levhari (1965) holds that one technique would not correspond to the same rates of profit throughout the entire economic system, which seems to further support Samuelson's conclusion.

However, Samuelson's conclusion relies crucially on the assumption that the capital-labour ratio of capital production is the same as that of consumption production, which is a singularly peculiar assumption. Pasinetti (1966) is the first to produce a counter-example to Levhari (1965)'s non-switching theorem and make clear that the surrogate production function provides no general foundation for economic analysis.[12] Without the assumption, the reswitching of techniques can occur if capital consists of a bundle of reproducible commodities, regardless of whether the techniques are decomposable.[13] Moreover, capital reversing

[12] See the set of papers published in the 'Paradoxes in Capital Theory' symposium in the *Quarterly Journal of Economics*: Bruno et al. (1966), Garegnani (1966), Levhari and Samuelson (1966), Morishima (1966), Pasinetti (1966), Samuelson (1966b).

[13] Burmeister (1980, pp. 114–115) asserts that the reswitching of techniques is an irrelevant phenomenon within the field of neo-classical economics. This is because it does not necessarily accompany the paradoxical

can occur. In other words, capital intensity may not be a monotonically decreasing function of the rate of profit. These phenomena contradict the principle of marginal productivity of capital.[14]

The Cambridge capital controversies provide sufficient reason to doubt the validity of the FPET. The FPET relies on Properties 1–4, which are based on capital as a primary factor of production and satisfied by the neo-classical production function.

3.2 The General Impossibility of FPE in Economies with Reproducible Capital: the case of $n = 2$

In the 1970s and 1980s, the HOS model was criticised by Mainwaring, Metcalfe, and Steedman exclusively on the basis of the controversies (Metcalfe and Steedman, 1972, 1973; Mainwaring, 1984; Steedman, 1979). Although they present a number of numerical examples and criticise the FPET, they do not rigorously show the (in)validity of the FPET under the Leontief production model. Thus, we shall derive a theorem, which states that the FPET is valid under the Leontief production model with $n = 2$. In this case, however, the necessary and sufficient conditions for factor price equalisation will be extremely restrictive.

The basic premise of the model is that labour is the only primary factor and that all physical input is composed of reproducible commodities. One technique is represented by the Leontief production model; consequently, the equilibrium price of commodity j $(j = 1, \cdots, n)$ is given as follows:

$$p_j = l_j w + (1 + r) \sum_{i=1}^{n} a_{ij} p_i,$$

where $l_j > 0, r \geqq 0$, and $w \geqq 0$ denote the labour coefficient to produce a unit of commodity j, the rate of profit, and the wage rate, respectively. For simplicity, we assume that capital is circulating. In general, there are some Leontief techniques available for the production of commodity j. The criterion for the choice of techniques is to minimise the production cost given a certain price system. Suppose that the following is the cost minimising technique under price system $(\boldsymbol{p}, w, r)$:

$$\Big(\left(a_{ij} \left(\boldsymbol{p}, w, r \right) \right)_{i=1,\dots,n}, l_j \left(\boldsymbol{p}, w, r \right) \Big) \text{ for } \forall j = 1, \cdots, n.$$

behaviour of consumption. Neo-classical economic thought maintains that the steady state consumption level per capita is a monotonically decreasing function of the rate of profit. This simple relation is a parable derived from a one-commodity model. According to Burmeister, any phenomenon which is not inconsistent with the neo-classical parable does not vitiate the essence of neo-classical economics.

[14]See Pasinetti (1977, chap. 6). There were three types of reactions from neo-classical economics against the critiques levelled by the neo-Ricardians. The first was to describe the phenomenon as a 'paradox', 'perverse', 'exceptional', 'inconvenient', or 'anomalous', that is, contending that the phenomenon was rarely observed in reality and therefore irrelevant (Blaug, 1975; Samuelson, 1966b). The second was to attempt to investigate the conditions under which capital reversing or reswitching of techniques would not take place (Burmeister and Dobell, 1970; Burmeister, 1980). The third was to assert that the neo-Ricardian model was merely a special case of the intertemporal general equilibrium model. The latter model could consequently be freed from the neo-Ricardian critiques (Bliss, 1975; Hahn, 1982). See Pasinetti (2000) concerning this topic in detail.

Then, the following theorem is valid.

Theorem 3: In the case of $n = 2$ under the Leontief production model, the commodity price and the rate of profit are global univalent if and only if there is no capital intensity reversal.

Proof: See the Appendix A. ■

The relationship between the factor price frontier and relative price is such that:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{l_1(\boldsymbol{p}, w, r)\{1 - (1+r)a_{22}(\boldsymbol{p}, w, r) + (1+r)l_2(\boldsymbol{p}, w, r)a_{21}(\boldsymbol{p}, w, r)\}}{2a_{21}(\boldsymbol{p}, w, r)}\frac{\mathrm{d}^2 w^1}{\mathrm{d}r^2},$$

which implies:

$$sign\left(\frac{\mathrm{d}^2 w^1}{\mathrm{d}r^2}\right) = -sign\left(\frac{\mathrm{d}p}{\mathrm{d}r}\right).$$

As Mainwaring (1984) points out, the relative capital intensity determines the sign of $\frac{\mathrm{d}p}{\mathrm{d}r}$, which in turn determines the form of the factor price frontier in the two-commodity Leontief production model. If the numéraire industry is more capital intensive (labour intensive) than the other industry, the relative price will be a decreasing (increasing) function of the rate of profit and the factor price frontier is concave (convex) to the origin. The inter-linkage between sectors determines whether the relative price is increasing or decreasing with respect to the rate of profit, which in turn determines whether the factor price frontier is concave or convex. This is a particular feature of the model in which capital consists of reproducible commodities and is never observed in the model examined in Section 2.

Under a convex production set, the technical change that reverses the size of $\frac{l_1(\boldsymbol{p},w,r)a_{12}(\boldsymbol{p},w,r)+l_2(\boldsymbol{p},w,r)a_{22}(\boldsymbol{p},w,r)}{l_2(\boldsymbol{p},w,r)}$ and $\frac{l_1(\boldsymbol{p},w,r)a_{11}(\boldsymbol{p},w,r)+l_2(\boldsymbol{p},w,r)a_{21}(\boldsymbol{p},w,r)}{l_1(\boldsymbol{p},w,r)}$ is not particular at all. In some limited cases, factor intensity reversal may not occur. Therefore, we may interpret Theorem 3 as a *de facto* impossibility theorem of factor price equalisation.

Indeed, the following example of the Leontief production model with $n = 2$ shows that factor intensity reversal occurs.

Example: Assume a Leontief production economy with two commodities in which only one technique is available to produce commodity 1 while two techniques, α and β, are available to produce commodity 2; $(\boldsymbol{a}_1, l_1) \equiv \left(\begin{bmatrix}0.001\\0.25\end{bmatrix}, 1\right)$ for the production of commodity 1; $(\boldsymbol{a}_2^\alpha, l_2^\alpha) \equiv \left(\begin{bmatrix}0.5\\0.002\end{bmatrix}, \frac{1}{3}\right)$ and $\left(\boldsymbol{a}_2^\beta, l_2^\beta\right) \equiv \left(\begin{bmatrix}1/6\\0.0015\end{bmatrix}, 2\right)$ for the production of commodity 2. Therefore, the techniques available to produce the commodities in the whole economy are given by $(\boldsymbol{A}^\alpha, \boldsymbol{l}^\alpha) \equiv ([\boldsymbol{a}_1, \boldsymbol{a}_2^\alpha], (l_1, l_2^\alpha)) = \left(\begin{bmatrix}0.001 & 0.5\\0.25 & 0.002\end{bmatrix}, (1, \frac{1}{3})\right)$ and $(\boldsymbol{A}^\beta, \boldsymbol{l}^\beta) \equiv \left(\left[\boldsymbol{a}_1, \boldsymbol{a}_2^\beta\right], \left(l_1, l_2^\beta\right)\right) = \left(\begin{bmatrix}0.001 & 1/6\\0.25 & 0.0015\end{bmatrix}, (1, 2)\right)$.

Let the price of commodity 1 be the numéraire. Then, the wage-profit curve of technique α is given as follows:

$$w^{\alpha}(r) \cong \frac{1.308 \times 10^{6} - 379494r - 187497r^{2}}{1.622 \times 10^{6} + 122000r},$$

and that of technique β is:

$$w^{\beta}(r) \cong \frac{5.73501 \times 10^{6} - 514982r - 249991r^{2}}{8.991 \times 10^{6} + 2.991 \times 10^{6}r}.$$

Figure 1 depicts the factor price frontier that is obtained from the above wage-profit curves.

Insert Figure 1 here.

It demonstrates that $r \cong 1.0364$, shown by point A in the figure, is a switching point. The cost minimising technique is α if $r \in [0, 1.0364]$ and β if $r \in [1.0364, 3.8692]$, where $r = 3.8692$ is the maximum rate of profit obtainable under technique β. Neither reswitching of techniques nor capital reversing can be observed.

Figure 2 depicts the capital intensities of commodity 1 and 2 in terms of the price of commodity 1. It demonstrates that commodity 1 is more labour intensive than commodity 2 if $r \in [0, 1.0364]$ but more capital intensive if $r \in [1.0364, 3.8692]$. In other words, factor intensity reversal does occur. Figure 3 depicts the relative price of commodity 2. It shows that the relative price and the rate of profit is not global univalent. It is an increasing function of the rate of profit if $r \in [0, 1.0364]$ since the numéraire is a labour intensive commodity under technique α, while it is a decreasing function of the rate of profit if $r \in [1.0364, 3.8692]$ since the numéraire is a capital intensive commodity under technique β.[15] ■

Insert Figures 2 and 3 here.

One can clearly recognise, in a two-commodity and two-factor model, a strict difference between the case where capital is a primary factor and the case where capital consists of a bundle of reproducible commodities. While factor intensity reversal is impossible in the former case (by Theorem 1), it is *possible* in the latter. Therefore, while factor price equalisation is always observed where capital is a primary factor, it is not generally observed where capital consists of a bundle of reproducible commodities. This difference is attributable, as noted earlier, to the fact that there is a complex inter-linkage between sectors in the latter case but not in the former.

[15] We obtain the relative prices, the capital intensities as the function of the rate of profit as follows:

$$\begin{cases} p^{\alpha}(r) \cong \frac{10.2418+6.14344r}{13.2951+r} & \text{if } r \in [0, 1.0364], \\ p^{\beta}(r) \cong \frac{12988+988r}{8991+2991r} & \text{if } r \in [1.0364, 3.8692], \end{cases}$$

$$\left\{ \begin{array}{l} k_1^{\alpha}(r) \cong \frac{2.57375+1.53686r}{13.2951+r} \\ k_2^{\alpha}(r) \cong \frac{10.0328+0.786861r}{13.2951+r} \end{array} \right. \quad \text{if } r \in [0, 1.0364],$$

$$\left\{ \begin{array}{l} k_1^{\beta}(r) \cong \frac{1.0886+0.0835811r}{3.00602+r} \\ k_2^{\beta}(r) \cong \frac{0.253758+0.0835811r}{3.00602+r} \end{array} \right. \quad \text{if } r \in [1.0364, 3.8692].$$

3.3 A Further Neo-Ricardian Critique of the FPET

The previous subsection demonstrates that factor intensity could be easily reversed in a two-commodity model when capital consists of reproducible commodities. Now let us make a strict distinction between intermediate goods and final goods and consider whether no factor intensity reversal is the sufficient condition for the global univalence between the relative prices and the rate of profit.

To answer this question, we utilise a two-integrated-sector model.[16] Suppose that each of *Sectors* 1 and 2 is composed of industries that produce a final good and an intermediate good. The final good, called *commodity* 1 hereafter, producing industry of *Sector* 1 is termed *Industry* 1 and the intermediate good, called *commodity* $1_{(\mathrm{m})}$ hereafter, producing industry of *Sector* 1 is *Industry* 2. Similarly, the final good, called *commodity* 2 hereafter, producing industry of *Sector* 2 is *Industry* 3, and the intermediate good, called *commodity* $2_{(\mathrm{m})}$ hereafter, producing industry of *Sector* 2 is *Industry* 4.

Let us assume that Industry 1 has three available techniques:

$$\left(a_{11}^{\alpha}, a_{1_{(\mathrm{m})}1}^{\alpha}, l_1^{\alpha}\right) = (0.38, 0.63, 0.06)\,,$$
$$\left(a_{11}^{\beta}, a_{1_{(\mathrm{m})}1}^{\beta}, l_1^{\beta}\right) = (0.4188, 0.424, 0.265)\,,$$
$$\left(a_{11}^{\gamma}, a_{1_{(\mathrm{m})}1}^{\gamma}, l_1^{\gamma}\right) = (0.52, 0.01, 0.65)\,,$$

where $a_{ij}^{\iota}, l_j^{\iota}$ denote the amount of commodity i and labour that are required to produce a unit of commodity j under technique ι $(\iota = \alpha, \beta, \gamma)$. On the other hand, Industry 2 has only one available technique:

$$\left(a_{11_{(\mathrm{m})}}, a_{1_{(\mathrm{m})}1_{(\mathrm{m})}}, l_{1_{(\mathrm{m})}}\right) = (0.08, 0, 1)\,.$$

Figure 4 depicts the factor price frontier that is the outer envelope of the wage-profit curves obtained under each technique. The vertical axis of the figure represents the wage rate in terms of the commodity produced by Industry 1, w_1.

Insert Figure 4 here.

There are four switching points: technique α is chosen if $0 \leqq r \leqq r_1 \cong 0.18$; technique β is chosen if $r_1 \leqq r \leqq r_2 \cong 0.317$; technique γ is chosen if $r_2 \leqq r \leqq r_3 \cong 0.503$; technique β is chosen again if $r_3 \leqq r \leqq r_4 \cong 0.9003$;and technique α is chosen again if $r_4 \leqq r \leqq R_{\alpha} \cong 1.066$, where R_{α} is the maximum rate of profit in Sector 1. The reswitching of techniques occurs.

Let $w_1^{\iota}(r)$ denote the wage rate measured by the final commodity produced in Industry 1 under technique ι. Let $k_1(r)$ denote the capital intensity in terms of the final commodity. $k_1(r)$ is defined as follows:[17]

[16] This model is a revised version of Kurose and Yoshihara (2015), which was written in Japanese.

[17] See Garegnani (1970) concerning the derivation in detail

$$k_1(r) = \begin{cases} \left| \frac{\mathrm{d}w_1^{\alpha}(r)}{\mathrm{d}r} \mid_{r=0} \right|, \text{ if } r = 0, \\ \frac{w_1^{\alpha}(0) - w_1^{\alpha}(r)}{r}, \text{ if } 0 < r \leqq r_1 \text{ and } r_4 \leqq r \leqq R_{\alpha}, \\ \frac{w_1^{\beta}(0) - w_1^{\beta}(r)}{r}, \text{ if } r_1 \leqq r \leqq r_2 \text{ and } r_3 \leqq r \leqq r_4, \\ \frac{w_1^{\gamma}(0) - w_1^{\gamma}(r)}{r}, \text{ if } r_2 \leqq r \leqq r_3, \end{cases} \tag{8}$$

The switch from γ to β at $r = r_3$ and from β to α at $r = r_4$ does not adhere to the monotonically decreasing relationship between the rate of profit and capital intensity. In other words, there is capital reversing.

With respect to Sector 2, let us assume that Industry 3 has two alternative techniques:

$$\left(a_{22}^{\delta}, a_{2_{(\mathrm{m})}2}^{\delta}, l_2^{\delta}\right) = (0.2, 0.485, 0.03),$$
$$\left(a_{22}^{\epsilon}, a_{2_{(\mathrm{m})}2}^{\epsilon}, l_2^{\epsilon}\right) = (0.3, 0.41, 0.02).$$

On the other hand, Industry 4 has only one available technique:

$$\left(a_{22_{(\mathrm{m})}}, a_{2_{(\mathrm{m})}2_{(\mathrm{m})}}, l_{2_{(\mathrm{m})}}\right) = (0.29, 0, 1.61).$$

Letting w_2 be the wage rate in terms of the final commodity produced in Industry 3, Figure 5 depicts the factor price frontier.

Insert Figure 5 here.

In Sector 2, ε switches to δ at $r = r_5 \cong 0.205$ but the techniques do not reswitch and there is no capital reversing. Using the same procedure described in (8), we can obtain capital intensity in terms of the final commodity produced in Industry 3, which is denoted as $\overline{k}_2(r)$.

Let p_1 and p_2 be the price of the final commodities produced in Industry 1 and Industry 3, respectively. In order to compare the capital intensity of both sectors, they must first be measured in terms of the same commodity price. Let $k_2(r)$ be the capital intensity of Sector 2 in terms of the final commodity produced in Industry 1. Then, $k_2(r) = \overline{k}_2(r) \times \frac{p_2}{p_1}$ is, by definition, given. Since the wage rate is uniform in both sectors, $\frac{p_2}{p_1} = \frac{w_1}{w_2}$, which implies that $k_2(r) = \overline{k}_2(r) \times \frac{w_1}{w_2}$.

Table 1 presents a summary of the above model.

Insert Table 1 here.

Table 1 shows that Sector 2 is always more capital intensive than Sector 1, which means that no capital intensity reversal occurs in this model (see Figure 6 as well). Moreover, it shows that the relative price, p_2/p_1, is not a monotonic function of the rate of profit (see Figure 7). This means that the relative price and the rate of profit are not global univalent.

Insert Figures 6 and 7 here.

In other words, the FPET does not necessarily hold, even in the absence of factor intensity reversal, if capital consists of a bundle of reproducible commodities. This is a problem that neo-classical economists, who treat capital merely as a primary factor, cannot neglect.

Our numerical example featured four commodities. Two are, so to speak, the intermediate goods, while the other two are final goods that are traded internationally. Then, the above numerical example shows that the global univalence between the relative price and the rate of profit does not necessarily hold, even in the absence of factor intensity reversal, if capital consists of a bundle of reproducible commodities. In other words, we demonstrate that no factor intensity reversal cannot be sufficient for the FPET when capital is composed of a bundle of heterogeneously reproducible commodities.

