Electronic Journal of Probability

Electron. J. Probab. **19** (2014), no. 30, 1–26.
ISSN: 1083-6489 DOI: 10.1214/EJP.v19-2647

The hitting time of zero for a stable process

A. Kuznetsov* A. E. Kyprianou† J. C. Pardo‡ A. R. Watson§

Abstract

For any α-stable Lévy process with jumps on both sides, where $\alpha \in (1,2)$, we find the Mellin transform of the first hitting time of the origin and give an expression for its density. This complements existing work in the symmetric case and the spectrally one-sided case; cf. [38, 19] and [33, 36], respectively. We appeal to the Lamperti–Kiu representation of Chaumont et al. [16] for real-valued self-similar Markov processes. Our main result follows by considering a vector-valued functional equation for the Mellin transform of the integrated exponential Markov additive process in the Lamperti–Kiu representation. We conclude our presentation with some applications.

Keywords: Lévy processes, stable processes, hitting times, positive self-similar Markov processes, Lamperti representation, real self-similar Markov processes, Lamperti–Kiu representation, generalised Lamperti representation, exponential functional, conditioning to avoid zero.
AMS MSC 2010: 60G52, 60G18, 60G51.
Submitted to EJP on March 3, 2013, final version accepted on January 24, 2014.
Supersedes arXiv:1212.5153v2.

1 Introduction

Let $X := (X_t)_{t\geq 0}$ be a one-dimensional Lévy process, starting from zero, with law P. The Lévy–Khintchine formula states that for all $\theta \in \mathbb{R}$, the characteristic exponent $\Psi(\theta) := -\log \mathrm{E}(e^{\mathrm{i}\theta X_1})$ satisfies

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbb{1}_{\{|x|\leq 1\}})\Pi(\mathrm{d}x) + q,$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure (the *Lévy measure*) concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. The parameter q is the killing rate. When $q = 0$, we say that the process X is *unkilled*, and it remains in $\mathbb{R}$ for all time; when $q > 0$, the process X is sent to a cemetery state at a random time, independent of the path of X and with an exponential distribution of rate q.

*Department of Mathematics and Statistics, York University, Canada.
E-mail: kuznetsov@mathstat.yorku.ca
†University of Bath, UK. E-mail: a.kyprianou@bath.ac.uk
‡CIMAT, Mexico. E-mail: jcpardo@cimat.mx
§University of Zurich, Switzerland. E-mail: alexander.watson@math.uzh.ch. This article was submitted while the fourth author was at the University of Bath.

The process (X, P) is said to be a *(strictly) α-stable process* if it is an unkilled Lévy process which also satisfies the *scaling property*: under P, for every $c > 0$, the process $(cX_{tc^{-\alpha}})_{t\geq 0}$ has the same law as X. It is known that $\alpha \in (0, 2]$, and the case $\alpha = 2$ corresponds to Brownian motion, which we exclude. In fact, we will assume $\alpha \in (1, 2)$. The Lévy-Khintchine representation of such a process is as follows: $\sigma = 0$, and Π is absolutely continuous with density given by

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{\{x>0\}} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{\{x<0\}}, \qquad x \in \mathbb{R},$$

where $c_+,\, c_- \geq 0$, and $a = (c_+ - c_-)/(\alpha - 1)$.

The process X has the characteristic exponent

$$\Psi(\theta) = c|\theta|^\alpha (1 - \mathrm{i}\beta \tan \tfrac{\pi\alpha}{2} \operatorname{sgn} \theta), \qquad \theta \in \mathbb{R}, \tag{1.1}$$

where $\beta = (c_+ - c_-)/(c_+ + c_-)$ and $c = -(c_+ + c_-)\Gamma(-\alpha)\cos(\pi\alpha/2)$. For more details, see Sato [35, Theorems 14.10 and 14.15].

For consistency with the literature that we shall appeal to in this article, we shall always parametrise our α-stable process such that

$$c_+ = \Gamma(\alpha + 1)\frac{\sin(\pi\alpha\rho)}{\pi} \quad \text{and} \quad c_- = \Gamma(\alpha + 1)\frac{\sin(\pi\alpha\hat{\rho})}{\pi},$$

where $\rho = \mathrm{P}(X_t \geq 0)$ is the positivity parameter, and $\hat{\rho} = 1 - \rho$. In that case, the constant c simplifies to just $c = \cos(\pi\alpha(\rho - 1/2))$.

We take the point of view that the class of stable processes, with this normalisation, is parametrised by α and ρ; the reader will note that all the quantities above can be written in terms of these parameters. We shall restrict ourselves a little further within this class by excluding the possibility of having only one-sided jumps. This gives us the following set of admissible parameters (see [6, §VII.1]):

$$\mathcal{A}_{\mathrm{st}} = \big\{(\alpha, \rho) : \alpha \in (1, 2),\, \rho \in (1 - 1/\alpha, 1/\alpha)\big\}.$$

Let us write P_x for the law of the shifted process $X + x$ under P. We are interested in computing the distribution of

$$T_0 = \inf\{t \geq 0 : X_t = 0\},$$

the first hitting time of zero for X, under P_x, for $x \neq 0$. When $\alpha > 1$, this random variable is a.s. finite, while when $\alpha \leq 1$, points are polar for the stable process, so $T_0 = \infty$ a.s. (see [35, Example 43.22]); this explains our exclusion of such processes. This paper consists of two parts: the first deals with the case where X is symmetric. Here, we may identify a positive, self-similar Markov process R and make use of the Lamperti transform to write down the Mellin transform of T_0. The second part concerns the general case where X may be asymmetric. Here we present instead a method making use of the generalised Lamperti transform and Markov additive processes.

It should be noted that the symmetric case can be deduced from the general case, and so in principle we need not go into details when X is symmetric; however, this case provides familiar ground on which to rehearse the arguments which appear in a more complicated situation in the general case. Let us also note here that, in the symmetric case, the distribution of T_0 has been characterised in Yano et al. [38, Theorem 5.3], and the Mellin transform appears in Cordero [19, equation (1.36)]; however, these authors proceed via a different method.

For the spectrally one-sided case, which our range of parameters omits, representations of law of T_0 have been given by Peskir [33] and Simon [36]. This justifies our exclusion of the one-sided case. Nonetheless, as we explain in Remark 3.8, our methodology can also be used in this case.

We now give a short outline of the coming material.

In section 2, we suppose that the stable process X is symmetric, that is $\rho = 1/2$, and we define a process R by

$$R_t = |X_t| \mathbb{1}_{\{t < T_0\}}, \qquad t \geq 0,$$

the *radial part of* X. The process R satisfies the α-scaling property, and indeed is a positive, self-similar Markov process, whose Lamperti representation, say ξ, is a Lévy process; see section 2 for definitions. It is then known that T_0 has the same distribution as the random variable

$$I(\alpha\xi) := \int_0^\infty \exp(\alpha\xi_t)\, \mathrm{d}t,$$

the so-called exponential functional of $\alpha\xi$.

In order to find the distribution of T_0, we compute the Mellin transform, $\mathbb{E}[I(\alpha\xi)^{s-1}]$, for a suitable range of s. The result is given in Proposition 2.3. This then characterises the distribution, and the transform here can be analytically inverted.

In section 3 we consider the general case, where X may not be symmetric, and our reasoning is along very similar lines. The process R still satisfies the scaling property, but, since its dynamics depend on the sign of X, it is no longer a Markov process, and the method above breaks down. However, due to the recent work of Chaumont et al. [16], there is still a type of Lamperti representation for X, not in terms of a Lévy process, but in terms of a so-called Markov additive process, say ξ. Again, the distribution of T_0 is equal to that of $I(\alpha\xi)$ (but now with the role of ξ taken by the Markov additive process), and we develop techniques to compute a vector-valued Mellin transform for the exponential function of this Markov additive process. Further, we invert the Mellin transform of $I(\alpha\xi)$ in order to deduce explicit series representations for the law of T_0.

After the present article was submitted, the preprint of Letemplier and Simon [29] appeared, in which the authors begin from the classical potential theoretic formula

$$\mathbb{E}_x[e^{-qT_0}] = \frac{u^q(-x)}{u^q(0)}, \qquad q > 0,\ x \in \mathbb{R},$$

where u^q is the q-potential of the stable process. Manipulating this formula, they derive the Mellin transform of T_0. Their proof is rather shorter than ours, but it appears to us that the Markov additive point of view offers a good insight into the structure of real self-similar Markov processes in general, and, for example, will be central to the development of a theory of entrance laws of recurrent extensions of rssMps.

In certain scenarios the distribution of T_0 is a very convenient quantity to have, and we consider some applications in section 4: for example, we give an alternative description of the stable process conditioned to avoid zero, and we give some identities in law similar to the result of Bertoin and Yor [7] for the entrance law of a pssMp started at zero.

2 The symmetric case

In this section, we give a brief derivation of the Mellin transform of T_0 for a symmetric stable process. As we have said, we do this by considering the Lamperti transform of the radial part of the process; let us therefore recall some relevant definitions and results.

A *positive self-similar Markov process* (*pssMp*) with *self-similarity index* $\alpha > 0$ is a standard Markov process $R = (R_t)_{t\geq 0}$ with associated filtration $(\mathscr{F}_t)_{t\geq 0}$ and probability laws $(\mathrm{P}_x)_{x>0}$, on $[0,\infty)$, which has 0 as an absorbing state and which satisfies the *scaling property*, that for every $x, c > 0$,

$$\text{the law of } (cR_{tc^{-\alpha}})_{t\geq 0} \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}.$$

Here, we mean "standard" in the sense of [8], which is to say, $(\mathscr{F}_t)_{t\geq 0}$ is a complete, right-continuous filtration, and R has càdlàg paths and is strong Markov and quasi-left-continuous.

In the seminal paper [28], Lamperti describes a one-to-one correspondence between pssMps and Lévy processes, which we now outline. It may be worth noting that we have presented a slightly different definition of pssMp from Lamperti; for the connection, see [37, §0].

Let $S(t) = \int_0^t (R_u)^{-\alpha}\,\mathrm{d}u$. This process is continuous and strictly increasing until R reaches zero. Let $(T(s))_{s\geq 0}$ be its inverse, and define

$$\eta_s = \log R_{T(s)} \qquad s \geq 0.$$

Then $\eta := (\eta_s)_{s\geq 0}$ is a Lévy process started at $\log x$, possibly killed at an independent exponential time; the law of the Lévy process and the rate of killing do not depend on the value of x. The real-valued process η with probability laws $(\mathbb{P}_y)_{y\in\mathbb{R}}$ is called the *Lévy process associated to* R, or the *Lamperti transform of* R.

An equivalent definition of S and T, in terms of η instead of R, is given by taking $T(s) = \int_0^s \exp(\alpha\eta_u)\,\mathrm{d}u$ and S as its inverse. Then,

$$R_t = \exp(\eta_{S(t)})$$

for all $t \geq 0$, and this shows that the Lamperti transform is a bijection.

Let X be the symmetric stable process, that is, the process defined in the introduction with $\rho = 1/2$. We note that this process is such that $\mathrm{E}(e^{\mathrm{i}\theta X_1}) = \exp(-|\theta|^\alpha)$, and remark briefly that it has Lévy measure $\Pi(\mathrm{d}x) = k|x|^{-\alpha-1}\,\mathrm{d}x$, where

$$k = \Gamma(\alpha+1)\frac{\sin(\pi\alpha/2)}{\pi} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)}.$$

Connected to X is an important example where the Lamperti transform can be computed explicitly. In Caballero and Chaumont [10], the authors compute Lamperti transforms of killed and conditioned stable processes; the simplest of their examples, given in [10, Corollary 1], is as follows. Let

$$\tau_0^- = \inf\{t > 0 : X_t \leq 0\},$$

and denote by ξ^* the Lamperti transform of the pssMp $\left(X_t \mathbb{1}_{\{t<\tau_0^-\}}\right)_{t\geq 0}$. Then ξ^* has Lévy density

$$ke^x|e^x - 1|^{-(\alpha+1)}, \qquad x \in \mathbb{R},$$

and is killed at rate k/α.

