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The holonomy groupoid of a singular foliation

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Abstract. We construct the holonomy groupoid of *any* singular foliation. In the regular case this groupoid coincides with the usual holonomy groupoid of Winkelnkemper ([30]); the same holds in the singular cases of [24], [2], [9], [10], which from our point of view can be thought of as being "almost regular". In the general case, the holonomy groupoid can be quite an ill behaved geometric object. On the other hand it often has a nice longitudinal smooth structure. Nonetheless, we use this groupoid to generalize to the singular case Connes' construction of the C^*-algebra of the foliation. We also outline the construction of a longitudinal pseudo-differential calculus; the analytic index of a longitudinally elliptic operator takes place in the K-theory of our C^*-algebra.

In our construction, the key notion is that of a *bi-submersion* which plays the role of a *local Lie groupoid* defining the foliation. Our groupoid is the quotient of germs of these bi-submersions with respect to an appropriate equivalence relation.

Introduction

A foliated manifold is a manifold partitioned into immersed submanifolds (leaves). Such a partition often presents singularities, namely the dimension of the leaves needs not be constant. Foliations arise in an abundance of situations, e.g. the fibers of a submersion, the orbits of a Lie group action or the structure of a manifold with corners. They are quite important tools in geometric mechanics; for instance, foliations appear in the problem of Hamiltonian reduction (cf. e.g. the monograph [21]). Also, every Poisson manifold is determined by its symplectic foliation (cf. [8], pages 112–113).

The relationship between foliations and groupoids is very well known: to any Lie groupoid corresponds a foliation, namely the leaves are the orbits of the groupoid. Conversely, to a regular foliation there corresponds its *holonomy groupoid* constructed by Ehresmann [11] and Winkelnkemper [30] (see also [2], [25]). The holonomy groupoid is a (not necessarily Hausdorff) Lie groupoid whose orbits are the leaves and which in a sense is minimal with this condition.

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This holonomy groupoid has been generalized to singular cases by various authors. In particular, a construction suggested by Pradines and Bigonnet [24], [2] was carefully analyzed, and its precise range of applicability found, by Claire Debord [9], [10]. These authors dealt with those singular foliations that can be defined by an *almost injective* Lie algebroid, namely a Lie algebroid whose anchor map is injective in a dense open subset of the base manifold. It was shown that every such Lie algebroid is integrable, and the Lie groupoid that arises is the holonomy groupoid of the foliation, in the sense that it is the smallest among all Lie groupoids which realize the foliation. Note that the integrability of almost injective Lie algebroids was reproved as a consequence of Crainic and Fernandes' characterization of integrable Lie algebroids in [7].

Both in the regular case and the singular case studied by Bigonnet, Pradines and Debord, the holonomy groupoid was required to be:

(1) *Lie*, namely the space of arrows to have a smooth structure such that the source and target maps are surjective submersions and all the other structure maps (multiplication, inversion, unit inclusion) to be appropriately smooth.

(2) *minimal*, namely every other Lie groupoid which realizes the foliation has an open subgroupoid which maps onto the holonomy groupoid by a smooth morphism of Lie groupoids.

Minimality is essential mainly because it does away with unnecessary isotropy and this ensures that the holonomy groupoid truly records all the information of the foliation. On the other hand, the smoothness requirement leaves room for discussion. As we will see, in order to construct the C^*-algebra it is enough that the s-fibers be smooth, and we can even get around with this requirement. In fact, smoothness of the full space of arrows is exactly what prevents generalizing their construction to a singular foliation which comes from an arbitrary Lie algebroid.

This paper addresses the problem of the existence of a holonomy groupoid for a singular foliation without assuming the foliation to be defined by some algebroid. The properties of this groupoid may be summarized in the following theorem:

Theorem 0.1. *Let $\mathcal{F}$ be a (possibly singular) Stefan-Sussmann foliation on a manifold M. Then there exists a topological groupoid $\mathcal{H}(\mathcal{F}) \rightrightarrows M$ such that:*

- *Its orbits are the leaves of the given foliation $\mathcal{F}$.*

- *$\mathcal{H}(\mathcal{F})$ is minimal in the sense that if $G \rightrightarrows M$ is a Lie groupoid which defines the foliation $\mathcal{F}$ then there exists an open subgroupoid G_0 of G (namely its s-connected component) and a morphism of groupoids $G_0 \to \mathcal{H}(\mathcal{F})$ which is onto.*

- *If $\mathcal{F}$ is regular or almost regular i.e. defined from an almost injective Lie algebroid, then $\mathcal{H}(\mathcal{F}) \rightrightarrows M$ is the holonomy groupoid given in* [30], [25], [24], [2], [9], [10].

Note that the only possible topology of $\mathcal{H}(\mathcal{F})$ is quite pathological in the non quasi-regular case. On the other hand its source-fibers are often smooth manifolds.

