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THE HOLOMORPHIC AUTOMORPHISM GROUPS OF TWISTED FOCK-BARGMANN-HARTOGS DOMAINS

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Abstract. We consider a certain class of unbounded nonhyperbolic Reinhardt domains which is called the twisted Fock-Bargmann-Hartogs domains. By showing Cartan's linearity theorem for our unbounded nonhyperbolic domains, we give a complete description of the automorphism groups of twisted Fock-Bargmann-Hartogs domains.

Keywords: holomorphic automorphism group; Bergman kernel; Reinhardt domain

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1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and let Φ be a positive continuous function on Ω. Let us first observe that the Hartogs domain $\widehat{\Omega}_m = \{(z,\zeta) \in \Omega \times \mathbb{C}^m\colon \|\zeta\|^2 < \Phi(z)\}$ can be rewritten as

$$\widehat{\Omega}_m = \{(z,\zeta) \in \Omega \times \mathbb{C}^m\colon \zeta \in \Phi(z)^{1/2}\mathbb{B}^m\}.$$

Based on this observation, Roos introduced the following domain in [19]:

$$\widehat{\Omega}_F = \{(z,\zeta) \in \Omega \times \mathbb{C}^m\colon \zeta \in \Phi(z)^{1/2}F\},$$

where F is an arbitrary circular domain in $\mathbb{C}^m$.

In recent years, the Hartogs domain $\widehat{\Omega}_m$ received lots of attention especially when the base domain Ω is an irreducible bounded symmetric domain $\mathscr{F}$ or $\mathbb{C}^n$. More precisely, the following two domains are investigated from many different aspects:

$$\Omega_{\rm CH} := \{(z,\zeta) \in \mathscr{F} \times \mathbb{C}^m\colon \|\zeta\|^2 < N(z,z)^{\mu}\},$$
$$\Omega_{\rm FBH} := \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^m\colon \|\zeta\|^2 < {\rm e}^{-\mu\|z\|^2}\},$$

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where N is the generic norm of $\mathscr{F}$ and $\mu > 0$. These domains are called the Cartan-Hartogs domain ($\Omega_{\rm CH}$) and the Fock-Bargmann-Hartogs domain ($\Omega_{\rm FBH}$), respectively. A remarkable fact on these domains is that explicit descriptions of the automorphism group and explicit forms of the Bergman kernels are known. These two objects are important research objects in several complex variables and usually hard to compute explicitly. The Bergman kernel of $\Omega_{\rm CH}$ is firstly computed by Yin in [29]. Another expression of the kernel in terms of the polylogarithm function is given by the second author in [24]. In a paper by Ahn, Byun and Park [1], the automorphism group of $\Omega_{\rm CH}$ is determined completely. The Bergman kernel of $\Omega_{\rm FBH}$ is firstly computed by Springer, see [20], for $m = n = 1$. This is generalized by the second author in [25] for general m and n. By using an explicit form of Bergman kernel, the automorphism group of $\Omega_{\rm FBH}$ is determined in [9] by the authors of the present paper and Ninh. For other works related to these domains, see [2], [10], [11], [14], [22], [30] and references therein.

As a natural question, one may ask which kinds of properties of the Hartogs domains remain true for Roos' domain $\widehat{\Omega}_F$ or more general domains. For the Hartogs domain $\widehat{\Omega}_m$, a series representation formula of the Bergman kernel, which is the so-called Forelli-Rudin construction, is known (see [13]). This formula is generalized for Roos' domain $\widehat{\Omega}_F$ when F is an irreducible bounded symmetric space or the complex ellipsoid

$$D_p = \{z \in \mathbb{C}^m\colon\ |z_1|^{2p_1} + \ldots + |z_m|^{2p_m} < 1\},$$

where $p_1, \ldots, p_m \in \mathbb{Z}_+$ (see [4] and [27]). Another direction for a generalization of the Hartogs domain has been also considered by several authors. Namely, we can construct an analogue of the Hartogs domain by using the complex ellipsoid D_p as follows:

$$\widehat{\Omega}_{m,p} = \{(z,\zeta) \in \Omega \times \mathbb{C}^m\colon\ |\zeta_1|^{2p_1} + \ldots + |\zeta_m|^{2p_m} < \Phi(z)\}, \quad p_1, \ldots, p_m \in \mathbb{Z}_+.$$

The Forelli-Rudin construction is generalized for $\widehat{\Omega}_{m,p}$ in [27]. If Ω is an irreducible bounded symmetric domain $\mathscr{F}$ and Φ is the generic norm of $\mathscr{F}$, then this domain is called the Hua domain. For works related to the Hua domain, see [17], [23]. In [28], the Bergman kernel and the automorphism group of $\widehat{\Omega}_{m,p}$ are studied when $\Omega = \mathbb{C}^n$ and $\Phi(z) = {\rm e}^{-\mu\|z\|^2}$.

Let us consider the domain

$$\widehat{\Omega}_{F,\Psi} = \{(z,\zeta) \in \Omega \times \mathbb{C}^{k_1} \times \ldots \times \mathbb{C}^{k_m}\colon\ (\Psi_1(z)^{-1/2}\zeta_1, \ldots, \Psi_m(z)^{-1/2}\zeta_m) \in F\},$$

where $\Psi_1, \ldots, \Psi_m$ are positive continuous functions on Ω and $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^{k_1} \times \ldots \times \mathbb{C}^{k_m}$, $(k_1, \ldots, k_m) \in \mathbb{Z}_+^m$. If $\Psi_1 = \ldots = \Psi_m$, then this domain coincides with Roos' domain $\widehat{\Omega}_F$. This domain $\widehat{\Omega}_{F,\Psi}$ is called the Hua construction when $F = \{(\xi_1, \ldots, \xi_m) \in \mathbb{C}^{k_1} \times \ldots \times \mathbb{C}^{k_m}\colon \|\xi_1\|^{2p_1} + \ldots + \|\xi_m\|^{2p_m} < 1\}$, $\Omega = \mathscr{F}$ and $\Psi_i = N^{\mu_i/(2p_i)}$ for $1 \leqslant i \leqslant m$ (cf. [31]). Although a complete description of the automorphism group of the Hua domain is already known (cf. [17], [23]), its generalization to the Hua construction, is still open. The main result of this paper gives a complete description of the automorphism group of the twisted Fock-Bargmann-Hartogs domain

$$D^{\mu,p}_{n,m,k} := \left\{(z, \zeta_1, \ldots, \zeta_m) \in \mathbb{C}^n \times \mathbb{C}^{k_1} \times \ldots \times \mathbb{C}^{k_m}\colon \frac{\|\zeta_1\|^{2p_1}}{\mathrm{e}^{-\mu_1\|z\|^2}} + \ldots + \frac{\|\zeta_m\|^{2p_m}}{\mathrm{e}^{-\mu_m\|z\|^2}} < 1\right\}$$

which is regarded as an unbounded counterpart of the Hua construction. Moreover, we also give a complete description of the automorphism group of the domain

$$\mathscr{D} = \{(z, \zeta) \in \mathbb{C}^3\colon \mathrm{e}^{\mu|z|^2}|\zeta_1|^2 + |\zeta_2|^2 < 1\}.$$

Recently, this domain was considered by Huo and its Bergman kernel was computed explicitly (see [5]). In contrast to the Fock-Bargmann-Hartogs domains, our argument here does not require an explicit form of the Bergman kernel.

The organization of the paper is described as follows: In Section 2, we recall basic properties of the Bergman kernel and the representative domain. In Section 3, we study Cartan's linearity theorem in a Bergman kernel theoretic way. The result in this section plays a substantial role in Section 4. We provide a description of the automorphism group of the degenerate case $\widetilde{D}^{\mu}_{n,m}$ (see Section 4.1.1 for the definition) in Theorem 4.9; $\mathrm{Aut}(\mathscr{D})$ is obtained as a corollary. Then we further give a description of the automorphism group of $D^{\mu,p}_{n,m,k}$ in Section 4.

2. Bergman kernel and representative domain

Since the Bergman kernels play a substantial role in our argument, let us first prepare some basic facts on this kernel function. Throughout this paper, we assume that D is a complex domain in $\mathbb{C}^n$ containing the origin.