Using this, we can analyse a pssMp which will give us the information we seek. Let

$$R_t = |X_t|\mathbb{1}_{\{t<T_0\}}, \qquad t \geq 0,$$

be the radial part of the symmetric stable process. It is simple to see that this is a pssMp. Let us denote its Lamperti transform by ξ.

In Caballero et al. [11], the authors study the Lévy process ξ; they find its characteristic function, and Wiener-Hopf factorisation, when $\alpha < 1$, as well as a decomposition into two simpler processes when $\alpha \le 1$. We will now demonstrate that their expression for the characteristic function is also valid when $\alpha > 1$, by showing that their decomposition has meaning in terms of the Lamperti transform.

Proposition 2.1. *The Lévy process ξ is the sum of two independent Lévy processes, ξ^{L} and ξ^{C}, such that*

(i) The Lévy process ξ^{L} has characteristic exponent

$$\Psi^*(\theta) - k/\alpha, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of the process ξ^*, which is the Lamperti transform of the stable process killed upon first passage below zero. That is, ξ^{L} is formed by removing the independent killing from ξ^*.

(ii) The process ξ^{C} is a compound Poisson process whose jumps occur at rate k/α, whose Lévy density is

$$\pi^{\mathrm{C}}(y) = k \frac{e^y}{(1+e^y)^{\alpha+1}}, \qquad y \in \mathbb{R}. \tag{2.1}$$

Proof. Precisely the same argument as in [27, Proposition 3.4] gives the decomposition into ξ^{L} and ξ^{C}, and the process ξ^{L} is exactly as in that case. The expression (2.1), which determines the law of ξ^{C}, follows from [11, Proposition 1], once one observes that the computation in that article does not require any restriction on α. □

We now compute the characteristic exponent of ξ. As we have mentioned, when $\alpha < 1$, this has already been computed in [11, Theorem 7], but whereas in that paper the authors were concerned with computing the Wiener-Hopf factorisation, and the characteristic function was extracted as a consequence of this, here we provide a proof directly from the above decomposition.

Theorem 2.2 (Characteristic exponent)**.** *The characteristic exponent of the Lévy process ξ is given by*

$$\Psi(\theta) = 2^\alpha \frac{\Gamma(\alpha/2 - \mathrm{i}\theta/2)}{\Gamma(-\mathrm{i}\theta/2)} \frac{\Gamma(1/2 + \mathrm{i}\theta/2)}{\Gamma((1-\alpha)/2 + \mathrm{i}\theta/2)}, \qquad \theta \in \mathbb{R}. \tag{2.2}$$

Proof. We consider separately the two Lévy processes in Proposition 2.1.

By [26, Theorem 1],

$$\Psi^{\mathrm{L}}(\theta) = \frac{\Gamma(\alpha - \mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\Gamma(\alpha/2 - \mathrm{i}\theta)\Gamma(1-\alpha/2+\mathrm{i}\theta)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)},$$

and via the beta integral,

$$\begin{aligned}
\Psi^{\mathrm{C}}(\theta) &= k \int_{-\infty}^{\infty} (1 - e^{\mathrm{i}\theta y}) \pi^C(y) \mathrm{d}y \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)} \left[\frac{1}{\alpha} - \frac{\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha+1)} \right].
\end{aligned}$$

Summing these, then using product-sum identities and [21, 8.334.2–3],

$$\begin{aligned}
\Psi(\theta) &= \Gamma(\alpha - \mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta) \left[\frac{\sin\bigl(\pi(\alpha/2 - \mathrm{i}\theta)\bigr)}{\pi} - \frac{\sin\bigl(\pi\alpha/2\bigr)}{\pi} \right] \\
&= \frac{2}{\pi} \Gamma(\alpha - \mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta) \cos\frac{\pi(\alpha - \mathrm{i}\theta)}{2} \sin\frac{-\mathrm{i}\theta\pi}{2} \\
&= 2\pi \frac{\Gamma(\alpha - \mathrm{i}\theta)}{\Gamma(1/2 + (\alpha - \mathrm{i}\theta)/2)} \frac{\Gamma(1+\mathrm{i}\theta)}{\Gamma(1/2 + (1+\mathrm{i}\theta)/2)} \frac{1}{\Gamma(-\mathrm{i}\theta/2)\Gamma((1-\alpha+\mathrm{i}\theta)/2)}.
\end{aligned}$$

Now, applying the Legendre–Gauss duplication formula [21, 8.335.1] for the gamma function, we obtain the expression in the theorem. □

We now characterise the law of the exponential functional

$$I(\alpha\xi) = \int_0^\infty e^{\alpha\xi_t}\,\mathrm{d}t$$

of the process $\alpha\xi$, which we do via the Mellin transform,

$$\mathcal{M}(s) = \mathbb{E}[I(\alpha\xi)^{s-1}],$$

for $s \in \mathbb{C}$ whose real part lies in some open interval which we will specify.

To begin with, we observe that the Laplace exponent ψ of the process $-\alpha\xi$, that is, the function such that $\mathbb{E}e^{-z\alpha\xi_1} = e^{\psi(z)}$, is given by

$$\psi(z) = -2^\alpha \frac{\Gamma(1/2 - \alpha z/2)}{\Gamma(1/2 - \alpha(1+z)/2)} \frac{\Gamma(\alpha(1+z)/2)}{\Gamma(\alpha z/2)}, \qquad \mathrm{Re}\, z \in (-1, 1/\alpha).$$

We will now proceed via the 'verification result' [26, Proposition 2]: essentially, this result says that we must find a candidate for $\mathcal{M}$ which satisfies the functional equation

$$\mathcal{M}(s+1) = -\frac{s}{\psi(-s)}\mathcal{M}(s), \tag{2.3}$$

for certain s, together with some additional conditions. Let us now state our result.

Proposition 2.3. *The Mellin transform of T_0 satisfies*

$$\mathrm{E}_1[T_0^{s-1}] = \mathbb{E}_0[I(\alpha\xi)^{s-1}] = \sin(\pi/\alpha) \frac{\cos\!\big(\frac{\pi\alpha}{2}(s-1)\big)}{\sin\!\Big(\pi\big(s-1+\frac{1}{\alpha}\big)\Big)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}, \tag{2.4}$$

for $\mathrm{Re}\, s \in \big(-\frac{1}{\alpha}, 2-\frac{1}{\alpha}\big)$.

Proof. Denote the right-hand side of (2.4) by $f(s)$. We begin by noting that $-\alpha\xi$ satisfies the Cramér condition $\psi(1/\alpha - 1) = 0$; the verification result [26, Proposition 2] therefore allows us to prove the proposition for $\mathrm{Re}\, s \in (0, 2-1/\alpha)$ once we verify some conditions on this domain.

There are three conditions to check. For the first, we require $f(s)$ is analytic and zero-free in the strip $\mathrm{Re}\, s \in (0, 2-1/\alpha)$; this is straightforward. The second requires us to verify that f satisfies (2.3). To this end, we expand $\cos$ and $\sin$ in gamma functions (via reflection formulas [21, 8.334.2–3]) and apply the Legendre duplication formula [21, 8.335.1] twice.

Finally, there is an asymptotic property which is needed. More precisely, we need to investigate the asymptotics of $f(s)$ as $\mathrm{Im}\, s \to \infty$. To do this, we will use the following formula

$$\lim_{|y|\to\infty} \frac{|\Gamma(x+\mathrm{i}y)|}{|y|^{x-\frac{1}{2}} e^{-\frac{\pi}{2}|y|}} = \sqrt{2\pi}. \tag{2.5}$$

This can be derived (see [1, Corollary 1.4.4]) from Stirling's asymptotic formula:

$$\log(\Gamma(z)) = \left(z - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + O(z^{-1}), \tag{2.6}$$

as $z \to \infty$ and $|\arg(z)| < \pi - \delta$, for fixed $\delta > 0$.

Since Stirling's asymptotic formula is uniform in any sector $|\arg(z)| < \pi - \delta$, it is easy to see that the convergence in (2.5) is also uniform in x belonging to a compact subset of $\mathbb{R}$. Using formula (2.5) we check that $|1/f(s)| = O(\exp(\pi|\mathrm{Im}\, s|))$ as $\mathrm{Im}\, s \to \infty$, uniformly in the strip $\mathrm{Re}\, s \in (0, 2-1/\alpha)$. This is the asymptotic result that we require.

The conditions of [26, Proposition 2] are therefore satisfied, and it follows that the formula in the proposition holds for $\mathrm{Re}\, s \in (0, 2-1/\alpha)$. Since $f(s)$ is analytic in the wider strip $\mathrm{Re}\, s \in (-1/\alpha, 2-1/\alpha)$, we conclude the proof by analytic extension. □

We note that the expression of Cordero [19, equation (1.36)] can be derived from this result via the duplication formula for the gamma function; furthermore, it is not difficult to deduce it from [38, Theorem 5.3].

Now, this Mellin transform completely characterises the law of T_0, and we could at this point invert the Mellin transform to find a series expansion for the density of T_0. However, as we will shortly perform precisely this calculation in section 3 for the general case, we shall leave the Mellin transform as it is and proceed to consider what happens when X may not be symmetric.

3 The asymmetric case

With the symmetric case as our model, we will now tackle the general case where X may be asymmetric. The ideas here are much the same, but the possibility of asymmetry leads us to introduce more complicated objects: our positive self-similar Markov processes become real self-similar Markov processes; our Lévy processes become Markov additive processes; and our functional equation for the Mellin transform (2.3) becomes vector-valued.

The section is laid out as follows. We devote the first two subsections to a discussion of Markov additive processes and their exponential functionals, and then discuss real self-similar Markov processes and the generalised Lamperti representation. Finally, in the last subsection, we apply the theory which we have developed to the problem of determining the law of T_0 for a general two-sided jumping stable process with $\alpha \in (1,2)$.

3.1 Markov additive processes

Let E be a finite state space and $(\mathscr{G}_t)_{t\geq 0}$ a standard filtration. A càdlàg process (ξ, J) in $\mathbb{R} \times E$ with law $\mathbb{P}$ is called a *Markov additive process (MAP)* with respect to $(\mathscr{G}_t)_{t\geq 0}$ if $(J(t))_{t\geq 0}$ is a continuous-time Markov chain in E, and the following property is satisfied, for any $i \in E$, $s, t \geq 0$:

$$\begin{aligned}&\text{given } \{J(t) = i\}, \text{ the pair } (\xi(t+s) - \xi(t), J(t+s)) \text{ is independent of } \mathscr{G}_t,\\ &\qquad\qquad \text{and has the same distribution as } (\xi(s) - \xi(0), J(s)) \text{ given } \{J(0) = i\}. \end{aligned} \tag{3.1}$$

Aspects of the theory of Markov additive processes are covered in a number of texts, among them [3] and [4]. We will mainly use the notation of [23], which principally works under the assumption that ξ is spectrally negative; the results which we quote are valid without this hypothesis, however.

