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The Helgason Fourier transform for compact Riemannian symmetric spaces of rank one

by

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0. Introduction

For Riemannian symmetric spaces (RSS) of noncompact type Helgason [2], [3], [5], found the analog of classical Fourier analysis. This paper concerns the counterpart of Helgason's theory for RSS of compact type. Together with classical Fourier theory these results constitute a unified Fourier analysis on RSS related to, but distinct from the established alternatives of representation theory and the spherical Fourier transform. An advantage of this style of Fourier theory is that the transform kernel has the same kind of simplicity as the functions $e^{ix\cdot y}$ of classical Fourier theory. In particular, the transform kernel employs scalar-valued eigenfunctions of first order differential operators.

On the other hand, the theory given here for the compact RSS involves a severe singularity in part of the transform kernel. One may avoid this singularity by confining consideration to the half of the RSS closest to the origin; call this the *local theory*. The local theory is given in Section 1 for any compact RSS. The rest of this paper is devoted to the global theory for compact RSS of rank one. (It is not clear that a global theory exists for the higher rank compact RSS.)

In broad outline, Helgason's transform comes from the wedding of the spherical Fourier transform with an integral formula for the Poisson kernel. In Helgason's notation ([5], p. 418) this formula is

$$\phi_\lambda(g^{-1}h) = \int_K e^{(-i\lambda+\varrho)(A(kg))} e^{(i\lambda+\varrho)(A(kh))} dk \quad (g, h \in G). \tag{0.1}$$

It has a generalization, Theorem 1.7, which, following Helgason, is combined in Section 1 with the spherical transform to obtain the local theory. At the end of Section 1 a statement of the Main theorem in the global theory for rank one RSS of compact type is given. The remaining sections prove this theorem.

These ideas have been worked out in [7] and [8] for the sphere. The sphere theory is summarized in Section 2 of this paper. Section 2 goes on to develop some refinements of the sphere theory which play an essential role in Section 3. These refinements have to do with the restriction of the Fourier transform to functions of a given K-type.

The main achievement of this paper is the establishment of the global theory for the rank one spaces which have double restricted roots. To this end we introduce a device which is of independent interest: a map ξ from the symmetric space S to the closed unit ball Ω in a Euclidean space of low dimension (3, 5, or 9 depending on the multiplicity of the double restricted root of S). Section 4 develops the rich theory of this map. Via ξ we reduce the main analytic difficulties to the study of a certain singular integral on Ω. Section 3 anticipates and treats this singular integral by separation-of-variables and the theory of Section 2. These results were announced in [9].

Notation. If L is a Lie group acting on a manifold M (e.g., $M=L$) and x is an element of the Lie algebra of L then D_x denotes the corresponding differential operator on M:

$$D_x f(m) = \frac{d}{dt} f(\exp(tx)\, m)_{t=0} \quad (f \in C^1(M),\ m \in M).$$

Thus $[D_x, D_y] = -D_{[x,y]}$. Extend the definition of D_x to x in the complex Lie algebra of L by complex linearity.

As a rule, the invariant measure on the homogeneous space of a compact group will be normalized to have total mass 1. This includes spheres with the single exception of the circle $\mathbf{R}/2\pi\mathbf{Z}$ where the conventional Lebesgue integral is used.

e_{ij} is used to denote a square matrix all of whose entries are 0 except for a 1 in the (i,j) position.

In Sections 2 and 3 we use the following notation: q is always ≥ 2 and

$$t_\diamond = \sqrt{1-|t|^2} \quad (|t| \leq 1),$$

$$P_{n,q}(t) = \frac{\Gamma(q/2)\,(-2)^{-n}}{\Gamma(n+q/2)} (t_\diamond)^{2-q} \left(\frac{d}{dt}\right)^n (t_\diamond)^{2n+q-2} \quad (n \in \mathbf{N}, |t| \leq 1). \tag{0.2}$$

On the other hand, when $z \in \mathbf{C}$ and $0<t\leq 1$ we take

$$P_{z,q}(t) = {}_2F_1(-z, z+q-1; q/2; (1-t)/2) \tag{0.3}$$

$$= {}_2F_1(-z/2, (z+q-1)/2; q/2; 1-t^2). \tag{0.4}$$

(0.2)–(0.4) are consistent (see [1], Vol. 2, Section 10.9). (0.2) is the Rodrigues formula defining the nth-degree orthogonal polynomial on $[-1, 1]$ with respect to the weight $(t_\diamond)^{q-2}$ and normalized so that $P_{n,q}(1)=1$. Except for this normalization, $P_{n,q}$ is the Gegenbauer polynomial $C_n^{(q-1)/2}$ (again see [1], Vol. 2, Section 10.9).

In Sections 2 and 3 the following constants are frequently used:

$$\omega_q\colon \quad \omega_q^{-1} = B(q/2, 1/2) = \int_{-1}^{1} (x_\diamond)^{q-2}\, dx. \tag{0.5}$$

$$\varpi(j, q; z) = (-i)^j \prod_{k=1}^{j} \left(\frac{z+k+q-2}{2k+q-2}\right) \quad (j \in \mathbf{N}, z \in \mathbf{C}). \tag{0.6}$$

In the degenerate case $j=0$ we take $\varpi(0, q; z)=1$.

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1. The local theory

Let $S=G/K$ be a Riemannian symmetric space with K a compact subgroup of the connected Lie group G. We begin by recalling the spherical Fourier transform theory for S and then explain the generalization of (0.1) which combines with the former to produce the Helgason–Fourier transform.

Let Φ denote the set of positive definite zonal spherical functions on S. To each K-invariant f in $L^1(S)$ is assigned its spherical Fourier transform $\tilde{f}\colon \Phi \to \mathbf{C}$ by

$$\tilde{f}(\phi) = \int_S f(s)\, \overline{\phi(s)}\, ds \quad (\phi \in \Phi).$$

Φ is given the weakest topology which makes all such $\tilde{f}$ continuous. Then Φ is locally compact and bears a Plancherel measure $d\phi$ such that for K-invariant f in $L^1(S) \cap L^2(S)$ we get $\|f\|_2 = \|\tilde{f}\|_2$. Moreover, for sufficiently nice K-invariant f, say $f \in C_c^\infty(S)$, we can recapture f from $\tilde{f}$ by

$$f(s) = \int_{\Phi} \tilde{f}(\phi)\, \phi(s)\, d\phi. \tag{1.1}$$

See [5], Chapter IV, especially the notes to §§ 5–7.

We can expand the scope of these facts to cover functions which are not K-invariant: To each $\phi \in \Phi$ there corresponds a function, also denoted ϕ, on $S \times S$ as follows

$$\phi(gK, hK) = \phi(g^{-1}h) \quad (g, h \in G).$$

Then for $f \in C_c^\infty(S)$, not necessarily K-invariant, we have

$$f(s) = \int_{\Phi} \int_{S} f(s')\, \phi(s', s)\, ds'\, d\phi. \tag{1.2}$$

To obtain a full Fourier theory the idea is to replace the $\phi(s', s)$ in (1.2) by something like the right side of (0.1). Helgason carried this out in [3] for G/K of noncompact type. In a similar way, classical Fourier analysis on $\mathbf{R}^n$ can—not that it should—be developed from (1.2) and the formula

$$\phi_{\|z\|}(y, x) = j(\|x-y\|\,\|z\|) = \int_{SO(n)} e^{ix\cdot kz} e^{-iy\cdot kz}\, dk \quad (x, y, z \in \mathbf{R}^n) \tag{1.3}$$

where

$$j(r) = \Gamma(n/2)\, r^{-(n-2)/2} J_{(n-2)/2}(r) \quad (r > 0)$$

and J_m is the mth order Bessel function of the first kind. (1.3) is the analog of (0.1) for $S = \mathbf{R}^n$. On $\mathbf{R}^n$, (1.1) becomes the Bessel–Hankel transform.

A major point of this section is that (0.1) and (1.3) have a common generalization, Theorem 1.7, that holds on all RSS.

To appreciate the generality of these ideas, assume only that G is a connected unimodular Lie group, K is a compact subgroup, and their complex Lie algebras are $\mathfrak{g}$, $\mathfrak{k}$. Assume also that $\mathfrak{g}$ contains a complex subalgebra $\mathfrak{b}$, complimentary to $\mathfrak{k}$:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{b}.$$

By $\hat{\mathfrak{b}}$ denote the set of Lie algebra homomorphisms λ of $\mathfrak{b}$ into $\mathbf{C}$, i.e., $\lambda: \mathfrak{b} \to \mathbf{C}$ is linear and $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$.

Lemma 1.1. *Let $\mathcal{O}$ be any connected, simply connected open subset of $S = G/K$*

such that on $\mathcal{O}$ the vector fields D_x $(x \in \mathfrak{b})$ are nonvanishing. Then for every $\lambda \in \hat{\mathfrak{b}}$ there is a nonzero function f on $\mathcal{O}$ such that on $\mathcal{O}$

$$D_x f = \lambda(x) f \quad (x \in \mathfrak{b}). \tag{1.4}$$

f is C^ω, nonvanishing on $\mathcal{O}$, and unique up to a scalar factor.

Proof. λ defines a closed, hence exact, 1-form dg on $\mathcal{O}$ by

$$dg(D_x) = \lambda(x) \quad (x \in \mathfrak{b}).$$

Take $f = e^g$. □

Let $\check{\mathfrak{b}}$ denote the subset of $\hat{\mathfrak{b}}$ consisting of λ for which (1.4) admits a global solution $f \neq 0$ in $C^\omega(S)$. Let s_0 denote the origin (i.e., the identity coset) in $S = G/K$.

COROLLARY 1.2. *For any $\lambda \in \check{\mathfrak{b}}$, (1.4) has a unique solution f such that $f(s_0) = 1$. This solution will be denoted $e(\mathfrak{b}, \lambda)$.*

$\check{\mathfrak{b}}$ is clearly a semigroup under addition and

$$e(\mathfrak{b}, \lambda_1)\, e(\mathfrak{b}, \lambda_2) = e(\mathfrak{b}, \lambda_1 + \lambda_2) \quad (\lambda_1, \lambda_2 \in \check{\mathfrak{b}}).$$

Example. If $S = G/K$ is an RSS of noncompact type and $G = NAK$ is its Iwasawa decomposition, take $\mathfrak{b}$ to be the complex Lie algebra of the group NA. Then

$$e(\mathfrak{b}, \lambda)(gs_0) = e^{\lambda(A(g))} \quad (g \in G)$$

where the notation on the right side is that used in (0.1): $A(g)$ is in the Lie algebra of A such that $g \in N \exp(A(g)) K$.

LEMMA 1.3. *For all $\lambda \in \check{\mathfrak{b}}$ the function*

$$\phi_\lambda(s) = \int_K e(\mathfrak{b}, \lambda)(ks)\, dk \quad (s \in S) \tag{1.5}$$

is a spherical function on S in the sense that it satisfies $\phi_\lambda(s_0) = 1$ and

$$\phi_\lambda(gK)\, \phi_\lambda(s) = \int_K \phi_\lambda(gks)\, dk \quad (g \in G, s \in S).$$

Proof. For any $s \in S$ define f_s on G/K by $f_s(gK) = \int_K e(\mathfrak{b}, \lambda)(gks)\, dk$. Then f_s satisfies (1.4) and so by Lemma 1.1,

$$f_s = \phi_\lambda(s)\, e(\mathfrak{b}, \lambda).$$

Consequently

$$\int_K \phi_\lambda(gks)\,dk = \int_K \int_K e(\mathfrak{b},\lambda)(k_1gk_2s)\,dk_1\,dk_2.$$

Reverse the order of integration to get

$$\int_K f_s(k_1gK)\,dk_1 = \int_K \phi_\lambda(s)\,e(\mathfrak{b},\lambda)(k_1gK)\,dk_1 = \phi_\lambda(s)\,\phi_\lambda(gK). \qquad \square$$

This lemma gives Harish-Chandra's formula for zonal spherical functions in case G/K is a RSS of noncompact type. Of course he also showed the much harder fact (for those spaces) that the ϕ_λ ($\lambda \in \tilde{\mathfrak{b}}$) exhaust the zonal spherical functions.

LEMMA 1.4. *If G is compact and $\mathfrak{b}$ is solvable then*

(i) *each irreducible G-submodule of $L^2(S)$ contains a function $e(\mathfrak{b},\lambda)$ for some $\lambda \in \tilde{\mathfrak{b}}$;*

(ii) *$L^1(K\backslash G/K)$ is commutative;*

(iii) *each zonal spherical function on S is of the form ϕ_λ (as given by (1.5)) for some $\lambda \in \tilde{\mathfrak{b}}$.*

Proof. Since G is compact any irreducible G-submodule V of $L^2(S)$ is finite dimensional. Then by Lie's theorem, $D_{\mathfrak{b}}$ has a nonzero eigenvector $f \in V$. In other words, f satisfies (1.4). Thus it is nonzero at s_0, so $f=f(s_0)\,e(\mathfrak{b},\lambda)$ where λ and f are related by (1.4). This proves (i).

The uniqueness of $e(\mathfrak{b},\lambda)$ asserted by Lemma 1.1 shows that the various irreducible G-submodules of $L^2(S)$ are inequivalent. This implies (ii) as follows: $L^1(K\backslash G/K)$ acts on $L^2(S)$ by convolution on the right. This action commutes with the regular representation of G on $L^2(S)$. Since each irreducible component of $L^2(S)$ is distinct as a G-module, right convolution by $L^1(K\backslash G/K)$ is a scalar on each irreducible component by Schur's lemma. Thus $L^1(K\backslash G/K)$ has a faithful representation by commuting operators, proving (ii).

(iii) follows from (i), (ii), and Lemma 1.3 because (ii) implies that every zonal spherical function ϕ belongs to an irreducible G-submodule V_ϕ of $L^2(S)$ and is unique in V_ϕ. Take $\lambda \in \tilde{\mathfrak{b}}$ such that $e(\mathfrak{b},\lambda) \in V_\phi$. Then we must have $\phi=\phi_\lambda$. $\square$

Of course if S is a compact RSS then it satisfies the hypothesis of Lemma 1.4, but there are some compact nonsymmetric spaces which also satisfy Lemma 1.4. An example is $U(n)/SU(n-1)$. For this space, $\mathfrak{g}=\mathfrak{gl}(n,\mathbf{C})$ and $\mathfrak{k}\approx\mathfrak{sl}(n-1,\mathbf{C})$ regarded as acting on coordinates $z_2,\ldots,z_n$ of $z \in \mathbf{C}^n$; $\mathfrak{b}$ is spanned by

$$I, \quad e_{12}-e_{21}, \quad e_{11}+e_{22}, \quad e_{11}-e_{22}+i(e_{12}+e_{21}),$$
$$e_{1j}+ie_{2j} \quad \text{and} \quad e_{j1}+ie_{j2} \quad (j=3, \dots, n).$$

In harmonic analysis on noncompact RSS a prominent role is played by ϱ, half the sum of positive roots. However, here we find it more convenient to use 2ϱ. It enters as the trace form τ on $\mathfrak{b}$:

$$\tau(x) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{b}}(x)) \quad (x \in \mathfrak{b}).$$

LEMMA 1.5. *If G is unimodular and K is compact and connected then $-\tau \in \tilde{\mathfrak{b}}$.*

Proof. Let $\omega_{\mathfrak{b}}$ and $\omega_{\mathfrak{k}}$ be nonzero, homogeneous elements of maximal degree in the Grassmann algebras of $\mathfrak{k}$ and $\mathfrak{b}$ respectively. Then the function f on G defined by

$$f(g)\,\omega_{\mathfrak{b}} \wedge \omega_{\mathfrak{k}} = \omega_{\mathfrak{b}} \wedge \mathrm{Ad}(g)\,\omega_{\mathfrak{k}}$$

is constant on right K-cosets in G and so may be regarded as defined on S. As such, it satisfies $f(s_0)=1$ and (1.4) with $\lambda=-\tau$ since by the unimodularity of G,

$$D_x f(g)\,\omega_{\mathfrak{b}} \wedge \omega_{\mathfrak{k}} = -(\mathrm{ad}(x)\,\omega_{\mathfrak{b}}) \wedge \mathrm{Ad}(g)\,\omega_{\mathfrak{k}} \quad (x \in \mathfrak{g})$$

and

$$\mathrm{ad}(x)\,\omega_{\mathfrak{b}} = \tau(x)\,\omega_{\mathfrak{b}} \quad (x \in \mathfrak{b}).$$

Consequently $f=e(\mathfrak{b}, -\tau)$. □

Note. In this argument the compact-connectedness of K really enters only to insure that K is strictly unimodular in the sense that $\det(\mathrm{Ad}_{\mathfrak{k}}(k))=1$ for $k \in K$. $O(2n)$ is an example of a compact group which is not strictly unimodular and for which the lemma fails. This point should have been made in [9].

Assume throughout the rest of this section that the conditions of Lemma 1.5 are met.

LEMMA 1.6. *If $s \in S$ is a point at which $e(\mathfrak{b}, -\tau)(s) \neq 0$ then $e(\mathfrak{b}, \lambda)(s) \neq 0$ for all $\lambda \in \tilde{\mathfrak{b}}$. Consequently there is a maximal connected, open, K-invariant neighborhood $S_{1/2}$ of $s_0 \in S$ on which $e(\mathfrak{b}, \lambda)$ are $\neq 0$ for all $\lambda \in \tilde{\mathfrak{b}}$.*

Proof. At points gK of S where $e(\mathfrak{b}, \tau) \neq 0$, we have (in the notation of Lemma 1.5) that

$$\omega_{\mathfrak{b}} \wedge \mathrm{Ad}(g)\,\omega_{\mathfrak{k}} = e(\mathfrak{b}, -\tau)(gK)\,\omega_{\mathfrak{b}} \wedge \omega_{\mathfrak{k}} \neq 0,$$

i.e. $\mathfrak{b}+\mathrm{Ad}(g)\mathfrak{k}=\mathfrak{g}$. At such points gK the operators D_x $(x\in\mathfrak{b})$ are nonvanishing and so by Lemma 1.1, $e(\mathfrak{b},\lambda)\neq 0$ at those points.

Since K is compact, the K-orbit of the zero set of $e(\mathfrak{b},-\tau)$ is closed. Take $S_{1/2}$ to be the component containing s_0 of the compliment of this K-orbit. □

In case S is a noncompact RSS, $S_{1/2}=S$. For rank 1 simply connected compact RSS, $S_{1/2}$ is the ball around s_0 whose radius is half the distance to the antipodal set, hence the notation.

$S_{1/2}$ is the set on which the local theory lives. On $S_{1/2}$ define

$$e_*(\mathfrak{b},\lambda)=e(\mathfrak{b},\lambda-\tau)^{-1}\quad(\lambda\in\tilde{\mathfrak{b}}).$$

The idea is that these functions and their K-conjugates comprise the Fourier transform kernel while the $e(\mathfrak{b},\lambda)$ (and their K conjugates) give the inverse transform kernel.

It is time to bring in the generalization of (0.1) which will combine with (1.2) to give the local theory.

THEOREM 1.7. *For all* $\lambda\in\tilde{\mathfrak{b}}$

$$\phi_\lambda(s',s)=\int_S e_*(\mathfrak{b},\lambda)(ks')\,e(\mathfrak{b},\lambda)(ks)\,dk\quad(s'\in S_{1/2},s\in S).\tag{1.6}$$

For RSS of compact type this may be proved by analytic continuation of (0.1). However a more general proof will be given which applies at once to all RSS and some other spaces like $U(n)/SU(n-1)$. Theorem 1.9 uses the same idea in its proof.

We need an intermediate result. Let $P_{\mathfrak{b}}$ and $P_{\mathfrak{k}}$ denote the complimentary projections of $\mathfrak{g}$ onto $\mathfrak{b}$ and $\mathfrak{k}$ respectively. For $x\in\mathfrak{g}$ define

$$\zeta_x(k)=P_{\mathfrak{k}}\mathrm{Ad}((k)x)\quad(k\in K).$$

Regard ζ_x as a vector field on K thus:

$$D_{\zeta_x}f(k)=\frac{d}{dt}f(\exp(t\zeta_x(k))\,k)_{t=0}\quad(k\in K,f\in C^\infty(K)).$$

If $\mathfrak{k}$ is unimodular (i.e., its trace form is 0) then $\mathrm{div}(\zeta_x)$ is a well-defined function on K. The following lemma does not require that K be compact or connected.

LEMMA 1.8. *If* $\mathfrak{g}$ *and* $\mathfrak{k}$ *are unimodular and* τ *is the trace form of* $\mathfrak{b}$ *then*

$$\mathrm{div}(\zeta_x)(k)=\tau(P_{\mathfrak{b}}(\mathrm{Ad}(k)x))\quad(x\in\mathfrak{g},k\in K).$$

Proof. Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{k}$ and $\eta, \ldots, \eta_n$ a dual basis of $\mathfrak{k}^*$. Then for any $x \in \mathfrak{g}$

$$\operatorname{div}(\zeta_x)(k) = \sum_{j=1}^{n} D_{x_j}(\eta_j(\zeta_x(k))) = \sum_{j=1}^{n} \eta_j(P_{\mathfrak{k}}([x_j, \operatorname{Ad}(k)x])) = -\operatorname{tr}(P_{\mathfrak{k}} \circ \operatorname{ad}(\operatorname{Ad}(k)x)). \quad (1.7)$$

Let $y = \operatorname{Ad}(k)x$. Since $\mathfrak{k}$ is unimodular,

$$\operatorname{tr}(P_{\mathfrak{k}} \circ \operatorname{ad}(P_{\mathfrak{k}} y)) = 0$$

and since $y = P_{\mathfrak{k}} y + P_{\mathfrak{b}} y$ we get that the right side of (1.7) is

$$-\operatorname{tr}(P_{\mathfrak{k}} \circ \operatorname{ad}(y)) = -\operatorname{tr}(P_{\mathfrak{k}} \circ \operatorname{ad}(P_{\mathfrak{b}} y)).$$

By the unimodularity of $\mathfrak{g}$ this is

$$\operatorname{tr}(P_{\mathfrak{b}} \circ \operatorname{ad}(P_{\mathfrak{b}}(y))) = \tau(P_{\mathfrak{b}}(y))$$

as was to be proved. □

Proof of Theorem 1.7. Denote the right side of (1.6) by $\psi(s', s)$. Lemma 1.3 already gives

$$\phi_\lambda(s_0, s) = \phi_\lambda(s) = \psi(s_0, s) \quad (s \in S)$$

so it suffices to show that

$$g \mapsto \psi(gs', gs) \quad (s' \in S_{1/2}, s \in S, g \in G) \tag{1.8}$$

is constant in g for g sufficiently near 1 in G. We do this by differentiating (1.8) with respect to $x \in \mathfrak{g}$ at $g=1$ and showing that the result is always 0. In fact, if we fix s' and s and define

$$f(g) = e_*(\mathfrak{b}, \lambda)(gs')\, e(\mathfrak{b}, \lambda)(gs)$$

then the aforementioned derivative of (1.8) is

$$\int_K (D_{\operatorname{Ad}(k)x} f)(k)\, dk.$$

Split $\operatorname{Ad}(k)x$ into its $\mathfrak{b}$ and $\mathfrak{k}$ components (the latter is $\zeta_x(k)$) and use the definition of $e_*(\mathfrak{b}, \lambda)$ to get

$$(D_{\operatorname{Ad}(k)x} f)(k) = (D_{\zeta_x} f)(k) + \tau(P_{\mathfrak{b}}(\operatorname{Ad}(k)x)) f(k)$$

which by Lemma 1.8 is

$$(D_{\zeta_x} f)(k) + (\mathrm{div}(\zeta_x) f)(k).$$

The integral of this over K is 0 by the divergence theorem. □

Theorem 1.7 has an easy generalization:

THEOREM 1.9. *For $\lambda_1, \ldots, \lambda_j \in \mathfrak{b}$ let $\lambda = \lambda_1 + \ldots + \lambda_j$ and define $\psi = \psi_{\lambda_1, \ldots, \lambda_j}$ on the j-fold Cartesian product S^j by*

$$\psi(s_1, \ldots, s_j) = \int_K e(\mathfrak{b}, \lambda_1)(ks_1) \cdot \ldots \cdot e(\mathfrak{b}, \lambda_j)(ks_j)\, dk \quad (s_1, \ldots \in S).$$

Then ψ is K-invariant in the sense that

$$\psi(ks_1, \ldots, ks_j) = \psi(s_1, \ldots, s_j) \quad (k \in K; s_1, \ldots \in S)$$

and we can define an analytic function Ψ on S^{j+1} by

$$\Psi(gs_0, s_1, \ldots, s_j) = \psi(g^{-1}s_1, \ldots, g^{-1}s_j) \quad (g \in G; s_1, \ldots \in S).$$

Moreover, for $s \in S_{1/2}$ and $s_1, \ldots \in S$ we have

$$\Psi(s, s_1, \ldots, s_j) = \int_K e_*(\mathfrak{b}, \lambda)(ks)\, e(\mathfrak{b}, \lambda_1)(ks_1) \cdot \ldots \cdot e(\mathfrak{b}, \lambda_j)(ks_j)\, dk.$$

Proof. The K-invariance of ψ is obvious. Use the previous proof with

$$f(g) = e_*(\mathfrak{b}, \lambda)(gs)\, e(\mathfrak{b}, \lambda_1)(gs_1) \cdot \ldots \cdot e(\mathfrak{b}, \lambda_j)(gs_j) \quad (g \in G)$$

to show, as was done there, that

$$D_x \int_K f(kg)\, dk = 0 \quad (x \in \mathfrak{g}).$$ □

Remark. This is Theorem 1.7 if we take $j=1$, $\psi = \phi_\lambda(s)$ and $\Psi(s, s_1) = \phi_\lambda(s, s_1)$. Just as Theorem 1.7 leads to the global Main theorem so also Theorem 1.9 leads to the global Theorem 1.13.