2.1. Bergman kernel. Let $A^2(D)$ be the space of square integrable holomorphic functions on D,

$$A^2(D) = \left\{f \in \mathcal{O}(D)\colon \int_D |f(z)|^2 \,\mathrm{d}V(z) < \infty\right\},$$

where $\mathrm{d}V(z)$ is the standard Lebesgue measure on $\mathbb{C}^n$. The space $A^2(D)$ is called the Bergman space of D. We equip $A^2(D)$ with the inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)} \,\mathrm{d}V(z).$$

The reproducing kernel of $A^2(D)$ is called the *Bergman kernel* and we denote it by K_D. Let $\{e_k\}_{k\in\mathbb{N}}$ be a complete orthonormal basis of $A^2(D)$. Then the Bergman kernel is given by

$$K_D(z,w) = \sum_{k\in\mathbb{N}} e_k(z)\overline{e_k(w)}. \tag{1}$$

Let $\varphi\colon D \to D'$ be a biholomorphism. A fundamental fact on the Bergman kernel is the transformation formula

$$K_D(z,w) = \overline{\det \mathrm{Jac}(\varphi, w)} K_{D'}(\varphi(z), \varphi(w)) \det \mathrm{Jac}(\varphi, z),$$

where $\mathrm{Jac}(\varphi, z)$ is the Jacobian matrix of $\varphi = {}^t(\varphi_1, \ldots, \varphi_n)$ at z:

$$\mathrm{Jac}(\varphi, z) := \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial z_1}(z) & \cdots & \dfrac{\partial \varphi_1}{\partial z_n}(z) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \varphi_n}{\partial z_1}(z) & \cdots & \dfrac{\partial \varphi_n}{\partial z_n}(z) \end{pmatrix}.$$

Except some special cases, it is hard to compute the Bergman kernel for a given domain. Some examples of domains with explicit Bergman kernels can be found in [8], Chapter 12, and references therein.

2.2. Representative domain. Define an $n \times n$ matrix $T_D(z,w)$ by

$$T_D(z,w) = \Big(\frac{\partial^2 \log K_D(z,w)}{\partial \overline{w}_\alpha \partial z_\beta}\Big)_{\alpha,\beta=1,\ldots,n},$$

for $K_D(z,w) \neq 0$. We denote

$$K_{\overline{\alpha}\beta}(z,w) := \frac{\partial^2 \log K_D(z,w)}{\partial \overline{w}_\alpha \partial z_\beta}.$$

Remark 2.1. For many purposes such as the study of the Bergman metric, it is sufficient to consider $K_{\overline{\alpha}\beta}(z,w)$ only on the diagonal points (i.e. $z = w$). On the other hand, there is also a property which requires consideration of $K_{\overline{\alpha}\beta}(z,w)$ with off-diagonal points.

In this paper, we use $K_{\overline{\alpha}\beta}(z,w)$ to prove Cartan's linearity theorem in a Bergman kernel theoretic way. Instead of posing the circularity for a domain, we pose the following condition on T_D for our revised Cartan's theorem:

$$T_D(z,0) \equiv T_D(0,0) \quad \text{for any } z \in D.$$

Obviously, we need T_D at the off-diagonal points to describe this condition. In our argument, we always consider a domain D such that $K_D(0,0) > 0$ and $K_D(z,0) \equiv K_D(0,0)$. These conditions give us the well-definedness of $T_D(z,0)$ for any $z \in D$. We note that K_D is not necessarily zero-free in general. For details of the zero set of the Bergman kernel, see [8], Chapter 12, and references therein.

Let φ be a biholomorphism $\varphi\colon D \to D'$. It is well-known that T_D and $T_{D'}$ satisfy the following transformation formula (cf. [6], equation (2.2), and [21], Lemma 1.1):

$$T_D(z,w) = \overline{{}^t\operatorname{Jac}(\varphi,w)}\,T_{D'}(\varphi(z),\varphi(w))\operatorname{Jac}(\varphi,z). \tag{2}$$

Recall that a domain $D \subset \mathbb{C}^n$ is called *circular* with its center at the origin (or simply, circular) if $\mathrm{e}^{\mathrm{i}\theta}z \in D$ for any $\theta \in \mathbb{R}$ and $z \in D$. A circular domain D is called *complete* if $\lambda z \in D$ whenever $z \in D$ and $\lambda \in \overline{\mathbb{D}}$. Using the transformation formula (2), we obtain the following lemma (for details of the proof, see [6]).

Lemma 2.2. *Let D be a bounded complete circular domain. Then we have $T_D(z,0) \equiv T_D(0,0)$ for any $z \in D$.*

Before proceeding further, let us give a definition (see also [15]).

Definition 2.3. A bounded domain D in $\mathbb{C}^n$ is called a *representative domain* if there exists a point $z_0 \in D$ such that $T_D(z,z_0)$ is a constant matrix for all $z \in D$. The point z_0 is called the *center* of D.

The above lemma tells us that every bounded complete circular domain is a representative domain with its center at the origin. We note that the notion of representative domain can be considered for any domains (possibly unbounded) whenever $T_D(z,z_0)$ is well-defined.

3. Cartan's theorem

Let us first recall a classical theorem due to Cartan:

Theorem 3.1. *Let D be a bounded complete circular domain and f an automorphism fixing the origin. Then f is linear.*

In our previous paper [9], by an observation from [6], we showed that this theorem remains true even for unbounded circular cases under certain conditions on K_D and T_D. In [9], the notion of the Bergman mapping played the key role in the proof. After publishing the paper [9], we noticed that there is an alternative proof by avoiding to use the Bergman mapping. The reader will see that Cartan's linearity theorem is quickly derived from the transformation formula (2) of T_D. The proof of the next proposition is essentially due to Lu, see [15]. However, we give a proof for the convenience of the reader.

Proposition 3.2. *Let D be a domain (not necessarily bounded) in $\mathbb{C}^n$ such that $K_D \not\equiv 0$. We suppose two conditions:*

(i) $K_D(z,0) \neq 0$ *for all* $z \in D$,

(ii) $T_D(z,0) \equiv T_D(0,0)$ *and* $T_D(0,0)$ *is positive definite.*

If f is an automorphism of D such that $f(0)=0$ then f is linear.

Proof. Let us first observe that $T_D(z,w)$ is well-defined if $K(z,w) \neq 0$. By (i), we see that $T_D(z,0)$ is well-defined for any $z \in D$. Let f be as above. Applying (2) and (ii), we obtain

$$T_D(0,0) = \overline{{}^t\,\mathrm{Jac}(f,0)}\, T_D(0,0)\,\mathrm{Jac}(f,z),$$

which implies that $\mathrm{Jac}(f,z)$ is constant. Thus $f(z) = Az + c$ with $A \in GL(n,\mathbb{C})$ and $c \in \mathbb{C}^n$, whereas $c = 0$ since $f(0) = 0$. □

Note that if D is a representative domain with its center at the origin, then we always have $T_D(z,0) \equiv T_D(0,0)$. Moreover, if it is circular, then we have $K_D(z,0) \equiv K_D(0,0)$. Let us recall that a domain $D \subset \mathbb{C}^n$ is called *Reinhardt* if D is invariant under $(z_1,\dots,z_n) \mapsto (\mathrm{e}^{\mathrm{i}\theta_1}z_1,\dots,\mathrm{e}^{\mathrm{i}\theta_n}z_n)$ for $\theta_1,\dots,\theta_n \in \mathbb{R}$. For Reinhardt domains, the following theorem is known.

Theorem 3.3. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain (possibly unbounded nonhyperbolic). Suppose that $\mathrm{Vol}(D) < \infty$ and $z_i \in A^2(D)$ for any $1 \leqslant i \leqslant n$. Then all automorphisms f with $f(0)=0$ are linear.*

This theorem follows from Proposition 3.2 after checking that $K_D(0,0) > 0$ and that $T_D(0,0)$ is positive definite as in [28].

Remark 3.4. As we mentioned in Lemma 2.2, all bounded complete circular domains are representative domains with their center at the origin. In [26], it is proved that there is a certain class of quasi-circular domains in $\mathbb{C}^2$ which is representative with their center at the origin. For instance, due to the main result in [26], one can see that Cartan's linearity theorem remains true for the following domain Ω, even though it is not circular:

$$\Omega = \{(z_1,z_2) \in \mathbb{B}^2\colon\ |z_1^3 + z_2^2| < 1\}.$$

For Cartan's linearity theorem and related results, see also [16], [18] and references therein.