Let us introduce some notation. We write $\mathbb{P}_i = \mathbb{P}(\cdot|\xi(0) = 0, J(0) = i)$; and if μ is a probability distribution on E, we write $\mathbb{P}_\mu = \mathbb{P}(\cdot|\xi(0) = 0, J(0) \sim \mu) = \sum_{i\in E} \mu(i)\mathbb{P}_i$. We adopt a similar convention for expectations.

It is well-known that a Markov additive process (ξ, J) also satisfies (3.1) with t replaced by a stopping time. Furthermore, it has the structure given by the following proposition; see [4, §XI.2a] and [23, Proposition 2.5].

Proposition 3.1. *The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$, there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$ and a sequence of iid* *random variables* $(U^n_{ij})_{n\geq 0}$*, independent of the chain* J*, such that if* $T_0 = 0$ *and* $(T_n)_{n\geq 1}$ *are the jump times of* J*, the process* ξ *has the representation*

$$\xi(t) = \mathbb{1}_{\{n>0\}}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n), \qquad t \in [T_n, T_{n+1}),\, n \geq 0.$$

For each $i \in E$, it will be convenient to define, on the same probability space, ξ_i as a Lévy process whose distribution is the common law of the ξ^n_i processes in the above representation; and similarly, for each $i, j \in E$, define U_{ij} to be a random variable having the common law of the U^n_{ij} variables.

Let us now fix the following setup. Firstly, we confine ourselves to irreducible Markov chains J. Let the state space E be the finite set $\{1, \dots, N\}$, for some $N \in \mathbb{N}$. Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j\in E}$. For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i, in the sense that $e^{\psi_i(z)} = \mathbb{E}(e^{z\xi_i(1)})$, for all $z \in \mathbb{C}$ for which the right-hand side exists. For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij}, where this exists; write $G(z)$ for the $N \times N$ matrix whose (i, j)th element is $G_{ij}(z)$.We will adopt the convention that $U_{ij} = 0$ if $q_{ij} = 0$, $i \neq j$, and also set $U_{ii} = 0$ for each $i \in E$.

A multidimensional analogue of the Laplace exponent of a Lévy process is provided by the matrix-valued function

$$F(z) = \operatorname{diag}(\psi_1(z), \dots, \psi_N(z)) + Q \circ G(z), \tag{3.2}$$

for all $z \in \mathbb{C}$ where the elements on the right are defined, where $\circ$ indicates elementwise multiplication, also called Hadamard multiplication. It is then known that

$$\mathbb{E}_i(e^{z\xi(t)}; J(t) = j) = \left(e^{F(z)t}\right)_{ij}, \qquad i,\, j \in E,$$

for all $z \in \mathbb{C}$ where one side of the equality is defined. For this reason, F is called the *matrix exponent* of the MAP ξ.

We now describe the existence of the *leading eigenvalue* of the matrix F, which will play a key role in our analysis of MAPs. This is sometimes also called the *Perron–Frobenius eigenvalue*; see [4, §XI.2c] and [23, Proposition 2.12].

Proposition 3.2. *Suppose that* $z \in \mathbb{C}$ *is such that* $F(z)$ *is defined. Then, the matrix* $F(z)$ *has a real simple eigenvalue* $\kappa(z)$*, which is larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector* $v(z)$ *may be chosen so that* $v_i(z) > 0$ *for every* $i \in E$*, and normalised such that*

$$\pi v(z) = 1 \tag{3.3}$$

where π *is the equilibrium distribution of the chain* J*.*

This leading eigenvalue features in the following probabilistic result, which identifies a martingale (also known as the Wald martingale) and change of measure analogous to the exponential martingale and Esscher transformation of a Lévy process; cf. [4, Proposition XI.2.4, Theorem XIII.8.1].

Proposition 3.3. *Let*

$$M(t, \gamma) = e^{\gamma\xi(t)-\kappa(\gamma)t}\frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \qquad t \geq 0,$$

for some γ *such that the right-hand side is defined. Then,* $M(\cdot, \gamma)$ *is a unit-mean martingale with respect to* $(\mathscr{G}_t)_{t\geq 0}$*, under any initial distribution of* $(\xi(0), J(0))$*.*

The following properties of κ will also prove useful.

Proposition 3.4. *Suppose that F is defined in some open interval D of $\mathbb{R}$. Then, the leading eigenvalue κ of F is smooth and convex on D.*

Proof. Smoothness follows from results on the perturbation of eigenvalues; see [23, Proposition 2.13] for a full proof. The convexity of κ is a consequence of the convexity properties of the entries of F. The proof follows simply from [5, Corollary 9]; see also [24, 31]. □

3.2 The Mellin transform of the exponential functional

In section 2, we studied the exponential functional of a certain Lévy process associated to the radial part of the stable process; now we are interested in obtaining some results which will assist us in computing the law of an integrated exponential functional associated to Markov additive processes.

For a MAP ξ, let

$$I(-\xi) = \int_0^\infty \exp(-\xi(t))\,\mathrm{d}t.$$

One way to characterise the law of $I(-\xi)$ is via its Mellin transform, which we write as $\mathcal{M}(s)$. This is the vector in $\mathbb{R}^N$ whose ith element is given by

$$\mathcal{M}_i(s) = \mathbb{E}_i[I(-\xi)^{s-1}], \qquad i \in E.$$

We will shortly obtain a functional equation for $\mathcal{M}$, analogous to the functional equation (2.3) which we saw in section 2. For Lévy processes, proofs of the result can be found in Carmona et al. [12, Proposition 3.1], Maulik and Zwart [30, Lemma 2.1] and Rivero [34, Lemma 2]; our proof follows the latter, making changes to account for the Markov additive property.

We make the following assumption, which is analogous to the Cramér condition for a Lévy process; recall that κ is the leading eigenvalue of the matrix F, as discussed in section 3.1.

Assumption 3.5 (Cramér condition for a Markov additive process)**.** *There exists $z_0 < 0$ such that $F(s)$ exists on $(z_0, 0)$, and some $\theta \in (0, -z_0)$, called the* Cramér number*, such that $\kappa(-\theta) = 0$.*

Since the leading eigenvalue κ is smooth and convex where it is defined, it follows also that $\kappa(-s) < 0$ for $s \in (0, \theta)$. In particular, this renders the matrix $F(-s)$ negative definite, and hence invertible. Furthermore, it follows that $\kappa'(0-) > 0$, and hence (see [4, Corollary XI.2.7] and [23, Lemma 2.14]) that ξ drifts to $+\infty$ independently of its initial state. This implies that $I(-\xi)$ is an a.s. finite random variable.

Proposition 3.6. *Suppose that ξ satisfies the Cramér condition (Assumption 3.5) with Cramér number $\theta \in (0,1)$. Then, $\mathcal{M}(s)$ is finite and analytic when $\operatorname{Re} s \in (0, 1+\theta)$, and we have the following vector-valued functional equation:*

$$\mathcal{M}(s+1) = -s(F(-s))^{-1}\mathcal{M}(s), \qquad s \in (0, \theta).$$

Proof. At the end of the proof, we shall require the existence of certain moments of the random variable

$$Q_t = \int_0^t e^{-\xi(u)}\,\mathrm{d}u,$$

and so we shall begin by establishing this.

Suppose that $s \in (0, \theta]$, and let $p > 1$. Then, by the Cramér condition, it follows that $\kappa(-s/p) < 0$, and hence for any $u \geq 0$, $e^{-u\kappa(-s/p)} \geq 1$.

Recall that the process

$$M(u,z) = e^{z\xi(u)-\kappa(z)u}\frac{v_{J(u)}(z)}{v_{J(0)}(z)}, \qquad u \geq 0$$

is a martingale (the Wald martingale) under any initial distribution $(\xi(0), J(0))$, and set

$$V(z) = \min_{j\in E} v_j(z) > 0,$$

so that for each $j \in E$, $v_j(z)/V(z) \geq 1$.

We now have everything in place to make the following calculation, which uses the Doob maximal inequality in connection with the Wald martingale in the third line, and the Cramér condition in the fourth.

$$\begin{aligned}
\mathbb{E}_i[Q_t^s] &\leq t^s \mathbb{E}_i\left[\sup_{u\leq t}\left[e^{-s\xi(u)/p}\right]^p\right] \\
&\leq t^s \mathbb{E}_i\left[\sup_{u\leq t}\left[M(u,-s/p)v_i(-s/p)(V(-s/p))^{-1}\right]^p\right] \\
&\leq t^s v_i(-s/p)^p V(-s/p)^{-p}\left(\frac{p}{p-1}\right)^p \mathbb{E}_i\left[M(t,-s/p)^p\right] \\
&\leq t^s V(-s/p)^{-p}\left(\frac{p}{p-1}\right)^p e^{-tp\kappa(-s/p)}\max_{j\in J} v_j(-s/p)^p \mathbb{E}_i\left[e^{-s\xi(t)}\right] < \infty.
\end{aligned}$$

Now, it is simple to show that for all $s > 0$, $t \geq 0$,

$$\left(\int_0^\infty e^{-\xi(u)}\,\mathrm{d}u\right)^s - \left(\int_t^\infty e^{-\xi(u)}\,\mathrm{d}u\right)^s = s\int_0^t e^{-s\xi(u)}\left(\int_0^\infty e^{-(\xi(u+v)-\xi(u))}\,\mathrm{d}v\right)^{s-1}\mathrm{d}u.$$

For each $i \in E$, we take expectations and apply the Markov additive property.

$$\begin{aligned}
&\mathbb{E}_i\left[\left(\int_0^\infty e^{-\xi(u)}\,\mathrm{d}u\right)^s - \left(\int_t^\infty e^{-\xi(u)}\,\mathrm{d}u\right)^s\right] \\
&\quad = s\sum_{j\in E}\int_0^t \mathbb{E}_i\left[e^{-s\xi(u)}; J(u)=j\right]\mathbb{E}_j\left[\int_0^\infty e^{-\xi(v)}\,\mathrm{d}v\right]^{s-1}\mathrm{d}u \\
&\quad = s\int_0^t \sum_{j\in E}\left(e^{F(-s)u}\right)_{ij}\mathbb{E}_j\left[I(-\xi)^{s-1}\right]\mathrm{d}u.
\end{aligned}$$

Since $0 < s < \theta < 1$, it follows that $\big||x|^s - |y|^s\big| \leq |x-y|^s$ for any $x, y \in \mathbb{R}$, and so we see that for each $i \in E$, the left-hand side of the above equation is bounded by $\mathbb{E}_i(Q_t^s) < \infty$. Since $\left(e^{F(-s)u}\right)_{ii} \neq 0$, it follows that $\mathbb{E}_i[I(-\xi)^{s-1}] < \infty$ also.

If we now take $t \to \infty$, the left-hand side of the previous equality is monotone increasing, while on the right, the Cramér condition ensures that $F(-s)$ is negative definite, which is a sufficient condition for convergence, giving the limit:

$$\mathcal{M}(s+1) = -s(F(-s))^{-1}\mathcal{M}(s), \qquad s \in (0,\theta).$$

Furthermore, as we know the right-hand side is finite, this functional equation allows us to conclude that $\mathcal{M}(s) < \infty$ for all $s \in (0, 1+\theta)$. It then follows from the general properties of Mellin transforms that $\mathcal{M}(s)$ is finite and analytic for all $s \in \mathbb{C}$ such that $\operatorname{Re} s \in (0, 1+\theta)$.

□

3.3 Real self-similar Markov processes

In section 2, we studied a Lévy process which was associated through the Lamperti representation to a positive, self-similar Markov process. Here we see that Markov additive processes also admit an interpretation as Lamperti-type representations of *real* self-similar Markov processes.