Now restrict attention to RSS of compact type. It is traditional to write U instead of G for the connected group of isometries of S and we do so:

$$S = U/K.$$

Let $\mathfrak{u}$ denote the real Lie algebra of U and $\mathfrak{g}$ its complexification. Let $\mathfrak{k}_0$ denote the real Lie algebra of K and $\mathfrak{k}$ its complexification. θ will denote the Cartan involution on $\mathfrak{u}$, $\mathfrak{g}$, or U. K contains no normal subgroup of U, U is semisimple, and

$$\mathfrak{u} = \mathfrak{p}_0 + \mathfrak{k}_0 \quad \text{where} \quad \theta|\mathfrak{p}_0 = -I.$$

Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$ and $\mathfrak{a}$ its complexification. Choose a Weyl chamber in $i\mathfrak{a}_0$ and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

be the corresponding complexified Iwasawa decomposition. Let Σ (respectively Σ_+) denote the set of restricted (respectively, positive) roots so that $\mathfrak{n}$ is the sum of $\mathfrak{g}^\alpha, \alpha \in \Sigma_+$. We take

$$\mathfrak{b} = \mathfrak{a} + \mathfrak{n}.$$

In this context Lemma 1.4 is well known. The functions $e(\mathfrak{b}, \lambda)$ are the highest weight vectors in the irreducible U-submodules of $L^2(S)$, normalized to be 1 at the origin s_0. The functionals

$$-\lambda|\mathfrak{a} \quad (\lambda \in \check{\mathfrak{b}})$$

belong to $\mathbf{\Lambda}_+$, the set of highest (restricted) weights of the representations

$$x \mapsto -D_x \quad (x \in \mathfrak{g})$$

of $\mathfrak{g}$ on the various irreducible U-submodules of $L^2(S)$.

For simply connected spaces S it is known that

$$\mathbf{\Lambda}_+ = \{\mathbf{n} \cdot \boldsymbol{\mu} = n_1\mu_1 + \ldots + n_l\mu_l | \mathbf{n} = (n_1, \ldots, n_l) \in \mathbf{N}^l\} \tag{1.9}$$

where the μ_j may be given simply and explicitly in terms of Σ_+ as follows. Let Σ^* denote the "unmultipliable" roots $\beta \in \Sigma$, i.e., those such that $2\beta \notin \Sigma$. If $\{\alpha_1, \ldots, \alpha_l\}$ is the basis of Σ which gives Σ_+ then $\{\beta_1, \ldots, \beta_l\}$ is a basis of the root system Σ^* where

$$\beta_i = \begin{cases} \alpha_i & \text{if} \quad 2\alpha_i \notin \Sigma \\ 2\alpha_i & \text{if} \quad 2\alpha_i \in \Sigma \end{cases}$$

(see [4], p. 475). Then $\{\mu_j\}$ is the basis of $\mathfrak{a}^*$ determined by

$$\frac{\langle \mu_j, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} = \delta_{ij}.$$

This is proved in [10]. The following simple proof is due to Helgason:

Denote the right side of (1.9) by $\mathbf{M}_+$ and let $\mathbf{M}$ be the full lattice generated by $\mathbf{M}_+$. Also let

$$\mathbf{\Lambda} = \left\{\mu \,\middle|\, \frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbf{Z} \text{ for all } \alpha \in \Sigma\right\}.$$

Then

$$\mathbf{M}_+ = \{\mu \in \mathbf{M} | \langle \mu, \beta_j\rangle \geq 0 \text{ for all } j = 1, \ldots, l\}$$

and by Corollary 4.2 on p. 538 of [5],

$$\mathbf{\Lambda}_+ = \{\mu \in \mathbf{\Lambda} | \langle \mu, \beta_j\rangle \geq 0 \text{ for all } j = 1, \ldots, l\}.$$

Thus it suffices to show that $\mathbf{\Lambda}=\mathbf{M}$. Since reflection in one of the β_j permutes the remaining β_i, $\mathbf{M}$ is stable under the action of such reflections, hence under the Weyl group W. Every β in Σ^* has the form $s\beta_i$ for some $s \in W$ and $1 \leq i \leq l$. Thus

$$\mathbf{M} = \left\{\mu \,\middle|\, \frac{\langle \mu, \beta\rangle}{\langle \beta, \beta\rangle} \in \mathbf{Z} \text{ for all } \beta \in \Sigma^*\right\} = \mathbf{\Lambda}.$$

To summarize the situation for simply connected spaces, if we identify $\lambda \in \tilde{\mathfrak{b}}$ with $\lambda | \mathfrak{a}$ (since $\lambda(\mathfrak{n})=0$) we get

$$\tilde{\mathfrak{b}} = \{-\mathbf{n}\cdot\boldsymbol{\mu} | \mathbf{n} \in \mathbf{N}^l\}.$$

Let M denote the centralizer of $\mathfrak{a}$ in K and, as with the dual noncompact symmetric space, let $B=K/M$. Recall that the points of B are in natural bijective correspondence with the K-conjugates of $\mathfrak{b}$. Also, each $e(\mathfrak{b}, \lambda)$ is M-invariant.

It is convenient to introduce an alternative to the notation $e(\mathfrak{b}, \lambda)$ as follows:

For $\mathbf{n} \in \mathbf{N}^l$ and $b=kM \in B$ write

$$e(b, \mathbf{n})(s) = e(\mathfrak{b}, -\mathbf{n}\cdot\boldsymbol{\mu})(k^{-1}s) \quad (s \in S).$$

(Thus $e(b, \mathbf{n})$ (with $b=kM$) is an eigenfunction of $\mathrm{Ad}(k)\mathfrak{b}$ in the same way that $e(\mathfrak{b}, -\mathbf{n}\cdot\boldsymbol{\mu})$ is an eigenfunction of $\mathfrak{b}$.)

The trace form τ of $\mathfrak{b}$ is

$$\tau = \sum_{\alpha \in \Sigma_+} m_\alpha \alpha = \sum_{j=1}^{l} m_{\tau j} \mu_j$$

where m_α is the multiplicity of α. Assuming again that S is simply connected (so K is connected and Lemma 1.5 applies), $-\tau$ is in $\check{\mathfrak{b}}$. Thus the numbers $m_{\tau j}$ are integers $\geqslant 0$. Let

$$\mathbf{m}_\tau = (m_{\tau 1}, \dots, m_{\tau l}) \in \mathbf{N}^l.$$

Then for $b=kM$ define $e_*(\mathfrak{b}, \mathbf{n})$ on the $S_{1/2}$ of Lemma 1.6 by

$$e_*(b, \mathbf{n})(s) = e_*(\mathfrak{b}, -\mathbf{n}\cdot\boldsymbol{\mu})(k^{-1}s) = e(b, \mathbf{n}+\mathbf{m}_\tau)^{-1}(s) \quad (s \in S_{1/2}).$$

If we write $\phi_{\mathbf{n}}$ for $\phi_{-\mathbf{n}\cdot\boldsymbol{\mu}}$ then (1.6) becomes

$$\phi_{\mathbf{n}}(s', s) = \int_B e_*(b, \mathbf{n})(s')\, e(b, \mathbf{n})(s)\, db \quad (s' \in S_{1/2}, s \in S). \tag{1.10}$$

To reformulate (1.2) in this notation, let $\mathscr{H}_{\mathbf{n}}(S)$ denote the irreducible U-submodule of $L^2(S)$ containing $\phi_{\mathbf{n}}$ ($\mathbf{n} \in \mathbf{N}^l$). Let $d_{\mathbf{n}}$ denote its dimension. Then $\mathbf{N}^l$ parametrizes the space Φ of (1.2) and the Plancherel measure at the point $\mathbf{n}$ is precisely $d_{\mathbf{n}}$. Thus (1.2) says

$$f(s) = \sum_{\mathbf{n}} d_{\mathbf{n}} \int_S f(s')\, \phi_{\mathbf{n}}(s', s)\, ds'. \tag{1.11}$$

The sum is over $\mathbf{N}^l$. If f is in $L^2(S)$ (resp., $C^\infty(S)$) then the series converges in $L^2(S)$ (resp., $C^\infty(S)$). See [10].

The next result is the local Fourier inversion theorem.

THEOREM 1.10. *For the compact RSS S and $f \in C_c(S_{1/2})$ (with $S_{1/2}$ as in Lemma 1.6) define its Fourier transform $\hat{f}$ on $B \times \mathbf{N}^l$ by*

$$\hat{f}(b, \mathbf{n}) = \int_S f(s)\, e_*(b, \mathbf{n})(s)\, ds.$$

Then

$$f(s) = \sum_{\mathbf{n}} d_{\mathbf{n}} \int_B \hat{f}(b, \mathbf{n})\, e(b, \mathbf{n})(s)\, db$$

with the series converging in $L^2(S_{1/2})$. If $f \in C_c^\infty(S_{1/2})$ then the series converges in $C^\infty(S_{1/2})$.

Proof. In view of (1.11) all we have to prove is

$$\int_S f(s')\,\phi_{\mathbf{n}}(s',s)\,ds' = \int_B \hat{f}(b,\mathbf{n})\,e(b,\mathbf{n})(s)\,db.$$

For this, replace $\hat{f}(b,\mathbf{n})$ by its defining integral so that the right side becomes

$$\int_B \int_S f(s')\,e_*(b,\mathbf{n})(s')\,e(b,\mathbf{n})(s)\,ds'\,db.$$

Then switch the order of integration and apply (1.10) and (1.11). □

By reversing the roles of $e(b,\mathbf{n})$ and $e_*(b,\mathbf{n})$ we obtain a weaker alternative Fourier theory. For $f \in L^1(S)$ define the transform $\check{f}$ by

$$\check{f}(b,\mathbf{n}) = \int_S f(s)\,e(b,\mathbf{n})(s)\,ds.$$

Then

$$f(s) = \sum_{\mathbf{n}} d_{\mathbf{n}} \int_B \check{f}(b,\mathbf{n})\,e_*(b,\mathbf{n})(s)\,db \tag{1.12}$$

with qualifications on f and the convergence as in Theorem 1.10. For $s \in S_{1/2}$ the proof of (1.12) is similar to Theorem 1.10 with a slight twist: In place of (1.11) we need

$$f(s) = \sum_{\mathbf{n}} d_{\mathbf{n}} \int_S f(s')\,\phi_{\mathbf{n}}(s,s')\,ds'. \tag{1.13}$$

Since $\phi_{\mathbf{n}}(s,s') = \overline{\phi_{\mathbf{n}}(s',s)}$, (1.13) follows from (1.11) by replacing f by $\bar{f}$ in (1.11) and conjugating both sides.

But now observe that each term in the series in (1.12) is an analytic function on S. Thus we can really recapture f globally from $\check{f}$ by (1.12): Compute each term on $S_{1/2}$ and then extend that term to all of S by analytic continuation, or better, by the knowledge that it lies in $\overline{\mathcal{H}_{\mathbf{n}}(S)}$. Then sum the series.

This is perhaps best regarded as a hopeful sign that a more satisfactory approach to a global theorem for higher rank compact RSS will be possible. As a further indication, observe that there is a Parseval formula that links $\hat{f}$ and $\check{f}$:

$$\int_S f_1(s) f_2(s)\,ds = \sum_{\mathbf{n}} d_{\mathbf{n}} \int_B \hat{f}_1(b,\mathbf{n})\check{f}_2(b,\mathbf{n})\,db \tag{1.14}$$

assuming, say, that $f_1 \in L^2(S_{1/2})$ and $f_2 \in L^2(S)$. Again, the proof is similar to Theorem 1.10. However, it is not clear how to exploit this to define $\hat{f}$ globally.

Space	m_1	m_2	μ_1	m_τ
S_q	$q-1$	0	α	$q-1$
$P_l(\mathbf{R})$	$l-1$	0	2α	$(l-1)/2$
$P_l(\mathbf{C})$	$2(l-1)$	1	2α	l
$P_l(\mathbf{H})$	$4(l-1)$	3	2α	$2l+1$
$P_2(\mathbf{Cay})$	8	7	2α	11

Table 1

Now we turn to the statement of the global definition of $\hat{f}$ for the rank one compact RSS. This is equivalent to defining $e_*(b,n)$ as a distribution on all of S, not just $S_{1/2}$. Our approach is to do this first for $e_*(b_0,n)$ where b_0 is the identity coset in $B=K/M$. (Note then that $e(b_0,n)$ is an eigenfunction of $\mathfrak{b}$). Then we define

$$e_*(kb_0,n)(s)=e_*(b_0,n)(k^{-1}s) \quad (k\in K).$$

This is consistent with our earlier definition of $e_*(b,n)$ on $S_{1/2}$. Of course, the definition of $e_*(kb_0,n)$ is interpreted via integration against a C^∞ test function:

$$\hat{f}(kb_0,n)=\int_S e_*(b_0,n)(s)f(ks)\,ds. \tag{1.15}$$

To define $e_*(b_0,n)$ globally we anticipate future developments and introduce the function $\xi_1\colon S\to\mathbf{R}$ by $\xi_1(s)=\Re(e(b_0,1)(s))$ $(s\in S)$. Here $\Re$ denotes the real part.

On the open set $\xi_1\neq 0$ in S define $e_*(b_0,n)$ as the function

$$\begin{cases} e(b_0,n+m_\tau)^{-1} & \text{if } S \text{ is a real projective space or odd dimension sphere;}\\ \operatorname{sgn}(\xi_1)\,e(b_0,n+m_\tau)^{-1} & \text{otherwise.}\end{cases}$$

The numbers m_τ used here are defined, as before, by $m_\tau\mu_1=\tau$ where μ_1 is the generator of $\tilde{\mathfrak{b}}$ and τ is the trace form on $\mathfrak{b}$. Σ_+ is either $\{\alpha\}$ or $\{\alpha,2\alpha\}$ with multiplicities m_1 for α and m_2 for 2α if it exists (otherwise take $m_2=0$). μ_1 is α for the spheres and 2α for the projective spaces. Thus we have Table 1.

For the projective spaces we have the general formula

$$m_\tau=m_1/2+m_2.$$

On $\xi_1\neq 0$ the function $e_*(b_0,n)$ is not in L^1 so to make sense of (1.15) we must regularize the integral. For this we need the distance function δ on S. $\delta(s)$ gives the geodesic distance to s from the origin s_0. It is convenient to normalize δ so that its maximal value is π. Thus $\{s\in S \mid \delta(s)=\pi\}$ is the antipodal set.

Definition 1.11. For $\varepsilon, \eta \geqslant 0$ define

$$S(\varepsilon,\eta)=\{s\in S \mid \cos(\delta(s))\geqslant \varepsilon-1, |\xi_1(s)|\geqslant \eta\}.$$

Then for $f\in L_1(S)$, $k\in K$ and $n\in \mathbf{N}$ define

$$\hat{f}(kb_0,n;\varepsilon,\eta)=\int_{S(\varepsilon,\eta)} f(ks)\, e_*(b_0,n)(s)\, ds$$

and, if the limit exists, define $\hat{f}(b,n)$ by

$$\begin{cases}\lim_{\eta\to 0^+}\hat{f}(b,n;0,\eta) & \text{if } S \text{ is a sphere;}\\ \lim_{\varepsilon\to 0^+}\hat{f}(b,n;\varepsilon,0) & \text{if } S \text{ is } P_l(\mathbf{R});\\ \lim_{\varepsilon\to 0^+}(\lim_{\eta\to 0^+}\hat{f}(b,n;\varepsilon,\eta)) & \text{otherwise}\end{cases}$$

MAIN THEOREM 1.12. *For a compact rank one Riemannian symmetric space S and for $f\in C^\infty(S)$, $\hat{f}(b,n)$ exists as defined above and, as a function of $b\in B$, is in $C^\infty(B)$. Moreover, f can be recaptured from $\hat{f}$ by*

$$f(s)=\sum_{n=0}^{\infty} d_n \int_B \hat{f}(b,n)\, e(b,n)(s)\, db$$

with the series converging in $C^\infty(S)$.

Remark. There is also a global Parseval formula (1.14). The proof of this is straight-forward once the theorem is established; it will be let to the reader.

The proof of Theorem 1.12 for the sphere was given in [8]. That work is summarized and extended in Section 2 where Theorem 1.13 is also proved and it is shown how the case $P_l(\mathbf{R})$ is covered. Section 4 contains the proof of the Main theorem for the remaining projective spaces.

Definition 1.11 and Theorem 1.12 make precise what we mean by a *global theory:* the unbounded function $e_*(b,n)$ is extended to an eigenfunction of $\mathrm{Ad}(k)\,\mathfrak{b}$ on the dense open set $e(b,1)\neq 0$ and then to a distribution on all S in such a way as to make possible the recapture of f as in Theorem 1.12. This is the crucial point—there will be many ways to extend the eigenfunction function to a distribution but most of them will not result in the recapture of f.

When the proof of the Main theorem is applied to Theorem 1.9 the result is Theorem 1.13. The case of Theorem 1.13 for the sphere is used (under the name Corollary 2.12) at a crucial point (Lemma 4.34) in the proof of the Main theorem in Section 4. Theorem 1.13 asserts a kind of Kronecker factorization of a U-module $\mathcal{H}_n(S)$ by expressing it explicitly as a submodule of the tensor product of modules

$$\mathcal{H}_{n_1}(S), \ldots, \mathcal{H}_{n_j}(S), \quad n_1 + \ldots + n_j = n.$$

THEOREM 1.13. *For a compact rank one Riemannian symmetric space S and positive integers $n_1, \ldots, n_j$ define $\Psi = \Psi_{n_1, \ldots, n_j}$ on S^{j+1} as in Theorem 1.9 by*

$$\Psi(us_0, s_1, \ldots, s_j) = \int_B e(b, n_1)(u^{-1}s_1) \cdot \ldots \cdot e(b, n_j)(u^{-1}s_j)\, db \quad (u \in U).$$

For $f \in L^2(S)$ let $f_{n_1, \ldots, n_j} \in \mathcal{H}_{n_1, \ldots, n_j}(S^j) = \mathcal{H}_{n_1}(S) \otimes \ldots \otimes \mathcal{H}_{n_j}(S)$ be defined by

$$f_{n_1, \ldots, n_j}(s_1, \ldots, s_j) = d_n \int_S f(s)\, \Psi(s, s_1, \ldots, s_j)\, ds$$

where $n = n_1 + \ldots + n_j$. The map $f \mapsto f_{n_1, \ldots, n_j}$ is a U-module homomorphism of $L^2(S)$ into $\mathcal{H}_{n_1, \ldots, n_j}(S^j)$ such that $f_{n_1, \ldots, n_j}(s, \ldots, s) = f_n(s) \in \mathcal{H}_n(S)$. Moreover, if f is in C^∞ then

$$f_{n_1, \ldots, n_j}(s_1, \ldots, s_j) = d_n \int_B \hat{f}(b, n)\, e(b, n_1)(s_1) \cdot \ldots \cdot e(b, n_j)(s_j)\, db. \tag{1.16}$$

Partial proof. The construction of Ψ makes it clear that when all but one of the $s_1, \ldots, s_j$ (say s_i) are fixed then $s_i \mapsto f_{n_1, \ldots, n_j}(s_1, \ldots, s_i, \ldots, s_j)$ is in $\mathcal{H}_{n_i}(S)$; thus $f_{n_1, \ldots, n_j}$ is in $\mathcal{H}_{n_1, \ldots, n_j}(S^j)$. $f \mapsto f_{n_1, \ldots, n_j}$ is U-equivariant because Ψ is U-invariant in that

$$\Psi(us, us_1, \ldots) = \Psi(s, s_1, \ldots) \quad (u \in U).$$

$$\Psi(us_0, s, \ldots, s) = \int_B e(b, n)(u^{-1}s)\, db = \phi_n(u^{-1}s)$$

by Lemma 1.3 from which we get $f_{n_1, \ldots, n_j}(s, \ldots, s) = f_n(s)$ where f_n is the projection of f into the subspace $\mathcal{H}_n(S)$ of $L^2(S)$.

It only remains to prove (1.16). This is not quite a corollary of the Main theorem, but rather of its proof. For this we must wait until Lemmas 2.13 and 4.36. □

The numbers $d_\mathbf{n} = \dim \mathcal{H}_\mathbf{n}(S)$ are polynomials in $\mathbf{n}$ for all ranks. For rank one spaces these numbers are as in Table 2.

Space	d_n
S_q	$\dfrac{(2n+q-1)(n+q-2)!}{n!(q-1)!}$
$P_l(\mathbf{R})$	$\dfrac{(4n+l-1)(2n+l-2)!}{(2n)!(l-1)!}=d_{2n}$ for S_l
$P_l(\mathbf{C})$	$\dfrac{(2n+l)}{l!(l-1)!}((n+l-1)!/n!)^2$
$P_l(\mathbf{H})$	$\dfrac{(2n+2l+1)(n+2l)!(n+2l-1)!}{(2l+1)!(2l-1)!(n+1)!n!}$
$P_2(\mathbf{Cay})$	$\dfrac{(2n+11)(n+10)!(n+7)!3!}{11!n!(n+3)!7!}$

Table 2

2. Fourier theory on the sphere

This section is a summary and extension of some results in [7] and [8]. The summary states that the Main theorem holds for spheres. The extension, which is of central importance to Section 3, involves a closer look at the Fourier transform of finite K-types in $L^2(S)$.

Since B is itself the sphere S_{q-1}, and since $K=SO(q)$, it makes sense to speak of $\mathcal{H}_j(B)\subset L^2(B)$, for $j\in\mathbf{N}$. The K-types in $L^2(S)$ correspond to the K-modules $\mathcal{H}_j(B)$. Specifically, let $L^2_{(j)}(S)$ denote the subspace of $L^2(S)$ whose functions transform like $\mathcal{H}_j(B)$ under the action of K. Then $L^2(S)$ is the Hilbert space direct sum of the $L^2_{(j)}(S)$, $j\in\mathbf{N}$ and we obtain, as Theorem 2.9, the first of the following three essentially equivalent statements:

(i) $f\mapsto\hat{f}(b,n)$ is a continuous linear map from $L^2_{(j)}(S)$ to $\mathcal{H}_j(B)$. The norm of this map is of polynomial growth in j.

(ii) For $f\in L^2(S)$ we have that $\hat{f}(b,n)$ is defined and continuous in $b\in B$ provided that the function from K to $L^2(S)$ given by $K\ni k\mapsto f^k$ is sufficiently differentiable.

(iii) For $f\in L^2(S), \hat{f}(\cdot,n)$ is a distribution on B.

After the summary statements there is a segment on the orthogonal polynomials $P_{n,q}$ (see (0.2)) which give the zonal spherical functions ϕ_n on $S=S_q$. $P_{n,q}$ is defined for $n<0$ in (0.3)–(0.4). We will find that the functions $P_{n-j,q+2j}$ enter into the computation of $\hat{f}(b,n)$ when $f\in L^2_{(j)}(S)$. Thus the behavior of $P_{n-j,q+2j}$ as $j\to\infty$ is important. At the end of the section we show how to compute $\hat{f}(b,n)$, prove Theorem 1.13 for $S=S_q$, and discuss how this section applies to real projective space.

For the duration of the section fix the integer $q \geqslant 2$ and let $S = S_q$, the unit sphere in $\mathbf{R}^{q+1}$. Take $s_0 = (1, 0, \ldots, 0) \in S$, $U = SO(q+1)$, and K the subgroup of U which fixes s_0. Of course $K \cong SO(q)$. Now $\mathfrak{g} = \mathfrak{so}(q+1, \mathbf{C})$ has the complexified Iwasawa decomposition $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ where we may take $\mathfrak{a}$ and $\mathfrak{n}$ thus:

$$\mathfrak{a} = \mathbf{C}x_0, \quad x_0 = e_{21} - e_{12}, \quad \mathfrak{n} = \operatorname{span}\{e_{1j} - e_{j1} + i(e_{2j} - e_{j2}) \mid j = 2, \ldots, q+1\}.$$

Then $\Sigma_+ = \{\alpha\}$ where $\alpha(x_0) = -i$. $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$ and $\tilde{\mathfrak{b}}$ is generated by $\mu_1 = \alpha$.