Note that Dixit (1981, pp. 291-292) argues that the existence of non-tradable goods as circulating capital inputs other than the two tradable, final consumption goods makes it impossible to derive a 'simple condition'[18] that ensures the univalent relation between prices of commodities and factor prices, since the expression for the elasticity of a factor price frontier involves indirect effects that work through induced changes in the prices of non-tradable goods. In comparison to Dixit's (1981) claim, what we have discussed above provides a strengthening of the impossibility of the univalence relation, since the standard condition of no capital intensity reversal is satisfied by the economy constructed in this subsection. It may suggest a more fundamental source of the impossibility of the univalence relation, which remains to a future research agenda.

4 The HOS Model after the Cambridge Capital Controversies

Regarding the introduction of capital as a bundle of reproducible commodities into the HOS model, Samuelson first argues:

> Now suppose there are uniform differences in factor intensity, so that for some two goods that are simultaneously produced in both countries, say goods 1 and 2, $p_1(r)/p_2(r) = p_{12}(r)$ is a monotone, strictly increasing (or decreasing) function of r [the interest rate]. *Then, the interest rate will be equalized by positive trade in those goods alone*' (Samuelson, 1965, p. 49).

Bliss (1967) criticises Samuelson (1965), arguing that the problem is the condition for the monotonic relationship between the relative price and the rate of profit. However, Samuelson said nothing of this.[19]

In light of Bliss' (1967) and Metcalfe and Steedman's (1972, 1973) critiques, Samuelson (1975) acknowledges the possibility that the FPET might not hold globally but rather just

[18] Here, the 'simple condition' would be the standard condition of no capital intensity reversal.

[19] See Samuelson (1978) for more on this point.

locally when capital consists of a bundle of reproducible commodities (i.e. the local factor equalisation theorem). However, Samuelson (1975, p. 351) believes that Metcalfe and Steedman's (1972, 1973) warning is non-academic and thus gives it little credence. Just as the neo-classicals argued about the Cambridge capital controversies, he contends that the phenomena described by Metcalfe and Steedman are unlikely to occur in reality.

Following the neo-Ricardian critiques, however, Burmeister (1978) constructs the most rigorous model with reproducible capital, as discussed in the following subsection.

4.1 Burmeister (1978)

As Samuelson frankly admits, the FPET does not necessarily hold if capital consists of a bundle of reproducible commodities. Therefore, stronger conditions must be imposed on the model in order for the FPET to hold.

To discuss this point in detail, let us define the following matrix:

Definition 2: A square matrix, $\boldsymbol{A}$, is termed a P-matrix if all the principal minors are positive.

Burmeister (1978) specifies the conditions by using a P-matrix, which generalised the Stolper–Samuelson Theorem put forward by Chipman (1969), Inada (1971), and others. Inada (1971) modifies the SSS (Strong Stolper-Samuelson) condition defined in section 2.2 of Kurose and Yoshihara (2016) as well as in Appendix B as follows:

Condition 2 (SSS-I): All the diagonal elements of $\boldsymbol{A}^{-1}$ are positive and all the off-diagonal elements are negative.

Condition 3 (SSS-II): All the diagonal elements of $\boldsymbol{A}^{-1}$ are negative and all the off-diagonal elements are positive.

Here, $\boldsymbol{A}$ has no restriction except to be a square and non-negative matrix in contrast with Chipman's (1969) SSS condition shown in section 2.2 of Kurose and Yoshihara (2016) as well as in Appendix B.

Burmeister (1978) assumes that there are n consumption goods, m reproducible capital goods, and h primary factors, where $h \leqq n$. First, let us consider an economy in which there is no opportunity to choose techniques. The capital good price vector is denoted by $\boldsymbol{p} \equiv [p_i]$ $(i = 1, 2, \cdots m)$, the consumption good price vector by $\boldsymbol{s} \equiv [s_i]$ $(i = m+1, m+2, \cdots, m+n)$, and the primary factor price vector by $\boldsymbol{w} \equiv [w_i]$ $(i = 1, 2, \cdots, h)$. Furthermore, the capital coefficient matrix is represented by a $m \times (m+n)$ matrix $\boldsymbol{A}$,[20] and the primary factor coefficient matrix is represented by a $h \times (m+n)$ matrix $\boldsymbol{e}$.[21]

[20]Note that $\boldsymbol{A} \equiv \begin{bmatrix} a_{11} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1,m+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{m,m+n} \end{bmatrix}$.

[21]Note that $\boldsymbol{e} \equiv \begin{bmatrix} e_{11} & \cdots & e_{1m} & e_{1,m+1} & \cdots & e_{1,m+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e_{h1} & \cdots & e_{hm} & e_{h,m+1} & \cdots & e_{h,m+n} \end{bmatrix}$.

Consequently, the price equation is given as follows:

$$[\boldsymbol{p},\boldsymbol{s}] = \boldsymbol{w}\,\boldsymbol{e} + (1+r)\,\boldsymbol{p}\boldsymbol{A}, \tag{9}$$

where $[\boldsymbol{p},\boldsymbol{s}] = [p_1, \cdots, p_m, s_{m+1}, \cdots, s_{m+n}]$, and r denotes the rate of interest here. It is assumed that n consumption goods and at least one capital good are freely traded internationally and that consumption good 1 is adopted as the numéraire ($s_{m+1} = 1$).

Following Sraffa's (1960) terminology, consumption goods are the 'non-basic' goods under the above assumptions, and as such, the production condition of those 'non-basic' goods has no effect on the interest rate or the prices of the 'basic' goods.[22] This implies that, without losing generality, we can assume that $n = h$. As such, matrices $\boldsymbol{A}$ and $\boldsymbol{e}$ can be compiled as a $(m+h) \times (m+n)$ matrix $\overline{\boldsymbol{A}}$.[23]

Let us assume that $\overline{\boldsymbol{A}}$ is non-singular and defined as follows:

$$\boldsymbol{B} \equiv \overline{\boldsymbol{A}}^{-1} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{B}_3 & \boldsymbol{B}_4 \end{bmatrix}.$$

$\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3$, and $\boldsymbol{B}_4$ denote a square matrix of order m, a $(m \times h)$ matrix, a $(h \times m)$ matrix, and a square matrix of order h, respectively. Letting $\overline{\boldsymbol{s}} \equiv \left[1, \frac{s_{m+2}}{s_{m+1}}, \cdots, \frac{s_{m+h}}{s_{m+1}}\right]$ be the consumption good price vector when $s_{m+1} = 1$ allows us to rewrite (9) as follows:

$$[\boldsymbol{p}, \overline{\boldsymbol{s}}] = [(1+r)\,\boldsymbol{p}, \boldsymbol{w}]\,\overline{\boldsymbol{A}}, \text{ or} \tag{10}$$

$$[\boldsymbol{p}, \overline{\boldsymbol{s}}]\,\overline{\boldsymbol{A}}^{-1} = [\boldsymbol{p}, \overline{\boldsymbol{s}}]\,\boldsymbol{B} = [(1+r)\,\boldsymbol{p}, \boldsymbol{w}]\,. \tag{11}$$

Here, $\overline{\boldsymbol{s}}$ is regarded as a given vector, as it is assumed to be determined by free trade. Because of the assumption that $\overline{\boldsymbol{A}}$ is non-singular, $\boldsymbol{p}\boldsymbol{B}_2 + \overline{\boldsymbol{s}}\boldsymbol{B}_4 = \boldsymbol{w}$ holds. Moreover, as at least one of m capital goods is traded (say capital good 1), the price p_1 is regarded as given. Then if every capital good price is monotonic in the rate of interest r, r is uniquely determined by the fixed p_1. The remaining $m-1$ capital good prices $\boldsymbol{p}_{-1}$ are also uniquely determined corresponding to the fixed r. Then, as $\boldsymbol{p}\boldsymbol{B}_2 + \overline{\boldsymbol{s}}\boldsymbol{B}_4$ is fixed, the primary factor prices $\boldsymbol{w}$ are also uniquely determined.

By transforming the first m equations of (11), we obtain:

[22] In order to properly analyse techniques and factor income distributions, it is important to clearly distinguish between basic and non-basic goods. According to Sraffa (1960), a 'basic' good is a commodity that is directly or indirectly required for the production all commodities, while 'non-basic' goods encompass all other commodities.

[23] Note that the matrix $\overline{\boldsymbol{A}}$ is given as follows:

$$\overline{\boldsymbol{A}} \equiv \begin{bmatrix} a_{11} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1,m+h} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{m,m+h} \\ e_{11} & \cdots & e_{1m} & e_{1,m+1} & \cdots & e_{1,m+h} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e_{h1} & \cdots & e_{hm} & e_{h,m+1} & \cdots & e_{h,m+h} \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{A}}_1 & \overline{\boldsymbol{A}}_2 \\ \overline{\boldsymbol{A}}_3 & \overline{\boldsymbol{A}}_4 \end{bmatrix}.$$

$$\boldsymbol{p}\left[\boldsymbol{B}_1-(1+r)\,\boldsymbol{I}\right]=-\overline{\boldsymbol{s}}\boldsymbol{B}_3, \tag{12}$$

where $\boldsymbol{I}$ is an identity matrix of order m. By (12), we can show that, given the consumption good price vector $\overline{\boldsymbol{s}}$ and the price p_1 of capital good 1, the FPET is validated whenever the prices of capital goods are a monotonic function of the rate of interest. This is summarized in the following theorem:

Theorem 4 (Burmeister, 1978): Let the production of m capital goods and h ($\leqq n$) consumption goods satisfy the SSS-II (the SSS-I) condition for $\overline{\boldsymbol{A}}$. Then, $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}<(>)\,\mathbf{0}$ holds, and trades in n consumption goods and any one of m capital goods alone will equalise the remaining $m-1$ capital good prices and $h+1$ factor prices, $\boldsymbol{w}$ and r.

Proof: See the Appendix A. ■

Theorem 4 implies that, given $\overline{\boldsymbol{A}}$ and the consumption good price vector, the capital good price vector is a monotonic function of the rate of interest if the SSS-I or SSS-II condition for $\overline{\boldsymbol{A}}$ is satisfied.

Next, let us consider an economy in which there is a choice of techniques, as in the neo-classical production function. The relationship between factor rent, q_i, and capital good prices, p_i, can be obtained in equilibrium: $p_i=\frac{q_i}{1+r}$. Differentiating (10) with respect to r yields:

$$\left[\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r},\frac{\mathrm{d}\overline{\boldsymbol{s}}}{\mathrm{d}r}\right]=\left[\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}r},\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}r}\right]\overline{\boldsymbol{A}}\left(\boldsymbol{q},\boldsymbol{w}\right)+\left[\boldsymbol{q},\boldsymbol{w}\right]\frac{\mathrm{d}\overline{\boldsymbol{A}}\left(\boldsymbol{q},\boldsymbol{w}\right)}{\mathrm{d}r},$$

where $\boldsymbol{q}\equiv[q_i]$ ($i=1,2,\cdots,m$). By the Shephard's Lemma, $[\boldsymbol{q},\boldsymbol{w}]\,\frac{\mathrm{d}\overline{\boldsymbol{A}}(\boldsymbol{q},\boldsymbol{w})}{\mathrm{d}r}=\mathbf{0}$ holds. Therefore, we obtain:

$$\left[\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r},\frac{\mathrm{d}\overline{\boldsymbol{s}}}{\mathrm{d}r}\right]\boldsymbol{B}\left(\boldsymbol{q},\boldsymbol{w}\right)=\left[\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}r},\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}r}\right]. \tag{13}$$

Here, we assume that $\boldsymbol{B}\left(\boldsymbol{q},\boldsymbol{w}\right)=\overline{\boldsymbol{A}}^{-1}\left(\boldsymbol{q},\boldsymbol{w}\right)$ holds for all the possible technique choices. In other words, when the change in prices leads to a change in optimally chosen techniques (and thus the elements of $\overline{\boldsymbol{A}}\left(\boldsymbol{q},\boldsymbol{w}\right)$ change), the change in techniques is limited so as to preserve $\overline{\boldsymbol{A}}\left(\boldsymbol{q},\boldsymbol{w}\right)$ as non-singular.

As before, given the consumption good price vector and at least one of the capital good prices, we can confirm that the FPET is validated whenever the prices of capital goods are a monotonic function of the rate of interest. This is summarized in the following theorem:

Theorem 5 (Burmeister, 1978): Let all countries produce m capital goods and h ($\leqq n$) consumption goods subject to the neo-classical production function (4), and let the SSS-II (or SSS-I) condition for $\overline{\boldsymbol{A}}\left(\boldsymbol{q},\boldsymbol{w}\right)$ be satisfied at every feasible factor price $(\boldsymbol{q},\boldsymbol{w})=(q_1,\cdots,q_m,w_1,\cdots,w_h)$ where $q_i=(1+r)\,p_i$. Then, $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}<(>)\,\mathbf{0}$, and trades in n consumption goods and any one of m capital goods alone will equalise the remaining $m-1$ capital good prices and $h+1$ factor prices, $\boldsymbol{w}$ and r.

Proof: See the Appendix A. ■

Theorem 5 implies that, given the consumption good price vector, the capital good price vector is a monotonic function of the interest rate if capital goods are produced on the basis of the neo-classical production function and the SSS-I or SSS-II condition is satisfied. This means that there is a one-to-one correspondence between capital good prices and capital rental rates, and thus, capital good rental rates are internationally equalised under the equilibrium price system. On the other hand, due to the implicit assumption that $\boldsymbol{B}_3$ always exists, there is a one-to-one correspondence between the primary factor price vector and the consumption good price vector, and thus, factor prices equalise.

Unlike the traditional HOS model, Theorem 5 treats capital as a bundle of reproducible commodities. This begs the question of how Burmeister's model is related to the outcome of the Cambridge capital controversies, which we can evaluate by simplifying the model.

The simplest case of Burmeister's model features one consumption good, one capital good, and one primary factor (i.e. labour).[24] In this case, (10) is rewritten as follows:

$$\begin{aligned}[p, s] &= [(1+r)\,p, w]\,\overline{\boldsymbol{A}}, \text{ where} \\ \overline{\boldsymbol{A}} &\equiv \begin{bmatrix} a_{11} & a_{12} \\ l_1 & l_2 \end{bmatrix}. \end{aligned} \tag{14}$$

Despite the fact that capital goods are reproducible, (14) is a *de facto* one-good model with respect to the determination of factor prices because the consumption goods are non-basic goods. Since $\frac{\mathrm{d}p}{\mathrm{d}r} > 0$ holds because of (14), the model maintains a one-to-one correspondence between the rate of interest (or factor rental) and the price of capital goods, given the price of consumption goods.[25] This implies that the factor prices equalise. However, it is obvious that the simplified model simply avoids the difficulties pointed out by the Cambridge capital controversies because it is a *de facto* one-good model.

In other words, Burmeister's (1978) model is structured to circumvent the issues pointed out in the Cambridge capital controversies as it assumed away several economic environments. For instance, unlike the Leontief production model, in Burmeister's (1978) model there are never reproducible goods, like corn, which can be used as both capital and consumption goods.

The simplest example in which there exist commodities that can be used as both capital and consumption goods is a two-good economy where both commodities are basic goods. In this case, we obtain:

[24] In fact, the modern dynamic HOS model features the same structure. See, for example, Chen (1992), Nishimura and Shimomura (2002, 2006), and Bond et al. (2011, 2012).

[25] In this case, not only the price of capital goods but also that of consumption goods has a one-to-one correspondence with the rate of interest. Due to $p = \frac{wl_1}{1-(1+r)a_{11}}$, we obtain:

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{wl_1a_{12}\,[1-(1+r)\,a_{11}] + (1+r)\,wl_1a_{12}a_{11}}{[1-(1+r)\,a_{11}]^2} > 0.$$

$$\boldsymbol{A} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > \boldsymbol{0};\ \boldsymbol{L} \equiv (l_1, l_2) > \boldsymbol{0}.$$

By applying Burmeister (1978) to this case, we find that $\overline{\boldsymbol{A}}$ is given as follows:

$$\overline{\boldsymbol{A}} \equiv \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ l_1 & l_2 & 0 \end{bmatrix}.$$

Here, $\det \overline{\boldsymbol{A}} = 0$, which means that $\overline{\boldsymbol{A}}$ is singular. Therefore, we cannot obtain the property that is guaranteed by Theorems 4 and 5.

5 Concluding Remarks

In this paper, we examined the FPET and its generalization within the framework of the HOS trade theory by focusing on its relevance to the Cambridge capital controversies. The lesson raised in the controversies was that capital should be seriously formulated as a bundle of reproducible commodities rather than as a primary factor. The HOS trade theory has not followed this path. Our review suggests that such a lesson can be applied even to the case of international trade theory.

Indeed, we have observed that the FPET, one of the fundamental theorems in the HOS trade theory, depends crucially on the HOS assumption of capital as a primary factor. First, we have considered production economies with two commodities where each is used as a capital good as well as a consumption good. Thus capital is defined as a bundle of reproducible commodities and labour is the unique primary factor. In such economies, the factor price equalisation (FPE, hereafter) holds if and only if no capital intensity reversal arises, as Theorem 3 states. Moreover, FPE cannot be observed in general as capital intensity reversal may easily arise, as suggested by Example. This is in sharp contrast to the case of the standard HOS model with two commodities and two factors, where capital intensity reversal never arises, as implied by Theorem 1.

Second, we have considered production economies with two final goods and two intermediate goods. Only the former are traded internationally and consumed as well as used as capital inputs, while the latter are used only as capital inputs. In such economies, even if there is no reversal of capital intensity, the global univalence between the rate of profit and the relative price may not hold, as observed in Section 3.3. In other words, even in a two final-commodity model, the FPET may not necessarily hold if capital is treated as a bundle of reproducible commodities. This implies that no factor intensity reversal cannot be a sufficient condition for the FPET. All of these arguments suggest that it is necessary to construct a basic theory of international trade by treating capital as a bundle of reproducible inputs, which may not rely on factor price equalisation.