The structure of real self-similar Markov processes has been investigated by Chybiryakov [18] in the symmetric case, and Chaumont et al. [16] in general. Here, we give an interpretation of these authors' results in terms of a two-state Markov additive process. We begin with some relevant definitions, and state some of the results of these authors.

A *real self-similar Markov process* with *self-similarity index* $\alpha > 0$ is a standard (in the sense of [8]) Markov process $X = (X_t)_{t\geq 0}$ with probability laws $(\mathrm{P}_x)_{x\in\mathbb{R}\setminus\{0\}}$ which satisfies the *scaling property*, that for all $x \in \mathbb{R} \setminus \{0\}$ and $c > 0$,

$$\text{the law of } (cX_{tc^{-\alpha}})_{t\geq 0} \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}.$$

In [16] the authors confine their attention to processes in 'class **C.4**'. An rssMp X is in this class if, for all $x \neq 0$, $\mathrm{P}_x(\exists t > 0 : X_t X_{t-} < 0) = 1$; that is, with probability one, the process X changes sign infinitely often. As with the stable process, define

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

Such a process may be identified, under a deformation of space and time, with a Markov additive process which we call the *Lamperti–Kiu representation* of X. The following result is a simple corollary of [16, Theorem 6].

Proposition 3.7. *Let X be an rssMp in class **C.4** and fix $x \neq 0$. Define the symbol*

$$[y] = \begin{cases} 1, & y > 0, \\ 2, & y < 0. \end{cases}$$

Then there exists a time-change σ, adapted to the filtration of X, such that, under the law P_x, the process

$$(\xi(t), J(t)) = (\log|X_{\sigma(t)}|, [X_t]), \qquad t \geq 0,$$

is a Markov additive process with state space $E = \{1, 2\}$ under the law $\mathbb{P}_{[x]}$. Furthermore, the process X under P_x has the representation

$$X_t = x \exp\big(\xi(\tau(t)) + \mathrm{i}\pi(J(\tau(t)) + 1)\big), \qquad 0 \leq t < T_0,$$

where τ is the inverse of the time-change σ, and may be given by

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s \exp(\alpha\xi(u))\, \mathrm{d}u > t|x|^{-\alpha} \right\}, \qquad t < T_0, \tag{3.4}$$

We observe from the expression (3.4) for the time-change τ that under P_x, for any $x \neq 0$, the following identity holds for T_0, the hitting time of zero:

$$|x|^\alpha T_0 = \int_0^\infty e^{\alpha\xi(u)}\, \mathrm{d}u.$$

Implicit in this statement is that the MAP on the right-hand side has law $\mathbb{P}_1$ if $x > 0$, and law $\mathbb{P}_2$ if $x < 0$. This observation will be exploited in the coming section, in which we put together the theory we have outlined so far.

3.4 The hitting time of zero

We now return to the central problem of this paper: computing the distribution of T_0 for a stable process. We already have in hand the representation of T_0 for an rssMp as the exponential functional of a MAP, as well as a functional equation for this quantity which will assist us in the computation.

Let X be the stable process with parameters $(\alpha, \rho) \in \mathcal{A}_{\text{st}}$, defined in the introduction. We will restrict our attention for now to X under the measures $\mathrm{P}_{\pm 1}$; the results for other initial values can be derived via scaling.

Since X is an rssMp, it has a representation in terms of a MAP (ξ, J); futhermore, under $\mathrm{P}_{\pm 1}$,

$$T_0 = \int_0^\infty e^{\alpha\xi(s)}\,\mathrm{d}s = I(\alpha\xi);$$

to be precise, under P_1 the process ξ is under $\mathbb{P}_1$, while under P_{-1} it is under $\mathbb{P}_2$.

In [16, §4.1], the authors calculate the characteristics of the Lamperti–Kiu representation for X, that is, the processes ξ_i, and the jump distributions U_{ij} and rates q_{ij}. Using this information, and the representation (3.2), one sees that the MAP $(-\alpha\xi, J)$ has matrix exponent

$$F(z) = \begin{pmatrix} -\dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\hat\rho + \alpha z)\Gamma(1-\alpha\hat\rho-\alpha z)} & \dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)} \\ \dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\rho+\alpha z)\Gamma(1-\alpha\rho-\alpha z)} \end{pmatrix},$$

for $\operatorname{Re} z \in (-1, 1/\alpha)$.

Remark 3.8. *It is well-known that, when X does not have one-sided jumps, it changes sign infinitely often; that is, the rssMp X is in [16]'s class **C.4**. When the stable process has only one-sided jumps, which corresponds to the parameter values $\rho = 1 - 1/\alpha$, $1/\alpha$, then it jumps over 0 at most once before hitting it; the rssMp is therefore in class **C.1** or **C.2** according to the classification of [16]. The Markov chain component of the corresponding MAP then has one absorbing state, and hence is no longer irreducible. Although it seems plain that our calculations can be carried out in this case, we omit it for the sake of simplicity. As we remarked in the introduction, it is considered in [33, 36].*

We now analyse F in order to deduce the Mellin transform of T_0. The equation $\det F(z) = 0$ is equivalent to

$$\sin(\pi(\alpha\rho + \alpha z))\sin(\pi(\alpha\hat\rho + \alpha z)) - \sin(\pi\alpha\rho)\sin(\pi\alpha\hat\rho) = 0,$$

and considering the solutions of this, it is not difficult to deduce that $\kappa(1/\alpha - 1) = 0$; that is, $-\alpha\xi$ satisfies the Cramér condition (Assumption 3.5) with Cramér number $\theta = 1 - 1/\alpha$.

Define

$$f_1(s) := \mathrm{E}_1[T_0^{s-1}] = \mathbb{E}_1[I(\alpha\xi)^{s-1}], \quad f_2(s) := \mathrm{E}_{-1}[T_0^{s-1}] = \mathbb{E}_2[I(\alpha\xi)^{s-1}],$$

which by Proposition 3.6 are defined when $\operatorname{Re} s \in (0, 2 - 1/\alpha)$. This proposition also implies that

$$\mathbf{B}(s) \times \begin{bmatrix} f_1(s+1) \\ f_2(s+1) \end{bmatrix} = \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix}, \qquad s \in (0, 1 - 1/\alpha), \tag{3.5}$$

where $\mathbf{B}(s) := -F(-s)/s$. Using the reflection formula for the gamma function we find that

$$\mathbf{B}(s) = \frac{\alpha}{\pi}\Gamma(\alpha - \alpha s)\Gamma(\alpha s)\begin{bmatrix} \sin(\pi\alpha(\hat\rho - s)) & -\sin(\pi\alpha\hat\rho) \\ -\sin(\pi\alpha\rho) & \sin(\pi\alpha(\rho - s)) \end{bmatrix},$$

for $\mathrm{Re}\, s \in (-1/\alpha, 1)$, $s \neq 0$, and

$$\det(\mathbf{B}(s)) = -\alpha^2 \frac{\Gamma(\alpha - \alpha s)\Gamma(\alpha s)}{\Gamma(1 - \alpha + \alpha s)\Gamma(1 - \alpha s)}, \qquad \mathrm{Re}\, s \in (-1/\alpha, 1),\ s \neq 0. \tag{3.6}$$

Therefore, if we define $\mathbf{A}(s) = (\mathbf{B}(s))^{-1}$, we have

$$\mathbf{A}(s) = -\frac{1}{\pi\alpha}\Gamma(1 - \alpha + \alpha s)\Gamma(1 - \alpha s) \begin{bmatrix} \sin(\pi\alpha(\rho - s)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & \sin(\pi\alpha(\hat{\rho} - s)) \end{bmatrix}$$

for $\mathrm{Re}\, s \in (1 - 2/\alpha, 1 - 1/\alpha)$, and may rewrite (3.5) in the form

$$\begin{bmatrix} f_1(s+1) \\ f_2(s+1) \end{bmatrix} = \mathbf{A}(s) \times \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix}, \qquad s \in (0, 1 - 1/\alpha). \tag{3.7}$$

The following theorem is our main result.

Theorem 3.9. *For* $-1/\alpha < \mathrm{Re}(s) < 2 - 1/\alpha$ *we have*

$$\mathrm{E}_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \frac{\sin\left(\pi\hat{\rho}(1 - \alpha + \alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1 - \alpha + \alpha s)\right)} \frac{\Gamma(1 + \alpha - \alpha s)}{\Gamma(2 - s)}. \tag{3.8}$$

The corresponding expression for $\mathrm{E}_{-1}[T_0^{s-1}]$ *can be obtained from (3.8) by changing* $\hat{\rho} \mapsto \rho$*.*

Let us denote the function in the right-hand side of (3.8) by $h_1(s)$, and by $h_2(s)$ the function obtained from $h_1(s)$ by replacing $\hat{\rho} \mapsto \rho$. Before we are able to prove Theorem 3.9, we need to establish several properties of these functions.

Lemma 3.10.

(i) There exists $\varepsilon \in (0, 1 - 1/\alpha)$ *such that the functions* $h_1(s)$, $h_2(s)$ *are analytic and zero-free in the vertical strip* $0 \le \mathrm{Re}(s) \le 1 + \varepsilon$*.*

(ii) For any $-\infty < a < b < +\infty$ *there exists* $C > 0$ *such that*

$$e^{-\pi|\mathrm{Im}(s)|} < |h_i(s)| < e^{-\frac{\pi}{2}(\alpha - 1)|\mathrm{Im}(s)|}, \quad i = 1, 2$$

for all s *in the vertical strip* $a \le \mathrm{Re}(s) \le b$ *satisfying* $|\mathrm{Im}(s)| > C$*.*

Proof. It is clear from the definition of $h_1(s)$ that it is a meromorphic function. Its zeroes are contained in the set

$$\{2, 3, 4, \dots\} \cup \{1 - 1/\alpha + n/(\alpha\hat{\rho}) :\ n \in \mathbb{Z}, n \neq 0\}$$

and its poles are contained in the set

$$\{1 + n/\alpha :\ n \ge 1\} \cup \{n - 1/\alpha :\ n \in \mathbb{Z}, n \neq 1\}.$$

In particular, $h_1(s)$ possesses neither zeroes nor poles in the strip $0 \le \mathrm{Re}(s) \le 1$. The same is clearly true for $h_2(s)$, which proves part (i).