$B = K/M$ is identifiable with the sphere $s_0^{\perp} \cap S$, the "equator" in S if we regard s_0 as the north pole. In this identification, take $b_0 = (0, 1, 0, .., 0)$. For $s \in S_q$ with $s \cdot s_0 \neq 0$ we have

$$e(b, n)(s) = (s \cdot s_0 + is \cdot b)^n,$$

$$e_*(b, n)(s) = (\operatorname{sgn}(s \cdot s_0))^{q-1}(s \cdot s_0 + is \cdot b)^{-n-q+1}.$$

Recall that $\mathscr{H}_n(S)$ denotes the irreducible U-submodule of $L^2(S)$ containing $e(b, n)$. It consists of the homogeneous harmonic polynomials of degree n on $\mathbf{R}^{q+1}$, restricted to S. Its zonal spherical function ϕ_n is given by:

$$\phi_n(s) = P_{n,q}(s \cdot s_0) \quad (s \in S)$$

where $P_{n,q}$ is defined in (0.2). Theorem 1.7 has the following easy extension on S_q:

LEMMA 2.1. *For $s, s' \in S$ with $s' \cdot s_0 \neq 0$,*

$$P_{n,q}(s \cdot s') = \int_B e(b, n)(s)\, e_*(b, n)(s')\, db.$$

Proof. Theorem 1.7 gives this in case $s' \cdot s_0 > 0$. Next, replace s' by $-s'$ and observe that the minus sign comes out of both sides as $(-1)^n$. □

Remark. This is [8], Key Lemma 3.9. It shows why the factor $\operatorname{sgn}(s' \cdot s_0)^{q-1}$ is needed in $e_*(b, n)(s')$.

If we consider $e(b, n)(s)$ as a function of $b \in B$ then it is easy to see that the space spanned by these functions as s varies in S is the space of polynomials on B of degree $\leqslant n$. From this and Lemma 2.1 we get a result which will be used to prove Lemma 2.11 which is in turn a foundation of Theorem 1.13 and the Main theorem.

COROLLARY 2.2. *Let Q be a polynomial of degree $\leqslant n$ on B. Then there is a unique function $F_Q \in \mathscr{H}_n(S)$ such that*

$$F_Q(s) = \int_B Q(b)\, e_*(b, n)(s)\, db \quad (s \in S,\, s \cdot s_0 \neq 0).$$

Turn now to the Main theorem for $S = S_q$. In this case the subset $S(0, \eta)$ in Definition 1.11 is simply

$$S(0, \eta) = \{s \in S \mid |s \cdot s_0| > \eta\}.$$

For $\eta > 0$, $e_*(b, n)$ is bounded on $S(0, \eta)$ uniformly in b so the $\hat{f}(b, n; 0, \eta)$ of Definition 1.11 makes sense for all $f \in L^1(S)$ and $\eta > 0$.

The essential content of Theorem 3.6 and Lemma 5.20 of [8] is provided in

THEOREM 2.3. *For an integer $n \geq 0$ and a function $f \in C^{n+q-2}(S)$ we have that $\hat{f}(b, n; 0, \eta)$ converges uniformly on B to a continuous function $\hat{f}(b, n)$ as $\eta \to 0^+$. The map $f \mapsto \hat{f}(b, n)$ is K-equivariant and continuous from the Banach space $C^{n+q-2}(S)$ to $C(B)$. Moreover, the component f_n of f in $\mathcal{H}_n(S)$ may be recaptured from $\hat{f}(b, n)$ by*

$$f_n(s) = d_n \int_B \hat{f}(b, n)\, e(b, n)(s)\, db.$$

(Thus the Main theorem holds for $S = S_q$.)

To meet needs which arise in Section 3 it is necessary to go further in the following respect: the smoothness which is demanded of f in the previous result need only be imposed in the K-direction. That is, we really only need that $k \mapsto f^k$ be smooth from K to $L^2(S)$. This is the thrust of Theorem 2.9, as can be seen from the equivalence of (i)–(iii) in the introductory remarks to this section. In preparation for these results we need to look more closely at the functions $P_{n,q}$ especially when $1 - q < n < 0$.

Recall that for arbitrary complex z we define $P_{z,q}(t)$ $(0 < t \leq 1)$ by (0.3) and (0.4). For negative integers n we also define $P_{n,q}(t)$ on $-1 \leq t < 0$ by

$$P_{n,q}(-t) = (-1)^n P_{n,q}(t).$$

Then for all integers n and all $0 < |t| \leq 1$ we have

$$P_{-n-q+1,q}(t) = (\operatorname{sgn}(t))^{q-1} P_{n,q}(t).$$

Note that when $1 - q < n < 0$, $P_{n,q}$ is not a polynomial, even when it is restricted to $(0, 1]$; however for such n it is strictly positive and monotone decreasing on $(0, 1]$. The limit

$P_{z,q}(0^+)$ exists for all z and is computable from (0.4) and Gauss' formula for ${}_2F_1(a, b; c; 1)$. Specifically,

$$P_{z,q}(0^+) = \sqrt{\pi}\,\Gamma(q/2)\,\mathscr{P}(z, q) \quad \text{where} \quad \mathscr{P}(z, q) = (\Gamma((1-z)/2)\,\Gamma((z+q)/2))^{-1}. \tag{2.1}$$

Note that $\mathscr{P}(z, q)$ is an entire function of z.

LEMMA 2.4. *For $z, w \in \mathbf{C}$ we have*

$$\begin{aligned}(z-w)(z+w+q-1)\int_0^1 P_{z,q}(t)\,P_{w,q}(t)\,(t_\diamond)^{q-2}\,dt \\ = 2\pi\Gamma(q/2)^2(\mathscr{P}(z, q)\,\mathscr{P}(w+1, q-2) - \mathscr{P}(w, q)\,\mathscr{P}(z+1, q-2)).\end{aligned} \tag{2.2}$$

This is a restatement of a known fact about Jacobi polynomials [6], p. 282. As a consequence we obtain

LEMMA 2.5. *For complex z in the strip $1-q<\Re(z)<0$*

$$\begin{aligned}(2z+q-1)\,\omega_q\int_0^1 P^2_{z,q}(t)\,(t_\diamond)^{q-2}\,dt \\ = (\Gamma(q)/(2\Gamma(-z)\,\Gamma(z+q-1)))\int_0^1 (t^{-z-1} - t^{z+q-2})/(t+1)\,dt.\end{aligned}$$

Proof. Divide both sides of (2.2) by $z-w$ and let $w \to z$ to get

$$\begin{aligned}(2z+q-1)\int_0^1 P^2_{z,q}(t)\,(t_\diamond)^{q-2}\,dt \\ = 2\pi\Gamma(q/2)^2\mathscr{P}(z, q)\,\mathscr{P}(z+1, q-2)\frac{d}{dz}\ln(\mathscr{P}(z, q)/\mathscr{P}(z+1, q-2))\end{aligned}$$

which, with the aid of the Duplication theorem for Γ and (0.5), reduces to

$$(\Gamma(q)/(2\omega_q\Gamma(-z)\,\Gamma(z+q-1)))\frac{d}{dz}\ln\left(\frac{\Gamma(-z/2)\,\Gamma((z+q-1)/2)}{\Gamma((1-z)/2)\,\Gamma((z+q)/2)}\right).$$

The conclusion follows from [1], Vol. 1 (1.7(1) and 1.8(2)). □

For all integers n define the quantity $d(n, q)$ by

$$d(n, q)^{-1} = \omega_q\int_{-1}^1 P^2_{n,q}(t)\,(t_\diamond)^{q-2}\,dt. \tag{2.3}$$

When $n \geq 0$ then

$$d(n,q)^{-1} = \int_S \phi_n^2(s)\,ds = d_n^{-1}$$

and this is given in Table 2 in Section 1:

$$d(n,q) = (2n+q-1)(n+q-2)!/(n!(q-1)!) \quad (n \geq 0). \tag{2.4}$$

LEMMA 2.6. *For integers* $1-q<n<0$ *there is a number* $0<\varepsilon<1$ *such that*

$$d(n,q) = (1+\varepsilon)\binom{q-1}{-n}^{-1}.$$

Proof. In Lemma 2.5 take $z=r$ real and such that $1-q<r<0$, $r \neq (1-q)/2$. Then the numerator in the right hand integrand in that lemma does not change sign so for $r \neq (1-q)/2$ the claim follows from the integral mean value theorem which lets us replace $1/(t+1)$ by $1/(\varepsilon+1)$. The excluded case follows by continuity. □

The following extension of Lemma 1.3 for $S=S_q$ is known ([1], Vol. 2, 10.9 (31)):

$$P_{z,q}(t) = \int_B (t+it_\diamond\, b\cdot b')^z\,db' = \omega_{q-1}\int_{-1}^{1} (t+it_\diamond x)^z (x_\diamond)^{q-3}\,dx \quad (0<t\leq 1, z\in \mathbf{C}). \tag{2.5}$$

(Recall ω_q from (0.5).) Note that if $z=n$ is an integer then (2.5) also holds for $-1 \leq t < 0$.

Recall ϖ from (0.6). (2.5) has the following generalization (which is closely related to Corollary 2.2):

LEMMA 2.7. *For any integer* $j \geq 0$, *function* $Q \in \mathscr{H}_j(B)$, *and numbers* $z \in \mathbf{C}$, $0<t\leq 1$

$$\int_B (t+it_\diamond\, b\cdot b')^z Q(b')\,db' = Q(b)\,\varpi(j,q;-z-q+1)\,(t_\diamond)^j P_{z,2j+q}(t). \tag{2.6}$$

Proof. The special case $j=0$, $Q=1$ is just (2.5). For $j>0$, the Funk-Hecke theorem (see [1], Vol. 2, p. 247) shows that the left side of (2.6) equals $f(t)\,Q(b)$ where

$$f(t) = \omega_{q-1}\int_{-1}^{1} (t+it_\diamond)^z P_{j,q-1}(x)\,(x_\diamond)^{q-3}\,dx.$$

By the Rodrigues formula (0.2) for $P_{j,q-1}$ and j-fold integration-by-parts, this becomes

$$f(t) = \varpi(j,q-1;-z-q+2)\,\omega_{q-1}(t_\diamond)^j \int_{-1}^{1} (t+it_\diamond)^{z-j}(x_\diamond)^{2j+q-3}\,dx.$$

From (2.5) (with z replaced by $z-j$ and q by $2j+q$) this is

$$\varpi(j,q-1;-z-q+2)\,(\omega_{q-1}/\omega_{2j+q-1})\,(t_\diamond)^j P_{z-j,2j+q}(t).$$

This times $Q(b)$ is the right side of (2.6) since

$$\varpi(j,q-1;z)\,\omega_{q-1}=\varpi(j,q;z-1)\,\omega_{2j+q-1}. \qquad \square$$

If we replace z by $-n-q+1$ we get

COROLLARY 2.8. *For integers $n,j\geq 0$, $Q\in\mathcal{H}_j(B)$, and $0<|t|\leq 1$ we have*

$$\operatorname{sgn}(t)^{q-1}\int_B (t+it_\diamond b\cdot b')^{n-q+1}Q(b')\,db'=Q(b)\,\varpi(j,q;n)\,(t_\diamond)^j P_{n-j,2j+q}(t).$$

Now introduce cylindrical coordinates $(t,b)\in[-1,1]\times B$ in S in order to make precise some of our preliminary remarks about K-types in $L^2(S)$ and their relation to the spaces $\mathcal{H}_j(B)$:

$$[-1,1]\times B\ni(t,b)\mapsto s=s(t,b)=ts_0+t_\diamond b\in S.$$

The space $L^2_{(j)}(S)$ is spanned by functions of the form

$$f(s(t,b))=g(t)\,Q(b)\quad(Q\in\mathcal{H}_j(B),\, g\in L^2([-1,1],(t_\diamond)^{q-2}\,dt)). \tag{2.7}$$

THEOREM 2.9. *For any integers $n,j\geq 0$ and $f\in L^2_{(j)}(S)$, $\hat f(\cdot,n)$ exists, is in $\mathcal{H}_j(B)$, and the map $f\mapsto\hat f(\cdot,n)$ is continuous from $L^2_{(j)}(S)$ to $\mathcal{H}_j(B)$ with norm bounded above by $(j+1)^{n+(q-3)/2}$. If f is given by (2.7) then $\hat f(b,n)=Q(b)\,\mathcal{M}_{n,j}(g)$ where*

$$\mathcal{M}_{n,j}(g)=\omega_q\,\varpi(j,q;n)\int_{-1}^{1} g(t)\,P_{n-j,2j+q}(t)\,(t_\diamond)^{j+q-2}\,dt.$$

For arbitrary $f\in L^2_{(j)}(S)$,

$$\hat f(b,n)=\mathcal{M}_{n,j}(t\mapsto f(s(t,b))).$$

Proof. First suppose f is given by (2.7). For $0<\eta<1$ define $\mathcal{I}(\eta)=\{t\mid\eta\leq|t|\leq 1\}$. Then

$$\begin{aligned}\hat f(b,n;0,\eta)&=\int_{S(0,\eta)} f(s)\,e_*(b,n)(s)\,ds\\ &=\omega_q\int_{\mathcal{I}(\eta)} g(t)\operatorname{sgn}(t)^{q-1}\int_B Q(b')\,(t+it_\diamond b\cdot b')^{-n-q+1}\,db'(t_\diamond)^{q-2}\,dt\\ &=Q(b)\,\omega_q\,\varpi(j,q;n)\int_{\mathcal{I}(\eta)} g(t)\,P_{n-j,2j+q}(t)\,(t_\diamond)^{j+q-2}\,dt\end{aligned}$$

by Corollary 2.8. Since the integrand is integrable on $[-1,1]$,

$$\hat{f}(b,n)=\lim_{\eta\to 0^+}\hat{f}(b,n;0,\eta)$$

exists and is computed as claimed in the theorem.

It is routine to reduce the case of the general element of $L^2_{(j)}(S)$ to this more special case by choosing an orthonormal basis $\{Q_1,\dots\}$ of the finite dimensional space $\mathscr{H}_j(B)$ and observing that every element of $L^2_{(j)}(S)$ is a finite sum of elements $f_1,\dots$ of the form given in (2.7) with $Q=Q_1,\dots$.

If we integrate both sides of the following over B

$$|\hat{f}(b,n)|^2\leqslant(\omega_q\varpi(j,q;n))^2\int_{-1}^{1}|f(s(t,b))|^2(t_\diamond)^{q-2}\,dt\int_{-1}^{1}P^2_{n-j,2j+q}(t)\,(t_\diamond)^{2j+q-2}\,dt$$
$$=J^2\omega_q\int_{-1}^{1}|f(s(t,b))|^2(t_\diamond)^{q-2}\,dt \tag{2.8}$$

we see that the norm of the map $f\to\hat{f}(\cdot,n)$ is $\leqslant J$. From (0.5), (0.6), (2.3), (2.8) and the Duplication theorem for Γ we have

$$J=\frac{(j+n+q-2)!}{(n+q-2)!}\left(\frac{(q-1)!}{(2j+q-1)!d(n-j,2j+q)}\right)^{1/2}.$$

From (2.3) and Lemma 2.6 (when $j>n$) or (2.4) (when $j\leqslant n$) we have

$$d(n-j,2j+q)^{-1}=(2j+q-1)!\begin{cases}1/((1+\varepsilon)(j-n)!(j+n+q-1)!) & \text{if } j>n;\\ (n-j)!/((2n+q-1)(j+n+q-2)!) & \text{if } j\leqslant n.\end{cases}$$

From this it is easy to see that

$$J\leqslant(j+1)^{n+(q-3)/2} \tag{2.9}$$

proving the claimed estimate on the norm of $f\mapsto\hat{f}(\cdot,n)$. $\square$

This result can be used to make explicit computations of $\hat{f}$, as the following restatement and proof of a result from [7], [8] show. Suppose f belongs to $\mathscr{H}_m(S)\cap L^2_{(j)}(S)$. Then it must have the form

$$f(s(t,b))=P_{m-j,2j+q}(t)\,(t_\diamond)^jQ(b)\quad(b\in B,\ |t|\leqslant1,\ Q\in\mathscr{H}_j(B)). \tag{2.10}$$

THEOREM 2.10. *For* $0\leqslant j\leqslant m$ *and* $0\leqslant n$ *and* f *as given in* (2.10) *we have*

$$\hat{f}(b,n)=Q(b)\,\omega_q\,\varpi(j,q;n)\,\psi(m-j,n-j,2j+q) \tag{2.11}$$

where for integers k, l at least one of which is ≥ 0,

$$\psi(k,l,q)=\int_{-1}^{1} P_{k,q}(t)\,P_{l,q}(t)\,(t_{\diamond})^{q-2}\,dt.$$

$\psi(k,l,q)$ is 0 if $k-l$ is odd or if $k\geq 0$ and $l\geq 0$ and $l\neq k$; it is $(\omega_q d_k)^{-1}$ if $k=l\geq 0$; otherwise it is

$$\frac{2\pi\Gamma(q/2)^2(\mathcal{P}(k,q)\,\mathcal{P}(l+1,q-2)-\mathcal{P}(l,q)\,\mathcal{P}(k+1,q-2))}{(k-l)\,(k+l+q-1)}$$

where $\mathcal{P}(z,q)$ is defined in (2.1).

Proof. (2.11) is immediate from Theorem 2.9. The various values of ψ are obtained as follows: The function $P_{k,q}P_{l,q}$ is odd if $k-l$ is. If $k, l\geq 0$ then $P_{k,q}$ and $P_{l,q}$ are orthogonal polynomials whose inner product is 0 if $k\neq l$ and is $(\omega_q d_k)^{-1}$ if $k=l$. The remaining cases have $k\neq l$ and so are covered by Lemma 2.4. □

Now come three results related to the proof of Theorem 1.13 for S_q and also the proof of Lemma 4.34, itself a crucial step in the proof of the Main theorem in Section 4. First we have an extended version of Theorem 1.9 for the sphere.

LEMMA 2.11. *For positive integers $n_1, \dots, n_j$ let $\Psi=\Psi_{n_1,\dots,n_j}$ be defined as in Theorem* 1.13. *Let $n=n_1+\dots+n_j$ and take $s\in S-B$ and $s_1, \dots\in S$. Then*

$$\Psi(s,s_1,\dots,s_j)=\int_B e_*(b,n)(s)\,e(b,n_1)(s_1)\dots e(b,n_j)(s_j)\,db. \tag{2.12}$$

Proof. Theorem 1.9 already asserts that (2.12) holds for all s in the hemisphere $S_{1/2}$ containing s_0. To see that it holds in the other hemisphere as well, fix $s_1, \dots, s_j$ and let Q be the polynomial of degree n on B defined by

$$Q(b)=e(b,n_1)(s_1)\cdot\ldots\cdot e(b,n_j)(s_j)\quad (b\in B).$$

Then using Corollary 2.2 we can express the right side of (2.12) as $F_Q(s)$. This is analytic in s so equality on $S_{1/2}$ implies equality wherever the right side is defined, i.e., on $S-B$. □

For the proof of Lemma 4.34 and Lemma 4.36 we need

COROLLARY 2.12. *For positive integers $n_1, \dots, n_j$ there is a polynomial*

$$F_{n_1,\dots,n_j}(y,x^{(1)},\dots,x^{(j)})\quad (y,x^{(i)}\in\mathbf{R}^{q+1};\ i=1,\dots,j)$$

which is homogenous of degree n_i and harmonic in $x^{(i)}$ for each $i=1,\dots,j$ and homogeneous of degree $n=n_1+\dots+n_j$ and harmonic in y and is such that if $y\cdot s_0 \neq 0$ (where $s_0=(1,0,\dots,0)$) then

$$F_{n_1,\dots,n_j}(y, x^{(1)},\dots,x^{(j)})\|y\|^{-2n-q+1}$$

$$=\int_B \operatorname{sgn}(y\cdot s_0)^{q-1}(y\cdot(s_0+ib))^{-n-q+1}(x^{(1)}\cdot(s_0+ib))^{n_1}\dots(x^{(j)}\cdot(s_0+ib))^{n_j}\,db.$$

Proof. The homogeneity is clear. The rest follows from Lemma 2.11 by taking y and each $x^{(i)}$ to lie on S. □

LEMMA 2.13. *Theorem 1.13 is valid for the sphere $S=S_q$.*

Proof. By the partial proof of Theorem 1.13 it remains only to show (1.16) for f in $C^\infty(S)$. By Theorem 2.3, $\hat f(b,n;0,\eta)$ converges uniformly to $\hat f(b,n)$ as $\eta\to 0^+$. Then the right side of (1.16) can be approximated to within $\varepsilon>0$ by choosing $\eta>0$ so that

$$|\hat f(b,n)-\hat f(b,n;0,\eta)|<\varepsilon \quad (b\in B)$$

and replacing $\hat f(b,n)$ in (1.16) by $\hat f(b,n;0,\eta)$ to get

$$d_n\int_B\int_{|s\cdot s_0|>\eta} f(s)\,e_*(b,n)(s)\,e(b,n_1)(s_1)\cdot\ldots\cdot e(b,n_j)(s_j)\,ds\,db.$$

Now interchange the order of integration and use Lemma 2.11 to see that this is

$$d_n\int_{|s\cdot s_0|>\eta} f(s)\,\Psi(s,s_1,\dots s_j)\,ds$$

which tends to $f_{n_1,\dots,n_j}(s_1,\dots,s_j)$ as $\eta\to 0^+$ □

Finally we wish to indicate how to prove the Main theorem for the projective space $P_q(\mathbf{R})$. The proof reduces to Theorem 2.3 by lifting functions on $P_q(\mathbf{R})$ to even functions on the sphere S_q in $\mathbf{R}^{q+1}$. This goes as one would expect, but it is interesting to note that whereas for the sphere $S=S_q$ we integrate over $S(0,\eta)$ (and let $\eta\to 0$), for the projective space $S=P_q(\mathbf{R})$ we integrate over $S(\varepsilon,0)$ (using $\varepsilon=2\eta^2$). Thus for these spaces it seems that one or the other of the parameters in $S(\varepsilon,\eta)$ is redundant; yet for the projective spaces over $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{Cay}$ both parameters are needed.

In a bit more detail: $s\mapsto\pm s$ is the projection from S_q to $P_q(\mathbf{R})$. The complex Lie algebras $\mathfrak{g}, \mathfrak{k}, \mathfrak{b}$, etc. for the two spaces are the same. The $e(b,n)$ on $P_q(\mathbf{R})$ lifts to $e(b,2n)$ on the sphere S_q. Likewise $e_*(b,n)$ on $P_q(\mathbf{R})$ lifts to $e_*(b,2n)$ on S_q. The function $\xi_1 = \Re(e(b,1))$ on S_q which is used to define $\hat{f}$ in Definition 1.11 is just

$$\xi_1(s) = s \cdot s_0 \quad (s \in S_q, s_0 = (1,0,\dots,0))$$

and this is $\cos(\delta_S(s))$ when $s \cdot s_0 > 0$ and δ_S denotes the function giving distance from s_0 on $S = S_q$, normalized so that $\max(\delta_S) = \pi$.

On $P_q(\mathbf{R})$ denote the distance function by δ_P so that these two distance functions are related by

$$\delta_P(\pm s) = 2\delta_S(s) \quad (s \cdot s_0 > 0, s \in S).$$

Consequently

$$\cos(\delta_P(\pm s)) = 2(s \cdot s_0)^2 - 1 \quad (s \cdot s_0 > 0, s \in S).$$

This is why the set $S(2\eta^2, 0)$ (when $S = P_q(\mathbf{R})$) lifts to the set $S(0, \eta)$ (when $S = S_q$) as was claimed above. Thus the definition of $\hat{f}(b,n)$ on $P_q(\mathbf{R})$ given in Definition 1.11 lifts to the definition of $\hat{f}(b,2n)$ on S_q. With this correspondence, the Main theorem and Theorem 1.13 for $P_q(\mathbf{R})$ follow directly from the same results for even functions on the sphere.

3. An intermediate result

This section is devoted to proving a result about a singular integral which is at the heart of the analysis in Section 4, as mentioned in the Introduction. The proof makes essential use of Section 2.

The part of this section which is referred to in Section 4 ends with the statement of Theorem 3.1. Everything in this section after that point is in the service of the proof of that theorem. Any references to $S, B, e_*(b,n)$, etc. in this section after that point are references to these objects as they exist in Section 2, not Section 4. On the other hand, notation up through the statement of Theorem 3.1 is designed for consistency with Section 4.