4. Twisted Fock-Bargmann-Hartogs domains

4.1. Twisted Fock-Bargmann-Hartogs domains: special cases. Before considering the general cases, let us first study special ones. We believe that this section will help the readers to grasp the key ideas of our paper. In this section, we investigate the automorphism groups of the following cases of twisted Fock-Bargmann-Hartogs domains:

$$D^{\mu}_{n,m} = \Big\{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^m \colon \frac{|\zeta_1|^2}{\mathrm{e}^{-\mu_1\|z\|^2}} + \ldots + \frac{|\zeta_m|^2}{\mathrm{e}^{-\mu_m\|z\|^2}} < 1\Big\},$$

where $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m_+$ with $\mu_i \neq \mu_j$ for $i \neq j$. If $m = 2$, $n = 1$ and $\mu_2 \to 0$, then this domain degenerates to the domain

$$\mathscr{D} = \Big\{(z,\zeta) \in \mathbb{C} \times \mathbb{C}^2 \colon \frac{|\zeta_1|^2}{\mathrm{e}^{-\mu|z|^2}} + |\zeta_2|^2 < 1\Big\}.$$

We note that an explicit form of the Bergman kernel of $\mathscr{D}$ is obtained by Huo in [5]. If $\mu_1 = \mu_2 = \ldots = \mu_m$, then $D^{\mu}_{n,m}$ is a Hartogs type domain, which is the so-called Fock-Bargmann-Hartogs domain:

$$D_{n,m} = \{(z,\zeta) \in \mathbb{C}^{n+m} \colon \|\zeta\|^2 < \mathrm{e}^{-\mu\|z\|^2}\}.$$

As is proved in [9], the automorphism group of $D_{n,m}$ is generated by the mappings

$$\begin{aligned} \varphi_{U_1} &\colon (z,\zeta) \mapsto (U_1 z, \zeta),\\ \varphi_{U_2} &\colon (z,\zeta) \mapsto (z, U_2\zeta),\\ \varphi_{v} &\colon (z,\zeta) \mapsto (z+v, \mathrm{e}^{-\mu\langle z,v\rangle - \mu\|v\|^2/2}\zeta), \end{aligned}$$

where $U_1 \in U(n)$, $U_2 \in U(m)$ and $v \in \mathbb{C}^n$. The aim of this section is to give a complete description of the automorphism group of $D^{\mu}_{n,m}$. We also include a description of the automorphism group for $\mathscr{D}$ as a degenerate case of $D^{\mu}_{n,m}$.

4.1.1. Cartan's linearity theorem for $D^{\mu}_{n,m}$. Our domain $D^{\mu}_{n,m}$ is unbounded nonhyperbolic and thus we cannot apply the classical Cartan's theorem (Theorem 3.1). Therefore we start our study with origin-preserving automorphisms of $D^{\mu}_{n,m}$. Since our domain is a Reinhardt domain, we use Theorem 3.3 as a main tool.

To apply the theorem to our case, let us estimate the L^2-norm of $z^{\alpha}\zeta^{\beta} := \prod_{k=1}^{n} z_k^{\alpha_k} \prod_{l=1}^{m} \zeta_l^{\beta_l}$. By $|z_k| \leqslant \|z\|$ and $|\zeta_l|^2 \leqslant \mathrm{e}^{-\mu_l\|z\|^2}$ for any $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$, we first observe the estimate

$$\|z^{\alpha}\zeta^{\beta}\|^2_{L^2(D^{\mu}_{n,m})} \leqslant \int_{D^{\mu}_{n,m}} \|z\|^{2(\alpha_1+\ldots+\alpha_n)} \mathrm{e}^{-(\beta_1\mu_1+\ldots+\beta_m\mu_m)\|z\|^2}\,\mathrm{d}V(z,\zeta).$$

For an arbitrary fixed $z \in \mathbb{C}^n$, let us define

$$\Omega_z^m = \Big\{\zeta \in \mathbb{C}^m\colon\ \frac{|\zeta_1|^2}{\mathrm{e}^{-\mu_1\|z\|^2}} + \ldots + \frac{|\zeta_m|^2}{\mathrm{e}^{-\mu_m\|z\|^2}} < 1\Big\}.$$

Then, by Fubini's theorem, we have

$$\begin{aligned}(3)\qquad \int_{D_{n,m}^{\mu}} G(z)\,\mathrm{d}V(z,\zeta) &= \int_{z\in\mathbb{C}^n} G(z)\Big(\int_{\Omega_z^m} 1\cdot \mathrm{d}V(\zeta)\Big)\,\mathrm{d}V(z)\\ &= \frac{\pi^m}{m!}\int_{z\in\mathbb{C}^n} G(z)\mathrm{e}^{-(\mu_1+\ldots+\mu_m)\|z\|^2}\,\mathrm{d}V(z),\end{aligned}$$

where we put $G(z) = \|z\|^{2(\alpha_1+\ldots+\alpha_n)}\mathrm{e}^{-(\beta_1\mu_1+\ldots+\beta_m\mu_m)\|z\|^2}$ for simplicity of notation. By using the polar coordinates, (3) can be computed as follows:

$$\begin{aligned}&\frac{\pi^m}{m!}\int_{z\in\mathbb{C}^n} G(z)\mathrm{e}^{-(\mu_1+\ldots+\mu_m)\|z\|^2}\,\mathrm{d}V(z)\\ &= \frac{\pi^m(2n)\pi^{2n/2}}{m!\frac{2n}{2}\Gamma(\frac{2n}{2})}\int_0^\infty r^{2(\alpha_1+\ldots+\alpha_n)}\mathrm{e}^{-(\beta_1\mu_1+\ldots+\beta_m\mu_m)r^2}r^{2n-1}\,\mathrm{d}r\\ &= \frac{\pi^{n+m}(\alpha_1+\ldots+\alpha_n+n-1)!}{m!\,(n-1)!\,\{(\beta_1+1)\mu_1+\ldots+(\beta_m+1)\mu_m\}^{\alpha_1+\ldots+\alpha_n+n}}.\end{aligned}$$

Thus we have $\|z^\alpha\zeta^\beta\|^2_{L^2(D_{n,m}^{\mu})} < \infty$, for any $(\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $(\beta_1,\ldots,\beta_m) \in \mathbb{Z}^m_{\geqslant 0}$. In particular, it implies that our domain satisfies the assumption of Theorem 3.3. In conclusion, we obtain Cartan's linearity theorem for our domain.

Theorem 4.1. *Let f be an automorphism of $D_{n,m}^{\mu}$ with $f(0) = 0$. Then f is linear.*

We note that the above argument works for any $\mu = (\mu_1,\ldots,\mu_m) \neq (0,\ldots,0) \in \mathbb{R}^m_{\geqslant 0}$ and thus this theorem is also true for $\mathscr{D}$.

4.1.2. Isotropy group. As we proved in the previous section, all automorphisms of $D_{n,m}^{\mu}$ with $f(0) = 0$ are linear. This implies that the study of the isotropy subgroup $\mathrm{Iso}_0(D_{n,m}^{\mu}) := \{f \in \mathrm{Aut}(D_{n,m}^{\mu})\colon\ f(0) = 0\} \subset \mathrm{Aut}(D_{n,m}^{\mu})$ is reduced to that of the linear automorphism group. We begin our study with the invariance of $\mathscr{U} = \{(z,0) \in \mathbb{C}^{n+m}\} \subset D_{n,m}^{\mu}$. One can prove the next lemma along the same lines as for the Fock-Bargmann-Hartogs domain case (see [9], Lemma 8).