We now make use of Stirling's formula, as we did in section 2. Applying (2.5) to $h_1(s)$ we find that as $s \to \infty$ (uniformly in the strip $a \le \mathrm{Re}(s) \le b$) we have

$$\log(|h_1(s)|) = -\frac{\pi}{2}(1 + \alpha - 2\alpha\hat{\rho})|\mathrm{Im}(s)| + O(\log(|\mathrm{Im}(s)|)).$$

Since for $\alpha > 1$, the admissible parameters α, ρ must satisfy $\alpha - 1 < \alpha\hat{\rho} < 1$, this shows that

$$\alpha - 1 < 1 + \alpha - 2\alpha\hat{\rho} < 3 - \alpha < 2,$$

and completes the proof of part (ii). □

Lemma 3.11. *The functions $h_1(s)$, $h_2(s)$ satisfy the system of equations (3.7).*

Proof. Denote the elements of the matrix $\mathbf{A}(s)$ by $A_{ij}(s)$. Multiplying the first row of $\mathbf{A}(s)$ by the column vector $[h_1(s), h_2(s)]^T$, and using identity $\sin(\pi\rho) = \sin(\pi\hat{\rho})$, we obtain

$$\begin{aligned} &A_{11}(s)h_1(s) + A_{12}(s)h_2(s) \\ &\quad = -\frac{1}{\pi\alpha}\frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})}\frac{\Gamma(1-\alpha s)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)}\left[\frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}\Gamma(1-\alpha+\alpha s)\right] \\ &\quad\quad \times \left\{ \sin(\pi\alpha(\rho-s))\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right) + \sin(\pi\alpha\hat{\rho})\sin\left(\pi\rho(1-\alpha+\alpha s)\right) \right\}. \end{aligned}$$

Applying identity $\Gamma(z+1) = z\Gamma(z)$ and reflection formula for the gamma function, we rewrite the expression in the square brackets as follows:

$$\left[\frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}\Gamma(1-\alpha+\alpha s)\right] = \frac{\alpha\Gamma(\alpha-\alpha s)}{\Gamma(1-s)}\Gamma(1-\alpha+\alpha s) = \frac{\pi\alpha}{\sin(\pi\alpha(1-s))\Gamma(1-s)}.$$

Applying certain trigonometric identities, we obtain

$$\begin{aligned} &\sin(\pi\alpha(\rho-s))\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right) + \sin(\pi\alpha\hat{\rho})\sin\left(\pi\rho(1-\alpha+\alpha s)\right) \\ &\qquad\qquad = \sin(\pi\alpha(1-s))\sin(\pi\hat{\rho}(1+\alpha s)). \end{aligned}$$

Combining the above three formulas we conclude

$$A_{11}(s)h_1(s) + A_{12}(s)h_2(s) = -\frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})}\frac{\sin\left(\pi\hat{\rho}(1+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)}\frac{\Gamma(1-\alpha s)}{\Gamma(1-s)} = h_1(s+1).$$

The derivation of the identity $A_{21}(s)h_1(s) + A_{22}(s)h_2(s) = h_2(s+1)$ is identical. We have now established that two functions $h_i(s)$ satisfy the system of equations (3.7). □

Proof of Theorem 3.9. Our goal now is to establish the uniqueness of solutions to the system (3.7) in a certain class of meromorphic functions, which contains both $h_i(s)$ and $f_i(s)$. This will imply $h_i(s) \equiv f_i(s)$. Our argument is similar in spirit to the proof of Proposition 2 in [26].

First of all, we check that there exists $\varepsilon \in (0, 1/2 - 1/(2\alpha))$, such that the functions $f_1(s)$, $f_2(s)$ are analytic and bounded in the open strip

$$\mathcal{S}_\varepsilon = \{s \in \mathbb{C} : \varepsilon < \mathrm{Re}(s) < 1 + 2\varepsilon\}$$

This follows from Proposition 3.6 and the estimate

$$|f_1(s)| = |\mathrm{E}_1[T_0^{s-1}]| \le \mathrm{E}_1[|T_0^{s-1}|] = \mathrm{E}_1[T_0^{\mathrm{Re}(s)-1}] = f_1(\mathrm{Re}(s)).$$

The same applies to f_2. Given results of Lemma 3.10, we can also assume that ε is small enough, so that the functions $h_i(s)$ are analytic, zero-free and bounded in the strip $\mathcal{S}_\varepsilon$.

Let us define $D(s) := f_1(s)h_2(s) - f_2(s)h_1(s)$ for $s \in \mathcal{S}_\varepsilon$. From the above properties of $f_i(s)$ and $h_i(s)$ we conclude that $D(s)$ is analytic and bounded in $\mathcal{S}_\varepsilon$. Our first goal is to show that $D(s) \equiv 0$.

If both s and $s+1$ belong to $\mathcal{S}_\varepsilon$, then the function $D(s)$ satisfies the equation

$$D(s+1) = -\frac{1}{\alpha^2}\frac{\Gamma(1-\alpha+\alpha s)\Gamma(1-\alpha s)}{\Gamma(\alpha-\alpha s)\Gamma(\alpha s)}D(s), \tag{3.9}$$

as is easily established by taking determinants of the matrix equation

$$\begin{bmatrix} f_1(s+1) & h_1(s+1) \\ f_2(s+1) & h_2(s+1) \end{bmatrix} = \mathbf{A}(s) \times \begin{bmatrix} f_1(s) & h_1(s) \\ f_2(s) & h_2(s) \end{bmatrix},$$

and using (3.6) and the identity $\mathbf{A}(s) = \mathbf{B}(s)^{-1}$.

Define also

$$G(s) := \frac{\Gamma(s-1)\Gamma(\alpha - \alpha s)}{\Gamma(1-s)\Gamma(-\alpha + \alpha s)} \sin\left(\pi\left(s + \frac{1}{\alpha}\right)\right).$$

It is simple to check that:

(i) G satisfies the functional equation (3.9);

(ii) G is analytic and zero-free in the strip $\mathcal{S}_\varepsilon$;

(iii) $|G(s)| \to \infty$ as $\mathrm{Im}(s) \to \infty$, uniformly in the strip $\mathcal{S}_\varepsilon$ (use (2.5) and $\alpha > 1$).

We will now take the ratio of D and G in order to obtain a periodic function, borrowing a technique from the theory of functional equations (for a similar idea applied to the gamma function, see [2, §6].) We thus define $H(s) := D(s)/G(s)$ for $s \in \mathcal{S}_\varepsilon$. The property (ii) guarantees that H is analytic in the strip $\mathcal{S}_\varepsilon$, while property (i) and (3.9) show that $H(s+1) = H(s)$ if both s and $s+1$ belong to $\mathcal{S}_\varepsilon$. Therefore, we can extend $H(s)$ to an entire function satisfying $H(s+1) = H(s)$ for all $s \in \mathbb{C}$. Using the periodicity of $H(s)$, property (iii) of the function $G(s)$ and the fact that the function $D(s)$ is bounded in the strip $\mathcal{S}_\varepsilon$, we conclude that $H(s)$ is bounded on $\mathbb{C}$ and $H(s) \to 0$ as $\mathrm{Im}(s) \to \infty$. Since H is entire, it follows that $H \equiv 0$.

So far, we have proved that for all $s \in \mathcal{S}_\varepsilon$ we have $f_1(s)h_2(s) = f_2(s)h_1(s)$. Let us define $w(s) := f_1(s)/h_1(s) = f_2(s)/h_2(s)$. Since both $f_i(s)$ and $h_i(s)$ satisfy the same functional equation (3.7), if s and $s+1$ belong to $\mathcal{S}_\varepsilon$ we have

$$\begin{aligned} w(s+1)h_1(s+1) &= f_1(s+1) \\ &= A_{11}(s)f_1(s) + A_{12}(s)f_2(s) \\ &= w(s)[A_{11}(s)h_1(s) + A_{12}(s)h_2(s)], \end{aligned}$$

and therefore $w(s+1) = w(s)$. Using again the fact that f_i and h_i are analytic in this strip and h_i is also zero free there, we conclude that $w(s)$ is analytic in $\mathcal{S}_\varepsilon$, and the periodicity of w implies that it may be extended to an entire periodic function. Lemma 3.10(ii) together with the uniform boundedness of $f_i(s)$ in $\mathcal{S}_\varepsilon$ imply that there exists a constant $C > 0$ such that for all $s \in \mathcal{S}_\varepsilon$,

$$|w(s)| < Ce^{\pi|\mathrm{Im}(s)|}.$$

By periodicity of w, we conclude that the above bound holds for all $s \in \mathbb{C}$. Since w is periodic with period one, this bound implies that w is a constant function (this follows from the Fourier series representation of periodic analytic functions; see the proof of Proposition 2 in [26]). Finally, we know that $f_i(1) = h_i(1) = 1$, and so we conclude that $w(s) \equiv 1$. Hence, $f_i(s) \equiv h_i(s)$ for all $s \in \mathcal{S}_\varepsilon$. Since $h_i(s)$ are analytic in the wider strip $-1/\alpha < \mathrm{Re}(s) < 2 - 1/\alpha$, by analytic continuation we conclude that (3.8) holds for all s in $-1/\alpha < \mathrm{Re}(s) < 2 - 1/\alpha$. □

Remark 3.12. *Since the proof of Theorem 3.9 is based on a verification technique, it does not reveal how we derived the formula on the right-hand side of (3.8), for which we took a trial and error approach. The expression in (3.8) is already known, or may be easily computed, for the spectrally positive case ($\rho = 1 - 1/\alpha$; in this case T_0 is the time of first passage below the level zero, and indeed has a positive $1/\alpha$-stable law, as may be seen from [35, Theorem 46.3]), the spectrally negative case ($\rho = 1/\alpha$; due to [36, Corollary 1]) and for the symmetric case ($\rho = 1/2$; due to [38, 19]), we sought a function which interpolated these three cases and satisfied the functional equation (3.7). After a candidate was found, we verified that this was indeed the Mellin transform, using the argument above.*

We turn our attention to computing the density of T_0. Let us define $\|x\| = \min_{n\in\mathbb{Z}}|x-n|$, and

$$\mathcal{L} = \mathbb{R} \setminus (\mathbb{Q} \cup \{x \in \mathbb{R} : \lim_{n\to\infty} \tfrac{1}{n} \log\|nx\| = 0\}).$$

This set was introduced in [22], where it was shown that $\mathcal{L}$ is a subset of the Liouville numbers and that $x \in \mathcal{L}$ if and only if the coefficients of the continued fraction representation of x grow extremely fast. It is known that $\mathcal{L}$ is dense, yet it is a rather small set: it has Hausdorff dimension zero, and therefore its Lebesgue measure is also zero.

For $\alpha \in \mathbb{R}$ we also define

$$\mathcal{K}(\alpha) = \{N \in \mathbb{N} : \|(N - \tfrac{1}{2})\alpha\| > \exp(-\tfrac{\alpha-1}{2}(N-2)\log(N-2))\}.$$

Proposition 3.13. *Assume that $\alpha \notin \mathbb{Q}$.*

(i) The set $\mathcal{K}(\alpha)$ is unbounded and has density equal to one:

$$\lim_{n\to\infty} \frac{\#\{\mathcal{K}(\alpha) \cap [1,n]\}}{n} = 1.$$

(ii) If $\alpha \notin \mathcal{L}$, the set $\mathbb{N} \setminus \mathcal{K}(\alpha)$ is finite.

Proof. Part (i) follows, after some short manipulation, from the well-known fact that for any irrational α the sequence $\|(N - \frac{1}{2})\alpha\|$ is uniformly distributed on the interval $(0, 1/2)$.

To prove part (ii), first assume that $\alpha \notin \mathcal{L}$. Since $\lim_{n\to+\infty} \frac{1}{n}\log\|n\alpha\| = 0$, there exists $C > 0$ such that for all n we have $\|n\alpha\| > C2^{-n}$. Then for all N we have

$$\|(N - \tfrac{1}{2})\alpha\| \geq \frac{1}{2}\|(2N-1)\alpha\| > C2^{-2N}.$$

Since for all N large enough it is true that

$$C2^{-2N} > \exp(-\tfrac{\alpha-1}{2}(N-2)\log(N-2)),$$

we conclude that all N large enough will be in the set $\mathcal{K}(\alpha)$, therefore the set $\mathbb{N} \setminus \mathcal{K}(\alpha)$ is finite. □

Theorem 3.14. *Let p be the density of T_0 under P_1.*

(i) If $\alpha \notin \mathbb{Q}$ then for all $t > 0$ we have

$$p(t) = \lim_{\substack{N\in\mathcal{K}(\alpha)\\ N\to\infty}} \Bigg[\frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi\sin(\pi\hat\rho)} \sum_{1\le k<\alpha(N-\frac{1}{2})-1} \sin(\pi\hat\rho(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \times \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}} - \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi\sin(\pi\hat\rho)} \sum_{1\le k<N} \frac{\sin(\pi\alpha\hat\rho k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k-1)} t^{-k-1+\frac{1}{\alpha}} \Bigg]. \tag{3.10}$$

The above limit is uniform for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$.