In this respect, the contribution of Burmeister (1978), among many in the literature of the HOS trade theory, is worth emphasising. He derives the conditions for factor price equalisation under the assumption that there exist reproducible capital goods. The modern dynamic HOS models that feature reproducible capital goods, such as Chen (1992), Nishimura and

Shimomura (2002, 2006), and Bond et al. (2011, 2012), have essentially the same structure as Burmeister (1978). However, as mentioned in Section 4.1, these models can circumvent the issues pointed out in the Cambridge capital controversies.

Finally, let us briefly remark on the recent works of the 'multiple-cone' HOS models in international trade, such as Bernhofen (2009a, 2009b), Helpman (1984), and Schott (2003).[26] The 'multiple-cone' HOS models address cases where the factor endowments of countries are so divergent that factor prices cannot be equalised in the equilibrium and thus all countries specialise in the production of different commodities. As Bernhofen (2009b) explicitly argued, multiple cones can be observed in economies with two primary factors and more than three commodities. This result *per se* does not refute the FPET, since it is already acknowledged by the necessary and sufficient condition of the FPE in Blackorby et al. (1993) (see section 2.5 of Kurose and Yoshihara (2016) or Theorem 5B in Appendix B of this paper) that the FPE would be unlikely observed in economies with two primary factors and more than two commodities. In such economies, the 'multiple-cone' HOS models would be useful to analyze the generation of trade patterns. In this respect, the arguments in the 'multiple-cone' HOS models are compatible with (or, even the complement of) the FPET of the standard HOS theory. Indeed, in the basic model of international trade with two productive factors and two commodities, FPE is always observed whereas no multiple cones can be observed in incompletely specialized equilibria. In contrast, in this paper we have provided a more fundamental criticism of the standard HOS trade theory. The FPE may not be generally observed, even in economies with two productive factors and two commodities, whenever capital is formulated as a bundle of reproducible commodities. Our criticism would still be relevant even if the standard HOS theory developed the analysis based on the 'multiple-cone' model.

6 References

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[26] Deardorff (1994) examines the condition for factor price equalisation, called the lens condition, under the assumption that the multiple cones exist. See also Demiroglu and Yun (1999).

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7 Appendix A

7.1 The Proof of the FPET in the two-factor and two-commodity model

Proof: In order to prove the global univalence of the cost function, c, it is sufficient to show that, for price equation $\boldsymbol{p} = \boldsymbol{w}\boldsymbol{A}(\boldsymbol{w})$, the relative price, $p_1'(\boldsymbol{w}) \equiv \frac{p_1}{p_2}$, and $\boldsymbol{w}$ are global univalent. In other words, when the factor prices change in such a way that $\boldsymbol{w} \to \boldsymbol{w} + \triangle \boldsymbol{w} \equiv (w_1 + \triangle w, w_2 - \triangle w)$ for $\triangle w > 0$, it must be confirmed that $p_1'(\boldsymbol{w})$ and $\boldsymbol{w}$ are global univalent if $p_1'(\boldsymbol{w}) \to p_1'(\boldsymbol{w} + \triangle \boldsymbol{w})$ is monotonic. It is necessary to show that $\frac{\partial p_1'(\boldsymbol{w})}{\partial w_1} \triangle w - \frac{\partial p_1'(\boldsymbol{w})}{\partial w_2} \triangle w > 0$ or $\frac{\partial p_1'(\boldsymbol{w})}{\partial w_1} \triangle w - \frac{\partial p_1'(\boldsymbol{w})}{\partial w_2} \triangle w < 0$ always holds for any $p_1'(\boldsymbol{w})$. Thanks to

$$\frac{\partial p_1'(\boldsymbol{w})}{\partial w_1} = \frac{a_{11}(\boldsymbol{w})\, p_2 - p_1 a_{12}(\boldsymbol{w})}{(p_2)^2} = \frac{w_2\left(a_{11}(\boldsymbol{w})\, a_{22}(\boldsymbol{w}) - a_{12}(\boldsymbol{w})\, a_{21}(\boldsymbol{w})\right)}{(p_2)^2} \text{ and}$$
$$\frac{\partial p_1'(\boldsymbol{w})}{\partial w_2} = \frac{a_{21}(\boldsymbol{w})\, p_2 - p_1 a_{22}(\boldsymbol{w})}{(p_2)^2} = \frac{w_1\left(a_{12}(\boldsymbol{w})\, a_{21}(\boldsymbol{w}) - a_{11}(\boldsymbol{w})\, a_{22}(\boldsymbol{w})\right)}{(p_2)^2},$$

we obtain:

$$\forall \boldsymbol{w} \geq \boldsymbol{0},\ \frac{\partial p_1'(\boldsymbol{w})}{\partial w_1} \triangle w - \frac{\partial p_1'(\boldsymbol{w})}{\partial w_2} \triangle w > 0 \Leftrightarrow \forall \boldsymbol{w} \geq \boldsymbol{0},\ a_{11}(\boldsymbol{w})\, a_{22}(\boldsymbol{w}) - a_{12}(\boldsymbol{w})\, a_{21}(\boldsymbol{w}) > 0 \text{ and}$$
$$\forall \boldsymbol{w} \geq \boldsymbol{0},\ \frac{\partial p_1'(\boldsymbol{w})}{\partial w_1} \triangle w - \frac{\partial p_1'(\boldsymbol{w})}{\partial w_2} \triangle w < 0 \Leftrightarrow \forall \boldsymbol{w} \geq \boldsymbol{0},\ a_{11}(\boldsymbol{w})\, a_{22}(\boldsymbol{w}) - a_{12}(\boldsymbol{w})\, a_{21}(\boldsymbol{w}) < 0.$$

From (6), we obtain:

$$\forall \boldsymbol{w} \geq \boldsymbol{0},\ \frac{\partial p_1'(\boldsymbol{w})}{\partial w_1} \triangle w - \frac{\partial p_1'(\boldsymbol{w})}{\partial w_2} \triangle w > 0; \text{ or } \forall \boldsymbol{w} \geq \boldsymbol{0},\ \frac{\partial p_1'(\boldsymbol{w})}{\partial w_1} \triangle w - \frac{\partial p_1'(\boldsymbol{w})}{\partial w_2} \triangle w < 0.$$

This implies that $p_1'(\boldsymbol{w})$ is monotonic in relation to the change in factor prices $\boldsymbol{w} \to (w_1 + \triangle w, w_2 - \triangle w)$. In other words, $p_1'(\boldsymbol{w})$ and $\boldsymbol{w}$ are global univalent. This means that the FPET holds under condition (6). ■

7.2 The Proof of Theorem 1

Suppose that both Υ_{hk} and Φ_{hk} are invariant with respect to any change of factor prices other than that of h and k, respectively. Then, Υ_{hk} and Φ_{hk} are functions of the factor prices of h and k alone, and are independent of others. Therefore, without loss of generality, we can write $\Upsilon_{hk}(w_h, w_k)$ and $\Phi_{hk}(w_h, w_k)$. If Properties 1–4 in footnote 6 are satisfied, the functions are homogeneous of degree zero in w_h and w_k. Letting $\omega \equiv \frac{w_h}{w_k}$, they can be rewritten: $\Upsilon_{hk}(w_h, w_k) = \Upsilon_{hk}(\omega, 1) \equiv \upsilon(\omega)$ and $\Phi_{hk}(w_h, w_k) = \Phi_{hk}(\omega, 1) \equiv \phi(\omega)$. Suppose there exists a factor price ratio $\omega_0 > 0$ in which case (a) of Condition 1 is satisfied:

$\upsilon(\omega_0) = \phi(\omega_0) = \frac{V_h}{V_k}$. Because of Property 3, $\frac{\mathrm{d}\upsilon}{\mathrm{d}\omega} \leqq 0$ and $\frac{\mathrm{d}\phi}{\mathrm{d}\omega} \leqq 0$ hold. Therefore, both $\upsilon(\omega)$ and $\phi(\omega)$ are greater than $\frac{V_h}{V_k}$ for any $\omega < \omega_0$ and are smaller than $\frac{V_h}{V_k}$ for any $\omega > \omega_0$. If $\frac{\mathrm{d}\upsilon}{\mathrm{d}\omega} < 0$, or $\frac{\mathrm{d}\phi}{\mathrm{d}\omega} < 0$, or both, then ω_0 is the unique factor price ratio satisfying (7) with $\rho \in (0, 1)$. Since this implies that cases (b) and (c) in Condition 1 never happen, factor intensity reversal is impossible. If $\frac{\mathrm{d}\upsilon}{\mathrm{d}\omega} = \frac{\mathrm{d}\phi}{\mathrm{d}\omega} = 0$ for all $\omega > 0$, the input coefficients are fixed (i.e. the production function is of the Leontief type). It immediately means that factor intensity reversal is impossible. If ω_0 does not exist, then the switch from case (b) to (c), or vice versa, never happens, which implies no factor intensity reversal. ■

7.3 The Proof of Theorem 2

If the technologies are of the CES type, the unit cost functions also take the same type. Therefore, let us define the unit cost function of Sector 1 as c_1 and that of Sector 2 as c_2 as follows:

$$c_1 = A\left(\sum_{j=1}^{m} \alpha_j w_k^{\beta}\right)^{1/\beta}, c_2 = B\left(\sum_{j=1}^{m} \theta_j w_k^{\sigma}\right)^{1/\sigma},$$

where $A, B > 0$ are parameters, $\alpha_j, \theta_j > 0$ and $\beta, \sigma < 1$. The input coefficients of Sectors 1 and 2 are obtained by using (5). Therefore, the factor intensities of Sector 1 and 2 for any pair of factors, h and k, are respectively given as follows:

$$\Upsilon_{hk} = \frac{\alpha_h}{\alpha_k}\left(\frac{w_k}{w_h}\right)^{1-\beta}, \Phi_{hk} = \frac{\theta_h}{\theta_k}\left(\frac{w_k}{w_h}\right)^{1-\sigma}.$$

Both factor intensities are solely functions of w_h and w_k and are independent of other factor prices. From Theorem 1, therefore, factor intensity reversal is impossible. ■

7.4 The Proof of Theorem 3

Proof: The factor price frontier in the case of $n = 2$ is given as:

$$w^1(r) = \frac{\{1-(1+r)\,a_{11}(\boldsymbol{p}, w, r)\}\{1-(1+r)\,a_{22}(\boldsymbol{p}, w, r)\} - (1+r)^2\,a_{12}(\boldsymbol{p}, w, r)\,a_{21}(\boldsymbol{p}, w, r)}{l_1\{1-(1+r)\,a_{22}(\boldsymbol{p}, w, r)\} + (1+r)\,l_2(\boldsymbol{p}, w, r)\,a_{21}(\boldsymbol{p}, w, r)},$$

where $w^1(r) \equiv \frac{w(r)}{p_1}$. The relative price of commodity 2 is given by:

$$p(r) = \frac{l_2(\boldsymbol{p}, w, r)\{1-(1+r)\,a_{11}(\boldsymbol{p}, w, r)\} + (1+r)\,l_1(\boldsymbol{p}, w, r)\,a_{12}(\boldsymbol{p}, w, r)}{l_1(\boldsymbol{p}, w, r)\{1-(1+r)\,a_{22}(\boldsymbol{p}, w, r)\} + (1+r)\,l_2(\boldsymbol{p}, w, r)\,a_{21}(\boldsymbol{p}, w, r)}.$$

Therefore, we obtain:

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = \frac{l_1(l_1 a_{12} + l_2 a_{22}) - l_2(l_1 a_{11} + l_2 a_{21})}{l_1\{1-(1+r)\,a_{22}\} + (1+r)\,l_2 a_{21}}.$$

If the techniques are productive, then the denominator is positive for all feasible rates of profit. Therefore, the sign of $\frac{\mathrm{d}p}{\mathrm{d}r}$ is solely dependent on the numerator. The relative price and the rate of profit have the following relationship if the techniques are productive:

$$\frac{\mathrm{d}p}{\mathrm{d}r} \lesseqqgtr 0 \Leftrightarrow \frac{a_{12}(\boldsymbol{p},w,r)+pa_{22}(\boldsymbol{p},w,r)}{l_2(\boldsymbol{p},w,r)} \lesseqqgtr \frac{a_{11}(\boldsymbol{p},w,r)+pa_{21}(\boldsymbol{p},w,r)}{l_1(\boldsymbol{p},w,r)}.$$

Here, $\frac{a_{11}(\boldsymbol{p},w,r)+pa_{21}(\boldsymbol{p},w,r)}{l_1(\boldsymbol{p},w,r)}$ is the capital intensity of Sector 1 and $\frac{a_{12}(\boldsymbol{p},w,r)+pa_{22}(\boldsymbol{p},w,r)}{l_2(\boldsymbol{p},w,r)}$ is that of Sector 2. In other words, whether the relative price is monotonically increasing or decreasing with respect to the rate of profit is dependent on the relative size of the capital intensities. The relative price is a monotonically decreasing function with respect to the rate of profit if and only if industry 1 is more capital intensive than Sector 2. Conversely, it is monotonically increasing if and only if Sector 2 is more capital intensive than Sector 1.[27] In other words, the relative price and the rate of profit have no monotonic relationship (i.e. global univalent) if and only if capital intensity reversal takes place at the threshold of the relative price. ■

7.5 The Proof of Theorem 4

Proof: Proving the theorem under the SSS-II condition for $\overline{\boldsymbol{A}}$ will be sufficient. Since we assume that $-\overline{\boldsymbol{s}}\boldsymbol{B}_3 < 0$ holds and matrix $[\boldsymbol{B}_1 - (1+r)\boldsymbol{I}]$ has all negative diagonal elements and all positive off-diagonal elements, the solution for (12) is $\boldsymbol{p} > \boldsymbol{0}$ for $r \in [0, r^*)$ where r^* is the maximum rate of interest as determined by the Frobenius root of $\boldsymbol{B}_1$ (Takayama, 1985, p. 393). This implies that $[\boldsymbol{B}_1 - (1+r)\boldsymbol{I}]^{-1} < \boldsymbol{0}$. Differentiating (12) with respect to r yields $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}[\boldsymbol{B}_1 - (1+r)\boldsymbol{I}] - \boldsymbol{p} = \boldsymbol{0}$, as $\frac{\mathrm{d}\overline{\boldsymbol{s}}}{\mathrm{d}r} = \boldsymbol{0}$ holds. In other words, we obtain:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r} = \boldsymbol{p}[\boldsymbol{B}_1 - (1+r)\boldsymbol{I}]^{-1} < 0.$$

We can prove the theorem for the SSS-I condition in a similar manner. ■

7.6 The Proof of Theorem 5

Proof: Proving the theorem under the SSS-II condition for $\overline{\boldsymbol{A}}(\boldsymbol{q},\boldsymbol{w})$ will be sufficient. The first m equations of (13) are given as $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}\boldsymbol{B}_1(\boldsymbol{q},\boldsymbol{w}) + \frac{\mathrm{d}\overline{\boldsymbol{s}}}{\mathrm{d}r}\boldsymbol{B}_3(\boldsymbol{q},\boldsymbol{w}) = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}r}$, while $\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}r} = \boldsymbol{p} + (1+r)\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}$ holds by the property of $\boldsymbol{q} = (1+r)\boldsymbol{p}$ in equilibrium. Thus, we have:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}\boldsymbol{B}_1(\boldsymbol{q},\boldsymbol{w}) + \frac{\mathrm{d}\overline{\boldsymbol{s}}}{\mathrm{d}r}\boldsymbol{B}_3(\boldsymbol{q},\boldsymbol{w}) = \boldsymbol{p} + (1+r)\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}.$$

Since consumption prices are exogenously given, $\frac{\mathrm{d}\overline{\boldsymbol{s}}}{\mathrm{d}r} = \boldsymbol{0}$ holds. Consequently, we obtain:

[27] Whether the relative price is monotonically increasing or decreasing can be determined by the technical coefficients, independently of the price:

$$\frac{\mathrm{d}p}{\mathrm{d}r} \lesseqqgtr 0 \Longleftrightarrow \frac{l_1(\mathbf{p},w,r)\,a_{12}(\mathbf{p},w,r)+l_2(\mathbf{p},w,r)\,a_{22}(\mathbf{p},w,r)}{l_2(\mathbf{p},w,r)} \lesseqqgtr \frac{l_1(\mathbf{p},w,r)\,a_{11}(\mathbf{p},w,r)+l_2(\mathbf{p},w,r)\,a_{21}(\mathbf{p},w,r)}{l_1(\mathbf{p},w,r)}.$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}\left[\boldsymbol{B}_1\left(\boldsymbol{q},\boldsymbol{w}\right)-\left(1+r\right)\boldsymbol{I}\right]=\boldsymbol{p}.$$

If the SSS-II condition is satisfied, then $\left[\boldsymbol{B}_1\left(\boldsymbol{q},\boldsymbol{w}\right)-\left(1+r\right)\boldsymbol{I}\right]^{-1}<\mathbf{0}$. Therefore, we obtain:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}r}=\boldsymbol{p}\left[\boldsymbol{B}_1\left(\boldsymbol{q},\boldsymbol{w}\right)-\left(1+r\right)\boldsymbol{I}\right]^{-1}<\mathbf{0}.$$

We can prove the theorem for the SSS-I condition in a similar manner. ■

r	w_1	w_2	p_2/p_1	k_1	k_2
0	0.826	0.854	0.966	1.451	1.565
0.1	0.692	0.705	0.981	1.336	1.36
0.2	0.58	0.577	1.004	1.073	1.392
0.3	0.488	0.481	1.015	1.021	1.125
0.4	0.407	0.396	1.0295	0.797	1.075
0.5	0.328	0.319	1.0277	0.796	1.015
0.6	0.258	0.250	1.030	0.894	0.967
0.7	0.193	0.187	1.0320	0.859	0.924
0.8	0.133	0.128	1.0323	0.827	0.884
0.9	0.076	0.074	1.031	0.798	0.846
1.0	0.029	0.023	1.238	0.796	0.978
1.04	0.011	0.004	2.723	0.783	2.118