The singular integral considered here may be described as follows: Let Ω denote the closed unit ball in $\mathbf{R}^{q+1}, q \geqslant 2$. On Ω define the function

$$\varepsilon_{*,n}(x) = (\operatorname{sgn}(x_1))^{q-1}(x_1 + ix_2)^{-n-q+1} \quad (x = (x_1, \dots) \in \Omega, x_1 \neq 0).$$

This notation is chosen to evoke $e_*(b, n)$. Indeed

$$\varepsilon_{*,n}(x) = \|x\|^{-n-q+1} e_*(b_0, n)(x/\|x\|) \quad (x \in \mathbf{R}^{q+1},\ x_1 \neq 0).$$

For the rest of this section fix an integer $m \geq 0$ and define the weight function

$$w(r) = r^{-1}(1-r)^m \quad (0 < r \leq 1).$$

Loosely speaking, the singular integral to be examined in this section is

$$\int_\Omega f(x)\, \varepsilon_{*,n}(x)\, w(\|x\|)\, dx \tag{3.1}$$

where f is required to satisfy a certain smoothness condition. Since $\varepsilon_{*,n}$ is not integrable we have to regularize (3.1) to make sense of it. The particular regularization studied here is dictated by the need to prove the Main theorem. It is defined using

$$\Omega(\varepsilon, \eta) = \{x \in \Omega \mid |x_1| \geq \eta,\ \|x\| + x_1 \geq \varepsilon\} \quad (0 \leq \varepsilon, \eta \leq 1).$$

The interpretation of (3.1) is

$$\lim_{\varepsilon \to 0^+} \lim_{\eta \to 0^+} \int_{\Omega(\varepsilon,\eta)} f(x)\, \varepsilon_{*,n}(x)\, w(\|x\|)\, dx \tag{3.2}$$

provided that the limit exists.

The work of this section is to explore this existence question and establish control over the limit process in terms of a certain norm on f.

This norm requires the introduction of three mutually commuting partial differential operators $\Lambda, \Theta, \Theta_1$ on Ω. Here and hereafter, r will denote the radial variable on $\mathbf{R}^{q+1}$ ($r = \|x\|$) and $\partial/\partial r$ the radial derivative. Then, motivated by Lemma 4.21, define the operator

$$\Lambda = \left(\frac{r\partial}{\partial r}\right)^2 + (m+q)\frac{r\partial}{\partial r} - r\Delta \tag{3.3}$$

where Δ is the usual Laplacian on $\mathbf{R}^{q+1}$. The operator Θ denotes the spherical part of Δ so that

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{q}{r}\frac{\partial}{\partial r} + (1/r^2)\Theta.$$

The operator Θ_1 is defined as the spherical part of the Laplacian on $\mathbf{R}^q$, acting on functions on $\mathbf{R}^{q+1}$ by holding x_1 fixed and differentiating with respect to the remaining variables.

Ω_0 will denote the interior of $\Omega\setminus\{0\}$. Let $\mathcal{D}_n(\Omega)$ denote the normed linear space of functions in $C^{3n+q-2}(\Omega_0)$ for which the norm $\mathcal{N}_n(f)$ is defined and finite, where

$$\mathcal{N}_n(f)=\sup\{|(I-\Theta)(I-\Theta_1)^{(n+q-4)/2}\Lambda^l f(x)|\,|\,x\in\Omega_0, 0\leq l\leq n\}$$

and we define the possibly fractional power of $I-\Theta_1$ by the Spectral theorem.

Note that none of the operators in this definition is elliptic at $x=0$. Thus $\mathcal{D}_n(\Omega)$ can and does contain functions which are not smooth at $x=0$, including all functions of the form $rf(x)$ with f smooth on Ω.

THEOREM 3.1. *For $n\in\mathbf{N}$ there is a constant $C_{n,m,q}$ such that for $f\in\mathcal{D}_n(\Omega)$ and $0<\varepsilon,\eta$ satisfying*

$$\eta<(\varepsilon/2)^{\max(1,n-1)}\quad and\quad \varepsilon+\eta<1$$

we have

$$\left|\int_{\Omega(\varepsilon,\eta)} f(x)\,\varepsilon_{*,n}(x)\,w(\|x\|)\,dx\right|\leq C_{n,m,q}\mathcal{N}_n(f).$$

Moreover, the limit (3.2) *exists and shares this bound.*

The proof is the remainder of this section. Since it is so long the main ideas will be presented first. In a phrase, the method is separation-of-variables. f is written as a sum of functions of the form

$$rs\mapsto g(r)\,h(s)\quad (0<r\leq 1, s\in S=\partial\Omega)$$

where the h range over an orthonormal basis of $L^2(S)$ with each h in some $\mathcal{H}_k(S)\cap L^2_{(j)}(S)$. g is essentially given by

$$g(r)=\int_S f(rs)\,\overline{h(s)}\,ds\quad (0<r\leq 1).$$

The finiteness of $\mathcal{N}_n(f)$ imposes control over $g(r)$ as $r\to 0$, about which more will be said in a moment.

We study the limit (3.2) applied to the individual terms $g(r)\,h(s)$ and note that

$$\int_{\Omega(\varepsilon,\eta)} g(r)\,h(s)\,\varepsilon_{*,n}(rs)\,w(r)\,r^q\,dr\,ds=\int_S f_{h,\varepsilon,\eta}(s)\,e_*(b_0,n)(s)\,ds=\hat{f}_{h,\varepsilon,\eta}(b_0,n)$$

where $b_0=(0,1,0,\ldots,0)$,

$$f_{h,\varepsilon,\eta}(s)=h(s)\int_{r(\varepsilon,\eta,s\cdot s_0)}^{1} g(r)\,r^{-n}(1-r)^m\,dr,$$

and

$$r(\varepsilon,\eta,t)=\min\left(1,\max\left(\frac{\eta}{|t|},\frac{\varepsilon}{1+t}\right)\right)\quad (t>-1,\, t\neq 0). \tag{3.4}$$

Theorem 2.9 is made-to-order for the evaluation of $\hat{f}_{h,\varepsilon,\eta}$. However there is still a major hurdle to be leapt. The function $g(r)\,r^{-n}$ may blow up badly (in fact like r^{k-n}) as $r\to 0$. This point requires careful analysis and brings us back to the issue of the control on $g(r)$ provided by $\mathcal{N}_n(f)$.

Remember that $g(r)=\int_S f(rs)\,\overline{h(s)}\,ds$ and that h belongs to some $\mathcal{H}_k(S)$. If $k\geq n$ then $\mathcal{N}_n(f)<\infty$ will be shown to imply

$$g(r)=O(r^n|\ln(r)|).$$

In this, the usual case, $g(r)\,r^{-n}$ presents little difficulty. However, when $k<n$ then $\mathcal{N}_n(f)<\infty$ gives

$$(1-r)^m g(r)=a_k r^k+\ldots+a_{n-1}r^{n-1}+O(r^n|\ln(r)|).$$

Ultimately this leads us to prove the existence of

$$\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\int_S\left(\int_{r(\varepsilon,\eta,s\cdot s_0)} r^{-l}\,dr\right)h(s)\,e_*(b_0,n)(s)\,ds$$

for $1\leq l\leq n-k$. This fact, proved in a somewhat specialized form in Lemma 3.11, is perhaps the most remarkable point in the entire proof of Theorem 3.1. One might say that, if the proof were going to fail at some point then this would be it, for it is here that the worst part of the singularity of $\varepsilon_{*,n}$ is finally confronted. This is also the part of the proof that imposes the peculiar relation between ε and η in Theorem 3.1.

In the preceeding discussion the emphasis has been on the existence of (3.2) for the individual terms $g(r)\,h(s)$ of the expansion of $f(rs)$. However, we must also provide bounds

$$|\hat{f}_{h,\varepsilon,\eta}(b_0,n)|\leq B_h$$

such that $C_{n,m,q}=\Sigma_h B_h<\infty$. (The sum is over the orthonormal basis of $L^2(S)$.) This necessity adds substantially to the detail of the proof.

Here is a brief catalog of the lemmas of this section.

Lemmas 3.2–3.7 establish control over g in terms of $\mathcal{N}_n(f)$.

A special orthonormal basis $\{h_{ijk}\}$ for $L^2(\partial\Omega)$ is introduced. Lemma 3.8 describes the integral in (3.2) in terms of a series using this basis.

In Lemma 3.9 the individual terms of the series are computed as Fourier transforms $\hat{f}_{ijk\varepsilon\eta}(b_0, n)$ using Theorem 2.9.

Lemma 3.10 bounds these terms for $k \geqslant n$.

Lemma 3.11 is the technical heart of Lemma 3.12 which bounds these terms when $k<n$.

These pieces are drawn together in a concluding statement.

Some frequently used notation peculiar to this section is listed now for ease of reference. The integers $m \geqslant 0$ and $q \geqslant 2$ remain fixed throughout the section. Many objects are used which depend on m and q without that dependence being made explicit in the notation. $\Lambda, \Theta, \Theta_1, w$, which were defined earlier in this section, illustrate this.

Regard $\partial\Omega$ as the S of Section 2. From Section 2 borrow the following notation:

$$\omega_q, \quad s_0, \quad t_\diamond, \quad s(t, b), \quad B, \quad e_*(b_0, n), \quad P_{n,q}, \quad \mathcal{H}_k(S), \quad L^2_{(j)}(S).$$

For f in $C(\Omega_0)$ and h in $L^2(S)$ define

$$g(f, h, r) = (1-r)^m \int_S f(rs)\,\overline{h(s)}\,ds \quad (0<r<1) \tag{3.5}$$

$$\mathcal{N}_n(f, h) = \sup\{|g(\Lambda^l f, h, r)| \mid l = 0, \ldots, n; 0<r<1\} \tag{3.6}$$

$$G_n(f, h, t) = \int_t^1 g(f, h, r)\, r^{-n}\, dr \quad (0 \leqslant t \leqslant 1). \tag{3.7}$$

For any integer $k \geqslant 0$ write

$$\tilde{k} = k(k+m+q) \tag{3.8}$$

$$L_k\, g(r) = -r^{1-k-q} \frac{d}{dr}\left(r^{2k+q}(1-r)^{m+1} \frac{d}{dr}(r^{-k}(1-r)^{-m} g(r))\right). \tag{3.9}$$

For $0 \leqslant a \leqslant 1$ define the integral operator

$$\mathcal{I}_{k,a}(g(r)) = -(1-r)^m r^k \int_a^r (1-u)^{-m-1} u^{-2k-q} \int_0^u v^{q+k-1} g(v)\, dv\, du \tag{3.10}$$

provided that the function g is such that the integral makes sense.

Δ_S and Δ_B are the Laplace–Beltrami operators on S and B. These are closely tied to the operators Θ and Θ_1 on $\mathbf{R}^{q+1}$: if f is in $C^2(\Omega_0)$ then

$$(\Theta f)(rs) = \Delta_S(s \mapsto f(rs)) \quad (s \in S)$$

$$(\Theta_1 f)(rs(t,b)) = \Delta_B(b \mapsto f(rs(t,b))) \quad (b \in B, |t|<1, 0<r<1)$$

where, as in Section 2,

$$s(t,b) = ts_0 + t_\diamond b \quad (|t| \leq 1, b \in B).$$

LEMMA 3.2. *Take f in $C^2(\Omega_0)$.*
(i) *If h is in $\mathscr{H}_k(S)$ then $g(\Theta f, h, r) = -k(k+q-1)\, g(f,h,r)$.*
(ii) *If h is in $L^2_{(j)}(S)$ then $g(\Theta_1 f, h, r) = -j(j+q-2)\, g(f,h,r)$.*

Proof. For (i)

$$\begin{aligned} g(\Theta f, h, r) &= (1-r)^m \int_S (\Delta_S f(rs))\, \overline{h(s)}\, ds \\ &= (1-r)^m \int_S f(rs)\, \overline{\Delta_S h(s)}\, ds \end{aligned}$$

and $\Delta_S h = -k(k+q-1)\, h$.

For (ii) the argument is similar if h is in $L^2_{(j)}(S) \cap C^2(S)$. The general case follows since both sides are continuous in $h \in L^2_{(j)}(S)$. □

LEMMA 3.3. *For f in $\mathscr{D}_n(\Omega)$ and $h \in \mathscr{H}_k(S) \cap L^2_{(j)}(S)$*

$$(1+j^2)^{(n+q-4)/2}(1+k^2)\, \mathscr{N}_n(f,h) \leq \mathscr{N}_n(f)\, \|h\|_2.$$

Proof. For all $0<r<1$ and $0 \leq l \leq n$ the previous lemma gives

$$\begin{aligned} &|(1+k(k+q-1))(1+j(j+q-2))^{(n+q-4)/2} g(\Lambda^l f, h, r)| \\ &= |g((I-\Theta)(I-\Theta_1)^{(n+q-4)/2} \Lambda^l f, h, r)| \leq \mathscr{N}_n(f)\, \|h\|_1 \end{aligned}$$

and the result follows from $\|h\|_1 \leq \|h\|_2$. □

LEMMA 3.4. *For any $f \in C^2(\Omega_0)$ and $h \in \mathscr{H}_k(S)$*

$$L_k g(f,h,r) = g(\Lambda f - \tilde{k} f, h, r). \tag{3.11}$$

Proof. From (3.5) and (3.9) the left side of (3.11) is

$$(1-r)^m\left(r(r-1)\frac{d^2}{dr^2}+((m+q+1)r-q)\frac{d}{dr}+\frac{k(k+q-1)}{r}-\tilde{k}\right)\int_S f(rs)\overline{h(s)}\,ds$$
$$=(1-r)^m\int_S(\Delta f-\tilde{k}f)(rs)\overline{h(s)}\,ds. \qquad \square$$

If $0<a<1$ the operator $\mathscr{I}_{k,a}$ in (3.10) is a right inverse to L_k on the space of continuous functions g on $(0,1)$ which are integrable near $r=0$. If

$$g(r)=O(r^k|\ln(r)|) \quad \text{as} \quad r\to 0^+$$

then $\mathscr{I}_{k,0}(g(r))$ also makes sense and is also a right inverse to L_k.

LEMMA 3.5. *Suppose g is of class C^2 on $(0,1)$ such that both g and $L_k g$ are bounded near $r=0$. Choose $a: 0<a<1$. Then for $0<r<1$*

$$g(r)=\left(\frac{r}{a}\right)^k\left(\frac{1-r}{1-a}\right)^m g(a)+\mathscr{I}_{k,a}(L_k g(r)). \tag{3.12}$$

If $L_k g(r)=O(r^k\ln(r))$ as $r\to 0$ we may take $a=0$ and get

$$g(r)=r^k(1-r)^m\left(\lim_{t\to 0^+}t^{-k}g(t)\right)+\mathscr{I}_{k,0}(L_k g(r)). \tag{3.13}$$

Proof. With g fixed let y denote the general solution of $L_k y=L_k g$. By inspection,

$$y(r)=r^k(1-r)^m\left(c_1-\int_a^r(1-u)^{m-1}u^{-2k-q}\left(c_2+\int_0^u v^{q+k-1}L_k g(v)\,dv\right)du\right).$$

This solution has the property

$$y(r)=c_2\left(\frac{r^{-k-q+1}}{k+q-1}(1+O(r))\right)+O(1) \quad \text{as} \quad r\to 0.$$

If $y=g$ is bounded near $r=0$ then $c_2=0$. (3.12) follows by taking $r=a$ to get c_1.

For (3.13) observe that $L_k g(r)=O(r^k\ln(r))$ implies

$$(1-u)^{-m-1}u^{-2k-q}\int_0^u v^{q+k-1}L_k g(v)\,dv=O(\ln(u))$$

which is integrable near $u=0$. This gives (3.13) as the limit of (3.12) as $a\to 0$. $\square$

From these last two lemmas comes

COROLLARY 3.6. *For $f\in\mathcal{D}_1(\Omega)$, $h\in\mathcal{H}_k(S)$, and $0<a<1$*

$$g(f,h,r)=\left(\frac{r}{a}\right)^k\left(\frac{1-r}{1-a}\right)^m g(f,h,a)+\mathcal{I}_{k,a}(g(\Lambda f-\tilde{k}f,h,r)). \tag{3.14}$$

If $g(\Lambda f-\tilde{k}f,h,r)=O(r^k|\ln(r)|)$ as $r\to 0$ then

$$g(f,h,r)=r^k(1-r)^m c+\mathcal{I}_{k,0}(g(\Lambda f-\tilde{k}f,h,r)) \tag{3.15}$$

where $c=\lim_{t\to 0}t^{-k}g(f,h,t)$.

LEMMA 3.7. *For any integer $n\geq 0$ there are positive constants A_n, B_n, C_n, D_n depending only on n, m and q such that for all integers $k\geq 0$ and functions $f\in C^{2n}(\Omega_0)$ and $h\in\mathcal{H}_k(S)$ with $\mathcal{N}_n(f,h)<\infty$ we have for $0<r<1$*

(i) *if $n<k$ then $|g(f,h,r)|\leq r^n A_n\mathcal{N}_n(f,h)$;*

(ii) *if $n=k$ then $|g(f,h,r)|\leq r^n(|\ln(r)|+1)A_n\mathcal{N}_n(f,h)$;*

(iii) *if $n>k$ then there are numbers $c_j(f,h)$ with $j=k,\ldots,n-1$ such that for $0<r<1$*

$$\left|g(f,h,r)+\sum_{j=k}^{n-1}c_j(f,h)\,r^j\right|\leq r^n(|\ln(r)|+D_n)\,B_n\mathcal{N}_n(f,h)$$

and

$$|c_j(f,h)|\leq C_n\mathcal{N}_n(f,h)\quad (j=k,\ldots,n-1).$$

Proof. Regard the integer k as fixed and proceed by induction on n. At each stage the idea is to use Corollary 3.6 with (3.16) below to trade an increase of the n in $\mathcal{N}_n(f,h)$ for more control over $g(f,h,r)$ (as $r\to 0$). The argument divides naturally into the cases (i)–(iii) with the transition between cases due mainly to the shift from using (3.14) to (3.15). In (i) an important point is that the A_n is independent of k while in (ii) and (iii) the emphasis is on controlling the coefficients $c_j(f,h)$; to secure these features the argument is given in some detail.

We make frequent use of $\Lambda f-\tilde{k}f$ which we abbreviate as $\tilde{f}$. From the definition (3.6) of the seminorm $\mathcal{N}_n(f,h)$.

$$\mathcal{N}_{n-1}(\tilde{f},h)\leq(1+\tilde{k})\,\mathcal{N}_n(f,h). \tag{3.16}$$

Case (i) ($n<k$): The case $n=0$ is trivial if we take $A_0=1$. Now suppose $n<k$ and that (i) holds for $n-1$. Clearly

$$|g(f,h,r)| \leqslant \mathcal{N}_0(f,h) \leqslant r^n 2^n \mathcal{N}_n(f,h) \quad (1/2 \leqslant r < 1) \tag{3.17}$$

so restrict attention to $0<r<\frac{1}{2}$. The induction hypothesis gives

$$|g(f,h,1/2)| \leqslant 2^{1-n} A_{n-1} \mathcal{N}_{n-1}(f,h),$$

$$|g(\tilde{f},h,r)| \leqslant r^{n-1} A_{n-1}(1+\tilde{k}) \mathcal{N}_n(f,h).$$

Use this in (3.14) with $a=\frac{1}{2}$ and $0<r\leqslant a$ to get

$$\begin{aligned}|g(f,h,r)| &= |r^k(1-r)^m 2^{k+m} g(f,h,1/2) + \mathcal{I}_{k,1/2}(g(\tilde{f},h,r))| \\ &\leqslant A_{n-1}\mathcal{N}_n(f,h)\,(r^k(1-r)^m 2^{k+m-n+1} + (1+\tilde{k})\,\mathcal{I}_{k,1/2}(r^{n-1})).\end{aligned} \tag{3.18}$$

(3.10) and routine calculation give

$$|\mathcal{I}_{k,1/2}(r^{n-1})| \leqslant \frac{(r^n - r^k 2^{k-n})\,2^{m+1}}{(k-n)\,(k+n+q-1)} \quad (0<r\leqslant 1/2). \tag{3.19}$$

Also note that $(1+\tilde{k})/((k-n)\,(k+n+q-1)) \leqslant 2+n+m/2$ and $\mathcal{N}_{n-1}(f,h) \leqslant \mathcal{N}_n(f,h)$ so we can reduce (3.18) to

$$|g(f,h,r)| \leqslant r^n 2^{m+1}(2+n+m/2)\,A_{n-1}\mathcal{N}_n(f,h) \quad (0\leqslant r\leqslant 1/2).$$

With (3.17) gives (i) if we take A_n so that

$$A_n \geqslant \max(2^n, 2^{m+1}(2+n+m/2)\,A_{n-1}). \tag{3.20}$$

Case (ii) $(n=k)$: The proof is similar to (i) except that (3.19) is replaced by

$$\begin{aligned}|\mathcal{I}_{k,1/2}(r^{n-1})| &= \left| r^n(1-r)^m \int_{1/2}^{r} (1-u)^{-m-1} u^{-1}\,du \right| \Big/ (2k+q-1) \\ &\leqslant \frac{r^n |\ln(r)| 2^{m+1}}{2k+q-1} \quad (0<r\leqslant 1/2)\end{aligned}$$

which ultimately leads to (ii) with A_n as in (3.20).

Case (iii) $(n>k)$: In the argument following this paragraph we prove a superficially weakened form of (iii) in which the constants B_n, C_n, D_n are replaced by constants $B_{n,k}$, etc. which may depend on k. Once this is done the original form of (iii) follows by holding n fixed and taking $B_n = \max\{B_{n,k} | k=0,\ldots,n-1\}$, etc.

Continue to hold k fixed and argue by induction on n which is now taken to be $>k$. This gives us

$$\mathcal{N}_{k+1}(f,h)\leq\mathcal{N}_n(f,h)<\infty$$

and so we have from (ii) and (3.15) that for all $n>k$

$$g(f,h,r)=c_k(f,h)\,r^k(1-r)^m+\mathcal{J}_{k,0}(g(\tilde{f},h,r)) \tag{3.21}$$

where, as before, $\tilde{f}=\Lambda f-\tilde{k}f$, and we define

$$c_k(f,h)=\lim_{t\to 0}t^{-k}g(f,h,t).$$

Also for $n>k$ we have in place of (3.19) that if $c\geq 0$ then

$$\mathcal{J}_{k,0}(r^{n-1}(|\ln(r)|+c))|=\frac{r^k(1-r)^m}{k+n+q-1}\int_0^r(1-u)^{-m-1}\,u^{n-k-1}\left(|\ln(u)|+c+\frac{1}{k+n+q-1}\right)du \tag{3.22}$$

$$\leq\frac{r^n(|\ln(r)|+c+2)}{(1-r)(n-k)(k+n+q-1)}\quad(0<r<1).$$

Using (3.16), (3.22) and (ii) we can estimate $c_k(f,h)$ by setting $r=\frac{1}{2}$ in (3.21):

$$\begin{aligned}|c_k(f,h)|&\leq 2^m r^{-k}(|g(f,h,r)|+A_k\,\mathcal{N}_k(\tilde{f},h)\,\mathcal{J}_{k,0}(r^k(|\ln(r)|+1)))|_{r=1/2}\\&\leq 2^m A_k\,\mathcal{N}_{k+1}(f,h)\left(1+\frac{1+\tilde{k}}{2k+q}(|\ln(1/2)|+3)\right)\\&\leq 2^m A_k\,\mathcal{N}_{k+1}(f,h)\,(2k+2m+q+1).\end{aligned} \tag{3.23}$$

Rewrite (3.21) as

$$g(f,h,r)=c_k(f,h)\,r^k+R_{k+1}(f,h,r) \tag{3.24}$$

where

$$R_{k+1}(f,h,r)=\mathcal{J}_{k,0}(g(\tilde{f},h,r))+c_k(f,h)\,r^k((1-r)^m-1).$$

Then (3.22), (3.16), (3.23) and (ii) give

$$R_{k+1}(f,h,r)\leq A_k\,\mathcal{N}_{k+1}(f,h)\left(\frac{1+\tilde{k}}{2k+q}\,\frac{(|\ln(r)|+3)}{1-r}+m2^m(2k+2m+q+1)\right)$$

which proves the desired bound on $R_{k+1}(f, h, r)$ on, say, $(0, \frac{1}{2}]$. On $[\frac{1}{2}, 1]$ we can bound $R_{k+1}(f, h, r)$ using (3.24):

$$|R_{k+1}(f, h, r)| \leq |c_k(f, h)| + \mathcal{N}_0(f, h).$$

Altogether this gives (iii) for $n=k+1$ with k-dependent constants $B_{n,k}, C_{n,k}, D_{n,k}$ in place of B_n, C_n, D_n.