(ii) If $\alpha = m/n$ (where m and n are coprime natural numbers) then for all $t > 0$ we have

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi\sin(\pi\hat\rho)} \sum_{\substack{k\ge1\\ k\not\equiv -1 \text{ (mod m)}}} \sin(\pi\hat\rho(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}}$$
$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi\sin(\pi\hat\rho)} \sum_{\substack{k\ge1\\ k\not\equiv 0 \text{ (mod n)}}} \frac{\sin(\pi\alpha\hat\rho k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k-1)} t^{-k-1+\frac{1}{\alpha}}$$
$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi^2\alpha\sin(\pi\hat\rho)} \sum_{k\ge1} (-1)^{km} \frac{\Gamma\left(kn-\frac{1}{\alpha}\right)}{(km-2)!} R_k(t) t^{-kn-1+\frac{1}{\alpha}}, \tag{3.11}$$

where

$$R_k(t) := \pi\alpha\hat{\rho}\cos(\pi\hat{\rho}km) \\ - \sin(\pi\hat{\rho}km)\left[\pi\cot\left(\tfrac{\pi}{\alpha}\right) - \psi\left(kn - \tfrac{1}{\alpha}\right) + \alpha\psi(km-1) + \log(t)\right]$$

and ψ *is the digamma function. The three series in (3.11) converge uniformly for* $t \in [\varepsilon, \infty)$ *and any* $\varepsilon > 0$*.*

(iii) For all values of α *and any* $c > 0$*, the following asymptotic expansion holds as* $t \downarrow 0$*:*

$$p(t) = \frac{\alpha \sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat{\rho})} \sum_{1 \le n < 1+c} \sin(\pi\alpha\hat{\rho}n) \frac{\Gamma(\alpha n + 1)}{\Gamma\left(n + \frac{1}{\alpha}\right)} (-1)^{n-1} t^{n-1+\frac{1}{\alpha}} + O(t^{c+\frac{1}{\alpha}}).$$

Proof. Recall that $h_1(s) = \mathrm{E}_1[T_0^{s-1}]$ denotes the function in (3.8). According to Lemma 3.10(ii), for any $x \in \mathbb{R}$, $h_1(x + \mathrm{i}y)$ decreases to zero exponentially fast as $y \to \infty$. This implies that the density of T_0 exists and is a smooth function. It also implies that $p(t)$ can be written as the inverse Mellin transform,

$$p(t) = \frac{1}{2\pi\mathrm{i}} \int_{1+\mathrm{i}\mathbb{R}} h_1(s) t^{-s} \mathrm{d}s. \tag{3.12}$$

The function $h_1(s)$ is meromorphic, and it has poles at points

$$\{s_n^{(1)} := 1 + n/\alpha \; : \; n \ge 1\} \cup \{s_n^{(2)} := n - 1/\alpha \; : \; n \ge 2\} \cup \{s_n^{(3)} := -n - 1/\alpha \; : \; n \ge 0\}$$

If $\alpha \notin \mathbb{Q}$, all these points are distinct and $h_1(s)$ has only simple poles. When $\alpha \in \mathbb{Q}$, some of $s_n^{(1)}$ and $s_m^{(2)}$ will coincide, and $h_1(s)$ will have double poles at these points.

Let us first consider the case $\alpha \notin \mathbb{Q}$, so that all poles are simple. Let $N \in \mathcal{K}(\alpha)$ and define $c = c(N) = N + \frac{1}{2} - \frac{1}{\alpha}$. Lemma 3.10(ii) tells us that $h_1(s)$ decreases exponentially to zero as $\mathrm{Im}(s) \to \infty$, uniformly in any finite vertical strip. Therefore, we can shift the contour of integration in (3.12) and obtain, by Cauchy's residue theorem,

$$p(t) = -\sum_n \mathrm{Res}_{s=s_n^{(1)}}(h_1(s)t^{-s}) - \sum_m \mathrm{Res}_{s=s_m^{(2)}}(h_1(s)t^{-s}) \\ + \frac{1}{2\pi\mathrm{i}} \int_{c(N)+\mathrm{i}\mathbb{R}} h_1(s)t^{-s}\mathrm{d}s, \tag{3.13}$$

where $\sum_n$ and $\sum_m$ indicate summation over $n \ge 1$ such that $s_n^{(1)} < c(N)$ and over $m \ge 2$ such that $s_m^{(2)} < c(N)$, respectively. Computing the residues we obtain the two sums in the right-hand side of (3.10).

Now our goal is to show that the integral term

$$I_N(t) := \frac{1}{2\pi\mathrm{i}} \int_{c(N)+\mathrm{i}\mathbb{R}} h_1(s)t^{-s}\mathrm{d}s$$

converges to zero as $N \to +\infty$, $N \in \mathcal{K}(\alpha)$. We use the reflection formula for the gamma function and the inequalities

$$|\sin(\pi x)| > \|x\|, \qquad x \in \mathbb{R},$$
$$|\sin(x)|\cosh(y) \le |\sin(x + \mathrm{i}y)| = \sqrt{\cosh^2(y) - \cos^2(x)} \le \cosh(y), \qquad x, y \in \mathbb{R},$$

to estimate $h_1(s)$, $s = c(N) + \mathrm{i}u$, as follows

$$|h_1(s)| = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \left| \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)} \frac{\sin(\pi s)}{\sin(\pi\alpha(s-1))} \frac{\Gamma(s-1)}{\Gamma(\alpha(s-1))} \right| \\ \le \frac{C_1}{\|\alpha(N-\frac{1}{2})\|} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} \left| \frac{\Gamma(s-1)}{\Gamma(\alpha(s-1))} \right|. \tag{3.14}$$

Using Stirling's formula (2.6), we find that

$$\frac{\Gamma(s)}{\Gamma(\alpha s)} = \sqrt{\alpha} e^{-s((\alpha-1)\log(s)+A)+O(s^{-1})}, \qquad s \to \infty, \ \mathrm{Re}(s) > 0, \tag{3.15}$$

where $A := 1 - \alpha + \alpha \log(\alpha) > 0$. Therefore, there exists a constant $C_2 > 0$ such that for $\mathrm{Re}\, s > 0$ we can estimate

$$\left| \frac{\Gamma(s)}{\Gamma(\alpha s)} \right| < C_2 e^{-(\alpha-1)\mathrm{Re}(s)\log(\mathrm{Re}(s)) + (\alpha-1)|\mathrm{Im}(s)|\frac{\pi}{2}}.$$

Combining the above estimate with (3.14) and using the fact that $N \in \mathcal{K}(\alpha)$ we find that

$$\begin{aligned} |h_1(c(N)+\mathrm{i}u)| &< \frac{C_1 C_2}{\|\alpha(N-\frac{1}{2})\|} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} e^{-(\alpha-1)(c(N)-1)\log(c(N)-1)+(\alpha-1)|u|\frac{\pi}{2}} \\ &< C_1 C_2 e^{-\frac{\alpha-1}{2}(N-2)\log(N-2)} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} e^{(\alpha-1)|u|\frac{\pi}{2}}. \end{aligned}$$

Note that the function in the right-hand side of the above inequality decreases to zero exponentially fast as $|u| \to \infty$ (since $\alpha\hat{\rho} + \frac{1}{2}(\alpha-1) - \alpha < 0$), and hence in particular is integrable on $\mathbb{R}$. Thus we can estimate

$$\begin{aligned} |I_N(t)| = \frac{t^{-c(N)}}{2\pi} \left| \int_{\mathbb{R}} h_1(c(N)+\mathrm{i}u)) t^{-\mathrm{i}u}\, \mathrm{d}u \right| &< \frac{t^{-c(N)}}{2\pi} \int_{\mathbb{R}} |h_1(c(N)+\mathrm{i}u))|\, \mathrm{d}u \\ &< \frac{t^{-c(N)}}{2\pi} C_1 C_2 e^{-\frac{\alpha-1}{2}(N-2)\log(N-2)} \int_{\mathbb{R}} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} e^{(\alpha-1)|u|\frac{\pi}{2}}\, \mathrm{d}u. \end{aligned}$$

When $N \to \infty$, the quantity in the right-hand side of the above inequality converges to zero for every $t > 0$. This ends the proof of part (i).

The proof of part (ii) is very similar, and we offer only a sketch. It also begins with (3.13) and uses the above estimate for $h_1(s)$. The only difference is that when $\alpha \in \mathbb{Q}$ some of $s_n^{(1)}$ and $s_m^{(2)}$ will coincide, and $h_1(s)$ will have double poles at these points. The terms with double poles give rise to the third series in (3.11). In this case all three series are convergent, and we can express the limit of partial sums as a series in the usual sense.

The proof of part (iii) is much simpler: we need to shift the contour of integration in (3.13) in the opposite direction ($c < 0$). The proof is identical to the proof of Theorem 9 in [25]. □

Remark 3.15. *We offer some remarks on the asymptotic expansion. When $\hat{\rho} = 1/\alpha$, all of its terms are equal to zero. This is the spectrally positive case, in which T_0 has the law of a positive $1/\alpha$-stable random variable, and it is known that its density is exponentially small at zero; see [35, equation (14.35)] for a more precise result.*

Otherwise, the series given by including all the terms in (iii) is divergent for all $t > 0$. This may be seen from the fact that the terms do not approach 0; we sketch this now. When $\alpha\hat{\rho} \in \mathbb{Q}$, some terms are zero, and in the rest the sine term is bounded away from zero; when $\alpha\hat{\rho} \notin \mathbb{Q}$, it follows that $\limsup_{n\to\infty} |\sin(\pi\alpha\hat{\rho}n)| = 1$. One then bounds the ratio of gamma functions from below by $\Gamma(\alpha n)/\Gamma((1+\varepsilon)n)$, for some small enough $\varepsilon > 0$ and large n. This grows superexponentially due to (3.15), so $t^n \Gamma(\alpha n + 1)/\Gamma(n + 1/\alpha)$ is unbounded as $n \to \infty$.

The next corollary shows that, for almost all irrational α, the expression (3.10) can be written in a simpler form.

Corollary 3.16. *If $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ then*

$$
\begin{aligned}
p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat\rho)} \sum_{k\geq 1} \sin(\pi\hat\rho(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!} (-1)^{k-1} t^{-1-\frac{k}{\alpha}} \\
- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi \sin(\pi\hat\rho)} \sum_{k\geq 1} \frac{\sin(\pi\alpha\hat\rho k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)} t^{-k-1+\frac{1}{\alpha}}.
\end{aligned} \tag{3.16}
$$

The two series in the right-hand side of the above formula converge uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$.

Proof. As we have shown in Proposition 3.13, if $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ then the set $\mathbb{N} \setminus \mathcal{K}(\alpha)$ is finite. Therefore we can remove the restriction $N \in \mathcal{K}(\alpha)$ in (3.10), and need only show that both series in (3.16) converge.

In [22, Proposition 1] it was shown that $x \in \mathcal{L}$ iff $x^{-1} \in \mathcal{L}$. Therefore, according to our assumption, both α and $1/\alpha$ are not in the set $\mathcal{L}$. From the definition of $\mathcal{L}$ we see that there exists $C > 0$ such that $\|\alpha n\| > C2^{-n}$ and $\|\alpha^{-1} n\| > C2^{-n}$ for all integers n. Using the estimate $|\sin(\pi x)| \geq \|x\|$ and Stirling's formula (2.6), it is easy to see that both series in in (3.16) converge (uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$), which ends the proof of the corollary. □

Remark 3.17. *Note that formula (3.16) may not be true if $\alpha \in \mathcal{L}$, as the series may fail to converge. An example where this occurs is given after Theorem 5 in [25].*

4 Applications

4.1 Conditioning to avoid zero

In [16, §4.2], Chaumont, Pantí and Rivero discuss a harmonic transform of a stable process with $\alpha > 1$ which results in *conditioning to avoid zero*. The results quoted in that paper are a special case of the notion of conditioning a Lévy process to avoid zero, which is explored in Pantí [32].