Table 1: The real wage rate, relative price, and capital intensity

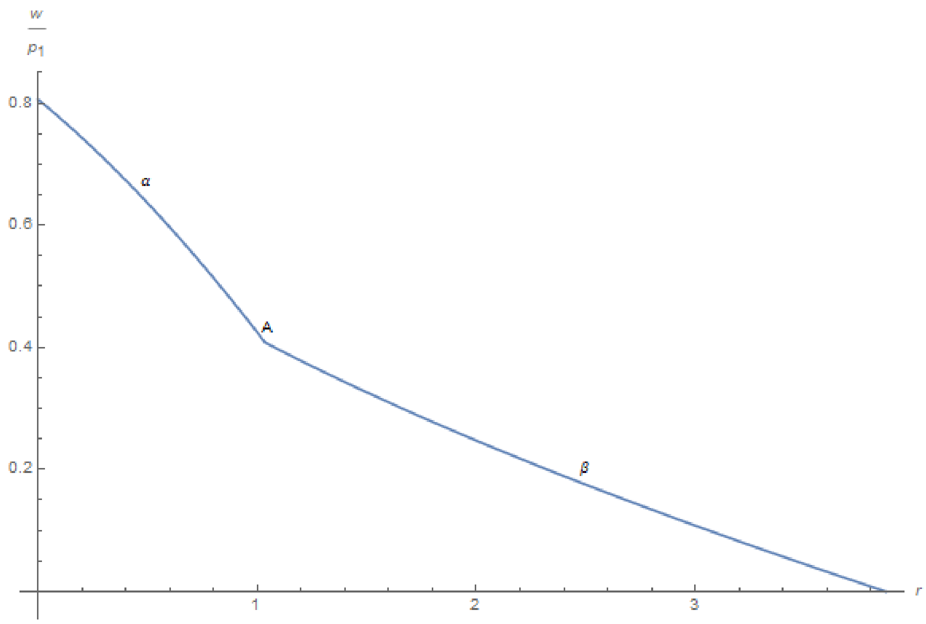

Figure 1：The factor price frontier

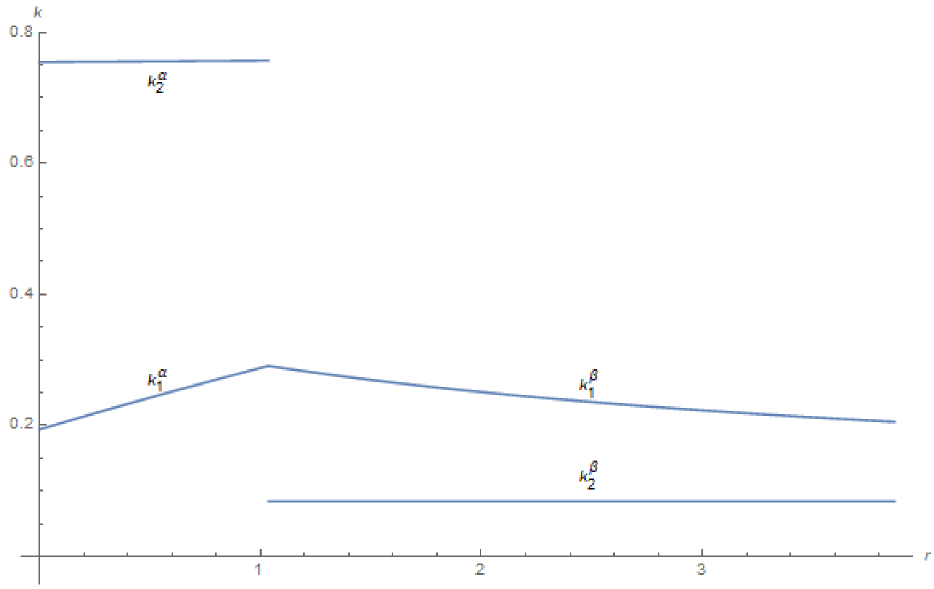

Figure 2: Capital intensities

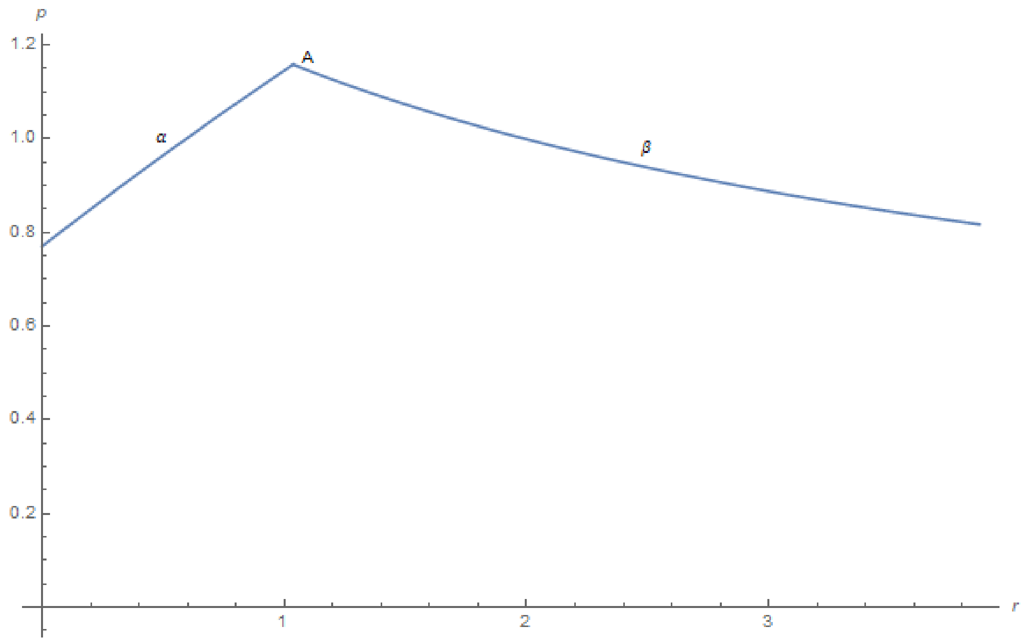

Figure 3: Relative price

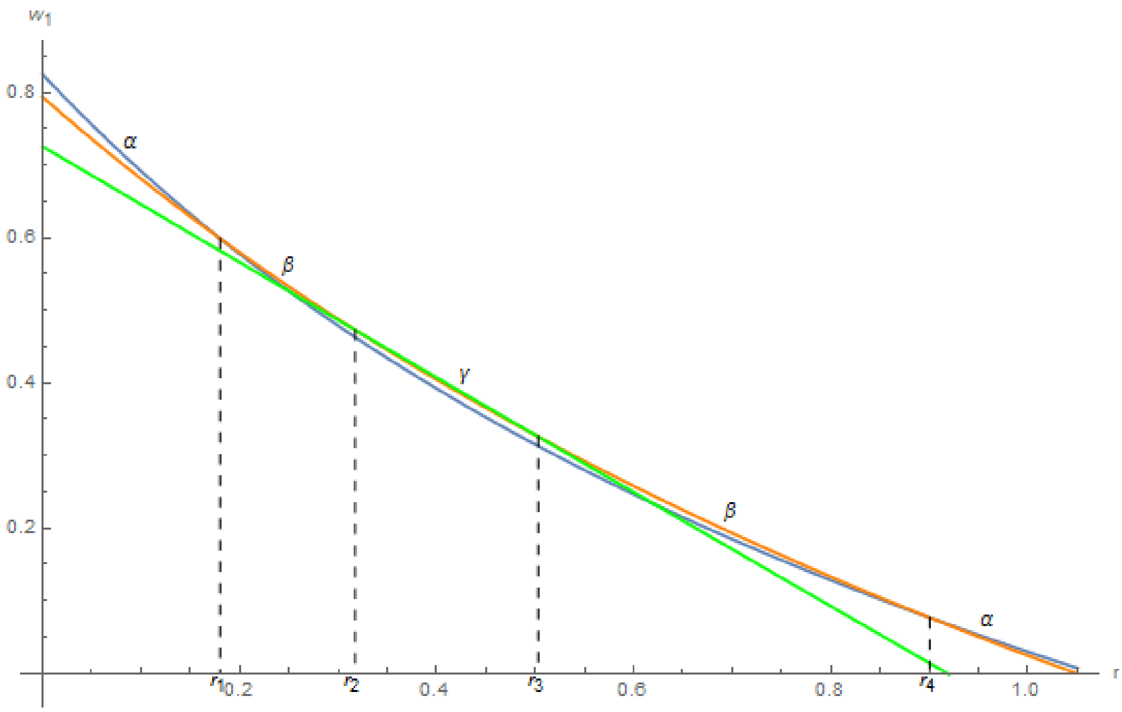

Figure 4: The factor price frontier of Sector 1

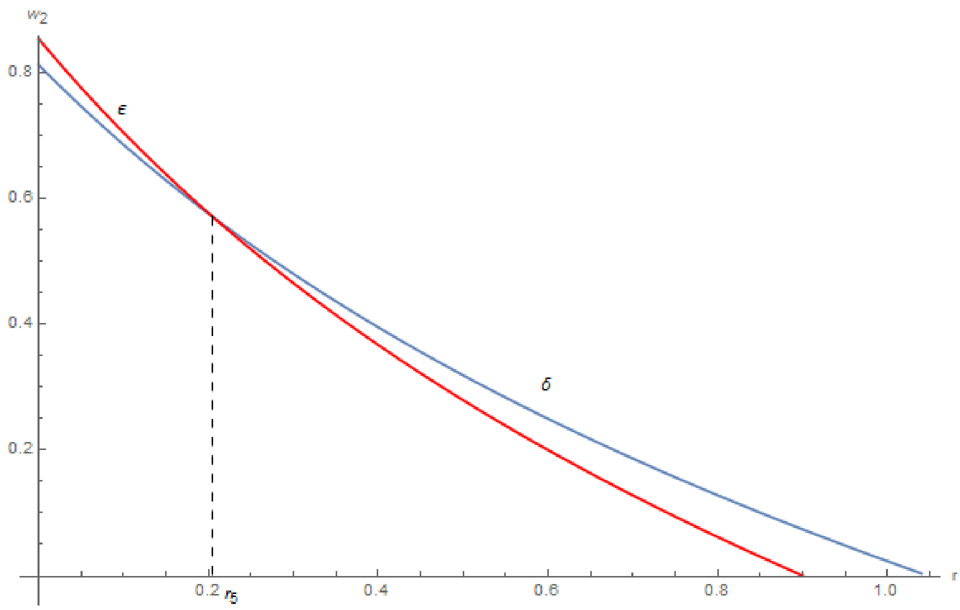

Figure 5: The factor price frontier of Sector 2

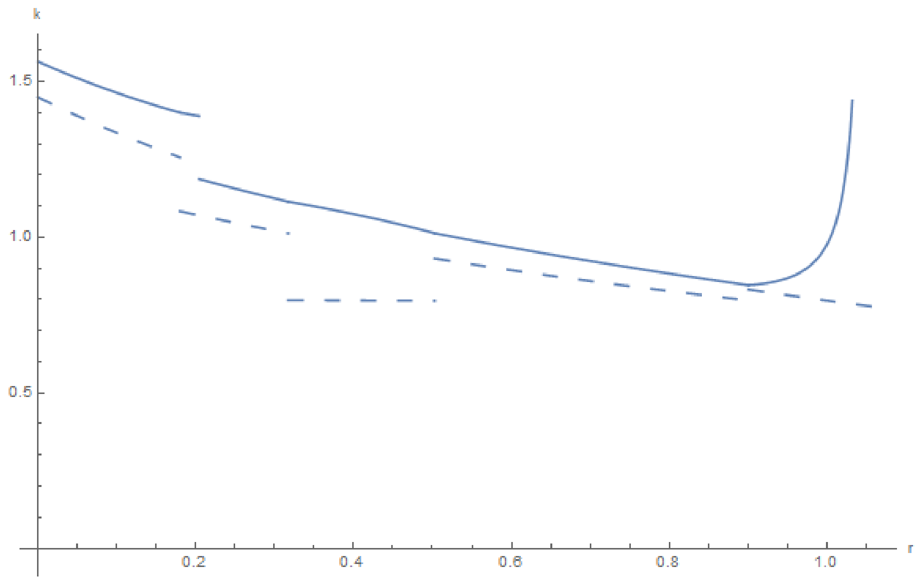

Figure 6: The capital intensity

(real line shows for Sector 2 and dashed line for Sector 1)

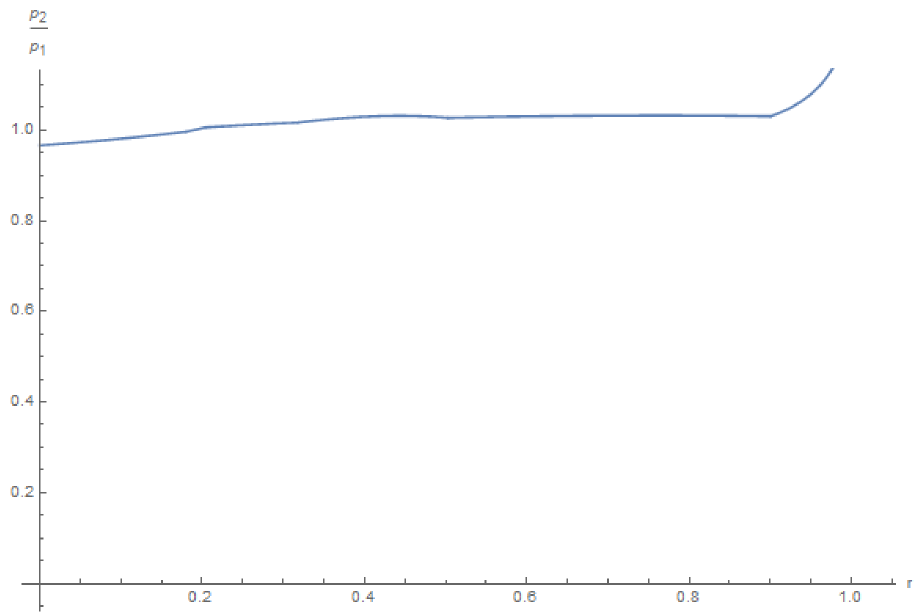

Figure 7: The relative price

Appendices B and C: The Heckscher–Ohlin–Samuelson Trade Theory and the Cambridge Capital Controversies: On the Validity of Factor Price Equalisation Theorem*

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Abstract

This addendum provides a supplement of the survey on the factor price equalisation theorem (FPET) in Heckscher–Ohlin–Samuelson (HOS) models.

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Keywords: factor price equalisation, global univalence, capital as a bundle of reproducible commodities, reswitching of techniques, capital reversing.

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8 Appendix B

8.1 On FPET in economies with more than the two-commodity case

In this Appendix B, we provide a survey on FPET with more general cases where the number of commodities is greater than two. Gale and Nikaido (1965) and Nikaido (1968) develop the Samuelsonian conjecture; Samuelson (1966a) and Nikaido (1972) define capital intensity as the relative share of factor costs; and Mas-Collel (1979a,b) further develops Nikaido's (1972) formulation.

While the above literature uses the Jacobian matrix of the unit cost functions to characterise the condition for the FPET, Kuga (1972) introduces a new method. He calls this method the 'differentiation method'. Blackorby et al. (1993) extend Kuga (1972) by allowing for decreasing returns to scale and intermediate goods.

8.1.1 The Analysis of FPET by Using the Jacobian Matrix

8.1.1.1 More General Case As already mentioned, the global univalence between commodity prices and factor prices can be characterised by the Jacobian matrix of the unit cost functions. Samuelson (1953) extends (6) to more general cases of $n \geqq 3$ and conjectures the sufficient condition for the validity of the FPET as follows:

$$\frac{\partial c_1(\boldsymbol{w})}{\partial w_1} \gtrless 0, \det \begin{bmatrix} \frac{\partial c_1(\boldsymbol{w})}{\partial w_1} & \frac{\partial c_2(\boldsymbol{w})}{\partial w_1} \\ \frac{\partial c_1(\boldsymbol{w})}{\partial w_2} & \frac{\partial c_2(\boldsymbol{w})}{\partial w_2} \end{bmatrix} \gtrless 0, \cdots, \det \begin{bmatrix} \frac{\partial c_1(\boldsymbol{w})}{\partial w_1} & \cdots & \frac{\partial c_n(\boldsymbol{w})}{\partial w_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial c_1(\boldsymbol{w})}{\partial w_n} & \cdots & \frac{\partial c_n(\boldsymbol{w})}{\partial w_n} \end{bmatrix} \gtrless 0. \tag{15}$$

(15) indicates that the sufficient condition for the validity of the FPET is that the successive principal minors of the Jacobian matrix be non-vanishing for any $\boldsymbol{w} \geq \mathbf{0}$.[28]

However, Gale and Nikaido (1965) and Nikaido (1968) point out that Samuelson's conjecture is not necessarily valid.[29] Before reviewing them in detail, let us define the following matrix in addition to Definition 2 given in Section 4:

Definition 3: A square matrix, $\boldsymbol{A}$, is termed an N-matrix if all the principal minors are negative. An N-matrix can be further divided into two categories:

[28] As related to (15) and Theorem 2, Chipman (1966) indicates that (15) is a sufficient as well as a necessary condition for the global univalence in the case of the CES production functions. As is obvious from the previous subsection, in the case, the unit cost minimising prices are in general expressed as $p_j^\beta = \sum_{i=1}^n a_{ij} w_i^\beta$ for $j = 1, \cdots, n$, which are linear in p_j^β and w_i^β, and these terms are in one-to-one correspondence with p_j and w_i, respectively. Therefore, one-to-one correspondence between p_j and w_i is obtained if the Jacobian determinant is non-vanishing.

[29] Gale and Nikaido (1965) provide a counter-example for condition (15). They suppose the mapping $F(\mathbf{x}) \equiv [f_i(\mathbf{x})]$ as defined below:

$$\begin{cases} f_1(x_1, x_2) = e^{2x_1} - x_2^2 + 3, \\ f_2(x_1, x_2) = 4e^{2x_1} x_2 - x_2^3. \end{cases}$$

The successive principal minors can then be given as:

i) An N-matrix is said to be of the *first category* if $\boldsymbol{A}$ has at least one positive element.
ii) An N-matrix is said to be of the *second category* if all of the elements are non-positive.