Lemma 4.2. *Let φ be an arbitrary automorphism of $D_{n,m}^{\mu}$. Then the space $\mathscr{U}$ is invariant under φ.*

Let us next consider the case when $\mu_{s+1} = \ldots = \mu_m = 0$ for some $s \geqslant 1$. Before doing so, let us maintain the notation. Put $\zeta = (\zeta_1, \ldots, \zeta_s) \in \mathbb{C}^s$ and $\widetilde{\zeta} = (\zeta_{s+1}, \ldots, \zeta_m) \in \mathbb{C}^{m-s}$. Define $\widetilde{D}^{\mu}_{n,m}$ by

$$\widetilde{D}^{\mu}_{n,m} := \Big\{(z, \zeta, \widetilde{\zeta}) \in \mathbb{C}^n \times \mathbb{C}^s \times \mathbb{C}^{m-s} \colon \frac{|\zeta_1|^2}{\mathrm{e}^{-\mu_1\|z\|^2}} + \ldots + \frac{|\zeta_s|^2}{\mathrm{e}^{-\mu_s\|z\|^2}} + \|\widetilde{\zeta}\|^2 < 1\Big\}.$$

Throughout what follows, we denote by $\mathcal{U}_a$ and $\mathcal{V}$ the sets defined by

$$\begin{aligned}\mathcal{U}_a &:= \{(z, 0, a) \in \mathbb{C}^n \times \mathbb{C}^s \times \mathbb{C}^{m-s} \colon z \in \mathbb{C}^n, \ a \in \mathbb{C}^{m-s}\},\\ \mathcal{V} &:= \{(z, 0, \tilde{\zeta}) \in \mathbb{C}^n \times \mathbb{C}^s \times \mathbb{C}^{m-s} \colon z \in \mathbb{C}^n, \ \tilde{\zeta} \in \mathbb{C}^{m-s}\} = \bigsqcup_{\tilde{\zeta} \in \mathbb{C}^{m-s}} \mathcal{U}_{\tilde{\zeta}}.\end{aligned}$$

Moreover, it immediately follows from the definition of $\mathscr{U}$ that $\mathscr{U} = \mathcal{U}_0$.

Lemma 4.3. *Let φ be an automorphism of $\widetilde{D}^{\mu}_{n,m}$. Then*

(i) *$\varphi(\mathcal{U}_0) \subset \mathcal{U}_a$ for a point $a \in \mathbb{B}^{m-s}$,*

(ii) *$\varphi(\mathcal{V}) \subset \mathcal{V}$.*

Proof. For each $\varphi \in \mathrm{Aut}(\widetilde{D}^{\mu}_{n,m})$, we let $\varphi(z, 0) = (\varphi_1(z), \varphi_2(z)) \in \mathbb{C}^n \times \mathbb{C}^m$ and $\varphi_2(z) = (\varphi_{21}(z), \ldots, \varphi_{2m}(z))$. In the same way as in [9], Lemma 8, one can see that $\varphi_{21}, \ldots, \varphi_{2m}$ are constant functions. However, since $\mu_{s+1} = \ldots = \mu_m = 0$, the argument of [9], Lemma 8, can be applied only to $\varphi_{21}, \ldots, \varphi_{2s}$. Thus we have $\varphi_{21}, \ldots, \varphi_{2s} \equiv 0$. This completes the proof of (i).

Let us next show (ii). For $Z \in \mathbb{B}^N$ and a fixed point $p \in \mathbb{B}^N_*$, define $h_p(Z)$ by

$$h_p(Z) := -\frac{1}{\|p\|^2} \frac{\sqrt{1 - \|p\|^2}(\|p\|^2 Z - \langle Z, p\rangle p) - \|p\|^2 p + \langle Z, p\rangle p}{1 - \langle Z, p\rangle}. \tag{4}$$

Here we set $h_0 := \mathrm{id}$. It is well-known that h_p is an automorphism of $\mathbb{B}^N$ and this mapping satisfies the relation (cf. [7], Chapter 2)

$$1 - \|h_p(Z)\|^2 = \frac{(1 - \|p\|^2)(1 - \|Z\|^2)}{|1 - \langle Z, p\rangle|^2}.$$

Using this fact, one can check that the following mapping is an automorphism of $\widetilde{D}^{\mu}_{n,m}$ for any $a \in \mathbb{B}^{m-s}$:

$$\varphi_a \colon (z, \zeta, \widetilde{\zeta}) \mapsto \Big(z, \frac{(1 - \|a\|^2)^{1/2}}{1 - \langle \widetilde{\zeta}, a\rangle}\,\zeta, h_a(\widetilde{\zeta})\Big).$$

By definition, one can see that this mapping sends $(z, 0, a)$ to $(z, 0, 0)$ for all $a \in \mathbb{B}^{m-s}$. Then $\varphi(z_0, 0, \widetilde{\zeta}_0)$ can be rewritten as $\varphi \circ (\varphi_{\widetilde{\zeta}_0})^{-1}(z_0, 0, 0)$ for any fixed element $(z_0, 0, \widetilde{\zeta}_0)$. Since $\varphi \circ (\varphi_{\widetilde{\zeta}_0})^{-1} \in \mathrm{Aut}(\widetilde{D}^{\mu}_{n,m})$, our desired conclusion now follows from (i). $\square$

Before giving a description of the linear automorphisms of $D^{\mu}_{n,m}$, we prepare one more lemma.

Lemma 4.4. *Let X and Y be elements of* $\mathrm{Mat}_{n\times n}(\mathbb{C})$*. Then:*

(i) *If $X \in U(n)$, then $Y = X^*YX$ if and only if $XY = YX$.*

(ii) *If Y is a diagonal matrix with pairwise distinct entries, then $XY = YX$ implies that X is a diagonal matrix.*

(iii) *If X is a diagonal unitary matrix, then X is* $\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}\dots\mathrm{e}^{\mathrm{i}\theta_n})$ *where are $\theta_1,\dots,\theta_n \in \mathbb{R}$.*

The proof is an exercise in elementary linear algebra and we omit it. We are now ready to study the linear automorphisms of $D^{\mu}_{n,m}$.

Proposition 4.5. *Let f be a linear automorphism of $D^{\mu}_{n,m}$. Then f is given by*

$$f = \begin{pmatrix} U_1 & 0 \\ 0 & \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}\dots\mathrm{e}^{\mathrm{i}\theta_m}) \end{pmatrix}$$

for some $U_1 \in U(n)$ and $(\theta_1,\dots,\theta_m) \in \mathbb{R}^m$.

Proof. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, $B \in \mathrm{Mat}_{n\times m}(\mathbb{C})$, $C \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ and $D \in \mathrm{Mat}_{m\times m}(\mathbb{C})$. Put

$$f(z,\zeta) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} z \\ \zeta \end{pmatrix}.$$

By Lemma 4.2, we see that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} Az \\ Cz \end{pmatrix} \in \mathscr{U}$$

for any $z \in \mathbb{C}^n$. Thus we have $C = 0$. We next show that $A \in U(n)$ and $B = 0$. Since our domain is a Reinhardt domain containing the origin, the set S defined by

$$\begin{aligned} S &:= \{z_1^{k_1}\dots z_n^{k_n}\zeta_1^{k'_1}\dots\zeta_m^{k'_m}\}_{k\in\mathbb{Z}^n_{\geqslant 0},k'\in\mathbb{Z}^m_{\geqslant 0}} \cap A^2(D^{\mu}_{n,m}) \\ &= \{z_1^{k_1}\dots z_n^{k_n}\zeta_1^{k'_1}\dots\zeta_m^{k'_m}\}_{k\in\mathbb{Z}^n_{\geqslant 0},k'\in\mathbb{Z}^m_{\geqslant 0}} \end{aligned}$$

forms a complete orthogonal basis of $A^2(D^{\mu}_{n,m})$ (see Section 4.1.1). Thus the normalized monomials form a complete orthonormal basis of $A^2(D^{\mu}_{n,m})$. Therefore, the Bergman kernel has the following form by (1):

$$K_{D^{\mu}_{n,m}}((z,\zeta),(z',\zeta')) = \sum_{k,k'} a_{k,k'} x_1^{k_1}\dots x_n^{k_n} y_1^{k'_1}\dots y_m^{k'_m},$$

where we put $x_i = z_i\overline{z}'_i$, $y_l = \zeta_l\overline{\zeta}'_l$ and $a_{k,k'} = \|z_1^{k_1}\dots z_n^{k_n}\zeta_1^{k'_1}\dots\zeta_m^{k'_m}\|^{-2}_{L^2(D^{\mu}_{n,m})}$. Using this form, we see that $T_{D^{\mu}_{n,m}}(0,0)$ has the form

$$T_{D^{\mu}_{n,m}}(0,0) = \begin{pmatrix} \lambda I_n & 0 \\ 0 & Q \end{pmatrix},$$

where $\lambda > 0$ and Q is a diagonal matrix with nonzero positive real entries. A direct computation shows that all diagonal entries of Q are pairwise distinct. For the convenience of the reader, we put the details of the computation of Q in Appendix separately. Then, by (2), we obtain

$$\begin{aligned} \begin{pmatrix} \lambda I_n & 0 \\ 0 & Q \end{pmatrix} &= \begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix}\begin{pmatrix} \lambda I_n & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \\ &= \begin{pmatrix} \lambda A^*A & \lambda A^*B \\ \lambda B^*A & \lambda B^*B + D^*QD \end{pmatrix}. \end{aligned} \tag{5}$$

