THE HOLONOMY COVERING SPACE IN PRINCIPAL FIBRE BUNDLES

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1. Introduction. Recently L. Auslander and L. Markus [2][1] have studied on flat affinely connected manifolds. They have defined the holonomy covering space over the manifold and have shown that it has similar nature of a minimal universal covering space.

In this paper we shall extend these to principal fibre bundles with locally flat connections.

We shall first recall the definition of a (differentiable) principal fibre bundle [5, 7, 9, 10, 11] and denote it by $P(M, \pi, G)$, where P is the bundle space, M the base space, π a projection of P onto M and G a Lie group (not necessarily connected) acting on P on the right. By differentiability we shall always understand that of class C^∞. We shall give in $P(M, \pi, G)$ a connection Γ by the distribution $\Gamma: u \rightarrow \Gamma_u$ (horizontal subspace at u), where $u \in P$, or equivalently by a connection form ω in P with values in the Lie algebra $\mathfrak{g}$ of G [1, 3, 5, 7, 9, 10]. It is well known that the structure equation is given by

$$d\omega(\overline{u_1}, \overline{u_2}) = -\frac{1}{2}[\omega(\overline{u_1}), \omega(\overline{u_2})] + \Omega(\overline{u_1}, \overline{u_2}) \tag{1}$$

on P with connection Γ, where $\overline{u_1}$ and $\overline{u_2}$ are any vector fields on P, the bracket is the bracket operation in the Lie algebra $\mathfrak{g}$ and Ω is the curvature form of the connection Γ.

Let $P(M, \pi, G)$ be a principal fibre bundle and let h be a mapping of a manifold M' into M. Let $h^{-1}(P)$ be the set of points (x', u) of $M \times P$ such that $\pi(u) = h(x')$. $h^{-1}(P)$ is clearly a principal fibre bundle and we call it the principal fibre bundle induced by h. The mapping $\bar{h}$ of $h^{-1}(P)$ into P defined by

$$\bar{h}(x', u) = u,$$

commutes with the right translation by G. Hence $\bar{h}$ is a bundle map of $h^{-1}(P)$ into P.

Let $\bar{h}$ be a bundle map of a principal fibre bundle $P(M, \pi, G)$ into another principal fibre bundle $P'(M', \pi', G)$ and Γ' a connection in P'. Then there exists a connection Γ in P which is naturally induced from Γ'. The form ω on P which defines Γ is given by

$$\omega = \omega' \circ \delta \bar{h}.$$

We shall Γ the connection induced from Γ' by $\bar{h}$ and ω the form induced from

1) Numbers in brackets refer to the bibliography at the end of the paper.

ω' by $\bar{h}$ [7, 8].

2. The holonomy covering space. In order to define the holonomy covering space, we shall first consider local flatness of connections given on P.

Let $P(M, \pi, G) = M \times G$ be a trivial principal fibre bundle. Then $P = \bigcup_{s \in G} \Gamma_s$, where Γ_s is a submanifold of P consisting of all points of the form $(x, s), x \in M$. A flat connection in P is defined by taking the tangent space Γ_s as the horizontal subspace at $u = (x, s) \in P$ defining a distribution Γ. In this case, the curvature form is obviously zero.

A connection in any principal fibre bundle $P(M, \pi, G)$ is called locally flat if every point $x \in M$ admits a neighbourhood U such that induced connection in $\pi^{-1}(U)$ by the injection $\pi^{-1}(U) \to P$ is isomorphic with the flat connection in $U \times G$.

We shall state now some propositions concerning with the locally flat connection.

PROPOSITION 1. *A connection in $P(M, \pi, G)$ is locally flat if and only if the curvature form is equal to zero.* [10]

PROPOSITION 2. *A connection in $P(M, \pi, G)$ with a compact structure group G is locally flat if the coordinate transformations of $P(M, \pi, G)$ are constant functions on a component of $U_\lambda \cap U_\mu$, where U_λ and U_μ are two coordinate neighbourhoods of the base space such that $U_\lambda \cap U_\mu \neq \phi$.* [6]

Coordinate neighbouhoods stated in Proposition 2 correspond to the affine covering described in [2] and this Proposition is an analogous result of Theorem 1 in [2].