Let the mapping $f : \Omega \to \mathbb{R}^n$ satisfy the following assumptions:

Assumption 1: Ω is a closed rectangular region in $\mathbb{R}^n$.[30]

Assumption 2: Given that the mapping $f(\mathbf{x}) \equiv [f_j(\mathbf{x})]$ $(\mathbf{x} \in \Omega, j = 1, 2, \cdots, n)$, $f_j(\mathbf{x})$ is monotonically increasing and totally differentiable on Ω:

$$\mathrm{d}f_j(\mathbf{x}) = \sum_{j=1}^{n} \frac{\partial f_j(\mathbf{x})}{\partial x_i} \mathrm{d}x_i, \ (j = 1, 2, \cdots, n).$$

Then, the following theorem holds:

Theorem 1B (Gale and Nikaido, 1965; Inada, 1971; Nikaido, 1968): For a given vector, $\boldsymbol{p} \equiv [p_j]$, mapping $\boldsymbol{p} = f(\mathbf{x})$ is global univalent if either (a) or (b) holds:

(a) The Jacobian matrix of $f(\mathbf{x})$, is everywhere a P-matrix in Ω.
(b) The Jacobian matrix is continuous and is everywhere an N-matrix in Ω.

Proof: See Nikaido (1968, pp. 370–371). ■

As Ethier (1984, p. 151) points out, Assumptions 1 and 2 are quite general and their conditions for global univalence are purely mathematical. Therefore, we still need economically meaningful assumptions for the cost functions to be globally univalent. Samuelson's (1966a) work corresponds to this subject. He conjectures that the factor intensity could be defined by the share of the increase in the cost of factor i relative to the increase in the cost of production per unit. For the price equation, $\boldsymbol{p} = c(\boldsymbol{w})$, the share of the cost of factor i relative to the unit cost of commodity j, α_{ij}, is given as:

$$\alpha_{ij}(\boldsymbol{w}) \equiv \frac{w_i a_{ij}(\boldsymbol{w})}{p_j}, \ (\forall i, j = 1, \ldots, n).$$

$$\frac{\partial f_1}{\partial x_1} = 2e^{2x_1} > 0,$$
$$\begin{vmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{vmatrix} = \begin{vmatrix} 2e^{2x_1} & -2x_2 \\ 8e^{2x_1}x_2 & 4e^{2x_1} - 3x_2^2 \end{vmatrix} = 2e^{2x_1}\left(4e^{2x_1} + 5x_2^2\right) > 0,$$

holds for any $\mathbf{x}$; therefore, (15) is satisfied. However, $F(0, 2) = F(0, -2) = (0, 0)$, which precludes global univalence.

[30] A closed rectangular region is defined as follows:

$$\Omega \equiv \{\mathbf{x} \,| p_i \leqq x_i \leqq q_i, \forall i = 1, 2, \cdots, n\},$$

where $-\infty < p_i < q_i < +\infty$.

As $a_{ij}(\boldsymbol{w}) = \frac{\partial c_j(\boldsymbol{w})}{\partial w_i}$, $\alpha_{ij}(\boldsymbol{w})$ also represents factor i's cost elasticity with respect to commodity j's price. Let us define matrix $\widetilde{\boldsymbol{A}}(\boldsymbol{w}) \equiv [\alpha_{ij}(\boldsymbol{w})]$ $(i, j = 1, 2, \cdots, n)$, which shall be termed the relative share matrix. Introduce the following condition on this matrix:

Assumption 3: $\widetilde{\boldsymbol{A}}(\boldsymbol{w})$ has successive principal minors whose absolute values are bounded from below by constant, positive numbers, δ_k $(k = 1, 2 \cdots, n)$, if its rows and columns are adequately renumbered:[31]

$$\left| \det \begin{bmatrix} \alpha_{11}(\boldsymbol{w}) & \cdots & \alpha_{1k}(\boldsymbol{w}) \\ \vdots & \ddots & \vdots \\ \alpha_{k1}(\boldsymbol{w}) & \cdots & \alpha_{kk}(\boldsymbol{w}) \end{bmatrix} \right| \geqq \delta_k > 0 \text{ for each } k = 1, 2 \cdots, n. \tag{16}$$

Assumption 3 can be considered to assume no factor intensity reversal in the general cases of $n \geqq 3$, as the following theorem holds:

Theorem 2B (Nikaido, 1972): If $c_j(\boldsymbol{w})$ satisfies Properties 1–4 and Assumption 3 for any $\boldsymbol{w} > \boldsymbol{0}$, then the price equation, $\boldsymbol{p} = c(\boldsymbol{w})$, is completely invertible for the given $\boldsymbol{p} > \boldsymbol{0}$.[32]

Proof: See Appendix C. ■

Theorem 2B verifies Samuelson's conjecture.[33] In this way, Nikaido (1972) demonstrates that a sufficient condition for the global univalence between commodity prices and factor prices shall be characterised by means of $\widetilde{\boldsymbol{A}}(\boldsymbol{w})$, not $\boldsymbol{A}(\boldsymbol{w})$ in (6) and (15), in the general cases of $n \geqq 3$.

Stolper and Samuelson (1941) also investigates the relationship between final commodity prices and factor prices; thus, the approach of using the cost function's Jacobian matrix is applied to the generalisation of the Stolper-Samuelson Theorem. Chipman (1969) proposes the following two conditions:

Condition 1B (Weak Stolper–Samuelson: WSS): An increase in p_j leads to a more than proportional increase in the price of the corresponding factor w_j. The price of factor w_i $(i \neq j)$ may increase, but the rate of increase is smaller than that of w_j:

$$\frac{\partial \ln w_j}{\partial \ln p_j} > 1, \ \frac{\partial \ln w_j}{\partial \ln p_j} > \frac{\partial \ln w_i}{\partial \ln p_j}.$$

Condition 2B (Strong Stolper–Samuelson: SSS): An increase in p_j decreases all factor prices except for that of w_j:

[31] (16) is equivalent to (6) if there are two-commodities and two-factors.

[32] 'Complete invertibility' means that $\boldsymbol{p} = c(\boldsymbol{w}) \neq c(\boldsymbol{w}') = \boldsymbol{p}'$ for arbitrarily positive vectors $\boldsymbol{w} \neq \boldsymbol{w}'$ and a unique $\boldsymbol{w} > \boldsymbol{0}$ exists such that $\boldsymbol{p} = c(\boldsymbol{w})$ for any $\boldsymbol{p} > \boldsymbol{0}$.

[33] Samuelson's own summary can be found in Samuelson (1967). Moreover, Stiglitz (1970) constructs a dynamic HOS model by introducing a relationship between savings and investment, and he derives the condition for the FPET. See Smith (1984) with respect to the dynamic HOS model in detail.

$$\frac{\partial \ln w_i}{\partial \ln p_j} < 0, \text{ if } i \neq j.$$

In order to satisfy the WSS condition, the inverse of $\widetilde{\boldsymbol{A}}(\boldsymbol{w})$ must exist and its diagonal elements must be greater than 1 and greater than its off-diagonal elements.[34] In other words, letting $\widetilde{\boldsymbol{A}}^{-1}(\boldsymbol{w}) \equiv [\alpha^{ij}(\boldsymbol{w})]$ $(i, j = 1, 2, \cdots, n)$, the WSS condition implies $\alpha^{jj}(\boldsymbol{w}) > 1$ and $\alpha^{jj}(\boldsymbol{w}) > \alpha^{ij}(\boldsymbol{w})$ $(i \neq j)$. Similarly, the SSS condition, in terms of $\widetilde{\boldsymbol{A}}^{-1}(\boldsymbol{w})$, implies that $\alpha^{ij}(\boldsymbol{w}) < 0$ $(i \neq j)$. While Chipman (1969) proves the case of $n \leqq 3$, Uekawa (1971) and Uekawa et al. (1972) rigorously prove the condition for the validity of the Stolper–Samuelson theorem in the case of $n \geqq 4$.[35]

8.1.1.2 Mas-Colell (1979a, b) Mas-Colell (1979a, b) uses the relative share matrix rather than the unit cost functions to characterise the condition for the FPET. In addition to Properties 1 and 2, he adds the following assumption to the unit cost functions:

Assumption 4: $c : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$.

Although the unit cost function is usually assumed to be concave with respect to $\boldsymbol{w}$, only homogeneity is assumed here. Because of Assumption 4, the iso-cost curve is unbounded. Then, the following theorem holds.

Theorem 3B (Mas-Colell, 1979a): Under Properties 1–2, and Assumption 4, if, for some $\varepsilon > 0$, $\left|\det \widetilde{\boldsymbol{A}}(\boldsymbol{w})\right| > \varepsilon$ holds for all $\boldsymbol{w} \in \mathbb{R}^n_{++}$, then $c(\boldsymbol{w})$ is a homeomorphism.

Proof: See the Appendix C. ■

[34] Chipman (1969) discusses the relationship between Gale and Nikaido's (1965) condition for the FPET and the WSS condition. When $n = 2$, the condition is equivalent to the WSS condition, but not if $n \geqq 3$. Suppose that $\boldsymbol{w}^0$ is determined for a given $\boldsymbol{p}$ and $\pi^0 = \varphi(\omega^0)$ (which is defined in the proof of Theorem 2B). Let us suppose that its Jacobian matrix is given as follows:

$$\varphi'(\boldsymbol{\omega}^0) = \begin{bmatrix} 0.55 & 0.40 & 0.05 \\ 0.05 & 0.50 & 0.45 \\ 0.25 & 0.35 & 0.40 \end{bmatrix}.$$

$\varphi'(\boldsymbol{\omega}^0)$ is a stochastic matrix. Furthermore, $\varphi(\boldsymbol{\omega})$ is a differentiable and monotonically increasing function, and all principal minors of $\varphi'(\boldsymbol{\omega}^0)$ are positive, namely $\varphi'(\boldsymbol{\omega}^0)$ is a P-matrix and satisfies Theorem 1B. This means that the FPET holds. However, we obtain:

$$[\varphi'(\boldsymbol{\omega}^0)]^{-1} = \begin{bmatrix} 0.77 & -2.59 & 2.82 \\ 1.68 & 3.77 & -4.45 \\ -1.95 & -1.68 & 4.64 \end{bmatrix}.$$

This means that the WSS condition does not hold.

[35] The Stolper–Samuelson theorem is similarly generalised by Inada (1971), Kemp and Wegge (1969), and Wegge and Kemp (1969). See Ethier (1984) for further research on the theorem.

In other words, Theorem 3B implies that for all $\boldsymbol{p} \in \mathbb{R}^n_{++}$ the equation $\boldsymbol{p} = c(\boldsymbol{w})$ has a unique solution $\boldsymbol{w} \in \mathbb{R}^n_{++}$ that continuously depends on $\boldsymbol{p}$.

The unit cost function assumed here is a generalisation of Theorem 2B (Nikaido, 1972). The difference lies in the fact that here the cost function is shown to be homeomorphic, even if it is assumed only to be linearly homogeneous. Remember that we assumed the linear homogeneity and concavity of the unit cost function to claim its complete invertibility, but complete invertibility does not require that the invertible mapping be continuous. Incidentally, global univalence does not require this continuity either.

Theorem 3B cannot say anything about the types of cost functions which allow bounded iso-cost curves. However, Mas-Colell (1979a) also shows that if the unit cost function is only differentiable and homogenous of degree one, and is defined as $c : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, then it can be a homeomorphism under the same condition for the successive principal minors of the relative sharing matrix $\widetilde{\boldsymbol{A}}(\boldsymbol{w})$ as in Theorem 2B of this paper.

8.1.2 The New Analysis of the FPET by the Differentiation Method

Here, we review the models into which a new method to characterise the FPET is introduced. The feature of this method is that it does not use the Jacobian matrix of the unit cost function.

8.1.2.1 Kuga (1972) The preceding analyses can only be applied to cases where the number of final commodities is equal to the number of factors. In order to overcome this limitation, Kuga (1972) uses a new method to characterise the condition for the FPET, which he terms the 'differentiation method'.

Let us assume that the general production possibility frontier is given as follows:

$$X_1 = T(\boldsymbol{V}, \boldsymbol{X}_{-1}),$$

where X_1 denotes the output of commodity 1, $\boldsymbol{V} \in \mathbb{R}^r_+$ is the factor endowment, and $\boldsymbol{X}_{-1} \equiv [X_j]$ is the output vector of commodity $j = 2, \cdots, n$. Moreover, T satisfies the following assumptions:

Assumption 5: T is positively homogeneous of degree one with respect to $(\boldsymbol{V}, \boldsymbol{X}_{-1})$.

Assumption 6: T is concave with respect to $(\boldsymbol{V}, \boldsymbol{X}_{-1})$.

Assumption 7: T is strictly concave with respect to $\boldsymbol{X}_{-1}$ for any fixed $\boldsymbol{V}$.

Assumption 8: T is twice differentiable with respect to $(\boldsymbol{V}, \boldsymbol{X}_{-1})$.

Let the price of commodity 1 be the numéraire. The problem is then expressed as follows:

$$\max T(\boldsymbol{V}, \boldsymbol{X}_{-1}) + \sum_{j=2}^{n} p_j X_j, \tag{17}$$

the solution of which is given by:

$$p_j = -\frac{\partial T(\boldsymbol{V}, \boldsymbol{X}_{-1})}{\partial X_j}, \quad j = 2, 3, \cdots, n, \tag{18}$$

Thanks to the Berge maximum theorem, we can see that the set of solutions to (17), X_j, is upper hemi-continuous with respect to $\boldsymbol{V}$ for a given $\boldsymbol{p}$. Thanks to Assumption 7, moreover, the set is a singleton. Therefore, the solution, X_j $(j = 2, \cdots n)$, is the continuous single valued function of $\boldsymbol{V}$:

$$\boldsymbol{X}_{-1}{=}X_{-1}\left(\boldsymbol{V}{,}\boldsymbol{p}\right).$$

The price of factor i is given by:

$$w_i = \frac{\partial T\left(\boldsymbol{V}, X_{-1}\left(\boldsymbol{V}{,}\boldsymbol{p}\right)\right)}{\partial V_i}, \quad i = 1, 2, \cdots, r. \tag{19}$$

The equalisation of factor prices in this model implies that factor price w_i is solely dependent on the commodity price that is determined by free trade, and thus the right-hand side of (19) is kept constant with respect to the variation of $\boldsymbol{V}$.

By partially differentiating (19) with respect to V_τ $(\tau = 1, 2, \cdots, r)$, we obtain:

$$\frac{\partial w_i}{\partial V_\tau} = \frac{\partial^2 T}{\partial V_\tau \partial V_i} + \sum_{j=2}^{n} \frac{\partial^2 T}{\partial X_j \partial V_i} \frac{\partial X_j}{\partial V_\tau}, \quad i, \tau = 1, 2, \cdots, r, \tag{20}$$

in matrix form (20) is written as

$$\boldsymbol{w}_V = \boldsymbol{M}_1 + \boldsymbol{M}_2 \boldsymbol{X}_V, \tag{21}$$

where $\boldsymbol{w}_V \equiv \begin{bmatrix} \frac{\partial w_1}{\partial V_1} & \cdots & \frac{\partial w_r}{\partial V_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_1}{\partial V_r} & \cdots & \frac{\partial w_r}{\partial V_r} \end{bmatrix}$, $\boldsymbol{M}_1 \equiv \begin{bmatrix} \frac{\partial^2 T}{\partial V_1^2} & \cdots & \frac{\partial^2 T}{\partial V_1 \partial V_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial V_r \partial V_1} & \cdots & \frac{\partial^2 T}{\partial V_r^2} \end{bmatrix}$,

$\boldsymbol{M}_2 \equiv \begin{bmatrix} \frac{\partial^2 T}{\partial V_1 \partial X_2} & \cdots & \frac{\partial^2 T}{\partial V_1 \partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial V_r \partial X_2} & \cdots & \frac{\partial^2 T}{\partial V_r \partial X_n} \end{bmatrix}$, $\boldsymbol{X}_V \equiv \begin{bmatrix} \frac{\partial X_2}{\partial V_1} & \cdots & \frac{\partial X_2}{\partial V_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial V_1} & \cdots & \frac{\partial X_n}{\partial V_r} \end{bmatrix}$.