In these works, in terms of the parameters used in the introduction, the authors define

$$
h(x) = \frac{\Gamma(2-\alpha)\sin(\pi\alpha/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\pi\alpha/2))} (1 - \beta \operatorname{sgn}(x))|x|^{\alpha-1}, \qquad x \in \mathbb{R}. \tag{4.1}
$$

If we write the function h in terms of the (α, ρ) parameterisation which we prefer, this gives

$$
h(x) = -\Gamma(1-\alpha)\frac{\sin(\pi\alpha\hat\rho)}{\pi}|x|^{\alpha-1}, \qquad x > 0,
$$

and the same expression with $\hat\rho$ replaced by ρ when $x < 0$.

In [32], Pantí proves the following proposition for all Lévy processes, and $x \in \mathbb{R}$, with a suitable definition of h. Here we quote only the result for stable processes and $x \neq 0$. Hereafter, $(\mathscr{F}_t)_{t\geq 0}$ is the standard filtration associated with X, and n refers to the excursion measure of the stable process away from zero, normalised (see [32, (7)] and [20, (4.11)]) such that

$$
n(1 - e^{-q\zeta}) = 1/u^q(0),
$$

where ζ is the excursion length and u^q is the q-potential density of the stable process.

Proposition 4.1 ([32, Theorem 2, Theorem 6]). *Let X be a stable process, and h the function in (4.1).*

(i) *The function* h *is invariant for the stable process killed on hitting* 0*, that is,*

$$\mathrm{E}_x[h(X_t), t < T_0] = h(x), \qquad t > 0,\, x \neq 0. \tag{4.2}$$

Therefore, we may define a family of measures $\mathrm{P}_x^{\updownarrow}$ *by*

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \frac{1}{h(x)} \mathrm{E}_x[h(X_t) \mathbb{1}_\Lambda, t < T_0], \qquad x \neq 0,\, \Lambda \in \mathscr{F}_t,$$

for any $t \geq 0$*.*

(ii) *The function* h *can be represented as*

$$h(x) = \lim_{q \downarrow 0} \frac{\mathrm{P}_x(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)}, \qquad x \neq 0,$$

where $\mathbf{e}_q$ *is an independent exponentially distributed random variable with parameter* q*. Furthermore, for any stopping time* T *and* $\Lambda \in \mathscr{F}_T$*, and any* $x \neq 0$*,*

$$\lim_{q \downarrow 0} \mathrm{P}_x(\Lambda, T < \mathbf{e}_q | T_0 > \mathbf{e}_q) = \mathrm{P}_x^{\updownarrow}(\Lambda).$$

This justifies the name 'the stable process conditioned to avoid zero' for the canonical process associated with the measures $(\mathrm{P}_x^{\updownarrow})_{x \neq 0}$. We will denote this process by $X^{\updownarrow}$.

Our aim in this section is to prove the following variation of Proposition 4.1(ii), making use of our expression for the density of T_0. Our presentation here owes much to Yano et al. [39, §4.3].

Proposition 4.2. *Let* X *be a stable process adapted to the filtration* $(\mathscr{F}_t)_{t \geq 0}$*, and* $h \colon \mathbb{R} \to \mathbb{R}$ *as in (4.1).*

(i) *Define the function*

$$Y(s, x) = \frac{\mathrm{P}_x(T_0 > s)}{h(x) n(\zeta > s)} \qquad s > 0,\, x \neq 0.$$

Then, for any $x \neq 0$*,*

$$\lim_{s \to \infty} Y(s, x) = 1, \tag{4.3}$$

and furthermore, Y *is bounded away from* 0 *and* ∞ *on its whole domain.*

(ii) *For any* $x \neq 0$*, stopping time* T *such that* $\mathbb{E}_x[T] < \infty$*, and* $\Lambda \in \mathscr{F}_T$*,*

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \lim_{s \to \infty} \mathrm{P}_x(\Lambda | T_0 > T + s).$$

Proof. We begin by proving

$$h(x) = \lim_{s \to \infty} \frac{\mathrm{P}_x(T_0 > s)}{n(\zeta > s)}, \tag{4.4}$$

for $x > 0$, noting that when $x < 0$, we may deduce the same limit by duality.

Let us denote the density of the measure $\mathrm{P}_x(T_0 \in \cdot)$ by $p(x, \cdot)$. A straightforward application of scaling shows that

$$\mathrm{P}_x(T_0 > t) = \mathrm{P}_1(T_0 > x^{-\alpha} t), \qquad x > 0,\, t \geq 0,$$

and so we may focus our attention on $p(1, t)$, which is the quantity given as $p(t)$ in Theorem 3.14. In particular, we have

$$p(1, t) = -\frac{\sin^2(\pi/\alpha)}{\pi \sin(\pi\hat{\rho})} \frac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha)} \frac{\Gamma(1 - 1/\alpha)}{\Gamma(\alpha - 1)} t^{1/\alpha - 2} + O(t^{-1/\alpha - 1}).$$

Denote the coefficient of $t^{1/\alpha-2}$ in the first term of this expression by P.

To obtain an expression for $n(\zeta > t)$, we turn to Fitzsimmons and Getoor [20], in which the authors compute explicitly the density of $n(\zeta \in \cdot)$ for a stable process; see p. 84 in that work, where n is denoted P^* and ζ is denoted R. The authors work with a different normalisation of the stable process; they have $c = 1$. In our context, their result says

$$n(\zeta \in \mathrm{d}t) = \frac{\alpha - 1}{\Gamma(1/\alpha)} \frac{\sin(\pi/\alpha)}{\cos(\pi(\rho - 1/2))} t^{1/\alpha-2}\,\mathrm{d}t, \qquad t \geq 0. \tag{4.5}$$

Denote the coefficient in the above power law by W.

We can now compute h. We will use elementary properties of trigonometric functions and the reflection identity for the gamma function. For $x > 0$,

$$\begin{aligned}\frac{\mathrm{P}_x(T_0 > t)}{n(\zeta > t)} &= \frac{P}{W} x^{\alpha-1} + O(t^{1-2/\alpha}) \\ &= -\frac{\cos(\pi(\rho - 1/2))\sin(\pi\alpha\hat{\rho})}{\Gamma(\alpha)\sin(\pi\hat{\rho})\sin(\pi\alpha)} x^{\alpha-1} + o(1) \\ &= -\frac{1}{\Gamma(\alpha)} \frac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha)} x^{\alpha-1} + o(1) \\ &= -\Gamma(1-\alpha)\frac{\sin(\pi\alpha\hat{\rho})}{\pi} x^{\alpha-1} + o(1).\end{aligned}$$

This proves (4.4) for $x > 0$, and it is simple to deduce via duality that this limit holds for $x \neq 0$.

We now turn our attention to the slightly more delicate result about Y. It is clear that the limit in (4.3) holds, so we only need to prove that Y is bounded. We begin by noting that, for fixed $x \neq 0$, $Y(t,x)$ is bounded away from 0 and ∞ in t since the function is continuous and converges to 1 as $t \to \infty$. Now, due to the expression (4.5) and the scaling property of X, we have the relation $Y(t,x) = Y(|x|^{-\alpha}t, \operatorname{sgn} x)$. This then shows that Y is bounded as a function of two variables.

With this in hand, we move on to the calculation of the limiting measure. This proceeds along familiar lines, using the strong Markov property:

$$\begin{aligned}\mathrm{P}_x(\Lambda | T_0 > T + s) &= \mathrm{E}_x\left[\frac{\mathrm{P}_x(\mathbb{1}_\Lambda, T_0 > T + s | \mathscr{F}_T)}{\mathrm{P}_x(T_0 > T + s)}\right] \\ &= \mathrm{E}_x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{T_0 > T\}} \frac{\mathrm{P}_{X_T}(T_0 > s)}{\mathrm{P}_x(T_0 > T + s)}\right] \\ &= \mathrm{E}_x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{T_0 > T\}} h(X_T) Y(s, X_T) \frac{n(\zeta > s)}{n(\zeta > T + s)} \frac{1}{h(x)Y(s+T, x)}\right].\end{aligned}$$

Now, as h is invariant for the stable process killed at zero, (4.2) also holds at T, and in particular the random variable $h(X_T)\mathbb{1}_{\{T_0 > T\}}$ is integrable; meanwhile, Y is bounded away from zero and ∞. Finally, $n(\zeta > s)/n(\zeta > T + s) = \left(1 + \frac{T}{s}\right)^{1-1/\alpha}$; the moment condition in the statement of the theorem permits us to apply the dominated convergence theorem, and this gives the result. □

We offer a brief comparison to conditioning a Lévy process to stay positive. In this case, Chaumont [13, Remark 1] observes that the analogue of Proposition 4.2(ii) holds under Spitzer's condition, and in particular for a stable process. However, it appears that in general, a key role is played by the exponential random variable analogous to that appearing in Proposition 4.1(ii).

4.2 The radial part of the stable process conditioned to avoid zero

For this section, consider X to be the symmetric stable process, as in section 2. There we computed the Lamperti transform ξ of the pssMp

$$R_t = |X_t| \mathbb{1}_{\{t<T_0\}}, \qquad t \geq 0,$$

and gave its characteristic exponent Ψ in (2.2).

Consider now the process

$$R_t^{\updownarrow} = \left|X_t^{\updownarrow}\right|, \qquad t \geq 0.$$

This is also a pssMp, and we may consider its Lamperti transform, which we will denote by $\xi^{\updownarrow}$. The characteristics of the Lamperti–Kiu representation of $X^{\updownarrow}$ have been computed explicitly in [16], and the characteristic exponent, $\Psi^{\updownarrow}$, of $\xi^{\updownarrow}$ could be computed from this information; however, the harmonic transform in Proposition 4.1(i) gives us the following straightforward relationship between characteristic exponents:

$$\Psi^{\updownarrow}(\theta) = \Psi(\theta - \mathrm{i}(\alpha - 1)).$$

This allows us to calculate

$$\Psi^{\updownarrow}(\theta) = 2^\alpha \frac{\Gamma(1/2 - \mathrm{i}\theta/2)}{\Gamma((1-\alpha)/2 - \mathrm{i}\theta/2)} \frac{\Gamma(\alpha/2 + \mathrm{i}\theta/2)}{\Gamma(\mathrm{i}\theta/2)}, \qquad \theta \in \mathbb{R}.$$

It is immediately apparent that $\xi^{\updownarrow}$ is the dual Lévy process to ξ. It then follows that R is a time-reversal of $R^{\updownarrow}$, in the sense of [14, §2]: roughly speaking, if one fixes $x > 0$, starts the process $R^{\updownarrow}$ at zero (as in [9], say) and runs it backward from its last moment below some level y, where $y > x$, simultaneously conditioning on the position of the left limit at this time taking value x, then one obtains the law of R under P_x.

We remark that this relationship is already known for Brownian motion, where $R^{\updownarrow}$ is a Bessel process of dimension 3. However, it seems unlikely that any such time-reversal property will hold for a general Lévy process conditioned to avoid zero.