For the induction step assume

$$g(\tilde{f}, h, r) = c_k(\tilde{f}, h)\, r^k + \ldots + c_{n-2}(\tilde{f}, h)\, r^{n-2} + R_{n-1}(\tilde{f}, h, r) \tag{3.25}$$

with

$$|c_j(\tilde{f}, h)| \leq C_{n-1,k}\, \mathcal{N}_{n-1}(\tilde{f}, h) \leq C_{n-1,k}(1+\tilde{k})\, \mathcal{N}_n(f, h)$$
$$R_{n-1}(\tilde{f}, h, r) \leq r^{n-1}(|\ln(r)| + D_{n-1,k})\, B_{n-1,k}(1+\tilde{k})\, \mathcal{N}_n(f, h).$$

(3.22) shows that

$$\left|\mathcal{I}_{k,0}(R_{n-1}(\tilde{f}, h, r))\right| \leq \frac{r^n(|\ln(r)| + D_{n-1,k}+2)}{(1-r)(n-k)(k+n+q-1)} B_{n-1,k}(1+\tilde{k})\, \mathcal{N}_n(f, h).$$

It is easy to see that $\mathcal{I}_{k,0}(r^j)$ (for $j \geq k$) is analytic around 0 with radius of convergence at least 1 and lowest nonzero coefficient of degree $j+1$. From this it follows that $\mathcal{I}_{k,0}$ applied to the polynomial on the right of (3.25) gives a polynomial with terms of degrees $k+1$ to $n-1$ plus a *remainder* which is $O(r^n)$ at 0; moreover we can bound the coefficients of the polynomial and (*remainder*)$/r^n$ in terms of $\mathcal{N}_n(f, h)$ and n, k, m, q, at least on $(0, \frac{1}{2}]$. When we add in $c_k(f, h)\, r^k(1-r)^m$ we get

$$g(f, h, r) = c_k(f, h)\, r^k + \ldots + c_{n-1}(f, h)\, r^{n-1} + R_n(f, h, r) \tag{3.26}$$

with

$$|c_j(f, h)| \leq C_{n,k}\, \mathcal{N}_n(f, h) \quad (k \leq j \leq n-1)$$

and, at least on the interval $(0, \frac{1}{2}]$,

$$|R_n(f, h, r)| \leq r^n(|\ln(r)| + D_{n,k})\, B_{n,k}\, \mathcal{N}_n(f, h) \tag{3.27}$$

for some choice of the constants $B_{n,k}$, etc. We can bound $R_n(f, h, r)$ on $[\frac{1}{2}, 1]$ by using (3.26) as we did in the case of $n=k+1$ to get

$$|R_n(f, h, r)| \leq \mathcal{N}_0(f, h) + |c_k(f, h)| + \ldots + |c_{n-1}(f, h)|$$

which, by possibly enlarging the constants $B_{n,k}$, etc., gives (3.27) on all of $(0, 1]$. Finally we eliminate dependence of the constants on k as indicated at the start of case (iii) of this proof. □

The next step is to choose the orthonormal basis of $L^2(S)$. We use a standard construction in which the basis functions h_{ijk} are indexed by

$$k \in \mathbf{N}; \quad j = 0, \ldots, k; \quad i = 1, \ldots, d(j, q-1) = \dim \mathscr{H}_j(B)$$

so that $\{h_{ijk} | i=1, \ldots, d(j, q-1)\}$ is an orthonormal basis of $L^2_{(j)}(S) \cap \mathscr{H}_k(S)$. Choose an orthonormal basis $\{Q_{1,j}, Q_{2,j}, \ldots\}$ of $\mathscr{H}_j(B)$ in such a way that

$$Q_{1,j}(b_0) > 0, \quad Q_{i,j}(b_0) = 0 \text{ for } i > 1.$$

Then define

$$h_{ijk}(s(t, b)) = a_{jk} P_{k-j, 2j+q}(t) (t_\diamond)^j Q_{i,j}(b) \quad (|t| \leq 1, b \in B) \tag{3.28}$$

where $P_{n,q}$ is defined in (0.2) and the constant a_{jk} is chosen >0 and to make h_{ijk} a unit vector in $L^2(S)$. This means that

$$1 = \omega_q a_{jk}^2 \int_{-1}^{1} P^2_{k-j, 2j+q}(t) (t_\diamond)^{2j+q-2} dt. \tag{3.29}$$

Abbreviate

$$\sum_{k=0}^{N} \sum_{j=0}^{k} \sum_{i=1}^{d(j,q-1)} \quad \text{by} \quad \sum_{kji}^{N} \quad (N = 0, 1, \ldots, \infty)$$

and let $\sigma_q = (q+1)\text{measure}(\Omega)$ so that for f in $L^1(\Omega)$,

$$\int_\Omega f(x)\, dx = \sigma_q \int_S \int_0^1 f(rs)\, r^q dr ds.$$

Recall the definitions of $G_n(f, h, t)$ in (3.7) and $r(\varepsilon, \eta, t)$ in (3.4).

LEMMA 3.8. For $\varepsilon, \eta > 0$ *and* f *a bounded, continuous function on* Ω_0,

$$\int_{\Omega(\varepsilon,\eta)} f(x)\, \varepsilon_{*,n}(x)\, w(\|x\|)\, dx = \sum_{kji}^{\infty} \sigma_q \int_S G_n(f, h_{ijk}, r(\varepsilon, \eta, s \cdot s_0))\, h_{ijk}(s)\, e_*(b_0, n)(s)\, ds. \tag{3.30}$$

Proof. For a given $0 < r \leq 1$, the series

$$\sum_{kji}^{\infty} g(f, h_{ijk}, r)\, h_{ijk}(s)$$

is simply the expansion in $L^2(S)$ of the continuous function

$$s \mapsto (1-r)^m f(rs) \quad (s \in S)$$

with respect to the orthonormal basis $\{h_{ijk}\}$. Thus the sequence of functions

$$\psi_N(r) = r^{q-1} \int_S \left| (1-r)^m f(rs) - \sum_{kji}^{N} g(f, h_{ijk}, r)\, h_{ijk}(s) \right|^2 ds$$

converges monotonically to 0 as $N \to \infty$ for $0<r\leqslant 1$. From this, and the boundedness of $\varepsilon_{*,n}$ on $\Omega(\varepsilon, \eta)$,

$$\sum_{kji}^{\infty} \int_{\Omega(\varepsilon,\eta)} g(f, h_{ijk}, r)\, h_{ijk}(s)\, \varepsilon_{*,n}(rs) r^{q-1} dr ds$$

converges to the left side of (3.30).

It remains to show that for h in $L^2(S)$

$$\int_{\Omega(\varepsilon,\eta)} g(f, h, r)\, h(s)\, \varepsilon_{*,n}(rs) r^{q-1} dr ds = \sigma_q \int_S G_n(f, h, r(\varepsilon, \eta, s \cdot s_0))\, h(s)\, e_*(b_0, n)(s)\, ds. \tag{3.31}$$

First observe that for any $\psi \in L^1(\Omega(\varepsilon, \eta))$

$$\int_{\Omega(\varepsilon,\eta)} \psi(x)\, dx = \sigma_q \int_S \int_{r(\varepsilon,\eta,s\cdot s_0)}^{1} \psi(rs)\, r^q dr ds \tag{3.32}$$

since

$$\begin{aligned} \Omega(\varepsilon, \eta) &= \{x \in \mathbf{R}^{q+1} \mid \|x\| \leqslant 1,\ |x_1| \geqslant \eta,\ x_1 + \|x\| \geqslant \varepsilon\} \\ &= \left\{ x = rs \mid s \in S,\ 0 \leqslant r \leqslant 1,\ r \geqslant \frac{\eta}{|s \cdot s_0|},\ r \geqslant \frac{\varepsilon}{1 + s \cdot s_0} \right\} \\ &= \{x = rs \mid s \in S,\ r(\varepsilon, \eta, s \cdot s_0) \leqslant r \leqslant 1\}. \end{aligned}$$

Also, by definition of $\varepsilon_{*,n}$,

$$\varepsilon_{*,n}(rs) = r^{-n-q+1} \varepsilon_*(b_0, n)(s) \quad (0 < r \leqslant 1,\ s \in S).$$

This in combination with (3.32) makes the left side of (3.31) equal

$$\sigma_q \int_S \int_{r(\varepsilon,\eta,s\cdot s_0)}^1 r^{-n} g(f,h,r)\,dr\,h(s)\,e_*(b_0,n)(s)\,ds.$$

In view of (3.7) this equals the right side of (3.31). □

Lemma 3.8 motivates the notation

$$f_{ijk\varepsilon\eta}(s) = \sigma_q G_n(f, h_{ijk}, r(\varepsilon,\eta,s\cdot s_0))\,h_{ijk}(s) \quad (s\in S)$$

for any bounded continuous function f on Ω_0. Then (3.30) can be rewritten as

$$\int_{\Omega(\varepsilon,\eta)} f(x)\,\varepsilon_{*,n}(x)\,\omega(\|x\|)\,dx = \sum_{kji}^{\infty} \hat{f}_{ijk\varepsilon\eta}(b_0,n). \tag{3.30 a}$$

LEMMA 3.9. *For bounded* $f\in C(\Omega_0)$,

$$\hat{f}_{ijk\varepsilon\eta}(b_0,n)=0 \quad (i\neq 1,\ 0\le j\le k).$$

If $i=1$ *then in the notation of Section* 2 *and* (3.28)

$$\hat{f}_{1jk\varepsilon\eta}(b_0,n) = C\int_{-1}^{1} G_n(f,h_{1jk},r(\varepsilon,\eta,t))\,P_{k-j,2j+q}(t)\,P_{n-j,2j+q}(t)\,(t_\diamond)^{2j+q-2}dt \tag{3.33}$$

where

$$C=\sigma_q\,\omega_q\,\varpi(j,q;n)\,a_{jk}\,Q_{1j}(b_0).$$

If $G_n(f,h_{1jk},\cdot)$ *is bounded then*

$$|\hat{f}_{1jk\varepsilon\eta}(b_0,n)| \le M_n(j+1)^{n+q-5/2}\,\|G_n(f,h_{1jk},\cdot)\|_\infty \tag{3.34}$$

where M_n *is a constant not depending on* j.

Proof. Except possibly for (3.34), this is immediate from Theorem 2.9, the definition of h_{ijk} in (3.28), and the definition of $\hat{f}_{ijk\varepsilon\eta}$ above. The value 0 for $f_{ijk\varepsilon\eta}(b_0,n)$ when $i>1$ is due to the choice of $Q_{ij}(b_0)=0$ in defining h_{ijk}.

From the proof of Theorem 2.9, (2.8) and (2.9) give us

$$|\hat{f}_{1jk\varepsilon\eta}(b_0,n)| \le (j+1)^{n+(q-3)/2}\sigma_q\,|Q_{1j}(b_0)|\,|a_{jk}|\left(\omega_q\int_{-1}^{1} P^2_{k-j,2j+q}(t)\,(t_\diamond)^{2j+q-2}dt\right)^{1/2}\|G_n(f,h_{1jk},\cdot)\|_\infty.$$

By (3.29) the coefficient of $\|G_n(f, h_{1jk}, \cdot)\|_\infty$ on the right side simplifies to

$$(j+1)^{n+(q-3)/2}\sigma_q Q_{1j}(b_0). \tag{3.35}$$

From the definition of Q_{1j} (right before (3.28)) it follows that $Q_{1j}(b)=cP_{j,q-1}(b\cdot b_0)$ where c is chosen to make Q_{1j} of norm 1 in $L^2(B)$. Since $P_{j,q-1}(1)=1$,

$$Q_{1j}(b_0) = c = d(j, q-1)^{1/2}.$$

From (2.4) we know that $d(j, q-1)$ is a polynomial of degree $q-2$ in j. Thus there is a constant M_n independent of j such that (3.35) is less than $M_n(j+1)^{n+q-5/2}$, proving (3.34). □

LEMMA 3.10. *For $k\geq n$ and $0\leq j\leq k$ there are constants $B_{jkn}>0$ such that*

$$\sum_{k=n}^{\infty}\sum_{j=0}^{k} B_{jkn} < \infty \tag{3.36}$$

and if f is in $\mathcal{D}_n(\Omega)$ then for all $0<\varepsilon,\eta<1$ we have

$$|\hat{f}_{1jk\varepsilon\eta}(b_0, n)| \leq B_{jkn}\mathcal{N}_n(f) \tag{3.37}$$

and the following limit exists:

$$\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\hat{f}_{1jk\varepsilon\eta}(b_0, n). \tag{3.38}$$

Proof. From Lemma 3.7 ((i), (ii)) we get

$$|G_n(f, h_{1jk}, r)| \leq 2A_n\mathcal{N}_n(f, h_{1jk}) \tag{3.39}$$

and that $G_n(f, h_{1jk}, r)$ is continuous in $r\in[0,1]$. From this, (3.4), (3.33), from the boundedness of $P_{k-j,2j+q}$ and $P_{n-j,2j+q}$, and from the dominated convergence theorem we get the existence of (3.38).

Lemma 3.3, (3.34) and (3.39) together give (3.37) with

$$B_{jkn} = 2A_n M_n(j+1)^{n+q-5/2}(j^2+1)^{-(n+q-4)/2}(k^2+1)^{-1}$$

from which (3.36) is evident. □

The previous lemma handles the part of $\Sigma_{kji}^{\infty}\hat{f}_{ijk\varepsilon\eta}(b_0, n)$ where $k\geq n$. Lemma 3.12 will control the finite number of terms for which $k<n$. Lemma 3.11 contains the heart of the argument in Lemma 3.12.

First, for integers $i,k,n \geqslant 0$ and real numbers $\varepsilon,\eta>0$ define numbers $\gamma(i,k,n,q,\varepsilon,\eta)$ and $\gamma(i,k,n,q)$ by:

$$\gamma(i,k,n,q,\varepsilon,\eta)=\int_{-1}^{1}\int_{r(\varepsilon,\eta,t)}^{1} r^{-i}dr\,P_{k,q}(t)\,P_{n,k}(t)\,(t_{\diamond})^{q-2}dt. \tag{3.40}$$

If $i>1$ then define $\gamma(i,k,n,q)=0$. If $i=1$ then define

$$\gamma(1,k,n,q)=\int_{-1}^{1}\ln(1+t)\,P_{k,q}(t)\,P_{n,q}(t)\,(t_{\diamond})^{q-2}dt.$$

LEMMA 3.11. *With $0\leqslant k<n$ and $1\leqslant i\leqslant n-k$ we have*

$$\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\gamma(i,k,n,q,\varepsilon,\eta)=\gamma(i,k,n,q).$$

In fact, if

$$0<\varepsilon,\eta,\quad \varepsilon+\eta<1,\quad \eta<\varepsilon/2\quad\text{and}\quad \eta<(\varepsilon/2)^{n-1} \tag{3.41}$$

then

$$|\gamma(i,k,n,q,\varepsilon,\eta)-\gamma(i,k,n,q)|<\varepsilon+\eta g_i(\varepsilon) \tag{3.42}$$

where

$$g_i(\varepsilon)=\frac{4}{3}\begin{cases}(2/\varepsilon)^i & \text{if } i>1;\\ 1-2\ln(\varepsilon)/\varepsilon & \text{if } i=1.\end{cases}$$

Proof. For $i\geqslant 1$, $t>-1$, and $\varepsilon,\eta>0$ define

$$\varrho_{i,\varepsilon,\eta}(t)=\int_{r(\varepsilon,\eta,t)}^{1} r^{-i}dr,$$

$$\varrho_{i,\varepsilon}(t)=\begin{cases}(((1+t)/\varepsilon)^{i-1}-1)/(i-1) & \text{if } i>1;\\ \ln(1+t)-\ln(\varepsilon), & \text{if } i=1.\end{cases}$$

For $i\geqslant 2$, $\varrho_{i,\varepsilon}(t)\,P_{k,q}(t)$ is a polynomial in t of degree $i-1+k<n$ and thus is orthogonal to $P_{n,q}(t)$ on $[-1,1]$ with respect to the weight $(t_{\diamond})^{q-2}$. The same is true for $\ln(\varepsilon)\,P_{k,q}(t)$; thus the left side of (3.42) equals

$$\left|\int_{-1}^{1}(\varrho_{i,\varepsilon,\eta}(t)-\varrho_{i,\varepsilon}(t))\,P_{k,q}(t)\,P_{n,q}(t)\,(t_{\diamond})^{q-2}dt\right|$$

and since $|P_{j,q}(t)|\leqslant 1$ on $[-1,1]$ $(j=k,n)$ it suffices to prove

$$\int_{-1}^{1}|\varrho_{i,\varepsilon,\eta}(t)-\varrho_{i,\varepsilon}(t)|dt\leqslant\varepsilon+\eta g_i(\varepsilon). \tag{3.43}$$

With (3.4) in mind, observe that

$$\varrho_{i,\varepsilon,\eta}(t)=\begin{cases}(\max(1,\min(|t|/\eta,(1+t)/\varepsilon))^{i-1}-1)/(i-1) & \text{if } i>1;\\ \ln(\max(1,\min(|t|/\eta,(1+t)/\varepsilon))) & \text{if } i=1.\end{cases}$$

Consequently, except for t in the disjoint (by (3.41)) intervals

$$I_1(\varepsilon)=[-1,-1+\varepsilon),\quad I_2(\varepsilon,\eta)=\left(\frac{-\eta}{\varepsilon+\eta},\frac{\eta}{\varepsilon-\eta}\right)$$

we have $\varrho_{i,\varepsilon,\eta}(t)=\varrho_{i,\varepsilon}(t)$. The contribution to the left side of (3.43) from $I_1(\varepsilon)$ is ε/i since $\varrho_{i,\varepsilon,\eta}(t)=0$ on $I_1(\varepsilon)$, and

$$\int_{-1}^{-1+e}\varrho_{i,\varepsilon}(t)\,dt=-\varepsilon/i.$$

On $I_2(\varepsilon,\eta)$ the condition $\eta<(\varepsilon/2)^{i-1}$ (implied by (3.41)) gives us that the maximum of the integrand in (3.43) occurs at $t=\eta$ and is (using $\eta<\varepsilon/2$ from (3.41))

$$|\varrho_{i,\varepsilon}(\eta)|\leqslant\begin{cases}(2/\varepsilon)^{i-1} & \text{if } i>1;\\ \varepsilon/2-\ln(\varepsilon) & \text{if } i=1.\end{cases}$$

Since the width of $I_2(\varepsilon,\eta)$ is

$$\frac{2\varepsilon\eta}{\varepsilon^2-\eta^2}<\frac{8\eta}{3\varepsilon}$$

the contribution to the left side of (3.43) from $I_2(\varepsilon,\eta)$ is less than $\eta g_i(\varepsilon)$. □

LEMMA 3.12. *For $0\leqslant j\leqslant k<n$ there are constants $B_{jkn}>0$ such that if f is in $\mathscr{D}_n(\Omega)$ then for all ε,η satisfying* (3.41) *we have*

$$|\hat{f}_{1jk\varepsilon\eta}(b_0,n)|\leqslant B_{jkn}\,\mathscr{N}_n(f) \tag{3.44}$$

and the double limit

$$\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\hat{f}_{1jk\varepsilon\eta}(b_0,n) \tag{3.45}$$

exists.

Proof. j and k are fixed so write h for h_{ijk}. By Lemma 3.7 (iii)

$$g(f,h,r)=\sum_{i=k}^{n-1} c_i(f,h)\,r^i+R_n(f,h,r)$$

where

$$|R_n(f,\,h,\,r)|\leqslant r^n(|\ln(r)|+D_n)\,B_n\,\mathcal{N}_n(f,h). \tag{3.46}$$

By definition of G_n (in (3.7))

$$G_n(f,h,t)=\sum_{i=k}^{n-1} c_i(f,h)\int_t^1 r^{i-n}dr+\int_t^1 r^{-n}R_n(f,h,r)\,dr.$$

From this, Lemma 3.9 and (3.40) give

$$\hat{f}_{1jk\varepsilon\eta}(b_0,n)=K_{jkn}\left(\sum_{i=k}^{n-1} c_i(f,h)\,\gamma(n-i,k-j,n-j,q+2j,\varepsilon,\eta)+\mathcal{G}(\varepsilon,\eta)\right)$$

where

$$K_{jkn}=Q_{1j}(b_0)\,\omega_q\varpi(j,q;n)\,a_{jk}\,\sigma_q$$

and

$$\mathcal{G}(\varepsilon,\eta)=\int_{-1}^{1}\int_{r(\varepsilon,\eta,t)}^{1} r^{-n}R_n(f,h,r)\,dr\,P_{k-j,2j+q}(t)\,P_{n-j,2j+q}(t)\,(t_\diamond)^{2j+q-2}dt$$

is uniformly bounded by $(1+D_n)B_n\mathcal{N}_n(f,h)$ (from (3.46)) and is convergent as $\varepsilon,\eta\to 0^+$ to

$$\mathcal{G}(0,0)=\int_0^1 r^{-n}R_n(f,h,r)\,dr\int_{-1}^{1} P_{k-j,2j+q}(t)\,P_{n-j,2j+q}(t)\,(t_\diamond)^{2j+q-2}dt=0$$

by the orthogonality of the polynomials $P_{i,2j+q}$ $(i=n-j,k-j)$. The boundedness and convergence of the terms

$$c_i(f,h)\,\gamma(n-i,k-j,n-j,2j+q,\varepsilon,\eta)$$

is established in Lemmas 3.7 and 3.11. The conclusion is that (3.45) exists with a bound like (3.44) but with $\mathcal{N}_n(f)$ replaced by $\mathcal{N}_n(f,h)$. However, (3.44) follows from this and Lemma 3.3. □

Proof of Theorem 3.1. Take C_{nmq} to be $\Sigma_{k=0}^{\infty} \Sigma_{j=0}^{\infty} B_{jkn}$ where the B_{jkn} are given in Lemmas 3.10 and 3.12. Take f in $\mathscr{D}_n(\Omega)$. Then by (3.30a) and Lemmas 3.8 and 3.9

$$\int_{\Omega(\varepsilon,\eta)} f(x)\, \varepsilon_{*,n}(x)\, w(\|x\|)\, dx = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \hat{f}_{1jk\varepsilon\eta}(b_0, n). \tag{3.30b}$$

Lemmas 3.10 and 3.12 give

$$|\hat{f}_{1jk\varepsilon\eta}(b_0, n)| \leq B_{jkn} \mathscr{N}_n(f),$$

the finiteness of C_{nmq}, and the convergence, as $\varepsilon \to 0^+, \eta \to 0^+$, of the individual terms in the series. This proves the boundedness and convergence of (3.30b), as asserted in the theorem. □

By paying closer attention to the details of the proof it is possible to give an explicit formula for C_{nmq} and to provide a uniform estimate on the rate of convergence of (3.30b) in terms of $\mathscr{N}_n(f)$ as $\varepsilon \to 0^+, \eta \to 0^+$.

4. Fourier theory for spaces with double roots

In this section $S = U/K$ is one of the projective spaces $P_l(\mathbf{C})$, $P_l(\mathbf{H})$ $(l \geq 2)$, or $P_2(\mathbf{Cay})$. As in Section 1 write $\mathfrak{g}$ for the complex Lie algebra of U. Again we have the complexified Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}.$$

The positive restricted roots of $(\mathfrak{g}, \mathfrak{a})$ are $\{\alpha, 2\alpha)$. There is associated with 2α a θ-stable subgroup U_2 of U which plays a central role in the definition and study of the important map

$$\xi: S \to \mathbf{R}^{q+1}$$

mentioned in the Introduction. Most of this section is devoted to this study. Then it is shown how ξ reduces the hardest problem of this section to Theorem 3.1.

The main ideas can be expressed briefly as follows. Write $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ where $\mathfrak{n}_j$ is the $\mathrm{ad}(\mathfrak{a})$ eigenspace corresponding to the restricted root $j\alpha$ $(j=1,2)$. Let $\mathfrak{g}_2$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}_2$ and $\theta(\mathfrak{n}_2)$. In the $\mathfrak{g}_2$-submodule $V(S)$ of $\mathscr{H}_1(S)$ generated by $e(b_0, 1)$ find an orthogonal basis of real-valued functions $\xi_1, \ldots, \xi_{q+1}$. Choose these functions so that

$$e(b_0, 1) = \xi_1 + i\xi_2$$

and also such that, as members of $L_2(S)$, they all have the same length. (This is possible by Lemma 4.2.) Then

$$\xi=(\xi_1,\ldots,\xi_{q+1})\colon S\to\Omega\subset\mathbf{R}^{q+1}$$

is our desired map.

Since $e(b_0,n)=e(b_0,1)^n$, both $e(b_0,n)$ and $e_*(b_0,n)$ can be factored through ξ. It is as important, but less obvious, that

$$\cos(\delta(s))=\xi_1(s)+\|\xi(s)\|-1 \quad (s\in S) \tag{4.1}$$

where $\delta(s)$ is the distance from s_0 to s on S. From this formula a cluster of facts emerge:

First,

$$S(\varepsilon,\eta)=\xi^{-1}(\Omega(\varepsilon,\eta)) \quad (0<\varepsilon,\eta)$$

(where $S(\varepsilon,\eta)$ is defined in Definition 1.11 and $\Omega(\varepsilon,\eta)$ is defined between (3.1) and (3.2).)

