

The Hellinger square-integrability of matrix-valued measures with respect to a non-negative hermitian measure

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Introduction

Let \mathcal{B} be a σ -algebra of subsets of a given space Ω , and let \mathbf{F} be a fixed non-negative hermitian measure on \mathcal{B} . For matrix-valued measures \mathbf{M} and \mathbf{N} the Hellinger integral $\int_{\Omega} (d\mathbf{M}d\mathbf{N}^*/d\mathbf{F})$ ($*$ = conjugate) is defined in such a way that the space of all matrix-valued measures \mathbf{M} for which $\int_{\Omega} (d\mathbf{M}d\mathbf{M}^*/d\mathbf{F})$ exist becomes a Hilbert space under the inner product $\tau \int_{\Omega} (d\mathbf{M}d\mathbf{N}^*/d\mathbf{F})$ (τ = trace). It will follow that $\int_{\Omega} (d\mathbf{M}d\mathbf{M}^*/d\mathbf{F})$ exists iff there exists a \mathcal{B} -measurable matrix-valued function Ψ on Ω such that $\Psi \in L_{2, \mathbf{F}}$ [9, p. 295], and for each $B \in \mathcal{B}$, $\mathbf{M}(B) = \int_B \Psi d\mathbf{F}$. These generalize the corresponding results [3, pp. 258–61] & [8, pp. 1414–18] concerning the Hellinger integrals $\int_{\Omega} (d\nu d\gamma/d\mu)$, where ν and γ are complex-valued measures and μ is a non-negative real-valued measure on \mathcal{B} .

For any matrix \mathbf{G} we write \mathbf{G}^- for the generalized inverse of \mathbf{G} [7, p. 407]. If μ is a σ -finite non-negative real-valued measure on \mathcal{B} with respect to (w.r.t.) which \mathbf{F} is absolutely continuous (a.c.), then it is easy to show that $(d\mathbf{F}/d\mu)^-$ is a \mathcal{B} -measurable matrix-valued function on Ω .

Lemma 1. *Let (i) \mathbf{M} and \mathbf{N} be matrix-valued measures on \mathcal{B} .*

(ii) μ and ν be σ -finite non-negative real-valued measures on \mathcal{B} w.r.t. which \mathbf{M} , \mathbf{N} and \mathbf{F} are a.c.¹ Then

$$(a) \quad \int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^- (d\mathbf{N}/d\mu)^* d\mu \quad \text{exists iff}$$

$$\int_{\Omega} (d\mathbf{M}/d\nu) (d\mathbf{F}/d\nu)^- (d\mathbf{N}/d\nu)^* d\nu \quad \text{exists.}$$

(b) *If these integrals exist, they are equal.*

Proof. (a) Let $\gamma = \mu + \nu$. If $\int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^- (d\mathbf{N}/d\mu)^* d\mu$ exists, then from the relations

¹ Each matrix-valued measure is a.c. w.r.t. the sum of the total variation measures of its components. Hence such a μ exists.

H. SALEHI, *Hellinger square-integrability of matrix-valued measures*

$$\begin{aligned} \int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^{-} (d\mathbf{N}/d\mu)^* d\mu &= \int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^{-} (d\mathbf{N}/d\mu)^* (d\mu/d\gamma) d\gamma \\ &= \int_{\Omega} (d\mathbf{M}/d\gamma) (d\mathbf{F}/d\gamma)^{-} (d\mathbf{N}/d\gamma)^* d\gamma \end{aligned} \quad (1)$$

follows that $\int_{\Omega} (d\mathbf{M}/d\gamma) (d\mathbf{F}/d\gamma)^{-} (d\mathbf{N}/d\gamma)^* d\gamma$ exists.

Conversely if $\int_{\Omega} (d\mathbf{M}/d\gamma) (d\mathbf{F}/d\gamma)^{-} (d\mathbf{N}/d\gamma)^* d\gamma$ exists,

again from (1) we infer that

$$\int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^{-} (d\mathbf{N}/d\mu)^* d\mu \text{ exists.}$$

Similar argument can be used to show that

$$\int_{\Omega} (d\mathbf{M}/d\nu) (d\mathbf{F}/d\nu)^{-} (d\mathbf{N}/d\nu)^* d\nu \text{ exists iff}$$

$$\int_{\Omega} (d\mathbf{M}/d\gamma) (d\mathbf{F}/d\gamma)^{-} (d\mathbf{N}/d\gamma)^* d\gamma \text{ exists.}$$

Hence (a) is proved.