Next we shall recall holonomy groups in $P(M, \pi, G)$ with a connection Γ. Let C be a piece-wise differentiable curve in M starting at x_0 and ending at x_1, there exists a unique horizontal curve starting at $u_0 \in P$ and covering the curve C, where $x_0 = \pi(u_0)$. Its end point is a certain point u_1 such that $\pi(u_1) = x_1$ and u_1 is called that it is obtained by the parallel displacement of u_0 along the curve C and will be denoted by Cu_0. The right translation by any element s of G and the parallel displacement along any curve C in M commute each other:

$$C(u_0 s) = (Cu_0)s.$$

Now we shall define the holonomy group of the connection. Let x be arbitrary point in M and let $\mathfrak{C}_x$ the set of all piece-wise differentiable closed curves at x. For each $C \in \mathfrak{C}_x$, the corresponding parallel displacement C is a homeomorphism of the fibre G_x over x onto itself. The set of all these homomorphisms corresponding to $C \in \mathfrak{C}_x$, forms a subgroup of G denoted by $H(M; \Gamma, x)$ is called the holonomy group at x of the given connection Γ. Since a curve starting at x and ending at x' determines an isomorphism of the group $H(M; \Gamma, x)$ onto $H(M; \Gamma, x')$, we shall denote it simply by $H(M; \Gamma)$.

Let $\mathfrak{C}_x^0$ be the set of all closed curves at x which are homotopic to zero. Corresponding to all curves in $\mathfrak{C}_x^0$, we obtain a subgroup of $H(M;\Gamma, x)$ denoted by $H^0(M;\Gamma, x)$ and we shall call it the restricted holonomy group at x. It is denoted simply by $H^0(M;\Gamma)$, too.

Let $P(M, \pi, G)$ be a principal fibre bundle with a connection Γ. Then we have the structure equation (1). On the other hand, it is known that for any vector fields $\bar{u}_1$ and $\bar{u}_2$ on P we have

$$2d\omega(\bar{u}_1, \bar{u}_2) = \bar{u}_1 \cdot \omega(\bar{u}_2) - \bar{u}_2 \cdot \omega(\bar{u}_1) - \omega([\bar{u}_1, \bar{u}_2])^{2)} \tag{2}$$

where $\bar{u}_1 \cdot \omega(\bar{u}_2)$ denotes the function obtained by applying $\bar{u}_1$ to the function $\omega(\bar{u}_2)$ and similarly for $\bar{u}_2 \cdot \omega(\bar{u}_1)$. If $\bar{u}_1$ and $\bar{u}_2$ are horizontal vector fields, the equation (2) reduces to

$$2d\omega(\bar{u}_1, \bar{u}_2) = -\omega([\bar{u}_1, \bar{u}_2]).$$

Hence for the same $\bar{u}_1$ and $\bar{u}_2$, the structure equation (1) reduces to

$$\omega([\bar{u}_1, \bar{u}_2]) = -2\Omega(\bar{u}_1, \bar{u}_2).$$

This shows that the vertical component of $[\bar{u}_1, \bar{u}_2]$ is given by $-2\Omega(\bar{u}_1, \bar{u}_2)$.

If a connection Γ given on $P(M, \pi, G)$ is locally flat, by virtue of Proposition 1, the curvature form is equal to zero. Hence in this case for any horizontal vector fields u_1 and $\bar{u}_2$ we have

$$\omega([\bar{u}_1, \bar{u}_2]) = 0,$$

which shows that $[\bar{u}_1, \bar{u}_2]$ is also horizontal. Then we have proved the following:

PROPOSITION 3. *If a connection Γ given on $P(M, \pi, G)$ is locally flat, then the distribution $\Gamma : u \to \Gamma u$ is involutive, where $u \in P$.*

Hence through every point $u \in P$, there passes a maximal integral manifold $M^{\#}(u)$ of the distribution.