Comparing both sides of this equality, we have $A \in U(n)$ and $B = 0$. We next show that $D \in U(m)$. To this end let us observe that $\{(0,\zeta) \in D^{\mu}_{n,m}\} = \{0\} \times \mathbb{B}^m$ and

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ D\zeta \end{pmatrix}.$$

This implies that f induces a linear automorphism $g\colon \zeta \mapsto D\zeta$ of $\mathbb{B}^m$. Thus we see that $D \in U(m)$. To complete the proof, it is enough to show that D is a diagonal matrix. By (5), we have $Q = D^*QD$. Since $D \in U(m)$ and Q is a diagonal matrix with pairwise distinct entries, our desired conclusion follows from Lemma 4.4. □

Before proceeding, we give a remark on $T_{D^{\mu}_{n,m}}(0,0)$.

Remark 4.6. In the proof of this proposition, a form of $T_{D^{\mu}_{n,m}}(0,0)$ played a substantial role. Forms of the $(1,1)$-block entry and the $(2,2)$-block entry can be determined by direct computation of integrals on $D^{\mu}_{n,m}$. We note that the form of the $(1,1)$-block entry can also be derived quickly from the transformation formula (2). An outline of the argument is described as follows:

Step 1. Show that $T_{D^{\mu}_{n,m}}(0,0)$ is a diagonal matrix by using the form of $K_{D^{\mu}_{n,m}}$ in the above proof.

Step 2. Apply (2) to $T_{D^{\mu}_{n,m}}(0,0)$ with the mapping F_{12} defined by

$$F_{12}\colon (z_1, z_2, \dots, z_n, \zeta) \mapsto (z_2, z_1, \dots, z_n, \zeta),$$

which is a unitary automorphism of $D^{\mu}_{n,m}$.

Step 3. By Step 2, one can see that $K_{\bar{1}1}(0,0) = K_{\bar{2}2}(0,0)$. In a similar way, one can define a unitary automorphism F_{ij} for $i \neq j$ and obtain $K_{\bar{1}1}(0,0) = \dots = K_{\bar{n}n}(0,0)$.

The proof of the above proposition does not work if $\mu_{s+1} = \ldots = \mu_m = 0$ and hence we shall consider this case separately.

Lemma 4.7. *Let g be a linear automorphism of $\widetilde{D}^{\mu}_{n,m}$. Then g is given as*

$$g = \begin{pmatrix} U_1 & 0 & 0 \\ 0 & \operatorname{diag}(e^{i\theta_1} \ldots e^{i\theta_s}) & 0 \\ 0 & 0 & U_2 \end{pmatrix},$$

where $U_1 \in U(n)$, $U_2 \in U(m-s)$ and $\theta_1, \ldots, \theta_s \in \mathbb{R}$.

Proof. Put

$$g = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}.$$

As we proved in Lemma 4.3, the subspace defined by $\mathcal{U}_0 := \{(z,0,0) \in \mathbb{C}^{n+s+(m-s)}\}$ is not invariant under actions of the automorphism group. Instead of the invariance, we have

$$\varphi(\mathcal{U}_0) \subset \mathcal{U}_a := \{(z,0,a) \in \mathbb{C}^{n+s} \times \mathbb{B}^{m-s}\},$$

where φ is an arbitrary fixed automorphism and a is a fixed point in the complex unit ball $\mathbb{B}^{m-s}$. It follows that

$$g \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X_{11}z \\ X_{21}z \\ X_{31}z \end{pmatrix} \in \mathcal{U}_a.$$

Thus we have $X_{31}z = a \in \mathbb{B}^{m-s}$ and $X_{21}z = 0$ for any $z \in \mathbb{C}^n$. Putting $z = 0$, we see that the fixed element a equals the zero vector in $\mathbb{C}^{m-s}$. Therefore we immediately have $X_{31} = 0$. It is clear that $X_{21} = 0$ if $X_{21}z = 0$ for all $z \in \mathbb{C}^n$. Moreover, we also obtain $X_{23} = 0$ by the invariance of $\mathcal{V}$ which is proved in Lemma 4.3.

Let us consider $T_{\widetilde{D}^{\mu}_{n,m}}(0,0)$. By using the idea mentioned in Remark 4.6, one can deduce that $T_{\widetilde{D}^{\mu}_{n,m}}(0,0)$ is a diagonal matrix and the $(1,1)$-block entry and the $(3,3)$-block entry are scalar matrices. Furthermore, we can verify by direct computation that the $(2,2)$-block entry is a diagonal matrix with pairwise distinct entries. Namely, $T_{\widetilde{D}^{\mu}_{n,m}}(0,0)$ has the form

$$T_{\widetilde{D}^{\mu}_{n,m}}(0,0) = \begin{pmatrix} \lambda_1 I_n & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & \lambda_2 I_{m-s} \end{pmatrix},$$

where $\lambda_1, \lambda_2 > 0$ and Q is a diagonal matrix with pairwise distinct positive entries. Then, by using an argument similar to that in the previous proposition, we see that

X_{12}, X_{13} and X_{32} are zero matrices. The rest of the proof also proceeds along the same lines as that of the previous proposition. Indeed, the transformation formula (2) and Lemma 4.4 ensure that $X_{11} \in U(n)$, $X_{33} \in U(m-s)$ and $X_{22} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1} \dots \mathrm{e}^{\mathrm{i}\theta_s})$ for $\theta_1, \dots, \theta_s \in \mathbb{R}$. □

4.1.3. Automorphism group. In Section 4.1.2, we provided a description of the linear automorphism group. In this section, we give a complete description of the holomorphic automorphism group of $D^{\mu}_{n,m}$.

Theorem 4.8. *The automorphism group of $D^{\mu}_{n,m}$ is generated by the following mappings:*

$$\begin{aligned}\psi_U\colon\ &(z,\zeta) \mapsto (Uz, \zeta),\\ \psi_\theta\colon\ &(z,\zeta) \mapsto (z, R_\theta\zeta),\\ \psi_v\colon\ &(z,\zeta) \mapsto (z+v, \mathrm{e}^{-\mu_1\langle z,v\rangle - \mu_1\|v\|^2/2}\zeta_1, \dots, \mathrm{e}^{-\mu_m\langle z,v\rangle - \mu_m\|v\|^2/2}\zeta_m),\end{aligned}$$

where $U \in U(n)$, $R_\theta = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1} \dots \mathrm{e}^{\mathrm{i}\theta_m})$, $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$ and $v \in \mathbb{C}^n$.

Proof. One can readily check that ψ_U, ψ_θ and ψ_v are automorphisms of $D^{\mu}_{n,m}$. Let F be an arbitrary automorphism of $D^{\mu}_{n,m}$. By the invariance of $\mathscr{U}$ (see Lemma 4.2), we put $F(0,0) = (v_0, 0)$ for some $v_0 \in \mathbb{C}^n$. Then we see that $\psi_{-v_0} \circ F$ preserves the origin. Therefore, we can conclude that $\psi_{-v_0} \circ F = \psi_\theta \circ \psi_U$. Since $(\psi_{-v_0})^{-1} = \psi_{v_0}$, our mapping F can be written as $\psi_{v_0} \circ \psi_\theta \circ \psi_U$. □

The next theorem gives a description of the automorphism group of $\widetilde{D}^{\mu}_{n,m}$.