Similarly, partially differentiating (18) with respect to V_τ yields:

$$\frac{\partial^2 T}{\partial X_j \partial V_\tau} + \sum_{l=2}^{n} \frac{\partial^2 T}{\partial X_l \partial X_j} \frac{\partial X_l}{\partial V_\tau} = 0, \quad j = 2, 3, \cdots, n, \tau = 1, 2, \cdots, r. \tag{22}$$

In matrix from, (22) is written as $\boldsymbol{M}_2^T + \boldsymbol{M}_3 \boldsymbol{X}_V = \boldsymbol{0}$, where $\boldsymbol{M}_3 \equiv \begin{bmatrix} \frac{\partial^2 T}{\partial X_2^2} & \cdots & \frac{\partial^2 T}{\partial X_2 \partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial X_n \partial X_2} & \cdots & \frac{\partial^2 T}{\partial X_n^2} \end{bmatrix}$ and the superscript T denotes the transpose. Thanks to Assumption 7, the Hessian matrix $\boldsymbol{M}_3$ has an inverse such that $\boldsymbol{X}_V = -\boldsymbol{M}_{32}^{-1}\boldsymbol{M}^T$ holds.[36] When this is combined with (21), we obtain:

[36] Assumption 7 means that T is strictly concave with respect to $\mathbf{X}_{-1}$. This implies that the Hessian matrix of T is negative definite. A square matrix is negative definite if and only if its inverse is negative definite (Mas-Colell et al., 1995, p. 936); therefore, $\mathbf{M}_3$ is invertible.

$$\boldsymbol{w}_V = \boldsymbol{M}_1 - \boldsymbol{M}_2^{-1}\boldsymbol{M}_3^T\boldsymbol{M}_2. \tag{23}$$

In order for the FPET to hold, $\boldsymbol{w}_V = \boldsymbol{0}$ must hold:

$$\boldsymbol{M}_1 = \boldsymbol{M}_2\boldsymbol{M}_{32}^{-1}\boldsymbol{M}_2^T \tag{24}$$

(23) and (24) have economic implications. Partially differentiating (18) with respect to p_j yields $1 = -\sum_{k=2}^{n} \frac{\partial^2 T}{\partial X_k \partial X_j}\frac{\partial X_k}{\partial p_j}$, which can be rewritten in matrix form as:

$$\boldsymbol{I} = -\boldsymbol{M}_3\boldsymbol{X}_p, \tag{25}$$

where $\mathbf{I}$ is an identity matrix of order $n-1$ and $\boldsymbol{X}_p \equiv \begin{bmatrix} \frac{\partial X_2}{\partial p_2} & \cdots & \frac{\partial X_2}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial p_2} & \cdots & \frac{\partial X_n}{\partial p_n} \end{bmatrix}$. Similarly, partially differentiating (19) with respect to p_j yields

$$\frac{\partial w_i}{\partial p_j} = \sum_{l=2}^{n} \frac{\partial^2 T}{\partial X_l \partial V_i}\frac{\partial X_l}{\partial p_j}, \quad i = 1, 2, \cdots, r, \; j = 1, 2, \cdots, n,$$

which can be written in matrix form as:

$$\boldsymbol{w}_p = \boldsymbol{M}_2\boldsymbol{X}_p, \tag{26}$$

where $\boldsymbol{w}_p \equiv \begin{bmatrix} \frac{\partial w_1}{\partial p_2} & \cdots & \frac{\partial w_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_r}{\partial p_2} & \cdots & \frac{\partial w_r}{\partial p_n} \end{bmatrix}$. Consequently, $\boldsymbol{M}_3^{-1} = -\boldsymbol{X}_p$ holds from (25), as does $\boldsymbol{M}_2 = \boldsymbol{w}_p\boldsymbol{X}_p^{-1}$ from (26). Because of (23), we therefore obtain:

$$\boldsymbol{w}_V = \boldsymbol{M}_1 + \boldsymbol{w}_p\boldsymbol{M}_2^T. \tag{27}$$

The variations in factor endowments, V_τ, tend to give rise to the variations in factor prices and output. The elements of $\boldsymbol{M}_1$, $\frac{\partial^2 T}{\partial V_i \partial V_\tau}$ $(i = 1, \cdots, r)$, indicate that the variation in factor prices varies in response to variation in V_τ by an amount of $\frac{\partial X_j}{\partial V_\tau}$ $(j = 1, \cdots, n)$ when there is no adjustment in X_j (i.e. $\frac{\partial X_j}{\partial V_\tau} = 0$). The elements of $\boldsymbol{M}_2$, $\frac{\partial^2 T}{\partial V_\tau \partial X_j}$, indicate the discrepancies between international prices, p_j, and domestic commodity production prices, $\frac{\partial T}{\partial X_j}$, vary in response to variations in V_τ when there is no adjustment in $\frac{\partial X_j}{\partial V_\tau}$ (i.e. $\frac{\partial X_j}{\partial V_\tau} = 0$). On the contrary, the elements of $\boldsymbol{w}_p$ convey the adjustment in the w_i's through the adjustments in the X_j's corresponding to the marginal discrepancies in international prices. Therefore, $\boldsymbol{w}_p\boldsymbol{M}_2^T$ in (27) can be interpreted as the potential amount of adjustment in the w_i's through the X_j's corresponding to the discrepancies in $\boldsymbol{M}_2^T$. Consequently, $\boldsymbol{M}_1$ and $\boldsymbol{w}_p\boldsymbol{M}_2^T$ shall be termed the 'direct effect' and the 'adjustment effect', respectively. In order for the FPET to hold in this model ($\boldsymbol{w}_V = \boldsymbol{0}$), the direct effect must be just offset by the adjustment effect. Summarising the above analysis, we obtain:

Theorem 4B: Under Assumptions 5–8, the FPET holds if and only if the direct effect is offset by the adjustment effect.

Kuga (1972) assures the validity of the FPET by keeping the factor price independent of the factor endowment, which was an entirely different approach than previous models had used.

8.1.2.2 Blackorby et al. (1993) Blackorby et al. (1993) characterise the necessary and sufficient condition for factor price equalisation, which is a generalisation of Kuga (1982) in that the possibility of joint production and decreasing returns to scale is allowed. Note that Blackorby et al. (1993) argue, as Kuga (1972) does, that factor prices must be solely dependent on commodity prices and, therefore, independent of factor endowments in all countries.

Suppose that the international economy consists of N countries, indexed as $\nu = 1, \cdots, N$. Moreover, there are n final commodities which are traded freely in international markets and m primary factors that each country is endowed with. Let $\boldsymbol{X}^{\nu} \in \mathbb{R}^n$ denote the production vector of country ν, the positive elements of which represent the outputs and the negative elements of which represent the inputs. $\boldsymbol{V}^{\nu} \in \mathbb{R}^m$ denotes factor endowments of country ν. Given the transformation function $T^{\nu} : \mathbb{R}^{n+m} \to \mathbb{R}$, a net output and factor endowment vector, $(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu})$, is feasible if and only if $T^{\nu}(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu}) \leqq 0$. Furthermore, it should be noted that T^{ν} satisfies the following assumptions:

Assumption 9: (i) $D^{\nu} \equiv \{(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu}) \in \mathbb{R}^{n+m} \mid T^{\nu}(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu}) \leqq 0\}$ is a non-empty and closed convex set, and $(\boldsymbol{0}, \boldsymbol{0}) \in D^{\nu}$; (ii) T^{ν} is increasing in $\boldsymbol{X}^{\nu} \in \mathbb{R}^n$ and decreasing in $\boldsymbol{V}^{\nu} \in \mathbb{R}^m$; (iii) T^{ν} is convex in $(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu}) \in \mathbb{R}^{n+m}$.

Assumption 10: T^{ν} is continuous and twice differentiable.

The transformation function is related to the production function, G^{ν}, as follows:

$$T^{\nu}(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu}) = 0 \Longleftrightarrow \boldsymbol{Y}^{\nu} = G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu}),$$

where $\boldsymbol{X}^{\nu} \equiv (\boldsymbol{Y}^{\nu}, \boldsymbol{Z}^{\nu})$. $\boldsymbol{Y}^{\nu}$ and $\boldsymbol{Z}^{\nu}$ denote the net output and net input, respectively. Due to Assumption 9, G^{ν} is concave and decreasing in $\boldsymbol{Z}^{\nu}$ as well as increasing in $\boldsymbol{V}^{\nu}$. Therefore, the two expressions shown below represent the same profit maximisation problem:

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) = \max_{\boldsymbol{X}^{\nu}} \{\boldsymbol{p}\boldsymbol{X}^{\nu} \mid T^{\nu}(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu}) \leqq 0\}, \tag{28}$$

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) = \max_{\boldsymbol{Y}^{\nu}, \boldsymbol{Z}^{\nu}} \{\boldsymbol{p}^{y}\boldsymbol{Y}^{\nu} + \boldsymbol{p}^{z}\boldsymbol{Z}^{\nu} \mid \boldsymbol{Y}^{\nu} \leqq G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu})\}, \tag{29}$$

where $\boldsymbol{p} \equiv (\boldsymbol{p}^{y}, \boldsymbol{p}^{z})$ is the price vector. $R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu})$ satisfies the same properties that the profit function generally does; that is, $R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu})$ is a homogeneous function of degree one, non-decreasing convex, and increasing concave in $\boldsymbol{V}^{\nu}$.

An equilibrium factor price vector $\boldsymbol{W}^{\nu} \in \mathbb{R}^m$ and a corresponding equilibrium production vector $\boldsymbol{X}^{\nu *} \in \mathbb{R}^n$ of country $\nu = 1, \cdots, N$, are respectively defined as follows:

$$
\begin{aligned}
&\boldsymbol{W}^{\nu}\begin{cases} = \nabla_{\boldsymbol{V}} R^{\nu}\left(\boldsymbol{p}, \boldsymbol{V}^{\nu}\right), \text{ if } R^{\nu} \text{ is differentiable,} \\ \in \partial_{\boldsymbol{V}} R^{\nu}\left(\boldsymbol{p}, \boldsymbol{V}^{\nu}\right), \text{ if } R^{\nu} \text{ is not differentiable,} \end{cases} \\
&\boldsymbol{X}^{\nu *}\begin{cases} = \nabla_{\boldsymbol{p}} R^{\nu}\left(\boldsymbol{p}, \boldsymbol{V}^{\nu}\right), \text{ if } R^{\nu} \text{ is differentiable,} \\ \in \partial_{\boldsymbol{p}} R^{\nu}\left(\boldsymbol{p}, \boldsymbol{V}^{\nu}\right), \text{ if } R^{\nu} \text{ is not differentiable,} \end{cases}
\end{aligned} \tag{30}
$$

where $\partial_i R^{\nu}\left(\boldsymbol{p}, \boldsymbol{V}^{\nu}\right)$ denotes the sub-gradient set at $(\boldsymbol{p}, \boldsymbol{V}^{\nu})$, where $i = \boldsymbol{p}, \boldsymbol{V}^{\nu}$.

Here, factor price equalisation is defined as follows:

Definition 4 (factor price equalisation: FPE): Equilibrium factor prices are equalised for countries $\nu = 1, \cdots, N$ if and only if there exists a non-empty, open, convex subset of commodity prices $\Pi \subseteq \mathbb{R}^n_+$, and for each $\boldsymbol{p} \in \Pi$, there exists a profile of non-empty, open, convex subsets of factor endowments, $\left(\Gamma^{\nu}\left(\boldsymbol{p}\right)\right)_{\nu=1,\cdots,N}$ such that for each $\boldsymbol{p} \in \Pi$ there exists a vector $\boldsymbol{W} \in \mathbb{R}^m_+$ such that $\boldsymbol{W} = \nabla_{\boldsymbol{V}} R^{\nu}\left(\boldsymbol{p}, \boldsymbol{V}^{\nu}\right)$ for each country $\nu = 1, \cdots, N$ and for an arbitrary profile of factor endowments, $\left(\boldsymbol{V}^{\nu}\right)_{\nu=1,\cdots,N} \in \underset{\nu=1,\cdots,N}{\times} \Gamma^{\nu}\left(\boldsymbol{p}\right)$.

As a preliminary step, let us introduce the following two concepts which play important roles in this model:

Definition 5 (Linear Segment): A vector $(\psi^{\nu}, \delta^{\nu}) = \left(\psi^{\nu}_y, \psi^{\nu}_z, \delta^{\nu}\right)$ is a linear segment of G^{ν} at $(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu})$ if and only if there exists an $\varepsilon > 0$ such that

$$
G^{\nu}\left(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu}\right) + \lambda \psi^{\nu}_y = G^{\nu}\left(\boldsymbol{Z}^{\nu} + \lambda \psi^{\nu}_z, \boldsymbol{V}^{\nu} + \lambda \delta^{\nu}\right),
$$

for all $\lambda \in (-\varepsilon, \varepsilon)$.

Definition 6 (Direction of linearity): A vector $(\psi^{\nu}, \delta^{\nu}) \in \mathbb{R}^{n+m}$ is a direction of linearity of T^{ν} at $(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu})$ if and only if there exists an $\varepsilon > 0$ such that

$$
T^{\nu}\left(\boldsymbol{X}^{\nu} + \lambda \psi^{\nu}, \boldsymbol{V}^{\nu} + \lambda \delta^{\nu}\right) = 0,
$$

for all $\lambda \in (-\varepsilon, \varepsilon)$.

While changes in factor endowments generally produce changes in the production vector, a direction of linearity means that a change in the feasible and efficient production vector, $\boldsymbol{X}^{\nu}$, precipitated by a change in factor endowments, δ^{ν}, is linear; in other words, $\partial T^{\nu}\left(\boldsymbol{X}^{\nu} + \lambda \psi^{\nu}, \boldsymbol{V}^{\nu} + \lambda \delta^{\nu}\right) = \partial T^{\nu}\left(\boldsymbol{X}^{\nu}, \boldsymbol{V}^{\nu}\right)$. Therefore, a change in the production vector along a direction of linearity does not change the gradient vector of T^{ν}.

According to Definitions 5 and 6, it is clear that $(\psi^{\nu}, \delta^{\nu})$ is a direction of linearity of T^{ν} if and only if $(\psi^{\nu}, \delta^{\nu})$ is a linear segment of G^{ν}. In what follows, for $\nu = 1, \cdots, N$, Π and $\Gamma^{\nu}\left(\boldsymbol{p}\right)$ are of full dimension. The notation $\Gamma^{\nu}_m\left(\boldsymbol{p}\right)$ is used to emphasise this. Here, the necessary and sufficient condition for FPE to hold is given by the following theorem.

Theorem 5B: Under Assumption 9, FPE holds for each $\boldsymbol{p} \in \Pi$ and each $\left(\boldsymbol{V}^{\nu}\right)_{\nu=1,\cdots,N} \in \underset{\nu=1,\cdots,N}{\times} \Gamma^{\nu}_m\left(\boldsymbol{p}\right)$ if and only if the following conditions hold:

1) there exist m vectors, $(\psi_i(\boldsymbol{p}), \delta_i(\boldsymbol{p}))$, for $i = 1, \cdots, m$ that are directions of linearity of T^ν at $(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)$;

2) for $i = 1, \cdots, m$, the vectors $\delta_i(\boldsymbol{p})$ are linearly independent and the same for all countries;

3) the mappings $\psi_i : \Pi \rightarrow \mathbb{R}^n$ for $i = 1, \cdots, m$ are the same for all countries.

Proof: See the Appendix C. ■

Suppose that the economy has a price vector, $\boldsymbol{p}$, and a factor endowment of $\boldsymbol{V}^\nu \in \Gamma_m^\nu(\boldsymbol{p})$. Now imagine a change in the economy's endowment of its ith factor, $\delta_i(\boldsymbol{p})$. The directions of linearity, $(\psi_i(\boldsymbol{p}), \delta_i(\boldsymbol{p}))$, at each $(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)$ allow the gradient vector of T^ν to remain constant if the net output, $\boldsymbol{X}^{\nu*}$, changes by the amount of $\psi_i(\boldsymbol{p})$. Since the m independent vectors ($\delta_i(\boldsymbol{p})$ for $i = 1, \cdots, m$) span an m dimensional space, any change in the economy's factor endowment can be allocated to m directions of linearity. Therefore, it is possible for the economy to adjust production to any local change in its factor endowment so that the gradient vector, $\nabla T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)$, remains constant. Based on (28) and (30) we can see that $\partial R^\nu / \partial V_i = \boldsymbol{p}(\partial X^{\nu*} / \partial V_i) = W_i$. Since Theorem 5B ensures that $\partial X^{\nu*} / \partial V_i = \psi_i(\boldsymbol{p})$ is the same for all countries, FPE holds.

Theorem 5B implies that even though the equilibrium production vectors differ from country to country (i.e. free trade is achieved), the factor prices can still be equalised. The following theorem emphasises this.

Theorem 6B: Under Assumptions 9 and 10, FPE holds for $\boldsymbol{p} \in \Pi$ and $(\boldsymbol{V}^\nu)_{\nu=1,\cdots,N} \in \underset{\nu=1,\cdots,N}{\times} \Gamma_m^\nu(\boldsymbol{p})$ if and only if there exist $\psi_i(\boldsymbol{p})$ for $i = 1, \cdots, m$ that are the same for all countries such that:

$$\nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) = -\nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\,\boldsymbol{\Psi}, \tag{31}$$

$$\nabla_{\boldsymbol{VV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) = \boldsymbol{\Psi}^T \nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\,\boldsymbol{\Psi}, \tag{32}$$

$$\nabla_{\boldsymbol{X}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\,\boldsymbol{\Psi} + \nabla_{\boldsymbol{V}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\,\boldsymbol{\Omega} = \boldsymbol{0}, \tag{33}$$

where $\boldsymbol{\Psi} \equiv \begin{bmatrix} \psi_1^1(\boldsymbol{p}) & \cdots & \psi_m^1(\boldsymbol{p}) \\ \vdots & \ddots & \vdots \\ \psi_1^n(\boldsymbol{p}) & \cdots & \psi_m^n(\boldsymbol{p}) \end{bmatrix}$, $\boldsymbol{\Omega}$ is an identity matrix of order N defined by $\boldsymbol{\Omega} \equiv (\delta_1, \cdots, \delta_m)$, with δ_i as the basis vector for $i = 1, \cdots, m$, $\nabla_{\boldsymbol{X}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) \equiv \begin{bmatrix} \frac{\partial T^\nu}{\partial X_1^\nu} & \frac{\partial T^\nu}{\partial X_2^\nu} & \cdots & \frac{\partial T^\nu}{\partial X_M^\nu} \end{bmatrix}$, $\nabla_{\boldsymbol{V}} T^\nu(\boldsymbol{X}^\nu, \boldsymbol{V}^\nu) \equiv \begin{bmatrix} \frac{\partial T^\nu}{\partial V_1^\nu} & \frac{\partial T^\nu}{\partial V_2^\nu} & \cdots & \frac{\partial T^\nu}{\partial V_m^\nu} \end{bmatrix}$, and $\nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu), \nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)$, and others denote the Hessian matrices of T^ν.

Proof: See the Appendix C. ■

(31) and (32) demonstrate that substitutability of the factor endowments, $\nabla_{\boldsymbol{VV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)$, and the interaction between endowments and commodities ($\nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)$) are determined by $\nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)$ and $\psi_i(\boldsymbol{p})$, $i = 1, \cdots, m$. This implies that there is no restriction to the substitutability between commodities. Therefore, if the price of commodities, $\boldsymbol{p}$,

changes, then the equilibrium production vector can vary from country to country. Since $\psi_i(\boldsymbol{p})$ are the same for all countries, however, the production vector's response to changes in factor endowments must be the same for every country.