(b) From the argument used in the proof of (a) we infer (b). (Q.E.D.)

Thus the following definition makes sense.

Definition 1. Let \mathbf{M} , \mathbf{N} , \mathbf{F} and μ be as in the previous lemma. Then (a) we say that (\mathbf{M}, \mathbf{N}) is *Hellinger integrable w.r.t. \mathbf{F}* if $\int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^{-} (d\mathbf{N}/d\mu)^* d\mu$ exists. We write

$$\int_{\Omega} \frac{d\mathbf{M} d\mathbf{N}^*}{d\mathbf{F}} = \int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^{-} (d\mathbf{N}/d\mu)^* d\mu.$$

(b) $\mathbf{H}_{2,\mathbf{F}}$ is the class of all matrix-valued measures \mathbf{M} on \mathcal{B} for which $\int_{\Omega} (d\mathbf{M} d\mathbf{M}^*/d\mathbf{F})$ exist.

It is easy to see that

- (1) $\mathbf{M}, \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}} \Rightarrow (\mathbf{M}, \mathbf{N})$ is Hellinger integrable w.r.t. \mathbf{F} ,
 $\mathbf{M} \in \mathbf{H}_{2,\mathbf{F}}$ and \mathbf{A} is a matrix $\Rightarrow \mathbf{A}\mathbf{M} \in \mathbf{H}_{2,\mathbf{F}}$,
 $\mathbf{M}, \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}} \Rightarrow \mathbf{M} + \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}}$.

By (1) $\int_{\Omega} (d\mathbf{M} d\mathbf{N}^*/d\mathbf{F})$ exists for $\mathbf{M}, \mathbf{N} \in \mathbf{H}_{2,\mathbf{F}}$. This matrix-valued integral behaves like an inner product. It is therefore convenient to write

$$(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = \int_{\Omega} \frac{d\mathbf{M} d\mathbf{N}^*}{d\mathbf{F}}.$$

We define the ordinary inner product for $\mathbf{H}_{2, \mathbf{F}}$ by

$$((\mathbf{M}, \mathbf{N}))_{\mathbf{F}} = \tau(\mathbf{M}, \mathbf{N})_{\mathbf{F}}.$$

Thus from (1) we immediately get:

Lemma 2. $\mathbf{H}_{2, \mathbf{F}}$ is an inner product space under $((\cdot, \cdot))_{\mathbf{F}}$, where for \mathbf{M} and $\mathbf{N} \in \mathbf{H}_{2, \mathbf{F}}$

$$((\mathbf{M}, \mathbf{N}))_{\mathbf{F}} = \tau(\mathbf{M}, \mathbf{N})_{\mathbf{F}}.$$

Let $\mathbf{L}_{2, \mathbf{F}}$ be the class of all matrix-valued functions Φ on Ω for which $\int_{\Omega} \Phi d\mathbf{F} \Phi^*$ exist (A detailed discussion of integrals $\int_{\Omega} \Phi d\mathbf{F} \Psi^*$ and $\int_{\Omega} \Phi d\mathbf{F}$ are given in [6] and [9]). It is known [9, p. 295] that $\mathbf{L}_{2, \mathbf{F}}$ is a Hilbert space under the inner product.

$$((\Phi, \Psi))_{\mathbf{F}} = \tau(\Phi, \Psi)_{\mathbf{F}} = \tau \int_{\Omega} \Phi d\mathbf{F} \Psi^*.$$

The following lemma is needed to establish an isomorphism between $\mathbf{L}_{2, \mathbf{F}}$ and $\mathbf{H}_{2, \mathbf{F}}$.

Lemma 3. Let (i) Φ and $\Psi \in \mathbf{L}_{2, \mathbf{F}}$.