Theorem 4.9. *The automorphism group of $\widetilde{D}^{\mu}_{n,m}$ is generated by the mappings*

$$\begin{aligned}\psi_{U,U'}\colon\ &(z,\zeta,\widetilde{\zeta}) \mapsto (Uz, \zeta, U'\widetilde{\zeta}), \quad U \in U(n),\ U' \in U(m-s),\\ \psi_\theta\colon\ &(z,\zeta,\widetilde{\zeta}) \mapsto (z, R'_\theta\zeta, \widetilde{\zeta}), \quad R'_\theta = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1} \dots \mathrm{e}^{\mathrm{i}\theta_s}),\ \theta = (\theta_1, \dots, \theta_s) \in \mathbb{R}^s,\\ \psi_{v,a}\colon\ &(z,\zeta,\widetilde{\zeta}) \mapsto (z+v, \mathrm{e}^{-\mu_1\langle z,v\rangle - \mu_1\|v\|^2/2} F_a(\widetilde{\zeta})\zeta_1, \dots,\\ &\qquad \mathrm{e}^{-\mu_s\langle z,v\rangle - \mu_s\|v\|^2/2} F_a(\widetilde{\zeta})\zeta_s, h_a(\widetilde{\zeta})),\end{aligned}$$

where $v \in \mathbb{C}^n$, $a \in \mathbb{B}^{m-s}$, $F_a(\widetilde{\zeta}) = (1-\|a\|^2)^{1/2}/(1-\langle\widetilde{\zeta}, a\rangle)$ and h_a is the mapping defined in (4)*.*

The proof of this theorem proceeds along the same lines as that of the previous one and we omit details of the proof. Note that we use Lemma 4.3 and Lemma 4.7 for the proof. As a special case of this theorem, we have a description of the automorphism group of $\mathscr{D}$.

Corollary 4.10. *The automorphism group of*

$$\mathscr{D} = \{(z,\zeta) \in \mathbb{C} \times \mathbb{C}^2 \colon \mathrm{e}^{\mu|z|^2}|\zeta_1|^2 + |\zeta_2|^2 < 1\}$$

is generated by the mappings

$$\varrho_\theta \colon (z,\zeta_1,\zeta_2) \mapsto (\mathrm{e}^{\mathrm{i}\theta_1}z, \mathrm{e}^{\mathrm{i}\theta_2}\zeta_1, \mathrm{e}^{\mathrm{i}\theta_3}\zeta_2), \quad \theta = (\theta_1,\theta_2,\theta_3) \in \mathbb{R}^3,$$
$$\psi_{v,a} \colon (z,\zeta) \mapsto \Big(z+v, \mathrm{e}^{-\mu z\overline{v}-\mu|v|^2/2}\frac{(1-|a|^2)^{1/2}}{1-\overline{a}\zeta_2}\zeta_1, \frac{\zeta_2-a}{1-\overline{a}\zeta_2}\Big),$$

where $v \in \mathbb{C}$ and $a \in \mathbb{D}$.

4.2. Twisted Fock-Bargmann-Hartogs domains: general cases. The remaining part of this paper is devoted to the study of the domain

$$D^{\mu,p}_{n,m,k} := \Big\{(z,\zeta_1,\ldots,\zeta_m) \in \mathbb{C}^n \times \mathbb{C}^{k_1} \times \ldots \times \mathbb{C}^{k_m} \colon \frac{\|\zeta_1\|^{2p_1}}{\mathrm{e}^{-\mu_1\|z\|^2}} + \ldots + \frac{\|\zeta_m\|^{2p_m}}{\mathrm{e}^{-\mu_m\|z\|^2}} < 1\Big\},$$

where $\mu_1,\ldots,\mu_m$, $p_1,\ldots,p_m$ are positive real numbers and $k_1,\ldots,k_m$ are positive integers. For simplicity of notation, we denote this domain by $\mathscr{T}$. Without loss of generality, in what follows we always assume that

$$p_i = p_j = 1, \quad \text{only if } \mu_i \neq \mu_j. \tag{6}$$

Indeed, if $p_i = p_j = 1$ and $\mu_i = \mu_j$ for some i and j, then we have

$$\frac{\|\zeta_i\|^{2p_i}}{\mathrm{e}^{-\mu_i\|z\|^2}} + \frac{\|\zeta_j\|^{2p_j}}{\mathrm{e}^{-\mu_j\|z\|^2}} = \frac{\|\zeta'\|^2}{\mathrm{e}^{-\mu_i\|z\|^2}}$$

for $z \in \mathbb{C}^n$, $\zeta_i \in \mathbb{C}^{k_i}$, $\zeta_j \in \mathbb{C}^{k_j}$ and $\zeta' := (\zeta_i,\zeta_j) \in \mathbb{C}^{k_i+k_j}$. This means that we can merge two factors $\mathbb{C}^{k_i}$ and $\mathbb{C}^{k_j}$ into $\mathbb{C}^{k_i+k_j}$ in the description of $D^{\mu,p}_{n,m,k}$. After all the possible mergers, we always have (6). Since the arguments for this domain are almost identical to that of the previous section except some points, we will only explain the key points of the arguments here.

We first observe that Cartan's theorem remains true for $\mathscr{T}$. For the proof, we can use an argument similar to that in Theorem 4.1. Indeed, we have $|z_k| \leqslant \|z\|$, $\|\zeta_l\|^2 \leqslant \mathrm{e}^{-\mu_l\|z\|^2/p_l}$ and $\mu_l/p_l > 0$ for all $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$. Although we need to modify the definition of Ω^m_z for this case, the final conclusion is the same (i.e. $\|z^\alpha\zeta^\beta\|^2_{L^2(\mathscr{T})} < \infty$).

Theorem 4.11. *Let f be an automorphism of $\mathscr{T}$ with $f(0) = 0$. Then f is linear.*

We next observe that Lemma 4.2 is also true for $\mathscr{T}$:

Lemma 4.12. *Let φ be an arbitrary automorphism of $\mathscr{T}$. Then the space $\{(z,0) \in \mathbb{C}^{n+k_1+\ldots+k_m}\} \subset \mathscr{T}$ is invariant under φ.*

In fact, using again $\|\zeta_l\|^2 \leqslant \mathrm{e}^{-\mu_l\|z\|^2/p_l}$ and $\mu_l/p_l > 0$, we can conclude that $\varphi_{2l} \equiv 0 \in \mathbb{C}^{k_l}$ for each $1 \leqslant l \leqslant m$ as in [9], Lemma 8. We next consider how Proposition 4.5 can be generalized to $\mathscr{T}$. For the convenience of the exposition, our study in this section will be divided into the following two cases.

Case I: $(p_1, \ldots, p_m) = (1, \ldots, 1)$,

Case II: $(p_1, \ldots, p_m) \neq (1, \ldots, 1)$.

The first case will be studied in the next subsection and the other in Section 4.2.2.

4.2.1. Case I: $(p_1, \ldots, p_m) = (1, \ldots, 1)$. Thanks to (6), we can always assume that $\mu_i \neq \mu_j$, $i \neq j$, in this case. We prepare the following lemma as a generalization of Proposition 4.5.

Lemma 4.13. *Every linear automorphism $g\colon (z, \zeta_1, \ldots, \zeta_m) \mapsto (\tilde{z}, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_m)$ of $\mathscr{T}$ is given as*

$$\tilde{z} = Uz, \quad \tilde{\zeta}_l = U_l\zeta_l,$$

where $U \in U(n)$, $U_l \in U(k_l)$, $1 \leqslant l \leqslant m$.

Proof. Let us use the same notation as in Proposition 4.5. Since the arguments for proving $A \in U(n)$, $B = 0$, $C = 0$ are the same as before, we discuss only D. In this case, $T_{\mathscr{T}}$ has the following form (see also Remark 4.6):

$$T_{\mathscr{T}}(0,0) = \begin{pmatrix} \lambda' I_n & 0 \\ 0 & Q' \end{pmatrix},$$

where $\lambda' > 0$. Here, by using a computation similar to that in Appendix, we see that Q' is a block diagonal matrix such that

$$Q' = \begin{pmatrix} c_1 I_{k_1} & 0 & \ldots & 0 \\ 0 & c_2 I_{k_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_m I_{k_m} \end{pmatrix},$$

where $c_1, \ldots, c_m$ are pairwise distinct positive numbers. In this case g induces a linear automorphism $\zeta \mapsto D\zeta$ of $\mathbb{B}^{k_1+\ldots+k_m}$ and thus $D \in U(k_1 + \ldots + k_m)$. Moreover, we can derive $Q' = D^*Q'D$ in the same way as in Proposition 4.5. Since $D \in U(k_1 + \ldots + k_m)$ and $Q' = D^*Q'D$ is such that Q' is a block diagonal matrix as above, one can deduce that D is a unitary block diagonal matrix by using the fact that $c_1, \ldots, c_m$ are pairwise distinct. Therefore we can conclude that D has our desired form. $\square$

Using the above arguments, we are now ready to state the following theorem.