Furthermore, this model can also be applied to the case where Π and $\Gamma^{\nu}(\boldsymbol{p})$ are not of full dimension, that is, when m dimensional space cannot be spanned while there are m primary factors. Let $\Gamma^{\nu}_{k}(\boldsymbol{p})$ for $\boldsymbol{p} \in \Pi$ denote a k dimensional subset of endowment space where $k \leqq m$. Furthermore, we assume that $\Gamma^{\nu}_{k}(\boldsymbol{p})$ is convex.

Definition 7: The vectors $\delta_i(\boldsymbol{p})$ for $i = 1, \cdots, k$ are said to locally span $\Gamma^{\nu}_{k}(\boldsymbol{p})$ at $\boldsymbol{V}^{\nu} \in \Gamma^{\nu}_{k}(\boldsymbol{p})$ if they span the k dimensional affine subset that contains $\Gamma^{\nu}_{k}(\boldsymbol{p})$.

As shown by Theorems 5B and 6B, and given the above concept, factor prices equalise in the k dimensional subspace (Blackorby et al., 1993).

Blackorby et al. (1993) is more general than Kuga (1972) and Mas-Collel (1979a, b) in that the assumed production technique allows for decreasing returns to scale, joint production, and the existence of inputs other than primary factors (i.e. intermediate goods). In order to derive the condition for factor price equalisation, it is crucial that the transformation functions, T^{ν}, are directions of linearity. Kuga (1972) is similar to Blackorby et al. (1993) in that they both allow the number of final commodities to differ from the number of primary factors and both of their conditions for factor price equalisation hold that factor prices are solely dependent on commodity prices and independent of factor endowments. It should be noted, however, that Kuga's definition of factor price equalisation differs slightly from that used by Blackorby et al.; specifically, the former is stronger than the latter. Additionally, the models rely on different mechanisms to equalise factor prices. Kuga (1972) uses the method presented in Theorem 4B because the direct effect would be offset by the adjustment effect, not because the gradient vector of the transformation function would remain constant in the face of changing factor endowments. The necessary and sufficient condition derived from Blackorby et al. (1993) is weaker than Kuga (1972) in that the class of production economies supposed in the former is broader than that in the latter and the definition of factor price equalisation is weaker.

It is unclear how broad the class of production economies that satisfy Blackorby et al. (1993) necessary and sufficient conditions are. However, we can check whether or not the factor prices equalise in a production economy by using Theorem 5B.

8.2 The HOS Model after the Controversies

8.2.1 Uekawa et al. (1972)

On first inspection, the economic meaning of the SSS-I and SSS-II conditions seems unclear. The SSS-I condition implies that $\boldsymbol{A}^{-1}$ is a Minkowski matrix, and the SSS-II condition implies that $\boldsymbol{A}^{-1}$ is a Metzler matrix. As Uekawa et al. (1972) point out, the SSS-I and the SSS-II conditions are equivalent to the SSS-I′ and the SSS-II″ conditions, respectively defined as follows.

Condition 3B (SSS-I′): The inverse of the non-negative matrix $\boldsymbol{A} \equiv [a_{ij}]$ is a Minkowski matrix if and only if, for any non-empty proper subset J of $\{1, 2, \cdots, n\}$ and any given $\overline{x}_i > 0$ for $i \in \{1, 2, \cdots, n\} \setminus J$, there exists $x_{i'} > 0$ for some $i' \in J$, such that

$$\sum_{i' \in J} a_{i'j} x_{i'} > \sum_{i \in \{1,2,\cdots,n\} \setminus J} a_{ij} \overline{x}_i \quad \text{for } j \in J,$$
$$\sum_{i' \in J} a_{i'j} x_{i'} < \sum_{i \in \{1,2,\cdots,n\} \setminus J} a_{ij} \overline{x}_i \quad \text{for } j \in J^C.$$

Condition 4B (SSS–II′): The inverse of the non-negative matrix $\boldsymbol{A} \equiv [a_{ij}]$ is a Metzler matrix if and only if, for $J \subset \{1, 2, \cdots, n\}$ and any given $\overline{w}_j > 0$ with $j \in \{1, 2, \cdots, n\} \setminus J$, there exists $w_{j'} > 0$ for some $j' \in J$, such that

$$\sum_{j' \in J} w_{j'} a_{ij'} < \sum_{j \in \{1,2,\cdots,n\} \setminus J} \overline{w}_j a_{ij} \quad \text{for } i \in J,$$
$$\sum_{j' \in J} w_{j'} a_{ij'} > \sum_{j \in \{1,2,\cdots,n\} \setminus J} \overline{w}_j a_{ij} \quad \text{for } i \in J^C.$$

According to Uekawa et al. (1972), Conditions 5B and 6B yield the following economic implications:

Condition 5B: Suppose that commodities are randomly grouped into two composite commodities, J and $\{1, 2, \cdots, n\} \setminus J$, and let x_i be the output of the ith commodity. Then, for any non-trivial J and any set of outputs $\overline{x}_i > 0$ with $i \in \{1, 2, \cdots, n\} \setminus J$, there exists $x_{i'} > 0$ with $i' \in J$, such that more of the jth factor ($j \in J$) and less of the jth factor ($j \in \{1, 2, \cdots, n\} \setminus J$) is used to produce the composite commodity J than to produce $\{1, 2, \cdots, n\} \setminus J$.

Condition 6B: Suppose that the primary factors are randomly grouped into two composite factors, J and $\{1, 2, \cdots, n\} \setminus J$, and let w_j be the price of factor j. Then, for any non-trivial J and any set of factor prices, $\overline{w}_j > 0$ with $j \in \{1, 2, \cdots, n\} \setminus J$, there exists $w_{j'} > 0$ with $j' \in J$, such that the composite factor J contributes less to the cost of production of the jth commodity ($j \in J$) and more to the cost of production of the jth commodity, ($j \in \{1, 2, \cdots, n\} \setminus J$).

Therefore, Conditions 5B and 6B characterise the factor intensity. Clearly, these conditions are extremely strong.

9 Appendix C

In Appendix C, we show the rigorous proofs of the theorems in Appendix B.

9.1 The Proof of Theorem 2B

First, we shall prove the following lemma.

Lemma 1: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and partially differentiable mapping, where $f_{ij} \equiv \frac{\partial f_j}{\partial x_i}$. Suppose that there are $2n$ positive numbers m_k, M_k, for $k = 1, 2 \cdots, n$, such that the absolute values of the successive principal minors satisfy:

$$m_k \leqq \left| \det \begin{bmatrix} f_{11} & \cdots & f_{1k} \\ \vdots & \ddots & \vdots \\ f_{k1} & \cdots & f_{kk} \end{bmatrix} \right| \leqq M_k, \text{ for } k = 1, 2, \cdots, n,$$

in the whole of $\mathbb{R}^n$. Then, the system of equations, $f(\boldsymbol{x}) = \boldsymbol{a}$, has a unique solution in $\mathbb{R}^n$ for any given vector, $\boldsymbol{a} > \boldsymbol{0}$.

Proof: See Nikaido (1972). ■

In order to prove Theorem 2B, it is useful to transform the variables in the cost function as follows: $\omega \equiv \ln \boldsymbol{w} = [\ln w_i]$ and $\pi \equiv \ln \boldsymbol{p} = [\ln p_i]$. Therefore, $\pi = \ln c(e^{\omega}) \equiv \varphi(\omega)$. Function φ is continuous and differentiable in $\mathbb{R}^n$. Therefore, $\frac{\partial \ln p_j}{\partial \ln w_i} = \frac{w_i}{c_j}\frac{\partial c_j}{\partial w_i} = \frac{w_i}{c_j} \times a_{ij}(\boldsymbol{w}) = \frac{w_i a_{ij}(\boldsymbol{w})}{p_j} = \alpha_{ij}$ holds. In other words, $\frac{\partial \ln p_j}{\partial \ln w_i}$ is the share of the cost of factor i relative to the production cost of commodity j. Therefore, matrix $\widetilde{\boldsymbol{A}}$ is the Jacobian matrix of $\pi = \varphi(\omega)$. As such, we can prove Theorem 2B.

Proof: The system of equations $f(\boldsymbol{x}) = \boldsymbol{a}$ in Lemma 1 corresponds to $\varphi(\omega) = \pi$. Because of $f_{ij} = \alpha_{ij} \geqq 0$, δ_k in (16) can serve as m_k in Lemma 1. Note that $\alpha_{ij} \geqq 0$ by its definition and $\sum_{i=1}^{n} \alpha_{ij} = 1$ is obtained by using Property 2 and the Euler theorem, which imply $\alpha_{ij} \in [0, 1]$. Since the determinant is a polynomial of its elements, it is clear that the principal minors are bounded from above, which justifies the existence of M_k. Therefore, the system of equations $\varphi(\omega) = \pi$ satisfies the premise of Lemma 1. This immediately shows that $\varphi(\omega) = \pi$ has a unique solution for any given vector, $\pi > \boldsymbol{0}$. Therefore, $c(\boldsymbol{w}) = \boldsymbol{p}$ has a unique solution for any given vector, $\boldsymbol{p} > \boldsymbol{0}$. ■

9.2 The Proof of Theorem 3B

Proof: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and differentiable function. Let $JF(\boldsymbol{x})$ be the Jacobian matrix of $F(\boldsymbol{x})$. It is known from Berger (1977, p. 222) that if $\det JF(\boldsymbol{x}) \neq 0$ and $\left\|(JF(\boldsymbol{x}))^{-1}\right\| \leqq k$ for some $k > 0$, then F is a homeomorphism. Note that, the sufficient condition for $\left\|(JF(\boldsymbol{x}))^{-1}\right\| \leqq k$ is that $|\det JF(\boldsymbol{x})| \geqq \varepsilon > 0$ holds and the absolute values of all elements of $JF(\boldsymbol{x})$ must be uniformly bounded, where $\|\cdot\|$ is a norm defined in $\mathbb{R}^n$ as $\|T\| \equiv \max_{\|\boldsymbol{x}\|=1} \|T\boldsymbol{x}\|$.

Here, the function $\pi(\omega) = \ln c(e^{\omega})$ from Theorem 2B is used. From Assumption 4, we know that $\pi : \mathbb{R}^n \to \mathbb{R}^n$ and $J\pi(\omega) = \widetilde{\boldsymbol{A}}(\boldsymbol{w}) = [\alpha_{ij}(\boldsymbol{w})]$, where $J\pi(\omega)$ is the Jacobian matrix of $\pi(\omega)$. Because of $\alpha_{ij}(\boldsymbol{w}) \in [0, 1]$, the absolute values of all elements of $J\pi(\omega)$ are uniformly bounded. Furthermore, we assume $\left|\det \widetilde{\boldsymbol{A}}(\boldsymbol{w})\right| > \varepsilon > 0$. Therefore, $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, which means that $c : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is also a homeomorphism. ■

9.3 The Proof of Theorem 5B

First, we shall present some lemmas necessary for the proof.

Lemma 2: Let $F : D \subset \mathbb{R}^k \to \mathbb{R}$ be a concave function and $\beta \in \partial F(\boldsymbol{X}_0)$ and $\beta \in \partial F(\boldsymbol{X}_1)$. Then, for $\forall\mu \in [0,1]$, $\beta \in \partial F(\boldsymbol{X}_\mu)$ where $\boldsymbol{X}_\mu = \mu\boldsymbol{X}_0 + (1-\mu)\boldsymbol{X}_1$. Furthermore, for $\forall\mu \in [0,1]$, $F(\boldsymbol{X}_\mu) = F(\boldsymbol{X}_0) + \beta(\boldsymbol{X}_\mu - \boldsymbol{X}_0)$ so that $\{\beta(\boldsymbol{X}_\mu - \boldsymbol{X}_0), (\boldsymbol{X}_\mu - \boldsymbol{X}_0)\}$ is a linear segment at $\boldsymbol{X}_0$.

Proof: Since β is the sub-gradient of F at $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$, the following inequalities hold for all $\boldsymbol{X} \in D$:

$$F(\boldsymbol{X}) \leqq F(\boldsymbol{X}_0) + \beta(\boldsymbol{X} - \boldsymbol{X}_0), \quad F(\boldsymbol{X}) \leqq F(\boldsymbol{X}_1) + \beta(\boldsymbol{X}_\mu - \boldsymbol{X}_1).$$

Multiplying the inequalities by μ and $1-\mu$, respectively, yields:

$$F(\boldsymbol{X}) \leqq \mu F(\boldsymbol{X}_0) + (1-\mu) F(\boldsymbol{X}_1) + \beta(\boldsymbol{X} - \boldsymbol{X}_\mu), \tag{34}$$

for $\forall\mu \in [0,1]$. On the other hand, the concavity of F ensures that

$$F(\boldsymbol{X}_\mu) \geqq \mu F(\boldsymbol{X}_0) + (1-\mu) F(\boldsymbol{X}_1). \tag{35}$$

From (34) and (35), we know that $F(\boldsymbol{X}) \leqq F(\boldsymbol{X}_\mu) + \beta(\boldsymbol{X} - \boldsymbol{X}_\mu)$, which implies that $\beta \in \partial F(\boldsymbol{X}_\mu)$. Consequently, we obtain:

$$F(\boldsymbol{X}_0) \leqq F(\boldsymbol{X}_\mu) + \beta(\boldsymbol{X}_0 - \boldsymbol{X}_\mu) \text{ and } F(\boldsymbol{X}_\mu) \leqq F(\boldsymbol{X}_0) + \beta(\boldsymbol{X}_\mu - \boldsymbol{X}_0),$$

from which $F(\boldsymbol{X}_\mu) = F(\boldsymbol{X}_0) + \beta(\boldsymbol{X}_\mu - \boldsymbol{X}_0)$. Therefore, $\{\beta(\boldsymbol{X}_\mu - \boldsymbol{X}_0), (\boldsymbol{X}_\mu - \boldsymbol{X}_0)\}$ is a linear segment at $\boldsymbol{X}_0$. ■

Lemma 3: Factor prices equalise for $\boldsymbol{p} \in \Pi$, $\boldsymbol{V}^\nu \in \Gamma^\nu(\boldsymbol{p})$ holds if and only if

$$R^\nu(\boldsymbol{p}, \boldsymbol{V}^\nu) = R_0^\nu(\boldsymbol{p}) + \sum_{i=1}^m R_i(\boldsymbol{p}) V_i^\nu, \tag{36}$$

where $\boldsymbol{V}^\nu \equiv [V_i^\nu]$ for $\nu = 1, \cdots, N$. In this case, R^ν is differentiable so that

$$\boldsymbol{W} = \nabla_{\boldsymbol{V}} R^\nu(\boldsymbol{p}, \boldsymbol{V}^\nu) = [R_1(\boldsymbol{p}), \cdots, R_m(\boldsymbol{p})]. \tag{37}$$

for all countries.

Proof: *Necessity.* If the profit function takes the form presented in (36), then it is differentiable and (37) holds, which is consistent with (30).

Sufficiency. From Definition 4, $(\Gamma_m^\nu(\boldsymbol{p}))_{\nu=1,\cdots,N}$ is defined for $\boldsymbol{p} \in \Pi$ where $\Pi \subseteq \mathbb{R}_+^m$ is a non-empty and open convex set. Moreover, $\boldsymbol{W} \in \mathbb{R}_+^m$ exists and $\boldsymbol{W} = \nabla R^\nu(\boldsymbol{p}, \boldsymbol{V}^\nu)$, for each

$\nu = 1, \cdots, N$, for an arbitrary profile $(\boldsymbol{V}^{\nu}(\boldsymbol{p}))_{\nu=1,\cdots,N} \in \bigtimes_{\nu=1,\cdots,N} \Gamma^{\nu}_m(\boldsymbol{p})$. For $\nu = 1, \cdots, N$, $i = 1, \cdots, m$, and $\boldsymbol{V}^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) \in \Gamma^{\nu}_m(\boldsymbol{p})$,

$$W_i = \lim_{\delta_i^{\nu} \to 0} \frac{R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu} + (\delta_i^{\nu}, \boldsymbol{0}_{-i})) - R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu})}{(V_i^{\nu} + \delta_i^{\nu}) - V_i^{\nu}}$$

holds, where $\boldsymbol{0}_{-i}$ stands for the vector of order $m-1$ excluding the ith element. Since, from Definition 4, the assumption for Lemma 2 is satisfied by $\boldsymbol{W}$ as shown above, the following is satisfied for arbitrary $\boldsymbol{V}^{\nu}, \overline{\boldsymbol{V}}^{\nu} \in \Gamma^{\nu}_m(\boldsymbol{p})$:

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) = R^{\nu}\left(\boldsymbol{p}, \overline{\boldsymbol{V}}^{\nu}\right) + \boldsymbol{W}\left(\boldsymbol{V}^{\nu} - \overline{\boldsymbol{V}}^{\nu}\right).$$

Since $\boldsymbol{W}$ is independent of $\boldsymbol{V}^{\nu}$, $R_i : \Pi \to \mathbb{R}_+$ exists for $i = 1, \cdots, m$. Since $W_i = R_i(\boldsymbol{p})$, we obtain:

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) = R^{\nu}\left(\boldsymbol{p}, \overline{\boldsymbol{V}}^{\nu}\right) + \sum_{i=1}^{m} R_i(\boldsymbol{p})\left(V_i^c - \overline{V}_i^c\right), \text{ for each } \nu = 1, \cdots N.$$

By treating $\overline{\boldsymbol{V}}^{\nu}$ as the vector fixed in $\Gamma^{\nu}_m(\boldsymbol{p})$, we obtain: $R_0^{\nu}(\boldsymbol{p}) \equiv R^{\nu}\left(\boldsymbol{p}, \overline{\boldsymbol{V}}^{\nu}\right) - \sum_{i=1}^{m} R_i(\boldsymbol{p}) \overline{V}_i^{\nu}$. ■

Lemma 4: Suppose that $F : D \subseteq \mathbb{R}^k \to \mathbb{R}$ is concave and $\left(\beta\delta^1, \delta^1\right)$ and $\left(\beta\delta^2, \delta^2\right)$ at $\boldsymbol{X}$ are linear segments where $\beta \in \partial F(\boldsymbol{X})$ in $\boldsymbol{X} \in D$. Then, $(\beta\delta, \delta)$ is also a linear segment at $\boldsymbol{X}$ for $\delta = \mu\delta^1 + (1-\mu)\delta^2$ for all $\mu \in [0, 1]$.