It is known that

PROPOSITION 4. *If a connection Γ given on $P(M, \pi, G)$ is locally flat, then there exists a homomorphism of* $\pi_1(M, x)$ onto $H(M;\Gamma, x)$ *is a fundamental group of M at x.* [5]

THEOREM 1. *The maximal integral manifold $M^{\#}(u)$ passing through a point $u \in P$ in a principal fibre bundle with a locally flat connection Γ forms a covering space of M whose covering map is $h = \pi | M^{\#}(u)$.*

PROOF. Since M is arc-wise connected, there exists a curve C in M which starts at a fixed point x and ends at any point x' of M. On the other hand, there exists a lift $C^{\#}$ of C in $M^{\#}$(u) ending at $u' \in M^{\#}(u)$ such that $h(u') = x'$. Hence, for any x' of M, there exists a point u' projected to x' by h, which shows that h is a differentiable mapping of $M^{\#}(u)$ onto M.

We take a suitably small neighbourhood U of $x_0 \in M$ in M to be simply

2) See p. 11 of [10].

connected. Let u_0 be a point of M (u) such that $h(u_0) = x_0$. For $x \in U$, we take a curve C_x, which starts at x_0 and ends at x_1, then there exists a unique lift $C^{\#}_{x_1}$ of C_{x_1} whose end point is a certain point u_1 of $M^{\#}(u)$ such that $h(u_1) = x_1$. Let $U^{\#}(u_0)$ be the set of such points as u_1 when x_1 runs all over U. It is clear that $h(U^{\#}(u_0)) = U$.

Next we shall show that h defines a (differentiable) homeomorphism of $U^{\#}(u_0)$ with U.

Suppose that, for $x' \in U$, there are two points u_1' and u_2' $(u_1' \neq u_2')$ of $U^{\#}(u_0)$ such that $h(u_1') = h(u_2') = x'$. Then we can join u_1' to u_2' by a piece-wise differentiable curve $C^{\#}$ contained in $U^{\#}(u_0)$. If we put $C' = h(C^{\#})$, C' is a closed curve at x' and is contained in U. Hence C' is homotopic to zero, since U is simply connected. On the other hand, the curve C' determines a parallel displacement and is contained in a class of the holonomy group $H(M; \Gamma, x')$. By virtue of Proposition 4, there exists a homomorphism of $\pi_1(M, x')$ onto $H(M; \Gamma, x')$. Since the curve C' is homotopic to zero, C' corresponds to the unit class of $H(M; \Gamma, x')$. Hence $C^{\#}$ must be closd. This is contrary to the assumption.

Therefore we can easily conclude that, for each $x \in M$, there exists a neighbourhood U of x such that h defines a differentiable homeomorphism of each component of $h^{-1}(U)$ with U. This shows that the maximal integral manifold $M^{\#}(u)$ forms a covering space of M whose covering map is $h = \pi | M^{\#}(u)$. Q.E.D.

The distribution $\Gamma: u \to \Gamma_u$ which gives a connection in $P(M, \pi, G)$ satisfies the condition that Γ is invariant under the right translations induced by the action of G, i.e., for every $s \in G$ and $u \in P$, Γ_{us} is the image of Γ_u by the right translation by s. Since, for u_1 and u_2 such that $\pi(u_1) = \pi(u_2)$, there is $s \in G$ such that $u_2 = u_2 s$, if a given connection is locally flat, we have

$$M^{\#}(u_2) = M^{\#}(u_1)s \,;\, h_2 = h_1 \circ s,$$

where $M^{\#}(u_i)$ is a maximal integral manifold passing through a point $u_i \in P$, which covers M by $h_i (i = 1, 2)$.

Hence a maximal integral manifold which covers M is unique in the above sense. Then we denote it by $M^{\#}$.

DEFINITION. If a connection Γ given on $P(M, \pi, G)$ is locally flat, a maximal integral manifold $M^{\#}$ of the distribution Γ, together with the restriction $h = \pi | M^{\#}$, is called a *holonomy covering space* of M for Γ.