But before giving an overview of our approach, let us first clarify the way a singular foliation is understood. The established definition of a foliation, following the work of Stefan [27] and Sussmann [28] is the partition of a manifold M to the integral submanifolds of a (locally) finitely generated module of vector fields which is integrable i.e. stable under the Lie bracket. In the regular case there is only one possible choice for this module: this is the module of vectors tangent along the leaves, it forms a vector subbundle and a Lie subalgebroid of TM. In the singular case though, the partition of M into leaves no longer defines uniquely the module of vector fields. Take for example the foliation of $\mathbb{R}$ into three leaves: $\mathbb{R}^-$, $\{0\}$ and $\mathbb{R}^+$. These can be regarded as integral manifolds of any of the vector fields $X_n = x^n \frac{\partial}{\partial x}$, yet the modules generated by each X_n are genuinely different.

Our construction depends on the choice of the prescribed submodule of vector fields defining the singular foliation. Actually, we call *foliation* precisely such an integrable (locally) finitely generated submodule of vector fields.

Note that the almost regular case is the case where the prescribed module of vector fields is isomorphic to the sections of a vector bundle—i.e. is locally free.

The construction of the holonomy groupoid by Winkelnkemper in the regular case is based on the notion of the holonomy of a path. The point of view of Bigonnet-Pradines is a little different: they first look at abstract holonomies as being local diffeomorphisms of local transversals, and obtain in this way a "big" groupoid. The holonomy groupoid is just the s-connected component of this big groupoid—or equivalently the smallest open subgroupoid i.e. the subgroupoid generated by a suitable open neighborhood of the space of units.

In a sense, our construction follows the ideas of Bigonnet-Pradines. So we will construct a big groupoid, as well as 'a suitable open neighborhood of the space of units'.

- Even without knowing what the (big) holonomy groupoid might be, it is quite easy to define what a submersion to this groupoid could mean: this is our notion of *bi-submersions*. A bi-submersion is understood merely as a manifold together with two submersions over M which generate (locally) the foliation $\mathcal{F}$ in the same way as a Lie groupoid would. The composition (pulling back) and inversion of bi-submersions are quite easily defined. Let us emphasize the fact that two different bi-submersions may very well be of different dimension. On the other hand, there is a natural criterion for two points of various bi-submersions to have the same image in the holonomy groupoid: this is the notion of equivalence of two (germs) bi-submersions at a point. The (big) holonomy groupoid is thus defined as being the quotient of the union of all possible bi-submersions by this equivalence relation.

- Exponentiating small vector fields defining the foliation gives rise to such a bi-submersion. The image of this bi-submersion is the desired 'suitable open neighborhood of the space of units' in the big groupoid. The (path) holonomy groupoid is thus the subgroupoid generated by this neighborhood.

The "big" holonomy groupoid takes into account all possible holonomies and the path holonomy groupoid is the smallest possible holonomy groupoid. In order to treat them simultaneously as well as all possible intermediate groupoids, we introduce a notion of *atlas* of bi-submersions: loosely speaking, an atlas is a family of bi-submersions which is stable up to equivalence by composition and inverse. The quotient of this atlas by the equivalence relation given above is algebraically a groupoid over M. The notion of atlas of bi-submersions we introduce here seems to be in the spirit of Pradine's ideas in [23].

Unless the foliation is almost regular, the (path) holonomy groupoid (endowed with the quotient topology) is topologically *quite an ugly space. Its topology is by no means Hausdorff: a sequence of points may converge to every point of a submanifold*. On the other hand, we show that when restricted over a leaf L with the leaf topology, it often carries a natural smooth structure making the source (and target) map a smooth submersion. When this happens, this restriction is actually a principal groupoid. The stabilizer of a point i.e. the holonomy group is a Lie group, whose Lie algebra is very naturally expressed in terms of the foliation $\mathscr{F}$. Every source (target) fiber has the same dimension as the underlying holonomy cover of the leaf. Of course, this dimension is not the same for different leaves—unless the foliation is almost regular.

In the (quasi) regular case, the holonomy groupoid of a foliation is the first step towards the construction of the C^*-algebra of the foliation which allows to obtain a number of different important results:

- Important information of any foliation lies in the space of leaves, namely the quotient of the manifold by the equivalence relation of belonging in the same leaf. This space presents considerable topological pathology, even in the regular case. A. Connes showed in [5] that it is appropriately replaced by the foliation C^*-algebra constructed from the holonomy groupoid.

- Using the holonomy groupoid and its C^*-algebra, A. Connes (and partly the second author) ([3], [4], [5], [6]) developed a longitudinal pseudodifferential calculus on foliations and an index theory for foliations.

- In another direction, by extending the construction of the C^*-algebra to an arbitrary Lie groupoid we get a formal deformation quantization of the Poisson structure on the dual of an integrable Lie algebroid (see [15] and [16]).

- Furthermore, the operators corresponding to the C^*-algebra of a Lie groupoid are families of pseudodifferential operators on the source fibers of the groupoid in hand, defined by Nistor, Weinstein and Xu [20] and independently by Monthubert and Pierrot [18]. In the case of the holonomy groupoid such operators play an important role in both index theory and deformation quantization.