Second, ξ carries the measure on S to a measure on Ω which can be computed using (4.1). We show its element to be $w(\|x\|)\,dx$ where w is a normalized version of the weight function used in Section 3:

$$w(r)=cr^{-1}(1-r)^m, \quad m=(\dim(\mathfrak{n}_1)-2)/2.$$

Closely related is the map E which carries functions from S to Ω. It is defined in Lemma 4.15 and is essentially a conditional expectation.

Third, and at a deeper level, but again depending on (4.1), is the relation

$$\Delta_S(f\circ\xi)=(\Lambda f)\circ\xi \quad (f\in C^\infty(\Omega))$$

where Λ is given in (3.3). From this we also get

$$E(\Delta_S f)=\Lambda E(f) \quad (f\in C^\infty(S)).$$

With this machinery developed in sufficient detail it is fairly easy to use E to reduce the problem of defining $\hat{f}(b,n)$ to Theorem 3.1.

From the preceeding it is clear that our first priority is to nail down the definition of ξ and prove (4.1). This occupies Lemmas 4.1 through 4.12. Then in Lemmas 4.13 to 4.34 we develop the facts mentioned above (and others), culminating in Theorem 4.35, the Main theorem for spaces of this section. As a byproduct we get Theorem 4.22

which gives a complete description of the eigenfunctions of Λ on Ω; this is a pretty subject with interesting ties, parallels and contrasts to classical orthogonal polynomial theory. We end the section by finishing the proof of Theorem 1.13.

Let $\mathfrak{u}_2=\mathfrak{u}\cap\mathfrak{g}_2$.

LEMMA 4.1. *$\mathfrak{u}_2$ is a real form of $\mathfrak{g}_2$. $\mathfrak{u}_2$ is the real Lie algebra of a closed, connected, θ-stable subgroup U_2 of U. If $\mathfrak{m}$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ then*

$$\mathfrak{g}_2=\mathfrak{n}_2+[\mathfrak{n}_2,\theta(\mathfrak{n}_2)]+\theta(\mathfrak{n}_2) \tag{4.2}$$

$$[\mathfrak{n}_2,\theta(\mathfrak{n}_2)]=\mathfrak{a}+(\mathfrak{g}_2\cap\mathfrak{m}). \tag{4.3}$$

Proof. Clearly $\supset$ holds in (4.2). On the other hand, the right side of (4.2) is a subalgebra as can be seen from

$$[\mathfrak{n}_2,\theta(\mathfrak{n}_2)]\subset\mathfrak{a}+(\mathfrak{g}_2\cap\mathfrak{m})$$

and

$$[\mathfrak{g}_2\cap\mathfrak{m},\mathfrak{n}_2]\subset\mathfrak{n}_2.$$

From this, $=$ holds in (4.2). Then both sides of (4.3) describe the centralizer of $\mathfrak{a}$ in $\mathfrak{g}_2$ so (4.3) holds.

Let σ denote conjugation on $\mathfrak{g}$ with respect to the real form $\mathfrak{u}$. Since α is pure imaginary on $\mathfrak{u}\cap\mathfrak{a}$ we have $\sigma(\mathfrak{n}_2)=\theta(\mathfrak{n}_2)$. Thus $\mathfrak{g}_2$ could also be defined as the smallest complex subalgebra of $\mathfrak{g}$ which contains $\mathfrak{n}_2$ and is σ-stable. Thus $\mathfrak{g}_2$ is the complexification of $\mathfrak{u}_2$.

The subgroup U_2 of U corresponding to $\mathfrak{u}_2$ is a semisimple connected subgroup of a compact group and so must be compact. The θ-stability of U_2 follows from that of $\mathfrak{u}_2$. □

Let $V=V(S)$ denote the complex $\mathfrak{g}_2$-submodule of $\mathscr{H}_1(S)$ generated by $e(b_0,1)$.

LEMMA 4.2. *The set V_r of real-valued functions in V is a real form of V. If we write*

$$e(b_0,1)=\xi_1+i\xi_2$$

where ξ_1,ξ_2 are real, then they belong to V_r and as elements of $L^2(S)$ they have the same length and are perpendicular.

Proof. Since V could also be described as the $\mathfrak{u}_2$-submodule of $\mathscr{H}_1(S)$ generated by $e(b_0,1)$, and since V_r is $\mathfrak{u}_2$-stable, we can show that V is V_r+iV_r by showing that $\overline{e(b_0,1)}$

is in V. Take any $y\neq 0$ in $\theta(\mathfrak{n}_2)$. Then $D_y^2 e(b_0, 1)$ and $\overline{e(b_0, 1)}$ both belong to the one-dimensional lowest weight space in V, proving that they are proportional. Thus ξ_1, $\xi_2 \in V_r$.

To prove the geometric assertion, take $x \in \mathfrak{a} \cap \mathfrak{u}$ such that $2\alpha(x) = -i$. Then

$$D_x(\xi_1 + i\xi_2) = D_x(e(b_0, 1)) = ie(b_0, 1) = -\xi_2 + i\xi_1.$$

The assertion follows from the skew-symmetry of the operator D_x. □

Definition 4.3. Extend the set $\{\xi_1, \xi_2\}$ of functions from Lemma 4.2 to an orthogonal basis $\{\xi_1, \ldots, \xi_{q+1}\}$ of V_r such that each ξ_j has the same L^2 length. Define $\xi: S \to \mathbf{R}^{q+1}$ as the map which has these functions as its components:

$$\xi(s) = (\xi_1(s), \ldots, \xi_{q+1}(s)) \quad (s \in S).$$

Remarks. (1) Lemma 4.2 shows that such a basis exists. (2) ξ_1 and ξ_2 are given but for the other ξ_j there is some freedom of choice subject to the requirement of orthogonality and equal length in V_r. Any choice will do; it turns out that all choices are conjugate under $U_2 \cap K$. In particular, we will show right away that $\|\xi(s)\|$ does not depend on the choice of the ξ_j, $j = 3, \ldots, q+1$.

LEMMA 4.4. *If $\mathcal{B} = \{\beta_1, \ldots, \beta_{q+1}\}$ is any orthonormal basis of V_r then the function*

$$f_{\mathcal{B}}(s) = \beta_1^2(s) + \ldots + \beta_{q+1}^2(s) \quad (s \in S)$$

is independent of the choice of the particular orthonormal basis $\mathcal{B}$.

Proof. This is an elementary general fact about any finite dimensional vector space of real-valued functions with an inner-product $\langle\, , \rangle$. For the proof, express $f_{\mathcal{B}}$ in a basis-independent way by defining h_s (for $s \in S$) to be the unique element of V_r such that

$$\langle h_s, v \rangle = v(s) \quad (v \in V_r).$$

Then $f_{\mathcal{B}}(s) = \langle h_s, h_s \rangle$ since both sides equal $\Sigma_{j=1}^{q+1} \langle \beta_j, h_s \rangle^2$. □

COROLLARY 4.5. *$\|\xi(s)\|$ is independent of the choice of ξ_j, $j = 3, \ldots, q+1$.*

Proof. For some real $c \neq 0$, $\beta_j = c\xi_j$ $(j = 1, \ldots, q+1)$ is an orthonormal basis $\mathcal{B}$ of V_r. Thus $\|\xi(s)\| = c^{-1} f_{\mathcal{B}}(s)^{1/2}$. □

The next goal is to prove (4.1). The idea is to reduce the proof to the case of $P_2(\mathbf{C})$

where it follows from a calculation in homogeneous coordinates. (The same calculation works in the other classical spaces and the reader may choose to pass over the oddly long reduction to $P_2(\mathbf{C})$ and consider only the calculation. Nevertheless it seemed important to give this argument to cover $P_2(\mathbf{Cay})$ and in anticipation of generalization to higher rank; even if the Helgason–Fourier transform itself does not generalize to higher rank, much of the analysis in this section regarding ξ may.)

There are two steps in the reduction to $P_2(\mathbf{C})$. Both steps use Helgason's $SU(1,2)$ theorem which is recorded here as Lemma 4.7. The first step shows that S is covered by totally geodesic copies $S_\#$ of $P_2(\mathbf{C})$ all of which share the geodesic $\exp(\mathfrak{a}\cap\mathfrak{u})\, s_0$. The second step (for which our proof seems too long) shows that $V(S)|S_\# = V(S_\#)$.

First introduce a family $\mathfrak{B}_\#$ of special copies of $\mathfrak{sl}(3,\mathbf{C})$ in $\mathfrak{g}$:

Definition 4.6. $\mathfrak{B}_\#$ denotes the family of all θ-stable complex subalgebras $\mathfrak{g}_\#$ of $\mathfrak{g}$ such that

(i) $\mathfrak{g}_\#$ is isomorphic to $\mathfrak{sl}(3,\mathbf{C})$ in such a way that $\theta|\mathfrak{g}_\#$ corresponds to conjugation of $\mathfrak{sl}(3,\mathbf{C})$ by the diagonal matrix $J=e_{11}-e_{22}-e_{33}$.

(ii) $\mathfrak{g}_\#\cap\mathfrak{u}$ is a real form $\mathfrak{u}_\#$ of $\mathfrak{g}_\#$ (necessarily isomorphic to $\mathfrak{su}(3)$).

(iii) $\mathfrak{g}_\#\supset\mathfrak{a}$.

Let σ denote conjugation on $\mathfrak{g}$ with respect to the real form $\mathfrak{u}$. (ii) is equivalent to the condition that $\sigma(\mathfrak{g}_\#)=\mathfrak{g}_\#$.

LEMMA 4.7. *Suppose that there are nonzero x_1, x_2 in $\mathfrak{g}$ such that*

$$x_j\in\mathfrak{n}_j,\quad \theta\sigma(x_j)=x_j\quad (j=1,2).$$

Then $\{x_1,x_2,\theta(x_1),\theta(x_2)\}$ generates a member $\mathfrak{g}_\#$ of $\mathfrak{B}_\#$.

This is a corollary of Helgason's $SU(1,2)$ theorem. That theorem speaks of the real form $\mathfrak{g}_0$ of $\mathfrak{g}$ which is dual to $\mathfrak{u}$ in the sense of symmetric space duality. $\mathfrak{g}_0$ is the fixed-point set of $\theta\sigma$. The x_1, x_2 of Lemma 4.7 lie in $\mathfrak{g}_0$. The $SU(1,2)$ theorem asserts that they generate an isomorph $\mathfrak{g}_{\#0}$ in $\mathfrak{g}_0$ of $\mathfrak{su}(1,2)$ with $\theta|\mathfrak{g}_{\#0}$ corresponding to conjugation by J as in (i) of Definition 4.6. (ii) is clear because

$$i(x_j-\sigma(x_j)),\quad x_j+\sigma(x_j)\quad (j=1,2)$$

generate a σ-stable real form of $\mathfrak{g}_\#$. (iii) is clear from $0\neq[x_1,\theta(x_1)]\in\mathfrak{a}$.

For each $\mathfrak{g}_\#$ in $\mathfrak{B}_\#$ we have $\mathfrak{u}_\#$ and the corresponding compact, connected, θ-stable subgroup $U_\#$ of U. $K_\#=K\cap U_\#$ has its Lie algebra isomorphic to $\mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(2))$.

Thus the orbit $S_\# = U_\# s_0$ is a totally geodesic submanifold of S, isomorphic to $U_\#/K_\# \cong P_2(\mathbf{C})$. Let us call $S_\#$ the *trajectory* of $\mathfrak{g}_\#$.

LEMMA 4.8. *Every point in S lies in the trajectory of some* $\mathfrak{g}_\#$ *in* $\mathfrak{B}_\#$.

Proof. Take any $s_\#$ in S and x in $\mathfrak{u}$ with $\theta(x) = -x$ such that $s_\# = \exp(x)\, s_0$. We need only show that x lies in some $\mathfrak{g}_\# \in \mathfrak{B}_\#$. Write

$$x = x_{-2} + x_{-1} + x_0 + x_1 + x_2$$

where

$$x_0 \in \mathfrak{a}, \quad x_j \in \mathfrak{n}_j, \quad x_{-j} \in \theta(\mathfrak{n}_j) \quad (j = 1, 2).$$

Then

$$\theta(x_j) = -x_{-j}, \quad \sigma(x_j) = x_{-j} \quad (j = \pm 1, \pm 2).$$

Consequently

$$\theta\sigma(x_j) = -x_j \quad (j = \pm 1, \pm 2) \tag{4.4}$$

It may be that one or more of the x_j is 0. If so, replace it with a nonzero x_j in $\mathfrak{n}_j$ satisfying (4.4).

Now Helgason's $SU(1,2)$ theorem, as Lemma 4.7, shows that

$$\{ix_1, ix_2, i\theta(x_1), i\theta(x_2)\}$$

generates an element $\mathfrak{g}_\#$ of $\mathfrak{B}_\#$ which contains x. □

Take any $\mathfrak{g}_\#$ in $\mathfrak{B}_\#$ with trajectory $S_\#$. Since $S_\# \cong P_2(\mathbf{C})$ it makes sense to speak of $V(S_\#)$. In fact we have

$$\mathfrak{g}_\# = \mathfrak{k}_\# + \mathfrak{a} + \mathfrak{n}_\#, \quad \mathfrak{k}_\# = \mathfrak{g}_\# \cap \mathfrak{k}, \quad \mathfrak{n}_\# = \mathfrak{g}_\# \cap \mathfrak{n}$$

and $e(b_0, 1)|S_\#$ is the $e(b_0, 1)$ of $S_\#$. Moreover, if we write

$$\mathfrak{n}_\# = \mathfrak{n}_{\#1} + \mathfrak{n}_{\#2}, \quad \mathfrak{n}_{\#j} = \mathfrak{g}_\# \cap \mathfrak{n}_j \quad (j = 1, 2)$$

and $\mathfrak{g}_{\#2} = \mathfrak{g}_\# \cap \mathfrak{g}_2$ then $\mathfrak{g}_{\#2}$ is generated by $\mathfrak{n}_{\#2}$ and $\theta(\mathfrak{n}_{\#2})$. As before, $V(S_\#)$ is the $\mathfrak{g}_{\#2}$-submodule of $\mathscr{H}_1(S_\#)$ generated by $e(b_0, 1)|S_\#$. Write

$$V(S)|S_\# = \{f|S_\# \mid f \in V(S)\}.$$

Note that $V(S_{\#}) \subset V(S)|S_{\#}$ since the latter contains $e(b_0, 1)|S_{\#}$ and is $\mathfrak{g}_{\#2}$-stable.

LEMMA 4.9. $V(S_{\#}) = V(S)|S_{\#}$. *As a* $\mathfrak{g}_{\#2}$*-module, this space is isomorphic to* $(\mathfrak{g}_{\#2}, \mathrm{ad})$ *and* $\mathfrak{g}_{\#2}$ *is isomorphic as a Lie algebra to* $\mathfrak{sl}(2, \mathbf{C})$. *Thus* $\dim(V(S_{\#})) = 3$.

Proof. This proof has two steps as follows:

Step 1. $V(S)|S_{\#} \subset \mathcal{H}_1(S_{\#}) \cong (\mathfrak{g}_{\#}, \mathrm{ad})$.

Step 2. $V(S_{\#})^{\perp} \cap V(S)\,|S_{\#} = 0$.

In each step we show that $\mathfrak{g}_{\#2}$-trival functions which seem as though they could occur in $V(S)|S_{\#}$ really cannot. Step 1 eliminates the possibility that $V(S)|S_{\#}$ contains constants. This leads to a $\mathfrak{g}_{\#2}$-module monomorphism of $V(S)|S_{\#}$ into $\mathfrak{g}_{\#2} + \mathfrak{m}_{\#}$. Step 2 eliminates the $\mathfrak{m}_{\#}$ part. The conclusion follows from $\mathfrak{g}_{\#2} \cong V(S_{\#})$.

For Step 1 first note that

$$\mathcal{H}_1(S)|S_{\#} \subset \mathcal{H}_0(S_{\#}) + \mathcal{H}_1(S_{\#}).$$

(This is clear from weight theory since the highest restricted weight of $\mathfrak{a}$ in $\mathcal{H}_1(S)|S_{\#}$ is 2α.) To complete Step 1 it must be shown that for any $f \in V(S)$,

$$\int_{S_{\#}} f(s)\, ds = 0.$$

There is, in any case, a unique function ψ in $\mathcal{H}_1(S)$ such that

$$\int_{S_{\#}} f(s)\, ds = \int_{S} f(s)\, \psi(s)\, ds \quad (f \in \mathcal{H}_1(S)).$$

From the $U_{\#}$-invariance of the measure on $S_{\#}$ and S, ψ is $U_{\#}$-invarient, i.e. $D_{\mathfrak{g}_{\#}} \psi = 0$. From this it can be shown that ψ is U_2-invariant. To do this it suffices to prove

$$D_{x_2} \psi = 0 = D_{\theta x_2} \psi \quad (x_2 \in \mathfrak{n}_2, \theta\sigma(x_2) = x_2)$$

since all such $x_2, \theta(x_2)$ generate $\mathfrak{g}_2$. Take such an x_2 and take also $x_1 \neq 0$ in $\mathfrak{n}_1 \cap \mathfrak{g}_{\#}$ satisfying $\theta\sigma(x_1) = x_1$. Then

$$D_{x_1} \psi = 0 = D_{\theta x_1} \psi \quad \text{and} \quad D_{\mathfrak{a}} \psi = 0.$$

By the $SU(1,2)$ theorem (Lemma 4.7), $\{x_1, x_2, \theta(x_1), \theta(x_2)\}$ generates a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(3, \mathbf{C})$. The only representations of $\mathfrak{s}$ which can occur in $\mathcal{H}_1(S)$ are (1) the trivial representation, (2) the natural representation of $\mathfrak{sl}(3, \mathbf{C})$ on $\mathbf{C}^3$, (3) the contragredient of (2), (4) the adjoint representation. In each of these representations we may verify directly that if x_1 and $\theta(x_1)$ kill a vector then so does the rest of $\mathfrak{sl}(3,\mathbf{C})\cong\mathfrak{s}$. Apply this to the component of ψ in each irreducible $\mathfrak{s}$-submodule of $\mathscr{H}_1(S)$ to see that $D_{\mathfrak{g}_2}\psi=0$. From this,

$$\int_{S_\#} V(S)=\int_S V(S)\,\psi=\int_S (D_{\mathfrak{g}_2}V(S))\,\psi=\int_S V(S)\,D_{\mathfrak{g}_2}\psi=0$$

proving $V(S)|S_\#\subset\mathscr{H}_1(S_\#)$.

Note the $\mathfrak{g}_\#$-module isomorphism $\iota\colon\mathscr{H}_1(S_\#)\to(\mathfrak{g}_\#,\mathrm{ad})$. (Both representations are irreducible, $\mathfrak{g}_\#$ contains a $K_\#$-invariant in the center of $\mathfrak{k}$ and it has the same highest restricted weight, 2α, as $\mathscr{H}_1(S_\#)$. Thus the two are isomorphic.) This completes Step 1.

In $(\mathfrak{g}_\#,\mathrm{ad})\cong(\mathfrak{sl}(3,\mathbf{C}),\mathrm{ad})$, the highest weight space is the one dimensional double root space $\mathfrak{n}_{\#2}$, i.e., it is $\iota(\mathbf{C}e(b_0,1)|S_\#)$. Consequently,

$$\iota(V(S_\#))=\mathfrak{g}_{\#2}$$

since both sides are irreducible $\mathfrak{g}_{\#2}$-modules and $\iota|V(S_\#)$ is a $\mathfrak{g}_{\#2}$-module isomorphism. The only eigenvalues of $\mathfrak{a}$ on $V(S)$ are $2\alpha,0,-2\alpha$. $\pm2\alpha$ each occur with multiplicity 1. Thus

$$\mathfrak{g}_{\#2}\subset\iota(V(S)|S_\#)\subset\mathfrak{g}_{\#2}+\mathfrak{m}_\#,$$

where $\mathfrak{m}_\#$ is the one-dimensional centralizer of $\mathfrak{a}$ in $\mathfrak{k}_\#$. Now suppose in contradiction to the claim of Step 2 that there were f in $V(S)$ such that

$$0\neq\iota(f|S_\#)\in\mathfrak{m}_\#. \tag{4.5}$$

To show that this cannot occur, take $x_1\neq0$ in $\mathfrak{n}_{\#1}$ such that $\theta\sigma(x_1)=x_1$. (For example, in the isomorphism of $\mathfrak{g}_\#$ with $\mathfrak{sl}(3,\mathbf{C})$ take x_1 corresponding to $i(e_{31}+e_{13})-(e_{32}+e_{23})$.) Then $\mathrm{ad}(\theta(x_1))^2\mathfrak{n}_{\#2}=\mathfrak{m}_\#$ so except for a constant that we may safely ignore,

$$f|S_\#=D^2_{\theta(x_1)}e(b_0,1)|S_\#.$$

On the other hand, since $f\in V(S)$, and we are assuming (4.5), we must have $f=D_{\theta(x_2)}e(b_0,1)$ for some $x_2\in\mathfrak{n}_2$. Consequently,

$$\int_{S_\#}|f|^2\,ds=\int_{S_\#}D^2_{\theta(x_1)}e(b_0,1)\,\overline{D_{\theta(x_2)}e(b_0,1)}\,ds. \tag{4.6}$$

Use $\theta(x_1)=\sigma(x_1)$ and the fact that the formal adjoint of D_x is $-D_{\sigma(x)}$ for any $x \in \mathfrak{g}_\#$; also use

$$D^2_{x_1} D_{\theta(x_2)} e(b_0, 1) = D_y e(b_0, 1), \quad y = [x_1, [x_1, \theta(x_2)]]$$

to see that the right side of (4.6) is

$$\int_{S_\#} e(b_0, 1) \overline{D_y e(b_0, 1)} \, ds.$$

But this y is in $\mathfrak{m}$ and so kills $e(b_0, 1)$. Thus the left side of (4.6) is 0 contradicting (4.5) and completing Step 2. □

LEMMA 4.10. *Let $\mathfrak{g}_\#$ and $S_\#$ be as in Lemma* 4.9. *Use Definition* 4.3 *to define the map $\xi: S_\# \to \mathbf{R}^3$ and denote this map by $\xi_\#$ Then*

$$\|\xi_\#(s)\| = \|\xi(s)\| \quad (s \in S_\#).$$

Proof. Let $V_\#$ denote the $\mathfrak{g}_{\#2}$-submodule of $V(S)$ generated by $e(b_0, 1)$ and V_0 its orthocompliment. Then, by the previous lemma, the restriction map to $S_\#$ is a isomorphism from $V_\#$ to $V(S_\#)$. Moreover, the kernel of this restriction map, as a map from $V(S)$, is V_0. From Corollary 4.5, $\|\xi(s)\|$ is independent of the choice of $\xi_3, \ldots, \xi_{q+1}$. Choose these so that $\xi_3 \in V_\#$ and for $j=4, \ldots, q+1$, $\xi_j \in V_0$. Then

$$\xi_\#(s) = (\xi_1(s), \xi_2(s), \xi_3(s)) \quad (s \in S_\#)$$

from which the conclusion is immediate. □

The next step in the proof of (4.1) is to show that it holds for $P_2(\mathbf{C})$. For this purpose express $P_2(\mathbf{C})$ in terms of homogeneous coordinates: Think of it as the unit sphere S_5 in $\mathbf{C}^3$ modulo multiplication by $\{u \in \mathbf{C} \,|\, |u|=1\}$. Denote the equivalence class of $z \in S_5$ by $[z]$. Take $s_0=[(1,0,0)]$.