(ii) For each $B \in \mathcal{B}$

$$\mathbf{M}(B) = \int_B \Phi d\mathbf{F} \quad \text{and} \quad \mathbf{N}(B) = \int_B \Psi d\mathbf{F}.$$

Then (\mathbf{M}, \mathbf{N}) is Hellinger-integrable w.r.t. \mathbf{F} and

$$(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = (\Phi, \Psi)_{\mathbf{F}}.$$

Proof. Let μ be any σ -finite non-negative measure w.r.t. which \mathbf{F} is a.c. Then for each $B \in \mathcal{B}$, $\mathbf{M}(B) = \int_B \Phi(d\mathbf{F}/d\mu) d\mu$ and $\mathbf{N}(B) = \int_B \Psi(d\mathbf{F}/d\mu) d\mu$. Hence

$$(d\mathbf{M}/d\mu) = \Phi(d\mathbf{F}/d\mu), \quad (d\mathbf{N}/d\mu) = \Psi(d\mathbf{F}/d\mu).$$

Therefore

$$\begin{aligned} \int_{\Omega} \frac{d\mathbf{M} d\mathbf{N}^*}{d\mathbf{F}} &= \int_{\Omega} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^{-} (d\mathbf{N}/d\mu)^* d\mu \\ &= \int_{\Omega} \Phi(d\mathbf{F}/d\mu) (d\mathbf{F}/d\mu)^{-} (d\mathbf{F}/d\mu) \Psi^* d\mu \\ &= \int_{\Omega} \Phi(d\mathbf{F}/d\mu) \Psi^* d\mu = (\Phi, \Psi)_{\mathbf{F}}. \end{aligned} \tag{1}$$

Since for Φ and $\Psi \in \mathbf{L}_{2, \mathbf{F}}$, $(\Phi, \Psi)_{\mathbf{F}}$ exists, from (1) it follows that $(\mathbf{M}, \mathbf{N})_{\mathbf{F}}$ is Hellinger integrable w.r.t. \mathbf{F} . Moreover $(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = (\Phi, \Psi)_{\mathbf{F}}$. (Q.E.D.)

If $\Psi \in \mathbf{L}_{2, \mathbf{F}}$, Then \mathbf{M}_Ψ will denote the matrix-valued measure in $\mathbf{H}_{2, \mathbf{F}}$ such that for each $B \in \mathcal{B}$, $\mathbf{M}_\Psi(B) = \int_B \Psi d\mathbf{F}$. Hence the following definition makes sense.

Definition 2. Let the transformation \mathbf{T} be defined on $\mathbf{L}_{2, \mathbf{F}}$ into $\mathbf{H}_{2, \mathbf{F}}$ as follows:

$$\mathbf{T}\Psi = \mathbf{M}_\Psi.$$

The important properties of \mathbf{T} are given in the following theorem:

Theorem 1. (a) \mathbf{T} is a linear operator on $\mathbf{L}_{2, \mathbf{F}}$ into $\mathbf{H}_{2, \mathbf{F}}$, i.e., if \mathbf{A} and \mathbf{B} are matrices and Φ and $\Psi \in \mathbf{L}_{2, \mathbf{F}}$, then

$$\mathbf{T}(\mathbf{A}\Phi + \mathbf{B}\Psi) = \mathbf{A}\mathbf{T}\Phi + \mathbf{B}\mathbf{T}\Psi.$$

(b) \mathbf{T} is an isometry on $\mathbf{L}_{2, \mathbf{F}}$ into $\mathbf{H}_{2, \mathbf{F}}$. In fact

$$(\mathbf{T}\Phi, \mathbf{T}\Psi)_\mathbf{F} = (\Phi, \Psi)_\mathbf{F}.$$

(c) \mathbf{T} is onto $\mathbf{H}_{2, \mathbf{F}}$, i.e., for each $\mathbf{M} \in \mathbf{H}_{2, \mathbf{F}}$, there exists a $\Psi \in \mathbf{L}_{2, \mathbf{F}}$ such that $\mathbf{M} = \mathbf{T}\Psi$. In fact we can take Ψ to be $(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^-$, where μ is any σ -finite non-negative real-valued measure w.r.t. \mathbf{M} and \mathbf{F} are a.c.

Proof. (a) and (b) follow from Lemma 3 and Definition 2.