Proof: Let us define $\boldsymbol{X}_1 = \boldsymbol{X} + \lambda_1\delta^1$ and $\boldsymbol{X}_2 = \boldsymbol{X} + \lambda_2\delta^2$ where $\lambda_1 \in (-\varepsilon_1, \varepsilon_1)$ and $\lambda_2 \in (-\varepsilon_2, \varepsilon_2)$. Then, $\beta \in \partial F(\boldsymbol{X}_1)$ and $\beta \in \partial F(\boldsymbol{X}_2)$ hold based on Definition 5. Since Lemma 2 can be applied, we find that $\beta \in \partial F(\boldsymbol{X}_\mu)$ holds, where $\boldsymbol{X}_\mu$ is the convex combination of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. Thanks to Lemma 2 $F(\boldsymbol{X}_\mu) = F(\boldsymbol{X}) + \beta(\boldsymbol{X}_\mu - \boldsymbol{X}_0)$ holds, which means that $(\beta\delta, \delta)$ is a linear segment of F in $\boldsymbol{X}$. ■

Lemma 5: Let R^{ν} be the profit function defined by (29). A linear segment of R^{ν} in $\boldsymbol{V}^{\nu}$ at $(\boldsymbol{p}, \boldsymbol{V}_0^{\nu})$ is equivalent to a linear segment of G^{ν} in $(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu})$ at $(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu})$, where $\boldsymbol{Z}_0^{\nu}$ is the profit maximiser in (29) given $(\boldsymbol{p}, \boldsymbol{V}_0^{\nu})$ for some $\boldsymbol{p} \in \Pi$.

Proof: If R^{ν} has a linear segment $(\boldsymbol{W}\delta, \delta)$ at $(\boldsymbol{p}, \boldsymbol{V}_0^{\nu})$, then $\boldsymbol{W} = \nabla_{\boldsymbol{V}} R^{\nu}(\boldsymbol{p}, \boldsymbol{V}_0^{\nu} + \lambda\delta)$ for all $\lambda \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Letting $\boldsymbol{p} = (\boldsymbol{p}^y, \boldsymbol{p}^z)$,

$$\nabla_{\boldsymbol{V}} R^{\nu}(\boldsymbol{p}, \boldsymbol{V}_0^{\nu}) = \sum_{i=1}^{m} p_i^y \frac{\partial G^{\nu}(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu})}{\partial V_i} = \boldsymbol{p}^y \nabla_{\boldsymbol{V}} G^{\nu}(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu}).$$

From the two equations shown above, we obtain:

$$\begin{aligned} \boldsymbol{W} = \nabla_{\boldsymbol{V}} R^{\nu}(\boldsymbol{p}, \boldsymbol{V}_0^{\nu} + \lambda\delta) &= \sum_{i=1}^{m} p_i^y \frac{\partial G^{\nu}(\boldsymbol{Z}^c(\lambda), \boldsymbol{V}_0^c + \lambda\delta)}{\partial V_i}, \\ &= \boldsymbol{p}^y \nabla_{\boldsymbol{V}} G^{\nu}(\boldsymbol{Z}^{\nu}(\lambda), \boldsymbol{V}_0^{\nu} + \lambda\delta), \end{aligned}$$

where $\boldsymbol{Z}^{\nu}(\lambda)$ is the profit maximiser at $(\boldsymbol{p}, \boldsymbol{V}_0 + \lambda\boldsymbol{\delta})$. Let $(\boldsymbol{Z}_1^{\nu}, \boldsymbol{V}_1^{\nu}) \equiv (\boldsymbol{Z}^{\nu}(\lambda), \boldsymbol{V}_0 + \lambda\delta)$ for all $\lambda \in (-\varepsilon, \varepsilon)$. Since

$$\boldsymbol{p}^y \nabla G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu}) = \left(\left(p_j^y \frac{\partial G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu})}{\partial Z_j}\right)_{j=1,\cdots,n}, \left(p_i^y \frac{\partial G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu})}{\partial V_j}\right)_{i=1,\cdots,m}\right)$$

and

$$\frac{\partial R^{\nu}}{\partial Z_j} = p_j^y \frac{\partial G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu})}{\partial Z_j} + p_j^z = 0 \ \ (\forall j = 1, \cdots n),$$

as such, we obtain:

$$\boldsymbol{p}^y \nabla_{\boldsymbol{Z}} G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu}) = \left(p_j^y \frac{\partial G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}^{\nu})}{\partial Z_j}\right)_{j=1,\cdots,n} = -\boldsymbol{p}^z.$$

Therefore, $\boldsymbol{p}^y \nabla G^{\nu}(\boldsymbol{Z}^{\nu}, \boldsymbol{V}_0^c) = (-\boldsymbol{p}^z, \boldsymbol{W})$ holds. Similarly, $\boldsymbol{p}^y \nabla G^{\nu}(\boldsymbol{Z}^{\nu}(\lambda), \boldsymbol{V}_0^{\nu} + \lambda\delta) = (-\boldsymbol{p}^z, \boldsymbol{W})$ also holds. Consequently, we obtain:

$$(-\boldsymbol{p}^z, \boldsymbol{W}) = \boldsymbol{p}^y \nabla G^{\nu}(\boldsymbol{Z}_0^c, \boldsymbol{V}_0^c) = \boldsymbol{p}^y \nabla G^{\nu}(\boldsymbol{Z}_1^c, \boldsymbol{V}_1^c).$$

From Lemma 2, therefore, we obtain: $(-\boldsymbol{p}^z, \boldsymbol{W}) = \boldsymbol{p}^y \nabla G^{\nu}\left(\boldsymbol{Z}_{\mu}^{\nu}, \boldsymbol{V}_{\mu}^{\nu}\right)$, where $\boldsymbol{Z}_{\mu}^{\nu} = \mu \boldsymbol{Z}_0^{\nu} + (1-\mu)\boldsymbol{Z}_1^{\nu}$, $\boldsymbol{V}_{\mu}^{\nu} = \mu \boldsymbol{V}_0^{\nu} + (1-\mu)\boldsymbol{V}_1^{\nu}$ for $\forall \mu \in [0,1]$. Here, letting $\beta = (-\boldsymbol{p}^z/\boldsymbol{p}^y, \boldsymbol{W}/\boldsymbol{p}^y)$ and $\delta = (\boldsymbol{Z}_1^{\nu}, \boldsymbol{V}_1^{\nu}) - (\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu})$,we see that $(\boldsymbol{\beta}\delta, \delta)$ is a linear segment of G^{ν} at $(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu})$.

In order to prove the converse, suppose that (ψ^y, ψ^z, δ) is a linear segment of G^{ν} at $(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu})$. As such, the following holds:

$$G^{\nu}(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu}) + \lambda\psi^y = G^{\nu}(\boldsymbol{Z}_0^{\nu} + \lambda\boldsymbol{\psi}^y, \boldsymbol{V}_0^{\nu} + \lambda\delta).$$

Therefore, the following relationships must hold for all $\lambda \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$:

$$\boldsymbol{p}^y \nabla_{\boldsymbol{Z}} G^{\nu}(\boldsymbol{Z}_0^{\nu} + \lambda\psi^x, \boldsymbol{V}_0^{\nu} + \lambda\delta) = \boldsymbol{p}^y \nabla_{\boldsymbol{Z}} G^{\nu}(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu}) = -\boldsymbol{p}^z,$$

and

$$\boldsymbol{p}^y \nabla_{\boldsymbol{V}} G^{\nu}(\boldsymbol{Z}_0^{\nu} + \lambda\psi^x, \boldsymbol{V}_0^{\nu} + \lambda\delta) = \boldsymbol{p}^y \nabla_{\boldsymbol{V}} G^{\nu}(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu}) = \boldsymbol{W}.$$

Because of $\nabla_{\boldsymbol{V}} R^{\nu}(\boldsymbol{p}, \boldsymbol{V}_0^{\nu}) = \boldsymbol{p}^y \nabla_{\boldsymbol{V}} G^{\nu}(\boldsymbol{Z}_0^{\nu}, \boldsymbol{V}_0^{\nu})$, $\boldsymbol{W} = \boldsymbol{\nabla}_{\boldsymbol{V}} R^{\nu}(\boldsymbol{p}, \boldsymbol{V}_0^c)$. Similarly,

$$\nabla_{\boldsymbol{V}} R^{\nu}(\boldsymbol{p}, \boldsymbol{V}_0^{\nu} + \lambda\delta) = \boldsymbol{p}^y \nabla_{\boldsymbol{V}} G^{\nu}(\boldsymbol{Z}_0^{\nu} + \lambda\psi^{\nu}, \boldsymbol{V}_0^{\nu} + \lambda\delta)$$

holds for ψ^z, thus satisfying $\boldsymbol{p}^y\psi^y = \boldsymbol{W}\delta - \boldsymbol{p}^z\psi^z$. Therefore, $\boldsymbol{W} = \boldsymbol{\nabla}_{\boldsymbol{V}} R^{\nu}(\boldsymbol{p}, \boldsymbol{V}_0^{\nu} + \lambda\delta)$ holds for all $\lambda \in (-\varepsilon, \varepsilon)$. This means that $(\boldsymbol{W}\delta, \delta)$ is a linear segment of R^{ν} in $\boldsymbol{V}$ at $(\boldsymbol{p}, \boldsymbol{V}_0^{\nu})$. ■

Now, let us proceed to the proof of Theorem 5B.

Proof: *Necessity.* Assume that the factor prices are equalised. As described in Lemma 3, the profit function, R^{ν}, has m linear segments at any point $(\boldsymbol{p}, \boldsymbol{V}^{\nu}) \in \Pi \times \Gamma_m^{\nu}(\boldsymbol{p})$. Let us denote these linear segments as $(R_i(\boldsymbol{p}), \delta_i)$, where δ_i for $i = 1, \cdots, m$ can be chosen as the standard basis vector for $\mathbb{R}^m$. Furthermore, let $\boldsymbol{Z}^{\nu *}$ be the equilibrium production vector given $(\boldsymbol{p}, \boldsymbol{V}_0^{\nu})$. From Lemma 5, we see that if $(R_i(\boldsymbol{p}), \delta_i)$ is a linear segment for R^{ν} at $(\boldsymbol{p}, \boldsymbol{V}^{\nu})$ there exists a vector, (ψ_i^{ν}, δ_i), that is a linear segment of G^{ν} at $(\boldsymbol{Z}^{\nu *}, \boldsymbol{V}^{\nu})$. Therefore, at any point $(\boldsymbol{p}, \boldsymbol{V}^{\nu}) \in \Pi \times \Gamma_m^{\nu}(\boldsymbol{p})$, there exist m linear segments $(\psi_i^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}), \delta_i)$ for G^{ν} at $(\boldsymbol{Z}^{\nu *}, \boldsymbol{V}_0^{\nu})$. Since (ψ_i, δ_i) are linear segments of G^{ν}, they are directions of linearity of T^{ν}.

Since δ_i for $i = 1, \cdots, m$ are the standard basis vectors, the following holds for $\lambda \in (-\varepsilon, \varepsilon)$ from Lemma 3:

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu} + \lambda(\delta_i, \boldsymbol{0}_{-i})) = R^{\nu}(\boldsymbol{p}) + \lambda R_i(\boldsymbol{p}).$$

Moreover, from the definitions of the profit function and linear segments, we can see that:

$$\boldsymbol{p}\{\boldsymbol{X}^{\nu *} + \lambda \psi_i^{\nu}(\boldsymbol{p}, \boldsymbol{V}^c)\} = R^{\nu}(\boldsymbol{p}) + \lambda R_i(\boldsymbol{p}).$$

Since $\boldsymbol{p}\boldsymbol{X}^{\nu *} = R^{\nu}(\boldsymbol{p})$ if $\lambda = 0$, we obtain:

$$\boldsymbol{p} \cdot \psi_i^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) = R_i(\boldsymbol{p}). \tag{38}$$

(38) must be satisfied for $\forall \boldsymbol{p} \in \Pi$. This implies that $\psi_i^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu})$ is independent of $\boldsymbol{V}^{\nu} \in \Gamma_m^{\nu}(\boldsymbol{p})$ and is the same for all countries.

Sufficiency. We assume that production function G^{ν} is m linear segments $(\psi_i(\boldsymbol{p}), \delta_i)$ for $i = 1, \cdots, m$, where δ_i are standard basis vectors. From Lemma 5, R^{ν} has m linear segments in $\boldsymbol{V}$ given $\boldsymbol{p}$. For $i = 1, \cdots, m$, therefore, $R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu} + \lambda(\delta_i, \boldsymbol{0}_{-i})) = R^{\nu}(\boldsymbol{p}) + \lambda R_i(\boldsymbol{p})$ holds for each $\lambda \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Based on Lemma 4, it is clear that $\left(\frac{1}{m}R(\boldsymbol{p}), \frac{\delta}{m}\right)$ is a linear segment of R^{ν} in $\boldsymbol{V}$ at $(\boldsymbol{p}, \boldsymbol{V}^{\nu})$. In other words, the following holds for each $\lambda \in (-\varepsilon/m, \varepsilon/m)$ for some $\varepsilon/m > 0$:

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu} + \lambda\delta) = R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) + \lambda \sum_{i=1}^{m} R_i(\boldsymbol{p})\,\delta_i.$$

By applying Lemma 2 and using the same logic as that used in the proof of Lemma 3, we obtain what follows:

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) = R_0^{\nu}(\boldsymbol{p}) + \sum_{i=1}^{m} R_i^{\nu}(\boldsymbol{p})\, V_i^{\nu}.$$

Here, $\psi_i : \Pi \to \mathbb{R}$, for $i = 1, \cdots, m$, are common to all countries and, as shown in (38), $\boldsymbol{p} \cdot \psi_i(\boldsymbol{p}) = R_i^{\nu}(\boldsymbol{p})$ for $\nu = 1, \cdots, N$, holds for $i = 1, \cdots, m$. In other words, we obtain:

$$R^{\nu}(\boldsymbol{p}, \boldsymbol{V}^{\nu}) = R_0^{\nu}(\boldsymbol{p}) + \sum_{i=1}^{m} R_i(\boldsymbol{p})\, V_i^{\nu}, \text{ for } \forall \nu = 1, \cdots, N.$$

From Lemma 3, we can see that the factor prices are equalised for arbitrary $\left(\boldsymbol{p}, (\boldsymbol{V}^{\nu})_{\nu=1,\cdots,N}\right) \in \Pi \times \left(\times_{\nu=1,\cdots,N} \Gamma_m^{\nu}\right)$. ■

9.4 The Proof of Theorem 6B

Proof: *Necessity.* We assume that factor prices are equalised. Theorem 5B shows that there exist m vectors, $(\psi_i(\boldsymbol{p}), \delta_i(\boldsymbol{p}))$, for $i = 1, \cdots, m$, that are directions of linearity for T^ν. In other words, there exists $\varepsilon > 0$ such that for $\lambda \in (-\varepsilon, \varepsilon)$:

$$T^\nu(\boldsymbol{X}^{\nu*} + \lambda\psi_i(\boldsymbol{p}), \boldsymbol{V}^\nu + \lambda\delta_i) = 0.$$

Differentiating with respect to λ yields:

$$\nabla_{\boldsymbol{X}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Psi} + \nabla_{\boldsymbol{V}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Omega} = \mathbf{0}.$$

This is simply a repetition of (33). From the above, we obtain:

$$\begin{aligned}
\nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Psi} + \nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Omega} = \mathbf{0}, \\
\nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Psi} + \nabla_{\boldsymbol{VV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Omega} = \mathbf{0}.
\end{aligned}$$

The former is (31) and pre-multiplying it by $\boldsymbol{\Psi}^T$, yields:

$$\boldsymbol{\Psi}^T \nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Psi} + \boldsymbol{\Psi}^T \nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Omega} = \mathbf{0}. \tag{39}$$

Pre-multiplying the latter by $\boldsymbol{\Omega}^T$ yields:

$$\boldsymbol{\Omega}^T \nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Psi} + \boldsymbol{\Omega}^T \nabla_{\boldsymbol{VV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Omega} = \mathbf{0},$$

transposing this yields:

$$\boldsymbol{\Psi}^T \nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Omega} + \boldsymbol{\Omega}^T \nabla_{\boldsymbol{VV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Omega} = \mathbf{0}. \tag{40}$$

Since $\boldsymbol{\Omega}$ is an identity matrix, $\nabla_{\boldsymbol{VV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) = \boldsymbol{\Psi}^T \nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu)\boldsymbol{\Psi}$ is obtained from (39) and (40).

Sufficiency. (31) and (32) imply the following relationships:

$$\begin{aligned}
\begin{bmatrix} \nabla_{\boldsymbol{XX}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) & \nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) \end{bmatrix} \begin{bmatrix} \psi_i(\boldsymbol{p}) \\ \delta_i(\boldsymbol{p}) \end{bmatrix} = \mathbf{0}, \\
\begin{bmatrix} \nabla_{\boldsymbol{XV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) & \nabla_{\boldsymbol{VV}} T^\nu(\boldsymbol{X}^{\nu*}, \boldsymbol{V}^\nu) \end{bmatrix} \begin{bmatrix} \psi_i(\boldsymbol{p}) \\ \delta_i(\boldsymbol{p}) \end{bmatrix} = \mathbf{0}.
\end{aligned}$$

These equations imply that the Hessian matrix of T^ν has m eigenvectors that satisfy (33) and that are associated with zero eigenvalues. The eigenvectors are given by $(\psi_i(\boldsymbol{p}), \delta_i(\boldsymbol{p}))$, $i = 1, \cdots, m$. These eigenvectors are directions of linearity of T^ν. Therefore, Theorem 5B implies that the FPET holds. ■