It we take the bundle of frames and the flat affine connection instead of the principal fibre bundle and the (infinitesimal) connection respectively, the definition of the holonomy covering space reduces to that described in [2].

3. Properties of the holonomy covering space. In this section we shall study various properties of the holonomy covering space. Now we shall prove the

PROPOSITION 5. *Let $\bar{h}$ be a bundle map of a principal fibre bundle $P'(M', \pi', G)$ into a principal fibre bundle $P(M, \pi, G)$ with a locally flat connection Γ. Then the connection Γ' induced from Γ by $\bar{h}$ is locally flat.*

PROOF. To prove the proposition, we shall show $\Omega'(\overline{u_1'}\ \overline{u_2'}) = 0$, where Ω' is the curvature form of the induced connection Γ' on P' and $\overline{u_1'}$ and $\overline{u_2'}$ are any vector fields on P'. Then

$$\Omega'(\overline{u_1'}, \overline{u_2'}) = d\omega'(\overline{u_1'}_H, \overline{u_2'}_H) = -\frac{1}{2}\omega'([\overline{u_1'}_H, \overline{u_2'}_H])$$
$$= -\frac{1}{2}\omega([\overline{u_1}_H, \overline{u_2}_H]),$$

where d denotes the usual exterior differentiation and $\overline{u_i'}_H$(resp. $\overline{u_i}_H$) the horizontal component of $\overline{u_i'}$ (resp. $\overline{u_i}$) $(i = 1, 2)$. Now $\overline{u_i}$ is horizontal. Since Γ is locally flat, $[\overline{u_1}, \overline{u_2}]$ is horizontal. Then we have

$$\Omega'(\overline{u_1'}, \overline{u_2'}) = 0. \qquad \text{Q.E.D.}$$

LEMMA. *Let h be a bundle map of a principal fibre bundle $P'(M', \pi', G)$ into a principal fibre bundle $P(M, \pi, G)$ with a connection Γ. A closed curve C' of M' starting at $x' \in M'$ is holonomous to zero for the connection Γ' induced from Γ by $\bar{h}$ if and only if the closed curve C is holonomous to zero for Γ, where $C = \bar{h}(C')$.*

PROOF. Suppose that a closed curve C' of M' starting at x' is holonomous to zero for Γ'. A point $u' \in P'$ such that $\pi'(u') = x'$ is parallely displaced to $C'u'$. Since C' is holonomous to zero, we have $C'u' = u'$. Let $\widetilde{C'}$ be the unique lift of C' starting at u'. It is clear that the image $\widetilde{C} = \bar{h}(\widetilde{C'})$ is a horizontal curve in P for Γ, which starts at $u = \bar{h}(u')$. Now

$$\bar{h}(C'u') = \bar{h}(u') = u,$$

hence $\widetilde{C}$ is a closed horizontal curve in P. Hence $h(C') = \pi(\widetilde{C}) = C$ is holonomous to zero for Γ.

Conversely let C' be a closed curve of M' at x'. Suppose that the image curve $C = h(C')$ at $x = h(x')$ is holonomous to zero for Γ, i.e., $Cu = u$, where $\pi(u) = x$. Let u' be a point of P' over x' such $\bar{h}(u') = u$. If $C'u' \neq u'$, there exist $s \in G$ ($s \neq$unit) such that $C'u' = u's$, since C' is closed. Since $\bar{h}$ is a bundle map, we have

$$\bar{h}(C'u') = \bar{h}(u')s = us.$$

But us must coincide with u, since C is holonomous to zero. Hence such s is the unit of G, which is contrary to the assumption. Hence $Cu' = u'$. This is true for all $u' \in P$ such that $\pi'(u') = x'$. Hence C' is holonomous to zero for Γ'. Q.E.D.