4.3 The entrance law of the excursion measure

It is known, from a more general result [17, (2.8)] on Markov processes in weak duality, that for any Borel function f, the equality

$$\int_0^\infty e^{-qt} n(f(X_t)) \, \mathrm{d}t = \int_{\mathbb{R}} f(x) \hat{\mathrm{E}}_x \left[e^{-qT_0} \right] \mathrm{d}x$$

holds, where n is the excursion measure of X from zero. (This formulation is from [39, (3.9)].) In terms of densities, this may be written

$$\begin{aligned} n(X_t \in \mathrm{d}x) \, \mathrm{d}t &= \hat{\mathrm{P}}_x(T_0 \in \mathrm{d}t) \, \mathrm{d}x \\ &= |x|^{-\alpha} \mathrm{P}_{\mathrm{sgn}(-x)}(T_0 \in |x|^{-\alpha} \mathrm{d}t) \, \mathrm{d}x \end{aligned}$$

Therefore, our expressions in Theorem 3.14 for the density of T_0 yield expressions for the density of the entrance law of the excursions of the stable process from zero.

4.4 Identities in law using the exponential functional

In a series of papers (Bertoin and Yor [7], Caballero and Chaumont [9], Chaumont et al. [15]) it is proved that under certain conditions, the laws $(\mathrm{P}_x)_{x>0}$ of an α-pssMp X admit a weak limit P_0 as $x \downarrow 0$, in the Skorokhod space of càdlàg paths. If ξ is the Lamperti transform of X under P_1, then provided that $\mathbb{E}|\xi_1| < \infty$ and $m := \mathbb{E}\xi_1 > 0$, it is known that the entrance law of P_0 satisfies

$$\mathrm{E}_0(f(X_t^\alpha)) = \frac{1}{\alpha m} \mathbb{E}(I(-\alpha\xi)^{-1} f(t/I(-\alpha\xi))),$$

for any $t > 0$ and Borel function f. Similar expressions are available under less restrictive conditions on ξ.

It is tempting to speculate that any rssMp may admit a weak limit P_0 along similar lines, but we do not propose any results in this direction; instead, we demonstrate similar formulae for the entrance law $n(X_t \in \cdot)$ of the stable process, and the corresponding measure $\mathrm{P}_0^{\updownarrow}$ for the stable process conditioned to avoid zero.

Let X be a stable process, possibly asymmetric. From the previous subsection, we have that

$$n(f(X_t)) = \int_{-\infty}^{\infty} |x|^{-\alpha} p(\operatorname{sgn}(-x), |x|^{-\alpha} t) f(x) \, \mathrm{d}x.$$

Substituting in the integral, and recalling that the law of T_0 for the stable process is equal to the law of the exponential functional $I(\alpha\xi)$ of the Markov additive process associated with it, we obtain

$$\begin{aligned} n(f(X_t)) &= \frac{1}{\alpha} \int_0^{\infty} p(1,u) f(-(u/t)^{-1/\alpha}) u^{-1/\alpha} t^{1/\alpha - 1} \, \mathrm{d}u \\ &\quad + \frac{1}{\alpha} \int_0^{\infty} p(-1,u) f((u/t)^{-1/\alpha}) u^{-1/\alpha} t^{1/\alpha - 1} \, \mathrm{d}u \\ &= \frac{1}{\alpha} \mathbb{E}_1 \bigl[f(-(t/I(\alpha\xi))^{1/\alpha}) I(\alpha\xi)^{-1/\alpha} t^{1/\alpha - 1} \bigr] \\ &\quad + \frac{1}{\alpha} \mathbb{E}_2 \bigl[f((t/I(\alpha\xi))^{1/\alpha}) I(\alpha\xi)^{-1/\alpha} t^{1/\alpha - 1} \bigr]. \end{aligned}$$

Recall from [32] that the law $\mathrm{P}_0^{\updownarrow}$ of the stable process conditioned to avoid zero is given by the following harmonic transform of the stable excursion measure n:

$$\mathrm{P}_0^{\updownarrow}(\Lambda) = n(\mathbb{1}_\Lambda h(X_t), t < \zeta), \qquad t \ge 0, \, \Lambda \in \mathscr{F}_t,$$

with h as in (4.1). Therefore, applying the above result to the Borel function hf, we obtain

$$\begin{aligned} \mathrm{E}_0^{\updownarrow}(f(X_t)) &= n(h(X_t) f(X_t)) \\ &= \Gamma(-\alpha) \frac{\sin(\pi\alpha\rho)}{\pi} \mathbb{E}_1 \bigl[I(\alpha\xi)^{-1} f(-(t/I(\alpha\xi))^{1/\alpha}) \bigr] \\ &\quad + \Gamma(-\alpha) \frac{\sin(\pi\alpha\hat{\rho})}{\pi} \mathbb{E}_2 \bigl[I(\alpha\xi)^{-1} f((t/I(\alpha\xi))^{1/\alpha}) \bigr], \end{aligned}$$

where we emphasise that $I(\alpha\xi)$ (under $\mathbb{E}_i$) is the exponential functional of the Markov additive process associated to X.

Acknowledgments. Some of this research was carried out whilst AEK and ARW were visiting ETH Zürich and CIMAT. Both authors would like to offer thanks to both institutions for their hospitality. AK acknowledges the support by the Natural Sciences and Engineering Research Council of Canada. JCP acknowledges the support by CONACYT (grant 128896). All authors would like to thank the referee for their detailed comments that led to an improved version of this paper.

References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999. ISBN 0-521-62321-9; 0-521-78988-5. MR-1688958

[2] E. Artin. *The gamma function*. Translated by Michael Butler. Athena Series: Selected Topics in Mathematics. Holt, Rinehart and Winston, New York, 1964. MR-0165148

[3] S. Asmussen. *Ruin probabilities*, volume 2 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000. ISBN 981-02-2293-9. MR-1794582

[4] S. Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. ISBN 0-387-00211-1. MR-1978607

[5] I. Ben-Ari and M. Neumann. Probabilistic approach to Perron root, the group inverse, and applications. *Linear and Multilinear Algebra*, 60(1):39–63, 2012. ISSN 0308-1087. MR-2869672

[6] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. ISBN 0-521-56243-0. MR-1406564

[7] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002. ISSN 0926-2601. MR-1918243

[8] R. M. Blumenthal and R. K. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968. MR-0264757

[9] M. E. Caballero and L. Chaumont. Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.*, 34(3):1012–1034, 2006. ISSN 0091-1798. MR-2243877

[10] M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.*, 43(4):967–983, 2006. ISSN 0021-9002. MR-2274630

[11] M. E. Caballero, J. C. Pardo, and J. L. Pérez. Explicit identities for Lévy processes associated to symmetric stable processes. *Bernoulli*, 17(1):34–59, 2011. ISSN 1350-7265. MR-2797981

[12] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pages 73–130. Rev. Mat. Iberoamericana, Madrid, 1997. MR-1648657

[13] L. Chaumont. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.*, 64(1):39–54, 1996. ISSN 0304-4149. MR-1419491

[14] L. Chaumont and J. C. Pardo. The lower envelope of positive self-similar Markov processes. *Electron. J. Probab.*, 11:no. 49, 1321–1341, 2006. ISSN 1083-6489. MR-2268546

[15] L. Chaumont, A. Kyprianou, J. C. Pardo, and V. Rivero. Fluctuation theory and exit systems for positive self-similar Markov processes. *Ann. Probab.*, 40(1):245–279, 2012. ISSN 0091-1798. MR-2917773

[16] L. Chaumont, H. Pantí, and V. Rivero. The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli*, 19(5B):2494–2523, 2013. ISSN 1350-7265. MR-3160562

[17] Z.-Q. Chen, M. Fukushima, and J. Ying. *Extending Markov processes in weak duality by Poisson point processes of excursions*, volume 2 of *Abel Symp.*, pages 153–196. Springer, Berlin, 2007. MR-2397787

[18] O. Chybiryakov. The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups. *Stochastic Process. Appl.*, 116(5):857–872, 2006. ISSN 0304-4149. MR-2218339

[19] F. Cordero. *On the excursion theory for the symmetric stable Lévy processes with index $\alpha \in]1,2]$ and some applications.* PhD thesis, Université Pierre et Marie Curie – Paris VI, 2010.

[20] P. J. Fitzsimmons and R. K. Getoor. Occupation time distributions for Lévy bridges and excursions. *Stochastic Process. Appl.*, 58(1):73–89, 1995. ISSN 0304-4149. MR-1341555

[21] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh edition, 2007. ISBN 978-0-12-373637-6. Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. MR-2360010

[22] F. Hubalek and A. Kuznetsov. A convergent series representation for the density of the supremum of a stable process. *Electron. Commun. Probab.*, 16:84–95, 2011. ISSN 1083-589X. MR-2763530

[23] J. Ivanovs. *One-sided Markov additive processes and related exit problems.* PhD thesis, Universiteit van Amsterdam, 2011.

[24] J. F. C. Kingman. A convexity property of positive matrices. *Quart. J. Math. Oxford Ser. (2)*, 12:283–284, 1961. ISSN 0033-5606. MR-0138632

[25] A. Kuznetsov. On extrema of stable processes. *Ann. Probab.*, 39(3):1027–1060, 2011. ISSN 0091-1798. MR-2789582

[26] A. Kuznetsov and J. C. Pardo. Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. *Acta Appl. Math.*, 123:113–139, 2013. ISSN 0167-8019. MR-3010227

[27] A. E. Kyprianou, J. C. Pardo, and A. R. Watson. Hitting distributions of α-stable processes via path censoring and self-similarity. *Ann. Probab.*, 42(1):398–430, 2014. ISSN 0091-1798. MR-3161489

[28] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972. MR-0307358

[29] J. Letemplier and T. Simon. Unimodality of hitting times for stable processes. Preprint, arXiv:1309.5321v2 [math.PR], 2013.

[30] K. Maulik and B. Zwart. Tail asymptotics for exponential functionals of Lévy processes. *Stochastic Process. Appl.*, 116(2):156–177, 2006. ISSN 0304-4149. MR-2197972

[31] H. D. Miller. A convexity property in the theory of random variables defined on a finite Markov chain. *Ann. Math. Statist.*, 32:1260–1270, 1961. ISSN 0003-4851. MR-0126886

[32] H. Pantí. On Lévy processes conditioned to avoid zero. Preprint, arXiv:1304.3191v1 [math.PR], 2013.

[33] G. Peskir. The law of the hitting times to points by a stable Lévy process with no negative jumps. *Electron. Commun. Probab.*, 13:653–659, 2008. ISSN 1083-589X. MR-2466193

[34] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. II. *Bernoulli*, 13(4):1053–1070, 2007. ISSN 1350-7265. MR-2364226

[35] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. ISBN 0-521-55302-4. MR-1739520

[36] T. Simon. Hitting densities for spectrally positive stable processes. *Stochastics*, 83 (2):203–214, 2011. ISSN 1744-2508. MR-2800088

[37] J. Vuolle-Apiala. Itô excursion theory for self-similar Markov processes. *Ann. Probab.*, 22(2):546–565, 1994. ISSN 0091-1798. MR-1288123

[38] K. Yano, Y. Yano, and M. Yor. On the laws of first hitting times of points for one-dimensional symmetric stable Lévy processes. In *Séminaire de Probabilités XLII*, volume 1979 of *Lecture Notes in Math.*, pages 187–227. Springer, Berlin, 2009. MR-2599211

[39] K. Yano, Y. Yano, and M. Yor. Penalising symmetric stable Lévy paths. *J. Math. Soc. Japan*, 61(3):757–798, 2009. ISSN 0025-5645. MR-2552915