Recall from the Introduction that e_{jk} denotes a matrix with all entries 0 but for a 1 at the jk-spot. For the space $U/K=S=P_2(\mathbf{C})$ we have $\mathfrak{u}=\mathfrak{su}(3)$. θ is conjugation by $J=e_{11}-e_{22}-e_{33}$. Take

$$\mathfrak{a} = \mathbf{C}x_0, \quad x_0 = (e_{21}-e_{12}),$$

$$\mathfrak{n}_1 = \mathbf{C}(e_{31}+ie_{32})+\mathbf{C}(e_{13}+ie_{23}),$$

$$\mathfrak{n}_2 = \mathbf{C}(e_{11}-e_{22}+i(e_{12}+e_{21})),$$

$$\begin{aligned} \alpha(x_0) &= -i, \\ e(b_0, 1)([z]) &= (z_1+iz_2)(\bar{z}_1+i\bar{z}_2) \\ &= |z_1|^2-|z_2|^2+i(z_1\bar{z}_2+\bar{z}_1 z_2) \quad (z \in S_5). \end{aligned} \tag{4.7}$$

(It is not hard to check that (4.7) defines a function on $P_2(\mathbf{C})$ and satisfies

$$D_{x_0} e(b_0, 1) = -2\alpha(x_0)\, e(b_0, 1)$$

$$D_{\mathfrak{n}_1+\mathfrak{n}_2} e(b_0, 1) = 0).$$

The distance function δ on $P_2(\mathbf{C})$ (which, recall, measures geodesic distance from s_0 and is normalized so that $\max(\delta)=\pi$) is given by

$$\delta([z]) = 2\arccos(|z_1|) \tag{4.8}$$

since this is K-invariant and gives the correct value, $2t$, at

$$\exp(tx_0) = [(\cos(t), \sin(t), 0)] \quad (0 \leqslant t \leqslant \pi/2).$$

LEMMA 4.11. *For $S=P_2(\mathbf{C})$, $\xi([z])$ is given for $z \in S_5$ by*

$$\xi_1([z]) = |z_1|^2-|z_2|^2,$$

$$\xi_2([z]) = z_1\bar{z}_2+z_2\bar{z}_1,$$

$$\xi_3([z]) = i(z_1\bar{z}_2-z_2\bar{z}_1).$$

In particular $\xi(S)$ is the closed unit ball Ω in $\mathbf{R}^3$ and (4.1) *holds for $P_2(\mathbf{C})$.*

Proof. The values of ξ_1 and ξ_2 are set by (4.7) and $e(b_0, 1)=\xi_1+i\xi_2$. The alleged value of ξ_3 is obtained by translating ξ_2 by an element of U_2:

$$\xi_3(s) = \xi_2(us), \quad u = \omega e_{11}+\bar{\omega} e_{22}+e_{33} \in U_2, \quad \omega = i^{1/2}.$$

Thus this ξ_3 is in $V(S)$ and has the same L^2 length as ξ_2. The orthogonality of ξ_3 to ξ_1 and ξ_2 follows from the more easily verified

$$0 = \int_{S_5} (z_1\bar{z}_2)^2\, ds, \quad 0 = \int_{S_5} |z_1|^2 z_1\bar{z}_2\, ds, \quad \text{etc.}$$

A simple calculation now shows

$$\|\xi([z])\|^2 = (|z_1|^2+|z_2|^2)^2 \leqslant 1 \quad (z \in S_5).$$

This proves that $\xi(S)\subset\Omega$ and, with (4.8) it gives

$$\xi_1([z])+\|\xi([z])\| = 2|z_1|^2 = \cos(\delta([z]))+1$$

proving (4.1) for $P_2(\mathbf{C})$.

Finally, for any $x\in\Omega$ we can solve $x=\xi([z])$ for z by taking

$$\begin{aligned} z_1 &= ((x_1+\|x\|)/2)^{1/2} \\ z_2 &= \begin{cases} 0 & \text{if } z_1=0 \\ (x_2+ix_3)/2z_1 & \text{if } z_1\neq 0 \end{cases} \\ z_3 &= (1-|z_1|^2-|z_2|^2)^{1/2} \end{aligned}$$

proving that $\xi(S)=\Omega$. □

Remark. For any projective space over $\mathbf{K}=\mathbf{C}$ or $\mathbf{H}$ we again have (4.7) and (4.8). Then it is best to regard ξ as mapping to $\mathbf{R}\oplus\mathbf{K}$ by

$$\xi([z]) = (|z_1|^2-|z_2|^2, 2z_1\bar{z}_2).$$

In this formulation, Lemma 4.11 proves (4.1) directly for these spaces. Despite this, it seemed worth the effort to develop a proof of (4.1) which was more intrinsic.

LEMMA 4.12. *For each of the spaces S of this section there is a representation π of U_2 on $\mathbf{R}^{q+1}$ such that*

(i) $\pi(U_2)=SO(q+1)$;

(ii) $\pi(u)\xi(s)=\xi(us)\,(u\in U_2, s\in S)$;

(iii) *π is equivalent to the natural representation of U_2 on $V_r(S)$ (=the real-valued functions in $V(S)$);*

Moreover,

(iv) (4.1) *holds;*

(v) $q=1+\dim_{\mathbf{C}}(\mathfrak{n}_2)$;

(vi) $\xi(S)=\Omega$, *the closed unit ball in* $\mathbf{R}^{q+1}$;

(vii) $\xi^{-1}(\partial\Omega)=U_2 s_0$;

(viii) $\xi\colon U_2 s_0\to\partial\Omega$ *is a diffeomorphism;*

(ix) $\Omega_0=\text{interior}(\Omega\setminus\{0\})$ *is the set of regular values of* ξ.

Proof. The existence of π satisfying (ii) and (iii) is automatic from the fact that

$$\{\xi_1,\dots,\xi_{q+1}\}$$

is an equal-length orthogonal basis of V_r. With $K_2 = U_2 \cap K$, U_2/K_2 is clearly a rank one symmetric space without double restricted roots; thus U_2/K_2 is either a sphere or real projective space. The latter is ruled out since $U_2/K_2 \cong U_2 s_0$ has on it the $\mathfrak{a}+\mathfrak{n}_2$ eigenfunction $e(b_0, 1)|U_2 s_0$. This corresponds to the highest weight 2α which is also the restricted root. This cannot happen on $P_l(\mathbf{R})$. This also shows that π is the fundamental class 1 representation of U_2, proving (i).

To prove (4.1), take any $s \in S$. By Lemma 4.8 choose $\mathfrak{g}_\# \in \mathfrak{B}_\#$ whose trajectory $S_\#$ contains s. Let $\xi_\#$ be the ξ of $S_\#$ as in Lemma 4.10. Then $\|\xi_\#(s)\| = \|\xi(s)\|$ by that lemma. Also,

$$\xi_{\#1}(s) = \Re e(b_0, 1)(s) = \xi_1(s).$$

Since $S_\#$ is totally geodesic in S and contains s_0, the distance function $\delta_\#$ of $S_\#$ is just $\delta|S_\#$ where δ is the distance function for S. (We also need to check that the two distance functions are normalized so as to have the same maximum value, π. This follows easily from the rank one geometry, since all complete geodesics in S are closed and have the same length.) In particular, $\delta_\#(s) = \delta(s)$. Consequently, (4.1) for S reduces to (4.1) for $S_\#$ and that in turn is settled in Lemma 4.11.

The same argument shows that $\|\xi\|$ can take on all values in $[0, 1]$ and only those values since the same is true of $\xi_\#$. Since $\xi(S)$ is $SO(q+1)$-invariant (by (i) and (ii), it must be Ω, proving (vi).

(v) follows from

$$q = \dim(U_2/K_2 = \dim_{\mathbf{C}}(\mathfrak{a}+\mathfrak{n}_2) = 1 + \dim_{\mathbf{C}}(\mathfrak{n}_2).$$

(vii) follows from the reduction to $S_\#$ and Lemma 4.11 which shows that

$$\|\xi(s)\| = \|\xi_\#(s)\| = 1 \quad \Leftrightarrow \quad s \in (U_2 \cap U_\#) s_0.$$

For (viii), both $U_2 s_0$ and $\partial\Omega$ are diffeomorphic to U_2/K_2, the latter via (i) and (ii).

It is clear that $\partial\Omega$ and 0 are not regular values. But for $x \in \Omega_0$ find $s \in \xi^{-1}(x)$ by (vi). (i) and (ii) make clear that the image of $d\xi$ at s contains at least the subspace $x^\perp$. The reduction to $S_\#$ and Lemma 4.11 show that the image of $d\xi$ also contains the line $\mathbf{R}x$; thus $d\xi$ is onto at s proving that it is a regular point. Since the argument applies to all s in $\xi^{-1}(x)$, x is a regular value, proving (ix). □

Lemma 4.12 is the foundation stone for our theory of ξ. Now we begin to erect that theory. Recall $S(\varepsilon, \eta)$ from Definition 1.11 and $\Omega(\varepsilon, \eta)$ from the beginning of Section 3.

LEMMA 4.13. $\xi^{-1}(\Omega(\varepsilon,\eta))=S(\varepsilon,\eta)$.

Proof.

$$S(\varepsilon,\eta)=\{s\in S \mid |\xi_1(s)|\geqslant\eta, \cos(\delta(s))\geqslant -1+\varepsilon\}$$
$$=\{s\in S \mid |\xi_1(s)|\geqslant\eta, \xi_1(s)+\|\xi(s)\|\geqslant\varepsilon\}$$

by (4.1). But this is precisely $\xi^{-1}(\Omega(\varepsilon,\eta))$. □

LEMMA 4.14. *Let* $m=(\dim(\mathfrak{n}_1)-2)/2$ *and*

$$w(r)=cr^{-1}(1-r)^m \quad (0<r\leqslant 1) \tag{4.9}$$

where c is chosen to give $\int_\Omega w(\|x\|)\,dx=1$. *Then for all* $f\in C(\Omega)$,

$$\int_S f\circ\xi(s)\,ds=\int_\Omega f(x)\,w(\|x\|)\,dx.$$

In other words, ξ carries the measure on S over to the measure on Ω whose element is $w(\|x\|)\,dx$.

Proof. Let μ denote the U-invariant measure on S, normalized as usual so that $\mu(S)=1$. Then $\mu\circ\xi^{-1}$ is a measure on Ω whose Radon-Nykodym derivative with respect to Lebesgue measure on Ω is

(1) smooth on Ω_0 (by (ix) of Lemma 4.12);

(2) rotation invariant (by (i), (ii) of Lemma 4.12).

Thus there is a smooth function w on $(0,1)$ such that

$$d\mu\circ\xi^{-1}=w(\|x\|)\,dx.$$

To prove that w is given by (4.9) recall the Cartan decomposition formula for the measure μ on U/K ([5], Theorem 5.10, p. 190):

$$\mu(\{s\in S \mid \delta(s)\leqslant t\})=c_1\int_0^t \sin^{m_1}(\theta/2)\sin^{m_2}(\theta)\,d\theta \quad (0<t<\pi)$$

where $m_j=\dim_{\mathbf{C}}(\mathfrak{n}_j)\,(j=1,2)$. By (4.1) and the definition of w this is the integral of $w(\|x\|)\,dx$ over

$$\{x\in\Omega \mid x_1+\|x\|-1>\cos(t)\}. \tag{4.10}$$

The latter integral may be simplified by means of a map

$$\Omega \ni x \mapsto y \in \Omega_1 = \Omega + (1, 0, \ldots, 0)$$

$$y_1 = x_1 + \|x\|, \quad y_j = x_j \quad (j = 2, \ldots, q+1).$$

If we write $\mathcal{I}$ for the line segment

$$\{(t, 0, \ldots, 0) \in \Omega \mid -1 \leqslant t \leqslant 0\}$$

then this map is easily seen to be a diffeomorphism from $\Omega \setminus \mathcal{I}$ to $\Omega_1 \setminus \{0\}$. The inverse map $y \mapsto x$ has

$$x_1 = (y_1^2 - y_2^2 - \ldots - y_{q+1}^2)/2y_1, \quad x_j = y_j \quad (j = 2, \ldots, q+1).$$

It is easy to see that

$$\|x\| = \|y\|^2/2y_1 \quad \text{and} \quad dx = (\|y\|^2/2y_1^2)\, dy$$

so the measure $w(\|x\|)\, dx$ on Ω goes over to

$$w(\|y\|^2/2y_1)\,(\|y\|^2/2y_1^2)\, dy \tag{4.11}$$

on Ω_1. (4.10) corresponds in Ω_1 to

$$\{y \in \Omega_1 \mid y_1 - 1 \geqslant \cos(t)\}. \tag{4.12}$$

If we write $v = y_1$, $u = (y_2^2 + \ldots + y_{q+1}^2)^{1/2}$ then the integral of (4.11) over (4.12) is

$$c_2 \int_{v=\cos(t)+1}^{2} \int_{u=0}^{(2v-v^2)^{1/2}} w((u^2+v^2)/2v)\,((u^2+v^2)/2v)\,((u^2+v^2)/2v^2)\, u^{q-1}\, du\, dv$$
$$= c_1 \int_0^t \sin^{m_1}(\theta/2) \sin^{m_2}(\theta)\, d\theta.$$

Differentiate both sides with respect to t to get (with $v = \cos(t) + 1$):

$$\int_{u=0}^{\sin(t)} w((u^2+v^2)/2v)\,((u^2+v^2)/2v^2)\, u^{q-1}\, du = c_3 \sin^{m_1}(t/2) \sin^{m_2}(t).$$

Rewrite this using $m_2 + 1 = q$ and $(m_1 - 2)/2 = m$ and a change of variables to get

$$\int_r^1 w(t)\, t(t-r)^{(q-2)/2}\, dt = c_3(1-r)^{m+q/2}.$$

$q \geqslant 2$ is even so we may differentiate both sides $q/2$ times to get (4.9). □

LEMMA 4.15. *There is a bounded linear map*

$$E: L^1(S) \to L^1(\Omega, w(\|x\|)\, dx)$$

such that:

(i) *For* $f \in L^1(S)$,

$$\int_S f\, ds = \int_\Omega E(f)\, w(\|x\|)\, dx.$$

(ii) *For* $f \in L^1(S)$ *and* $g \in L^\infty(\Omega)$,

$$E((g \circ \xi) f) = gE(f) \quad \text{a.e.}$$

(iii) *For* $f \in C^n(S)$, $E(f)$ *is in* $C^n(\Omega_0)$ $(n=0, 1, \ldots, \infty, \omega)$.

(iv) *In* (ii), *if f and g are continuous then the conclusion holds* everywhere *on* Ω (*not just a.e.*).

(v) *If* $f \geq 0$ *then* $E(f) \geq 0$.

(vi) $E(1)=1$.

(vii) *The norm of E as a map from* $L^p(S)$ *to* $L^p(\Omega, w(\|x\|)\, dx)$ *is* 1 $(1 \leq p \leq \infty)$.

(viii) *If* π *is the representation of* U_2 *on* $\mathbf{R}^{q+1}$ *in Lemma* 4.11 *then for* $f \in C(S)$ *and* $u \in U_2$

$$E(f \circ u)(x) = E(f)(\pi(u)x) \quad (x \in \Omega).$$

This sort of thing is fairly well known from probability theory since E is essentially a conditional expectation operator. See also [8].

The map ξ *carries over* the Laplace-Beltrami operator Δ_S on S to the operator Λ (defined in (3.3)) on Ω in the sense of

Definition 4.16. Let M be a compact C^ω manifold, $F: M \to \mathbf{R}^k$ a C^ω map with at least one regular point, and L a differential operator on M of order n. Then F carries over L to $F(M)$ if there exists a differential operator L on $F(M)$ such that

$$L(f \circ F) = (\tilde{L} f) \circ F \quad (f \in C^n(F(M))).$$

Note that since F has at least one regular point, the set of all regular points is dense and open in M and $F(M)$ is the closure of its interior.

LEMMA 4.17. *A necessary and sufficient condition that F carry over the differential operator L to F(M) is that for every polynomial p on* $\mathbf{R}^k$ *of degree* $\leq n=$order(L)

there is a function $\bar{p}$ on $F(M)$ such that

$$L(p\circ F)=\bar{p}\circ F.$$

Moreover, $\bar{L}$ is uniquely determined by $\bar{L}p=\bar{p}$ for all such polynomials p.

The proof is left to the reader.

Lemma 4.17 is easily stated and proved but it is rarely useful. The generic map F does not satisfy Definition 4.16 when $\dim(M)>k>1$. For example, $e(b_0, 1)$ carries Δ_S over to $\mathbf{C}\cong\mathbf{R}^2$ when S is a sphere, but not when it is one of the spaces of this section. The first hint that ξ satisfies Definition 4.16 comes from

LEMMA 4.18. *If h is a homogeneous, harmonic polynomial of degree n on $\mathbf{R}^{q+1}$ then $h\circ\xi$ is in $\mathscr{H}_n(S)$.*

Proof. The space of such h is spanned by $SO(q+1)$-translates of ε_n where

$$\varepsilon_n(x)=(x_1+ix_2)^n \quad (x\in\mathbf{R}^{q+1}). \tag{4.13}$$

Then by (i) and (ii) of Lemma 4.12, the space of functions $h\circ\xi$ is spanned by U_2-translates of $\varepsilon_n\circ\xi=e(b_0, n)$, all of which lie in $\mathscr{H}_n(S)$. □

The h of Lemma 4.18 that have degree ≤ 2 comprise all but one dimension of the space of polynomials of degree ≤ 2 on $\mathbf{R}^{q+1}$. Thus Lemma 4.18 (with Lemma 4.17) almost establishes that ξ carries Δ_S over to an operator on $\mathbf{R}^{q+1}$. What is missing is a statement expressing $\Delta_S(\|\xi\|^2)$ in terms of ξ. This is next.

Definition 4.19. Let $a=m=m_1/2-1$ and $b=m_2$ where $m_j=\dim(\mathfrak{n}_j)$ $(j=1,2)$. Let

$$Q_{n,j}(t)=P_{n-j}^{(a,\,b+2j)}(2t-1) \quad (0\leq t\leq 1)$$

where $P_k^{(a,b)}$ denotes the kth degree Jacobi polynomial corresponding to the weight

$$(1-t)^a(1+t)^b \quad (|t|\leq 1).$$

Write simply Q_n for $Q_{n,0}$. Thus

$$Q_n(t)=P_n^{(a,b)}(2t-1)=(\text{const.})_2F_1(-n, n+a+b+1; b+1; t) \quad (0\leq t\leq 1). \tag{4.14}$$

LEMMA 4.20. *$Q_n(\|\xi\|)\in\mathscr{H}_n(S)$ for $n\in\mathbf{N}$.*

Proof. Recall that δ is the distance function on S. Then the zonal spherical function $\phi_n\in\mathscr{H}_n(S)$ is given by

$$\phi_n(s) = p_n(\cos(\delta(s))) \quad (s \in S)$$

where p_n is a polynomial of degree n on $\mathbf{R}$ ([5], Theorem 4.5, p. 543). By (4.1)

$$\phi_n(s) = p_n(\xi_1(s) + \|\xi(s)\| - 1) \quad (s \in S).$$

Since ϕ_n is in $\mathscr{H}_n(S)$, so also is ψ_n where

$$\psi_n(s) = \int_{U_2} \phi_n(us)\, du = \int_{U_2} p_n(\xi_1(us) + \|\xi(s)\| - 1)\, du.$$

Let $q_n: [0, 1] \to \mathbf{R}$ be given by

$$q_n(\|x\|) = \int_{SO(q+1)} p_n((ux)_1 + \|x\| - 1)\, du.$$

Then $\psi_n = q_n(\|\xi\|)$ and q_n is a polynomial of degree n.

By Lemma 4.15 ((i), (ii), (vi)) the functions $q_n(\|x\|) = E(q_n(\|\xi\|))(x)$ are orthogonal on Ω relative to the measure $w(\|x\|)\, dx$. Thus the polynomials q_n are orthogonal on $[0, 1]$ relative to the weight

$$w(r) r^{q-1} = (1-r)^a r^b \quad (0 \leqslant r \leqslant 1)$$

where a, b are as in Definition 4.19. From this, $q_n = (\text{const.}) Q_n$. $\square$

As in Section 1, Lemma 4.14 and Definition 4.19, let

$$m_\tau = m_1/2 + m_2 = m + q = a + b + 1. \tag{4.19}$$

LEMMA 4.21. *ξ carries Δ_S over to the operator Λ on Ω defined in* (3.3):

$$\Lambda = \left(\frac{r\partial}{\partial r}\right)^2 + m_\tau \frac{r\partial}{\partial r} - r\Delta.$$

Proof. Any polynomial p on $\mathbf{R}^{q+1}$ of degree $\leqslant 2$ may be written

$$p(x) = h_0 + h_1(x) + h_2(x) + c_1 Q_1(\|x\|) + c_2 Q_2(\|x\|)$$

where Q_j is as in Definition 4.19, specifically (4.14), and h_j is harmonic, homogeneous on $\mathbf{R}^{q+1}$ of degree j. Then Lemmas 4.19 and 4.20 show that

$$h_j \circ \xi, \quad Q_j(\|\xi\|) \in \mathscr{H}_j(S).$$

From this, Lemma 4.17 shows that $F = \xi$ carries $L = \Delta_S$ over to a differential operator $\tilde{L}$ on $\Omega = \xi(S)$.

$\tilde{L}$ is determined by the fact that h_j and $x \to Q_j(\|x\|)$ are eigenfunctions of $\tilde{L}$ with the same eigenvalues as Δ_S has on $\mathcal{H}_j(S)$. If we normalize Δ_S so that

$$\Delta_S | \mathcal{H}_1(S) = (1+m_\tau) I$$

then

$$\Delta_S | \mathcal{H}_j(S) = j(j+m_\tau) I.$$

From this and the fact that $\Delta h_j = 0$ we get

$$\tilde{L} h_j = j(j+m_\tau) h_j = \frac{r\partial}{\partial r}\left(\frac{r\partial}{\partial r}+m_\tau\right) h_j = \Lambda h_j.$$

Similarly, if we use (4.14) and the fact that the radial part of Λ is the ordinary differential operator which has Q_j as an eigenfunction with eigenvalue $j(j+m_\tau)$, we get that

$$\tilde{L} Q_j(\|x\|) = j(j+m_\tau) Q_j(\|x\|) = \Lambda Q_j(\|x\|).$$

Thus $\tilde{L} = \Lambda$ on all polynomials of degree $\leqslant 2$ thereby proving the lemma by the uniqueness clause of Lemma 4.17. □

As a consequence we can give a complete description of the space of functions f on Ω such that $f \circ \xi \in \mathcal{H}_n(S)$.

THEOREM 4.22. *Let $\mathcal{F}_n(\Omega)$ denote the space of functions on Ω which are polynomials of degree $\leqslant n$ in x and $\|x\|$ $(x \in \Omega)$. Let $\mathcal{H}_n(\Omega, w)$ denote the orthocompliment of $\mathcal{F}_{n-1}(\Omega)$ in $\mathcal{F}_n(\Omega)$ with respect to the inner product*

$$\langle f_1, f_2 \rangle = \int_\Omega f_1(x)\, \overline{f_2(x)}\; w(\|x\|)\, dx.$$

Then for any $n \in \mathbf{N}$

(i) $\Lambda | \mathcal{H}_n(\Omega, w) = n(n+m_\tau) I$.

(ii) *$L^2(\Omega, w(\|x\|)\, dx)$ is the orthogonal direct sum of the subspaces $\mathcal{H}_n(\Omega, w)$.*

(iii) *$\mathcal{H}_n(\Omega, w)$ is precisely the space of functions f on Ω such that $f \circ \xi \in \mathcal{H}_n(S)$.*

(iv) *$\mathcal{H}_n(\Omega, w)$ is itself the orthogonal direct sum of subspaces $\mathcal{H}_{n,j}(\Omega, w)$ $(0 \leqslant j \leqslant n)$ each of which is $SO(q+1)$-irreducible. Functions in $\mathcal{H}_{n,j}(\Omega, w)$ all have the form*

$$h(x)\, Q_{n,j}(\|x\|) \quad (x \in \Omega)$$

where h is harmonic, homogeneous of degree j and $Q_{n,j}$ is as in Definition 4.19.

Proof. By consideration of the highest order terms in $\mathscr{F}_n(\Omega)$ observe that

$$\Lambda\mathscr{F}_0(\Omega)=\{0\},\quad (\Lambda-n(n+m_\tau))\,\mathscr{F}_n(\Omega)\subset\mathscr{F}_{n-1}(\Omega)\quad (n\geqslant 1).$$

Let $\mathscr{K}_n$ denote the kernel of $\Lambda-n(n+m_\tau)I$ in $\mathscr{F}_n(\Omega)$. By induction on n, $\Lambda-n(n+m_\tau)I$ is nonsingular on $\mathscr{F}_{n-1}(\Omega)$ and $\mathscr{F}_n(\Omega)$ is the linear direct sum of the subspaces $\mathscr{K}_0,\ldots,\mathscr{K}_n$ with the natural map of $\mathscr{K}_n$ to $\mathscr{F}_n(\Omega)/\mathscr{F}_{n-1}(\Omega)$ a linear isomorphism.

For (i) we must show that $\mathscr{H}_n(\Omega,w)=\mathscr{K}_n$. This follows from the pairwise orthogonality of the spaces $\mathscr{K}_j$ $(j\in\mathrm{N})$ which can be proved as follows: For $f\in\mathscr{K}_j$

$$\Delta(f\circ\xi)=(\Lambda f)\circ\xi=j(j+m_\tau)f\circ\xi$$

by Lemma 4.21; thus

$$f\circ\xi\in\mathscr{H}_j(S). \tag{4.15}$$

Then for $f_j\in\mathscr{K}_j$ and $f_n\in\mathscr{K}_n$ with $j\neq n$,

$$\begin{aligned}\langle f_j,f_n\rangle&=\int_\Omega f_j(x)\,\overline{f_n(x)}\,w(\|x\|)\,dx\\&=\int_S(f_j\circ\xi)\,(\overline{f_n\circ\xi})\,ds=0.\end{aligned}\tag{4.16}$$

For (ii), the sum of the $\mathscr{H}_n(\Omega,w)$ is $\mathscr{F}_\infty(\Omega)$, the union of the $\mathscr{F}_n(\Omega)$, which in turn contains all polynomials, certainly a dense subspace of $L^2(\Omega,w(\|x\|)\,dx)$.