Let $P(M, \pi, G)$ be a principal fibre bundle with a locally flat connection Γ. Then there is a holonomy covering space $M^{\#}$ whose covering map is

$h=\pi|M^{\#}$. Now we can construct an induced bundle $h^{-1}(P)$ with the induced map $\bar{h}$ of $h^{-1}(P)$ onto P. Next we can define a connection $\Gamma^{\#}$ on $P^{\#}=h^{-1}(P)$ induced from Γ by $\bar{h}$, i.e.,

$$\omega^{\#}=\omega\circ\delta\bar{h},$$

where $\omega^{\#}$ is a connection form on $P^{\#}$ and ω a given connection form on P.

By virtue of the above Lemma and Proposition 5, we have the

THEOREM 2. *Let $P(M,\pi,G)$ be a principal fibre bundle with a locally flat connection Γ and let $M^{\#}$ be the holonomy covering space of the base space M whose covering map is h. Then the connection $\Gamma^{\#}$ in the induced bundle $P^{\#}=h^{-1}(P)$ induced from Γ in P by induced map $\bar{h}$ if $P^{\#}$ onto P, is locally flat.*

Next we shall consider the holonomy covering space. By virtue of the Lemma it is easily verified for the induced connection the following

PROPOSITION 6. *Let Γ' be the connection in P' induced from a connection Γ in P by $\bar{h}: P'\to P$ and let $H(M';\Gamma')$, $H^0(M';\Gamma')$, $H(M;\Gamma)$ and $H^0(M;\Gamma)$ be the holonomy groups and the restricted holonomy groups of Γ' and Γ respectively. Then $H(M';\Gamma')$ (resp. $H^0(M';\Gamma')$) is a (not necessarily closed) subgroup of $H(M;\Gamma)$ (resp. $H^0(M;\Gamma)$).* [7. 8]

COROLLARY. *Let $P(M,\pi,G)$ be a principal fibre bundle with a locally flat connection Γ and let M' be a covering space of M whose covering map is h'. Then the induced connection Γ' in $h'^{-1}(P)$ from Γ by $\bar{h}':h'^{-1}(P)\to P$ is locally flat and $H(M';\Gamma')$ is a subgroup of $H(M;\Gamma)$.*

The corollary is true by virtue of Proposition 5 and Proposition 6. But if a given connection Γ in P is locally flat, Ω is equal to zero. Hence by the holonomy theorem [11], the restricted holonomy group consists of only the unit, which is the reason why in the corollary we do not consider the restricted holonomy group.

We shall easily see the following proposition as a corollary of Proposition 4.

PROPOSITION 7. *Let $P(M,\pi,G)$ be a principal fibre bundle with a locally flat connection Γ and let $\tilde{M}$ be a universal covering space of M whose covering map is $\tilde{h}$. Let $\tilde{\Gamma}$ be the locally flat connection in $\tilde{h}^{-1}(P)$ from a connection Γ in P by the induced map $\bar{\tilde{h}}^{-1}(P)\to P$. Then we have $H(\tilde{M};\tilde{\Gamma})=\{0\}$*[1].

If a connection Γ given in $P(M,\pi,G)$ is locally flat, we can define the holonomy covering space $M^{\#}$. Though $M^{\#}$ is not necessarily simply connected, we have the followng

THEOREM 3. *Let $P(M,\pi,G)$ be a principal fibre bundle with a locally flat connection Γ and let $M^{\#}$ be the holonomy covering space of M whose covering map is h. Then $H(M^{\#},\Gamma^{\#})=\{0\}$, where $\Gamma^{\#}$ is the locally flat connection on $P^{\#}$*

5) c. f. p. 41 of [10].

induced from Γ *by the induced map* $P^{\#}\to P$.

PROOF. Let $C^{\#}$ be a closed curve at $x^{\#}$ on $M^{\#}$. Since $M^{\#}$ is the holonomy covering space of M, the curve $C^{\#}$ is horizontal for Γ in P. If we put $C=h(C^{\#})$, $Cx^{\#}=x^{\#}$, since $C^{\#}$ is closed. Hence C is holonomous to zero for Γ in P. By virtue of the Lemma, we can conclude that the curve $C^{\#}$ is holonomous to zero for $\Gamma^{\#}$ in P. Q.E.D.