Theorem 4.14. *The automorphism group of* $\mathscr{T}$ *is generated by the linear mapping as in Lemma* 4.13 *and the mapping*

$$\psi_v\colon\ (z,\zeta)\mapsto(z+v,\mathrm{e}^{-\mu_1\langle z,v\rangle-\mu_1\|v\|^2/2}\zeta_1,\dots,\mathrm{e}^{-\mu_m\langle z,v\rangle-\mu_m\|v\|^2/2}\zeta_m),$$

where $v\in\mathbb{C}^n$.

The proof of this theorem is identical to that of Theorem 4.8.

4.2.2. Case II: $(p_1,\dots,p_m)\neq(1,\dots,1)$**.** Without loss of generality, we relabel the indices so that $p_i=1$, $i\leqslant t$, and $p_j\neq 1$, $j>t$, for some $t=0,1,\dots,m-1$. With this notation, we investigate the following domain in this section:

$$\mathscr{T}=\Big\{(z,\zeta_1,\dots,\zeta_m)\in\mathbb{C}^{n+k_1+\dots+k_m}\colon\ \sum_{i=1}^{t}\frac{\|\zeta_i\|^2}{\mathrm{e}^{-\mu_i\|z\|^2}}+\sum_{j=t+1}^{m}\frac{\|\zeta_j\|^{2p_j}}{\mathrm{e}^{-\mu_j\|z\|^2}}<1\Big\}.$$

We prepare the following lemma instead of Lemma 4.13.

Lemma 4.15. *Let t be as above. Then every linear automorphism*

$$g\colon\ (z,\zeta_1,\dots,\zeta_m)\mapsto(\tilde z,\tilde\zeta_1,\dots,\tilde\zeta_m)$$

of $\mathscr{T}$ *is given as*

$$\tilde z=Uz,\quad \tilde\zeta_i=\widehat{U}_i\zeta_i,\quad \tilde\zeta_j=U_j\zeta_{\sigma(j)}\quad \text{for } i\leqslant t \text{ and } j>t. \tag{7}$$

Here $U\in U(n)$, $\widehat{U}_i\in U(k_i)$, $U_j\in U(k_j)$, *and* σ *is a permutation of* $\{t+1,\dots,m\}$ *satisfying* $k_j=k_{\sigma(j)}$, $p_j=p_{\sigma(j)}$ *and* $\mu_j=\mu_{\sigma(j)}$ *for each* $j>t$.

Proof. Again, let us use the same notation as in Proposition 4.5. For $A\in U(n)$, $B=0$, $C=0$, we can use the same argument even for $\mathscr{T}$. In the rest of the proof, we discuss only D. In Proposition 4.5, we used the fact that D induces a linear automorphism of $\{(0,\zeta)\in D^{\mu}_{n,m}\}=\{0\}\times\mathbb{B}^m$. In the case of this lemma, D induces a linear automorphism of

$$E(k',k_{t+1},\dots,k_m;1,p_{t+1},\dots,p_m)$$
$$:=\{(\zeta',\zeta_{t+1},\dots,\zeta_m)\in\mathbb{C}^{k'+k_{t+1}+\dots+k_m}\colon\ \|\zeta'\|^2+\|\zeta_{t+1}\|^{2p_{t+1}}+\dots+\|\zeta_m\|^{2p_m}<1\},$$

where $k' := k_1 + \ldots + k_t$. On the other hand, we first note from Kodama in [12], Lemma, that D can be decomposed into the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where $D_1 \in \mathrm{Mat}_{k' \times k'}(\mathbb{C})$ and $D_2 \in \mathrm{Mat}_{k'' \times k''}(\mathbb{C})$ for $k'' := k_{t+1} + \ldots + k_m$. In addition, since $(z, \zeta') \mapsto (Az, D_1\zeta')$ induces a linear automorphism of

$$\Big\{(z, \zeta_1, \ldots, \zeta_t) \in \mathbb{C}^{n+k_1+\ldots+k_t} \colon \frac{\|\zeta_1\|^2}{\mathrm{e}^{-\mu_1\|z\|^2}} + \ldots + \frac{\|\zeta_t\|^2}{\mathrm{e}^{-\mu_t\|z\|^2}} < 1\Big\},$$

an argument similar to that in Lemma 4.13 after replacing m by t implies that

$$D_1 = \begin{pmatrix} \widehat{U}_1 & 0 & \ldots & 0 \\ 0 & \widehat{U}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \widehat{U}_t \end{pmatrix},$$

where $\widehat{U}_i \in U(k_i)$ for each $i \leqslant t$. If $t = 0$, then we have $D = D_2$.

Next we note that since $\zeta'' \mapsto D_2\zeta''$ induces a linear automorphism of a generalized complex ellipsoid

$$\begin{aligned} &E(k_{t+1}, \ldots, k_m; p_{t+1}, \ldots, p_m) \\ &\quad := \{\zeta'' = (\zeta_{t+1}, \ldots, \zeta_m) \in \mathbb{C}^{k_{t+1}+\ldots+k_m} \colon \|\zeta_{t+1}\|^{2p_{t+1}} + \ldots + \|\zeta_m\|^{2p_m} < 1\}, \end{aligned}$$

D_2 can be described by using the relation $\tilde{\zeta}_j = U_j\zeta_{\sigma(j)}$, $j > t$, in (7) (cf. [12], Lemma, and [23], Theorem 1.B). It clearly follows from the above conditions on σ that $k_j = k_{\sigma(j)}$ and $p_j = p_{\sigma(j)}$ for each $j > t$.

We now prove that $\mu_j = \mu_{\sigma(j)}$ for $j \neq \sigma(j)$. Aiming at contradiction, for the case when $\mu_j \neq \mu_{\sigma(j)}$ for some $j \neq \sigma(j)$, suppose that the associated mapping g of the form (7) is contained in the automorphism group of $\mathscr{T}$. Without loss of generality, we may assume that $j < \sigma(j)$. Then we take a point

$$\begin{aligned} (z, \zeta_0) := \Big(&z, 0, \ldots, 0, \mathrm{e}^{-\mu_j\|z\|^2/(2p_j)}\Big(\frac{1}{2}\Big)^{1/(2p_j)} a_j, 0, \ldots, 0, \\ &\mathrm{e}^{-\mu_{\sigma(j)}\|z\|^2/(2p_{\sigma(j)})}\Big(\frac{1}{2}\Big)^{1/(2p_{\sigma(j)})} a_{\sigma(j)}, 0, \ldots, 0\Big) \in \partial\mathscr{T}, \end{aligned}$$

where $z \in \mathbb{C}^n$, $a_j \in \partial\mathbb{B}^{k_j}$ and $a_{\sigma(j)} \in \partial\mathbb{B}^{k_{\sigma(j)}}$ for $j > t$. Since $g(\partial\mathscr{T}) = \partial\mathscr{T}$ as a set, we infer that

$$\begin{aligned} g(z, \zeta_0) = \Big(&Az, 0, \ldots, 0, U_j\mathrm{e}^{-\mu_{\sigma(j)}\|z\|^2/(2p_{\sigma(j)})}\Big(\frac{1}{2}\Big)^{1/(2p_{\sigma(j)})} a_{\sigma(j)}, 0, \ldots, 0, \\ &U_{\sigma(j)}\mathrm{e}^{-\mu_j\|z\|^2/(2p_j)}\Big(\frac{1}{2}\Big)^{1/(2p_j)} a_j, 0, \ldots, 0\Big) \in \partial\mathscr{T}, \end{aligned}$$

which contradicts the fact that

$$\frac{\frac{1}{2}\mathrm{e}^{-\mu_{\sigma(j)}\|z\|^2}}{\mathrm{e}^{-\mu_j\|z\|^2}} + \frac{\frac{1}{2}\mathrm{e}^{-\mu_j\|z\|^2}}{\mathrm{e}^{-\mu_{\sigma(j)}\|z\|^2}} \neq 1$$

for all $z \in \mathbb{C}^n \setminus \{0\}$ if $\mu_j \neq \mu_{\sigma(j)}$. This completes the proof of the assertion. □

Once the above arguments are established, we have the following theorem.