(c) Let $\mathbf{M} \in \mathbf{H}_{2, \mathbf{F}}$. If μ is any σ -finite non-negative real-valued measure on \mathcal{B} w.r.t. which \mathbf{M} and \mathbf{F} are a.c., then

$$\begin{aligned} (\mathbf{M}, \mathbf{N})_\mathbf{F} &= \int_\Omega (d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^- (d\mathbf{M}/d\mu)^* d\mu \\ &= \int_\Omega [(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^-] (d\mathbf{F}/d\mu) [(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^-]^* d\mu, \end{aligned}$$

where the first equality follows from the definition of $(\mathbf{M}, \mathbf{N})_\mathbf{F}$ and the second one is a consequence of $(d\mathbf{F}/d\mu)^- (d\mathbf{F}/d\mu) (d\mathbf{F}/d\mu)^- = (d\mathbf{F}/d\mu)^-$. Hence $(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^-$ is in $\mathbf{L}_{2, \mathbf{F}}$. Let $\mathbf{N}(B) = \int_B (d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^- d\mathbf{F}$. Then $(\mathbf{M}, \mathbf{M})_\mathbf{F} = (\mathbf{N}, \mathbf{N})_\mathbf{F}$ and $(\mathbf{M}, \mathbf{N})_\mathbf{F} = (\mathbf{N}, \mathbf{M})_\mathbf{F}$. Hence $(\mathbf{N} - \mathbf{M}, \mathbf{N} - \mathbf{M})_\mathbf{F} = 0$, i.e., \mathbf{N} and \mathbf{M} as elements of $\mathbf{H}_{2, \mathbf{F}}$ are equal. By Definition 2, $\mathbf{T}(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^- = \mathbf{N}$. Therefore $\mathbf{T}(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^- = \mathbf{M}$. (Q.E.D.)

We immediately obtain the following result.

Theorem 2. (a) $\mathbf{H}_{2, \mathbf{F}}$ is a Hilbert space under the inner-product $((\cdot, \cdot))_\mathbf{F}$.

(b) $\mathbf{M} \in \mathbf{H}_{2, \mathbf{F}}$ iff there exists a \mathcal{B} -measurable matrix-valued function Ψ on Ω such that $\Psi \in \mathbf{L}_{2, \mathbf{F}}$ and for each $B \in \mathcal{B}$,

$$\mathbf{M}(B) = \int_B \Psi d\mathbf{F}.$$

Moreover if μ is a σ -finite non-negative real-valued measure w.r.t. which \mathbf{M} and \mathbf{F} are a.c., then $(d\mathbf{M}/d\mu)(d\mathbf{F}/d\mu)^- (d\mathbf{F}/d\mu) = (d\mathbf{M}/d\mu)$ a.e. μ .

Proof. (a) (b). (a) and the first part of (b) are immediate consequences of Theorem 1. For the second part of (b) we have

$$\begin{aligned} (d\mathbf{M}/d\mu) (d\mathbf{F}/d\mu)^- (d\mathbf{F}/d\mu) &= \Psi(d\mathbf{F}/d\mu) (d\mathbf{F}/d\mu)^- (d\mathbf{F}/d\mu) \\ &= \Psi(d\mathbf{F}/d\mu) = (d\mathbf{M}/d\mu) \text{ a.e. } \mu, \end{aligned}$$

where the first and the third equalities are consequences of $\mathbf{M}(B) = \int_B \Psi d\mathbf{F}$ and the second one is a consequence of $(d\mathbf{F}/d\mu) = (d\mathbf{F}/d\mu) (d\mathbf{F}/d\mu)^- (d\mathbf{F}/d\mu)$ a.e. μ . (Q.E.D.)

Remark. The significance of the Hellinger integrals $\int_{\Omega} (d\nu d\gamma^-/d\mu)$, where ν and γ are complex-valued measures and μ is a non-negative real-valued measure, in univariate stochastic processes has been pointed out by H. Cramér [1, p. 487] and U. Grenander [2, p. 207]. Our Hellinger integrals play an important role in q -variate stochastic processes. In particular, they give rise to an extension of P. Masani's work on q -variate full-rank minimal processes [5, pp. 145–150] which in turn is a generalization of a well-known result of A. N. Kolmogorov on univariate minimal sequences [4, Thm. 24]. These and other results will be published separately.

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