By the reduction theorem for connections [10], we have the following

COROLLARY. *The principal fibre bundle* $P^{\#}$ *induced from* P *by* $h: M^{\#}\to M$ *reduces to the product bundle.*

PROPOSITION 8. *The covering transformation group of the holonomy covering space* $M^{\#}$ *of* M *is* $H(M;\Gamma)$.

PROOF. Since $M^{\#}$ is a covering space over M, the covering transformation group S equals to $\pi_1(M,x)/K$ at $x\in M$, where $K=\bigcap_{u\in h^{-1}(x)} h_*(\pi^{\#},u))$[6]. There is a homomorphism ϕ of $\pi_1(M,x)$ onto $H(M;\Gamma,x)$ by Proposition 4, hence we need only show that K is identified to the kernel of ϕ.

Let C be a piece-wise differentiable closed curve which is a representative of $\alpha\in K$. For C, we can take the lifts $\widetilde{Cu}$ of C starting at $u\in h^{-1}(x)$. Since $\widetilde{C}_u$ is a representative of an element of $\pi_1(M^{\#}, u)$, $\widetilde{Cu}$ is closed for every $u\in h^{-1}(x)$. Hence C is holonomous to zero for Γ. Hence α which contains C belongs to the kernel of ϕ.

Conversely let C' be a peice-wise differentiable closed curve which is a representative of an element β of the kernel of ϕ. Now C' is holonomous to zero for Γ. Hence for all $u\in h^{-1}(x)$, $C'u=u$. Hence for all $u\in h^{-1}(x)$, the lifts $\widetilde{C}'_u$ of C' are closed in $M^{\#}$. Therefore β which contains C' belongs to K. Q.E.D.

The following theorem shows that a holonomy covering space $M^{\#}$ of M is a minimal universal covering space in a certain sense.

THEOREM 4. *Let* $P(M,\pi,G)$ *be a principal fibre bundle with a locally flat connection* Γ *and let* M' *be a covering space of* M *whose covering map is* h'. *Then* $H(M';\Gamma')=\{0\}$ *if and only if* M' *covers the holonomy covering space* $M^{\#}$ *of* M *with the covering map* $h'_{\#}$ *such that* $h'=h\circ h'_{\#}$, *where* Γ' *is the induced connection in* $h'^{-1}(P)$ *from* Γ *by the induced map* $\bar{h}': h'^{-1}(P)\to P$ *and* h *the covering map of* $M^{\#}$.

PROOF. Let M' be a covering space of $M^{\#}$ with the covering map $h'^{\#}$. Any peice-wise differentiable closed curve C' in M' at x' is projected on a piece-wise differentiable closed curve $C^{\#}$ at $x^{\#}=h'_{\#}(x')$. By virtue of Theorem 3 any curve in $M^{\#}$ at any point $x^{\#}$ is holonomous to zero for $\Gamma^{\#}$, where $\Gamma^{\#}$ is the induced connection in $P^{\#}$ from Γ by the induced map $\bar{h}: P^{\#}\to P$. Since

6) See p. 68 of [11].

M' covers $M^{\#}$ with h, the induced map $\overline{h'_{\#}} : h'^{-1}_{\#}(P^{\#}) \to P^{\#}$ induces a connection Γ' in $h'^{-1}_{\#}(P^{\#})$ as follows:

$$\omega' = \omega^{\#} \circ \delta h'_{\#},$$

where $\omega^{\#}$ is the connection form of $\Gamma^{\#}$ in $P^{\#}$. By virtue of the Lemma, C' is holonomous to zero for Γ', since $C^{\#}$ is holonomous to zero for $\Gamma^{\#}$. Hence $H(M'; \Gamma') = \{0\}$.