For (iii), (4.15) is half the claim. Conversely, if f on Ω has $f\circ\xi$ in $\mathscr{H}_n(S)$ then $f|\Omega_0$ must be continuous and bounded, and so is in $L^2(\Omega,w(\|x\|)\,dx)$. By (4.16) with $f=f_n$, $f\perp\mathscr{H}_j(\Omega,w)$ for all $j\neq n$. Then (ii) implies f must be in $\mathscr{H}_n(\Omega,w)$.

For (iv) observe that $\mathscr{F}_n(\Omega)$ is $SO(q+1)$-stable so $\mathscr{H}_n(\Omega,w)$ is too. Let $\mathscr{K}$ be a nonzero, irreducible $SO(q+1)$-submodule of $\mathscr{H}_n(\Omega,w)$. For $0<r\leqslant 1$ let η_r be the $SO(q+1)$-module homomorphism of $\mathscr{K}$ into one of the irreducible spaces $\mathscr{H}_j(\partial\Omega)$ by

$$(\eta_r f)(s)=f(rs)\quad (s\in\partial\Omega,f\in\mathscr{K}).$$

Since $\mathscr{K}$ is irreducible, each η_r is either null or an isomorphism and all the image $\mathscr{H}_j(\partial\Omega)$'s are the same; moreover by Schur's lemma the η_r differ by a scalar factor. Thus for $f\in\mathscr{K}$ we have that $f(x)=g(\|x\|)\,h(x)$ where h is homogenous, harmonic of degree j.

Then from $\Lambda f=n(n+m_\tau)f$ and

$$\Lambda=r(r-1)\frac{\partial^2}{\partial r^2}+((m_\tau+1)r-q))\frac{\partial}{\partial r}+\frac{1}{r}\Theta$$

(where Θ is the spherical part of Δ as in Section 3) and $\Theta h = -j(j+q-1)h$ we get

$$r(r-1)g''(r)+((m_\tau+1)r-q)g'(r)-(n-j)(n+j+m_\tau)g(r)=0.$$

Thus g is a multiple of

$${}_2F_1(j-n, n+j+m_\tau; 2j+q; r) = (\text{const.})\, Q_{n,j}(r)$$

as advertised. However, this is a polynomial only if $j \leq n$, establishing the limitation on j in (iv). Conversely, the same calculation shows that all functions $h(x)\,Q_{n,j}(\|x\|)\,(0 \leq j \leq n$, and h as above) are in $\mathscr{H}_n(\Omega, w)$. □

COROLLARY 4.23. *Λ is essentially self-adjoint on $\mathscr{F}_\infty(\Omega)$, the union of the $\mathscr{F}_n(\Omega)$.*

Remark. This corollary could be proved directly and then used to establish (i), (ii), and (iv) of the theorem without making use of ξ in the proof.

As a corollary of the corollary we have

LEMMA 4.24. *For $g_1 \in C^2(\Omega_0)$ and $g_2 \in C_c^2(\Omega_0)$ we have $\langle \Lambda g_1, g_2 \rangle = \langle g_1, \Lambda g_2 \rangle$.*

Proof. By a standard trick (of multiplying g_1 by a function in $C_c^2(\Omega_0)$ which is 1 on the support of g_2) reduce to the case where both g_1 and g_2 have support in Ω_0. Then approximate both with elements from $\mathscr{F}_n(\Omega)$ (n sufficiently large) in the norm

$$(\langle g, g \rangle + \langle \Lambda g, \Lambda g \rangle)^{1/2} \quad (g \in C^2(\Omega_0)).$$

Finally, use the symmetry of Λ on $\mathscr{F}_n(\Omega)$ given by Corollary 4.23 (or really, by (i) and (ii) of the theorem). □

This makes it easy to prove the following promised extension of Lemma 4.21:

LEMMA 4.25. *For $f \in C^2(S)$*

$$E(\Delta_S f) = \Lambda E(f) \quad \text{on } \Omega_0. \tag{4.17}$$

Proof. Both sides of (4.17) are continuous on Ω_0 so it suffices to prove (4.17) in the weak sense that

$$\langle E(f), \Lambda g \rangle = \langle E(\Delta_S f), g \rangle \quad (g \in C_c^2(\Omega_0)). \tag{4.18}$$

The left side of (4.18) is

$$\int_\Omega E(f)(x)\,\overline{\Lambda g(x)}\,w(\|x\|)\,dx = \int_S \overline{(\Lambda g \circ \xi)}\,f\,ds = \int_S \Delta_S\,\overline{(g \circ \xi)}\,f\,ds$$

by Lemma 4.21. From the symmetry of Δ_S on $C^2(S)$, this is

$$\begin{aligned}\int_S \overline{(g \circ \xi)}\,\Delta_S f\,ds &= \int_\Omega E(\overline{(g \circ \xi)}\,\Delta_S f)(x)\,w(\|x\|)\,dx \\ &= \int_\Omega E(\Delta_S f)(x)\,\bar{g}(x)\,w(\|x\|)\,dx\end{aligned}$$

by Lemma 4.15 (i), (ii). But this is the right side of (4.18). □

COROLLARY 4.26. $E(\mathcal{H}_n(S)) = \mathcal{H}_n(\Omega, w)$.

Proof. Lemma 4.25 and Lemma 4.15 (vii) shows that $E(\mathcal{H}_n(S))$ consists of L^2 eigenfunctions of Λ with eigenvalue $n(n+m_\tau)$ and thus is inside $\mathcal{H}_n(\Omega, w)$ by Theorem 4.22. It is all of $\mathcal{H}_n(\Omega, w)$ by Lemma 4.15 (ii), (vi) (which shows that $E(g \circ \xi) = g$). □

Remark. Alternately, Corollary 4.26 could be proved along with (iii) of Theorem 4.22 and then used to prove Lemma 4.25. This proof of Corollary 4.26 would use (i) and (ii) of Lemma 4.15 to argue that $E(\mathcal{H}_n(S))$ was orthogonal to all $\mathcal{H}_j(\Omega, w)\,(j \neq n)$.

Corollary 4.26 lets us answer a question left open in Lemmas 4.15 and 4.25.

LEMMA 4.27. *For f in $C(S)$, $E(f)$ is continuous on all of Ω (not just Ω_0). Moreover, (4.17) holds on Ω for $f \in C^2(S)$.*

Proof. For the first assertion approximate f uniformly by a sequence of $\{f_n\}$ in the linear (i.e. nontopological) sum of the spaces $\mathcal{H}_j(S)\,(j \in \mathbf{N})$. Then $E(f_n)$ lies in $\mathcal{F}_\infty(\Omega)$ and is thus a continuous function on Ω. Now $E(f)$ is the uniform limit of the $E(f_n)$ by Lemma 4.15 (vii) with $p = \infty$. Consequently $E(f)$ is continuous on Ω.

The proof of (4.17) is similar except that f_n must now approximate f in the norm $\|f\|_\infty + \|\Delta_S f\|_\infty$. Then

$$\Lambda E(f_n) = E(\Delta_S f_n) \to E(\Delta_S f), \quad E(f_n) \to E(f)$$

uniformly on Ω. Since (4.17) already holds on Ω_0 the preceeding shows that it must also hold on Ω. □

To complete the tie-in between Section 3 and the problem of proving the Main theorem we need the operators on S that go over (under ξ) to Θ and Θ_1 on $\mathbf{R}^{q+1}$.

Recall that Θ was the spherical part of Δ, the Laplacian on $\mathbf{R}^{q+1}$. Θ_1 is the spherical part of the Laplacian on $\mathbf{R}^q$, regarded as acting on functions on $\mathbf{R}^{q+1}$ by holding the first coordinate fixed.

For a compact semisimple Lie group C let Δ_C denote the negative of its Casimir operator.

Recall the group U_2 introduced in Lemma 4.1 and its subgroup $K_2=U_2\cap K$. Δ_{U_2} and Δ_{K_2} act on functions on S in the obvious way. For example,

$$(\Delta_{U_2}f)(x)=\Delta_{U_2}(u\to f(us))|_{u=1} \quad (f\in C^2(S)).$$

LEMMA 4.28. *For* $f\in C^2(S)$
(i) $\Theta E(f)=E(\Delta_{U_2}f)$;
(ii) $\Theta_1 E(f)=E(\Delta_{K_2}f)$.

Proof. (viii) of Lemma 4.15 and (i) of Lemma 4.12 give

$$E(\Delta_{U_2}f)=\Delta_{SO(q+1)}E(f)=\Theta E(f),$$
$$E(\Delta_{K_2}f)=\Delta_{SO(q)}E(f)=\Theta_1 E(f). \qquad \square$$

Now recall the space $\mathscr{D}_n(\Omega)$ from Section 3; it was a normed space of functions on Ω_0 with norm $\mathscr{N}_n(\cdot)$.

LEMMA 4.29. *E is a bounded linear map from the Banach space* $C^{3n+q-2}(S)$ *to* $\mathscr{D}_n(\Omega)$.

Proof. For all f in $C^{3n+q-2}(S)$ and $0\leq l\leq n$,

$$(I-\Delta_{U_2})(I-\Delta_{K_2})^{(n+q-4)/2}\Delta_S^l f \tag{4.19}$$

is bounded on S with a sup norm less than some constant (independent of f) times the norm of $C^{3n+q-2}(S)$. If we apply E to (4.19) then Lemma 4.28 and Lemma 4.25 give

$$(I-\Theta)(I-\Theta_1)^{(n+q-4)/2}\Lambda^l E(f)$$

which must be continuous and bounded on Ω_0 by the sup norm of (4.19). (This is from Lemma 4.15 (iii) and (vii). In fact, Lemma 4.27 shows that it is continuous on all of Ω.) Thus $E(f)$ is in $\mathscr{D}_n(\Omega)$ and $\mathscr{N}_n(E(f))$ is bounded by a constant times the $C^{3n+q-2}(S)$-norm. $\square$

Recall $S(\varepsilon,\eta)$ from Definition 1.11 and $\varepsilon_{*,n}$ and $\Omega(\varepsilon,\eta)$ from early in Section 3.

LEMMA 4.30. *For* $f \in L^1(S)$ *and* $0<\varepsilon, \eta<1$

$$\int_{S(\varepsilon,\eta)} f(s) e_*(b_0, n)(s)\, ds = \int_{\Omega(\varepsilon,\eta)} E(f)(x)\, \varepsilon_{*,n+m+1}(x)\, w(\|x\|)\, dx$$

(*where, as usual in this section,* $m=m_1/2-1$ *and* $m_1=\dim \mathfrak{n}_1$).

Proof. This is immediate from Lemma 4.13 (i) and (ii) of Lemma 4.15, and the fact that

$$e_*(b_0, n) = \varepsilon_{*,n+m+1} \circ \xi$$

is bounded on $S(\varepsilon, \eta)$ when $\eta>0$. □

LEMMA 4.31. *For* $f \in C^{3n+3m+q-2}(S)$ *and* $0<\varepsilon, \eta<1$

$$\hat{f}(kb_0, n; \varepsilon, \eta) = \int_{S(\varepsilon,\eta)} f(ks)\, e_*(b_0, n)(s)\, ds \quad (k \in K)$$

is bounded by a constant times the norm of f *in* $C^{3n+3m+q-2}(S)$. *Also, the limit*

$$\hat{f}(b, n) = \lim_{\varepsilon\to 0^+} \lim_{\eta\to 0^+} \hat{f}(b, n; \varepsilon, \eta)$$

exists for all $b \in B$ *and shares the bound on* $\hat{f}(b, n; \varepsilon, \eta)$.

Proof. Write $f^k(s)=f(ks)$ for $k \in K$. Then $k \to f^k$ is a continuous, hence uniformly bounded map from K to $C^{3n+3m+q-2}(S)$ and

$$\hat{f}(kb_0, n; \varepsilon, \eta) = \widehat{f^k}(b_0, n; \varepsilon, \eta), \quad \hat{f}(kb_0, n) = \widehat{f^k}(b_0, n).$$

Thus it suffices to prove the bound on $\hat{f}(b_0, n; \varepsilon, \eta)$ and the existence of $\hat{f}(b_0, n)$. This follows immediately from Lemmas 4.29 and 4.30 and Theorem 3.1. □

We have the following corollary of the proof:

COROLLARY 4.32. *For* $f \in C^{3n+3m+q-2}(S)$

$$\hat{f}(kb, n) = \widehat{f^k}(b, n) \quad (k \in K, b \in B).$$

COROLLARY 4.33. *For* $f \in C^{3n+3m+q-2}(S)$, $\hat{f}(b, n)$ *is continuous in* $b \in B$.

Proof. $k \to f^k$ is continuous from K to $C^{3n+3m+q-2}(S)$ and $f \mapsto \hat{f}(b_0, n)$ is continuous from $C^{3n+3m+q-2}(S)$ to $\mathbf{C}$ so $k \mapsto \hat{f}(kb_0, n)$ is continuous on K. □

Recall that ϕ_n denotes the zonal spherical function in $\mathcal{H}_n(S)$ and that we also use ϕ_n to denote the corresponding function defined on $S \times S$ by

$$\phi_n(us_0, s) = \phi_n(u^{-1}s) \quad (u \in U, s \in S).$$

LEMMA 4.34. *For* $f \in C^{3n+3m+q-2}(S)$

$$\int_S f(s')\,\phi_n(s', s)\,ds' = \int_B \hat{f}(b, n)\,e(b, n)(s)\,ds. \tag{4.20}$$

Proof. For $\varepsilon > 0$ let

$$\hat{f}(b, n; \varepsilon, 0) = \lim_{\eta \to 0^+} \hat{f}(b, n; \varepsilon, \eta) \quad (b \in B).$$

Then

$$\lim_{\varepsilon \to 0^+} \int_B |\hat{f}(b, n; \varepsilon, 0) - \hat{f}(b, n)|\,db = 0$$

since as $\varepsilon \to 0^+$, $\hat{f}(b, n; \varepsilon, 0)$ converges to $\hat{f}(b, n)$ pointwise on B with a uniform bound. Thus for any $r > 0$ there is an $\varepsilon > 0$ such that

$$\int_B |\hat{f}(b, n; \varepsilon, 0) - \hat{f}(b, n)|\,db < r.$$

By the same argument there is $\eta_1 > 0$ such that for all $0 < \eta < \eta_1$,

$$\int_B |\hat{f}(b, n; \varepsilon, \eta) - \hat{f}(b, n; \varepsilon, 0)|\,db < r$$

so we have that the right side of (4.20) is within $2r$ of

$$\int_B \hat{f}(b, n; \varepsilon, \eta)\,e(b, n)(s)\,db = \int_K \int_{S(\varepsilon, \eta)} f(ks')\,e_*(b_0, n)(s')\,e(kb_0, n)(s)\,ds'\,dk. \tag{4.21}$$

Now it is tempting to try to simply change the order of integration while moving the k from $f(ks)$ to

$$e_*(b_0, n)(k^{-1}s') = e_*(kb_0, n)(s')$$

and then argue as in Theorem 1.10 using Theorem 1.7. The trouble with this is that $S(\varepsilon, \eta)$ is not K-invariant. However, it is K_2-invariant so we can at least interchange integration over K_2 with integration over $S(\varepsilon, \eta)$. This turns out to be enough to smooth out

$$\int_{K_2} e_*(kb_0, n)(s')\, e(kb_0, n)(s)\, dk$$

for s' outside the antipodal set.

In more detail, the right side of (4.21) becomes

$$\int_{K_2}\int_K\int_{S(\varepsilon,\eta)} f(kk_2 s')\, e_*(b_0, n)(s')\, e(k_2 b_0, n)(k^{-1}s)\, ds'\, dk\, dk_2$$

$$= \int_K\int_{S(\varepsilon,\eta)} f(ks') \int_{K_2} e_*(k_2 b_0, n)(s')\, e(k_2 b_0, n)(k^{-1}s)\, dk_2\, ds'\, dk$$

by using

$$e_*(b_0, n)(k_2^{-1} s') = e_*(k_2 b_0, n)(s') \quad (k_2 \in K_2, s' \in S(\varepsilon, \eta)).$$

Now Lemma 4.12 shows that in the representation π of U_2 on $\mathbf{R}^{q+1}, \pi(K_2)$ is the subgroup of $SO(q+1)$ which preserves $x_1 + \|x\| - 1$, i.e. it is $SO(q)$ (regarded as the subgroup of $SO(q+1)$ which fixes $(1, 0, \ldots, 0)$). Thus, with ε_n defined in (4.13),

$$\begin{aligned}\int_{K_2} e_*(k_2 b_0, n)(s')\, e(k_2 b_0, n)(s)\, dk_2 &= \int_{SO(q)} \varepsilon_{*, n+m+1}(u\xi(s'))\, \varepsilon_n(u\xi(s))\, du \\ &= F_{n, m+1}(\xi(s')), \xi(s), (1, 0, \ldots, 0))\, \|\xi(s')\|^{-2n-2m-q-1} \\ &= \mathscr{F}(s', s)\end{aligned}$$

by Corollary 2.12, where $F_{n,m+1}(y, x^{(1)}, x^{(2)})$ is the polynomial introduced there. $\mathscr{F}(s', s)$ therefore extends by continuity to an analytic function on $S \times S(\varepsilon, 0)$ for any $\varepsilon > 0$. Thus we can find $\eta_2 > 0$ such that for any $0 < \eta < \eta_2$, (4.21) is within r of

$$\int_K\int_{S(\varepsilon,0)} f(ks')\, \mathscr{F}(s', k^{-1}s)\, ds'\, dk = \int_{S(\varepsilon,0)} f(s') \int_K \mathscr{F}(k^{-1}s', k^{-1}s)\, dk\, ds' \tag{4.22}$$

since $S(\varepsilon, 0)$ is K-invariant. However, for $s' \in S(1/2, 0)$

$$\begin{aligned}\int_K \mathscr{F}(k^{-1}s', k^{-1}s)\, dk &= \int_K e(b_0, n)(k^{-1}s)\, e_*(b_0, n)(k^{-1}s')\, dk \\ &= \int_K e(b, n)(s)\, e_*(b, n)(s')\, db = \phi_n(s', s)\end{aligned}$$

by Theorem 1.7, so this also holds for $s' \in S(\varepsilon, 0)$ since both sides are analytic on $S(\varepsilon, 0)$. Thus (4.22) becomes

$$\int_{S(\varepsilon,0)} f(s')\,\phi_n(s', s)\,ds'.$$

Thus we can choose ε small enough so that there is η_3 for which if $0<\eta<\eta_3$ then the right side of (4.20) is within $4r$ of the left side. Since $r>0$ was arbitrary, this proves the lemma. □

Drawing these results together, we have the following expression of the Main theorem for spaces of this section:

THEOREM 4.35. *For $f \in C^\infty(S)$ and $n \in \mathbf{N}$*

$$\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+} \hat{f}(b, n; \varepsilon, \eta) = \hat{f}(b, n)$$

with the convergence in $C^\infty(B)$. Moreover, the component f_n of f in $\mathscr{H}_n(S)$ is

$$f_n(s) = d_n \int_B \hat{f}(b, n)\, e(b, n)(s)\, ds$$

(where $d_n = \dim(\mathscr{H}_n(S))$ is given in Table 2 of Section 1, and $\hat{f}(b, n; \varepsilon, \eta)$ is defined in Definition 1.11). The series $\Sigma_0^\infty f_n$ converges to f in $C^\infty(S)$.

Proof. Lemma 4.31 proves that the limit exists and $f \mapsto \hat{f}(b, n)$ is a continuous functional on the Banach space $C^{n'}(S)$ where $n' = 3n+3m+q-2$. For C^∞ functions f, the function $k \mapsto f^k$ is a C^∞ map from K to $C^{n'}(S)$. Then Corollary 4.32 shows that $k \mapsto \hat{f}(kb, n)$ is C^∞ on K. Each $\hat{f}(b, n; \varepsilon, \eta)$ is also in $C^\infty(B)$ and from Definition 1.11 satisfies

$$\widehat{f^k}(b, n; \varepsilon, \eta) = \hat{f}(kb, n; \varepsilon, \eta) \quad (k \in K).$$

Thus

$$\Delta_B \hat{f}(b, n; \varepsilon, \eta) = (\widehat{\Delta_K f})(b, n; \varepsilon, \eta).$$

If we iterate this and take the limit at each stage we get

$$\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+} \Delta_B^l \hat{f}(b, n; \varepsilon, \eta) = \Delta_B^l \hat{f}(b, n)$$

uniformly in $b \in B$ for all $l \in \mathbf{N}$. Thus the convergence is in $C^\infty(B)$.

The recovery of f_n from $\hat{f}(b,n)$ is immediate from Lemma 4.34 combined with Cartan's theorem that

$$f_n(S) = d_n \int_S \phi_n(s', s) f(s')\, ds'.$$

The C^∞ convergence of $\Sigma_0^\infty f_n$ to f for C^∞ functions f is well known. □

We end by completing the proof of Theorem 1.13.

LEMMA 4.36. *Theorem* 1.13 *is valid for the spaces of this section.*

Proof. We have to establish (1.16). For $f \in C^\infty(S)$ approximate the right side of (1.16) by

$$d_n \int_K \hat{f}(kb_0, n; \varepsilon, \eta)\, e(kb_0, n_1)(s_1) \dots e(kb_0, n_j)(s_j)\, dk \tag{4.23}$$

for small $\varepsilon, \eta > 0$. Write $\hat{f}(kb_0, n; \varepsilon, \eta)$ as

$$\int_{\Omega(\varepsilon,\eta)} E(f^k)(x)\, \varepsilon_{*,n+m+1}(x)\, w(\|x\|)\, dx$$

and replace $\int_K$ by $\int_K \int_{K_2}$ in (4.23) to get

$$\begin{aligned} d_n \int_K \int_{K_2} \int_{\Omega(\varepsilon,\eta)} & E((f^k)(x)\, \varepsilon_{*,n+m+1}(\pi(k_2^{-1})\, x)\, w(\|x\|)\, dx \\ & \times \varepsilon_{n_1}(\pi(k_2^{-1})\, \xi(k^{-1}s_1)) \dots \varepsilon_{n_j}(\pi(k_2^{-1})\, \xi(k^{-1}s_j))\, dk_2\, dk \end{aligned} \tag{4.24}$$

where $\pi: U_2 \to SO(q+1)$ is the representation in Lemma 4.12 and ε_n is defined in (4.13).

Since $\Omega(\varepsilon, \eta)$ is stable under $\pi(K_2)$ we may interchange the order of integration over K_2 and $\Omega(\varepsilon, \eta)$ in (4.24) and apply Corollary 2.12 to get

$$\begin{aligned} &\int_{K_2} \varepsilon_{*,n+m+1}(\pi(k^{-1})\, x)\, \varepsilon_{n_1}(\pi(k^{-1})\, x^{(1)}) \dots \varepsilon_{n_{j+1}}(\pi(k^{-1})\, x^{(j+1)})\, dk \\ &\qquad = F_{n_1,\dots,n_{j+1}}(x, x^{(1)}, \dots, x^{(j+1)})\, \|x\|^{-2n-2m-q-1} \end{aligned}$$

where $F_{n_1,\dots,n_{j+1}}$ is the polynomial introduced in Corollary 2.12, and

$$x^{(j+1)} = (1, 0, \dots, 0), \quad n_{j+1} = m+1$$

so that

$$\varepsilon_{n_{j+1}}(\pi(k_2^{-1})\,x^{(j+1)}) = 1.$$

Thus (4.24) becomes

$$d_n \int_K \int_{\Omega(\varepsilon,\eta)} E(f^k)(x)\, F_{n_1,\ldots,n_{j+1}}(x, \xi(k^{-1}s_1), \ldots, \xi(k^{-1}s_j), x^{(j+1)})\, w(\|x\|)\, dx\, dk.$$

For $\varepsilon>0$ the integrand is bounded regardless of η. Let $\eta\to 0^+$. $\Omega(\varepsilon, 0)$ is K-invariant so we can interchange the integrals over K and $\Omega(\varepsilon, 0)$ and obtain

$$d_n \int_{S(\varepsilon,0)} f(s)\,\Psi(s, s_1, \ldots, s_{n_j})\, ds$$

where Ψ is as in Theorems 1.9 and 1.13. Let $\varepsilon\to 0^+$ and this becomes the left side of (1.16). □

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