Theorem 4.16. *The automorphism group of* $\mathscr{T}$ *is generated by the linear mapping as in Lemma* 4.15 *and the mapping*

$$\psi_{v,p}\colon\ (z,\zeta) \mapsto (z+v, \mathrm{e}^{-\mu_1\langle z,v\rangle/p_1-\mu_1\|v\|^2/(2p_1)}\zeta_1, \ldots, \mathrm{e}^{-\mu_m\langle z,v\rangle/p_m-\mu_m\|v\|^2/(2p_m)}\zeta_m),$$

where $v \in \mathbb{C}^n$.

Since the proof of this theorem is identical to that of Theorem 4.8 except the definition of $\psi_{v,p}$, we omit it.

Appendix

In this appendix we give the details of the computation of the matrix Q in the procedure of proving Proposition 4.5.

As in the proof of Proposition 4.5, the Bergman kernel of $D^{\mu}_{n,m}$ can be written as

$$K_{D^{\mu}_{n,m}}((z,\zeta),(z',\zeta')) = \sum_{k,k'} a_{k,k'}(z_1\overline{z}'_1)^{k_1}\ldots(z_n\overline{z}'_n)^{k_n}(\zeta_1\overline{\zeta}'_1)^{k'_1}\ldots(\zeta_m\overline{\zeta}'_m)^{k'_m},$$

where $a_{k,k'} = \|z_1^{k_1}\ldots z_n^{k_n}\zeta_1^{k'_1}\ldots\zeta_m^{k'_m}\|^{-2}_{L^2(D^{\mu}_{n,m})}$. Using the definition of Q, we compute the values of

$$\Big(\frac{\partial^2}{\partial\overline{\zeta}'_\gamma\partial\zeta_\nu}\log K_{D^{\mu}_{n,m}}((z,\zeta),(z',\zeta'))\Big)_{1\leqslant\gamma,\nu\leqslant m}$$

at $((z,\zeta),(z',\zeta')) = (0,0)$. By a direct computation, one can deduce that

$$\frac{\partial^2}{\partial\overline{\zeta}'_\gamma\partial\zeta_\nu}\log K_{D^{\mu}_{n,m}}((z,\zeta),(z',\zeta'))\Big|_{((z,\zeta),(z',\zeta'))=(0,0)} = \frac{\delta_{\gamma\nu}a_{0,\widehat{k'_\nu}}}{a_{0,0}},$$

where $\widehat{k'_\nu}$ denotes the multi-index k' where the νth component is one and all other ones vanish. Indeed, the precise value of $a_{0,\widehat{k'_\nu}}$ can be obtained by considering the change of variables

$$\zeta_l = \varrho_l\mathrm{e}^{-\mu_l\|z\|^2/2}\mathrm{e}^{\mathrm{i}\theta_l} \quad \text{with } \sum_{i=1}^m \varrho_i^2 < 1,\ \varrho_l \geqslant 0,\ 0 \leqslant \theta_l \leqslant 2\pi \text{ for } l = 1,\ldots,m.$$

Then we get

$$\mathrm{d}V(\zeta) = \mathrm{e}^{-|\mu|\|z\|^2} \prod_{l=1}^{m} \varrho_l \, \mathrm{d}\varrho_1 \dots \mathrm{d}\varrho_m \, \mathrm{d}\theta_1 \dots \mathrm{d}\theta_m,$$

where $|\mu| := \sum_{l=1}^{m} \mu_l$. Putting $R := \Big\{(\varrho_1, \dots, \varrho_m) \in [0,1)^m \colon \sum_{l=1}^{m} \varrho_l^2 < 1\Big\}$, one can deduce that

$$\begin{aligned}
(a_{0,\widehat{k'_\nu}})^{-1} &= \int_{D^\mu_{n,m}} |\zeta_\nu|^2 \, \mathrm{d}V(z,\zeta) \\
&= (2\pi)^m \int_{\mathbb{C}^n} \int_R \mathrm{e}^{-\mu_\nu\|z\|^2} \varrho_\nu^2 \mathrm{e}^{-|\mu|\|z\|^2} \prod_{l=1}^{m} \varrho_l \, \mathrm{d}\varrho_1 \dots \mathrm{d}\varrho_m \, \mathrm{d}V(z) \\
&= (2\pi)^m \int_{\mathbb{C}^n} \mathrm{e}^{-(\mu_\nu+|\mu|)\|z\|^2} \, \mathrm{d}V(z) \int_R \varrho_\nu^2 \prod_{l=1}^{m} \varrho_l \, \mathrm{d}\varrho_1 \dots \mathrm{d}\varrho_m.
\end{aligned}$$

Due to the symmetry of coordinates, the second factor $\int_R \varrho_\nu^2 \prod_{l=1}^{m} \varrho_l \, \mathrm{d}\varrho_1 \dots \mathrm{d}\varrho_m$ is independent of the choice of ν. Moreover, since

$$\begin{aligned}
&\int_R x_1^{2a_1+1} \dots x_{m-1}^{2a_{m-1}+1} x_m^{2a_m+1} \, \mathrm{d}x_1 \dots \mathrm{d}x_{m-1} \, \mathrm{d}x_m \\
&\quad = \frac{\Gamma(a_1+1)\dots\Gamma(a_{m-1}+1)\Gamma(a_m+1)}{2^m\Gamma(a_1+\dots+a_{m-1}+a_m+m+1)}
\end{aligned}$$

for all constants $a_l > -1$ where $1 \leqslant l \leqslant m$ (cf. [3], Lemma 1), we obtain

$$\int_R \varrho_\nu^2 \prod_{l=1}^{m} \varrho_l \, \mathrm{d}\varrho_1 \dots \mathrm{d}\varrho_m = \frac{1}{(m+1)!\,2^m}.$$

On the other hand, we have

$$\int_{\mathbb{C}^n} \mathrm{e}^{-(\mu_\nu+|\mu|)\|z\|^2} \, \mathrm{d}V(z) = \prod_{k=1}^{n} \int_{\mathbb{C}} \mathrm{e}^{-(\mu_\nu+|\mu|)|z_k|^2} \, \mathrm{d}V(z_k) = \frac{\pi^n}{(\mu_\nu+|\mu|)^n}.$$

Therefore, we get

$$(a_{0,\widehat{k'_\nu}})^{-1} = \frac{\pi^{n+m}}{(m+1)!\,(\mu_\nu+|\mu|)^n}. \tag{8}$$

Now let us consider $(a_{0,0})^{-1}$ that is exactly the Euclidean volume $\mathrm{Vol}(D^\mu_{n,m})$ of $D^\mu_{n,m}$. This indeed follows from a special case of equation (3) in Section 4. More precisely, if we let $\alpha_1 = \ldots = \alpha_n = \beta_1 = \ldots = \beta_m = 0$ in (3), then the discussion right after (3) shows that

$$\int_{D^{\mu}_{n,m}} G(z)\,\mathrm{d}V(z,\zeta) = \int_{D^{\mu}_{n,m}} 1\cdot \mathrm{d}V(z,\zeta) = \mathrm{Vol}(D^{\mu}_{n,m}) = \frac{\pi^{n+m}}{m!\,|\mu|^n}. \tag{9}$$

Thus, combining (8) with (9), we obtain

$$\frac{\partial^2}{\partial\bar{\zeta}'_\gamma \partial\zeta_\nu} \log K_{D^{\mu}_{n,m}}((z,\zeta),(z',\zeta'))\Big|_{((z,\zeta),(z',\zeta'))=(0,0)} = \frac{\delta_{\gamma\nu} a_{0,\widehat{k'_\nu}}}{a_{0,0}} = \frac{\delta_{\gamma\nu}(m+1)(\mu_\nu + |\mu|)^n}{|\mu|^n}.$$

Hence this relation ensures that Q is a diagonal matrix with pairwise distinct positive diagonal entries.

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