Conversely let M' covers M with the covering map h' and we shall assume $H(M'; \Gamma') = \{0\}$, where Γ' is the induced connection in $h'^{-1}(P)$ from Γ by the induced map $h' : h'^{-1}(P) \to P$. Let $x \in M$, $x' \in M'$ and $x^{\#} \in M^{\#}$ be points such that $h'(x') = x$ and $h(x^{\#}) = x$. For each point $y' \in M'$ we can join x' to y' by a curve C' in M' and let $h'(C') = C$. Let $C^{\#}$ be the lift starting at $x^{\#}$ of C by Γ. Then $C^{\#}$ ends at $y^{\#}$ over $y = h'(y')$. Now introduce the map

$$h'_{\#} : M' \to M$$

defined by

$$h'_{\#}(y') = y^{\#}.$$

The map $h'_{\#}$ is independent from the choice of cuves in M'. In fact, let C'_1 and C'_2 be two curve $C'^{-1}_1 \circ C'_2$ is closed in M'. Hence $C^{-1}_1 \circ C_2 = h'(C'^{-1}_1 \circ C'_2)$ is closed in M. By the assumption $C'^{-1}_1 \circ C'_2$ is holonomous to zero for Γ', hence $C'^{-1}_1 \circ C'_2$ is holonomous to zero for Γ', hence $C^{-1}_1 \circ C_2$ is holonomous to zero for Γ by virtue of the Lemma. Hence the lift starting at $x^{\#}$ of $C^{-1}_1 \circ C$ is closed, i.e., $y^{\#}$ is uniquely determined.

It is clear that the map $h'_{\#}$ is onto. Next for $x^{\#} \in M^{\#}$, we take a suitably small simply connected neighbourhood $U^{\#}$ in $M^{\#}$ of $x^{\#}$. Let U' be a component of $h'^{-1}_{\#}(U^{\#})$ which contains an arbitrary point x' such that $h'_{\#}(x') = x^{\#}$. Suppose that two points y'_1 and y'_2 of U' are projected on a point $y^{\#} \in U^{\#}$, where $y'_1 \neq y'_2$. Then we can join y'_1 to y'_2 by a curve $C' \subset U'$. Let $C^{\#} = h'_{\#}(C')$, hence $C^{\#}$ is closed in $U^{\#}$. By virtue of Theorem 3, $C^{\#}$ is holonomous to zero for $\Gamma^{\#}$. By virtue of the lemma, C is holonomous to zero for Γ. Hence again by virtue of the Lemma, C' is holonomous to zero for Γ', since there is the covering map h' such that $h'(C') = C$. Hence C' must be closed. This is contrary to the assumption. Therefore $h'_{\#}$ maps U' homeomorphically onto $U^{\#}$. Q.E.D.

We can prove the following proposition by the same way as Theorem 4 of [2] and the proof is omitted.

PROPOSITION 9. *For each subgroup H of $H(M; \Gamma)$, there is a covering space M^0 of M whose covering map is h^0 with locally flat connection Γ^0 induced from a locally flat connection Γ such that $H(M^0; \Gamma^0)$ is isomorphic with H.*

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BIBLIOGRAPHY

[1] W. AMBROSE and I. SINGER: A theorem on holonomy, Trans. Amer. Math. Soc., 75(1953), 428-443.
[2] L. AUSLANDER and L. MARKUS: Holonomy of flat affinely connected manifolds, Ann. of Math., 62(1955), 139-151.
[3] S. S. CHERN: Topic in differential geometry, 1951.
[4] C. CHEVALLEY: Theory of Lie groups, 1959.
[5] C. EHRESMANN: Les connexions infinitésimales dans un espace fibré differentiable, Collodue de Topologie (Espace Fibrés), 1950, 29-55.
[6] S. ISHIHARA: Fibred Riemannian spaces with isometric parallel fibres, Tôhoku Math. Journ., 6(1954), p. 243-252.
[7] S. KOBAYASHI: Theory of connections, Annali di Mat., 43(1957), 119-194.
[8] ——————: Induced connections and imbedded Riemannian spaces, Nagoya Math. Journ., 10(1956), 15-25.
[9] A. LICHNEROWICZ; Théorie globale des connexions et des groupes d'holonomie, 1955.
[10] K. NOMIZU: Lie groups and differential geometry, 1956.
[11] N. STEENROD: The topology of fibre bundles, 1951.

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