Connes' construction of the C^*-algebra of the foliation involves working with the algebra of smooth functions on the arrow space of the holonomy groupoid (first defining involution and convolution and then completing it appropriately). Therefore it can no longer be applied for this holonomy groupoid, since the topology of its arrow space is very pathological, as a quotient topology, and the functions on it are highly non-continuous.

Nevertheless, we may still define some C^*-algebra(s). In fact we just push one small step further Connes' idea for the construction of the C^*-algebra of the foliation in the case the groupoid is not Hausdorff (cf. [4], [5]; see also [14]). Every bi-submersion is a smooth manifold, so we may work with the functions defined on it instead. It is then quite easy to decide when two functions defined on different bi-submersions should be identified, i.e. we define a suitable quotient of the vector space spanned by smooth compactly supported functions on bi-submersions. The inversion and composition operations of the atlas are translated to an involution and convolution at the quotient level, making it a $*$-algebra. One has to be somewhat careful with the formulation of this, in order to keep track of the correct measures involved in the various integrations. For this reason we work with suitable half-densities (following Connes cf. [5]). It is finally quite easy to produce an L^1-estimate which allows us to construct the full C^*-algebra. Actually, we describe all the representations of this C^*-algebra as being the integrated form of representations of the groupoid, pretty much in the spirit of [26] (cf. also [12]).

To construct the reduced C^*-algebra, we have to choose a suitable family of representations which should be the 'regular' ones. In 'good' cases, our groupoid carries a natural longitudinal smooth structure which allows us to define the associated family of regular representations. An alternate set of representations which exist in all cases and which can be thought in a sense as the regular ones are the natural representations on the space of L^2-functions on a leaf. To treat all possibilities in the same frame, we introduce a notion of a holonomy pair which consists of an atlas together with a longitudinally smooth quotient of the associated holonomy groupoid. The associated regular representations are those on $L^2(G_x)$ where $x \in M$.

With these in hand we outline the analytic index map following by and large the pattern of the tangent groupoid introduced by A. Connes (cf. [5]).

The text is structured as follows:

- Section 1 is an account of well known fundamental facts about singular foliations and the various groups and pseudogroups of (local) diffeomorphisms which respect them. In particular, we recall how to a Stefan-Sussmann foliation is associated the partition into leaves and discuss the longitudinal smooth structure and topology.

- In Section 2 we study bi-submersions and their bisections. We discuss the composition of bi-submersions, equivalence of germs of bi-submersions and those bi-submersions which have bisections inducing the identity map.

- In Section 3 we construct the holonomy groupoid of a foliation. We first define the notion of an atlas of bi-submersions, discuss several possible atlases, and then construct the groupoid of an atlas.

- In section 4 we construct the reduced C^*-algebras of the foliation.

- In section 5 we define the representations of the holonomy groupoid and show that they are in a one to one correspondence with the representations of (the full) C^*-algebra of the foliation.

- We conclude with a short discussion about further developments and in particular the analytic index map (Section 6).

Actually, this pseudodifferential calculus is in a sense our main motivation for generalizing the C^*-algebra of a foliation to the singular case. Indeed, as in the regular case, our C^*-algebra is designed to deal with longitudinal differential operators that are elliptic along the foliation:

- It is a receptacle for resolvants of such operators.

- Its K-theory is a receptacle for index problems.

In a forthcoming paper we intend to give the complete formulation of the pseudodifferential calculus and its correspondence with the C^*-algebras given here. We also intend to extend our construction to a case of 'longitudinally smooth foliations' in the spirit of continuous family groupoids of Paterson.

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1. Preliminaries

In this section, we recall the definitions and the results on foliations that will be used in the sequel. For the reader's convenience we include (or just sketch) most proofs.

1.1. Bundles and submodules. Let M be a manifold and let E be a smooth real vector bundle over M.

(1) Denote by $C^\infty(M;\mathbb{R})$ or just $C^\infty(M)$ the algebra of smooth real valued functions on M. Denote by $C_c^\infty(M)$ the ideal in $C^\infty(M)$ consisting of smooth functions with compact support in M.

(2) Denote by $C^\infty(M;E)$ the $C^\infty(M)$-module of smooth sections of the bundle E. Denote by $C_c^\infty(M;E)$ the submodule of $C^\infty(M;E)$ consisting of smooth sections with compact support in M.

(3) Let $\mathscr{E}$ be a submodule of $C_c^\infty(M;E)$. The submodule $\hat{\mathscr{E}} \subset C^\infty(M,E)$ of *global sections* of $\mathscr{E}$ is the set of sections $\xi \in C^\infty(M;E)$ such that, for all $f \in C_c^\infty(M)$, we have $f\xi \in \mathscr{E}$.

The module $\mathscr{E}$ is said to be *finitely generated* if there exist global sections $\xi_1, \dots, \xi_n$ of $\mathscr{E}$ such that $\mathscr{E} = C_c^\infty(M)\xi_1 + \cdots + C_c^\infty(M)\xi_n$.

(4) Let N be a manifold and $p : N \to M$ be a smooth map. Denote by $p^*(E)$ the pull-back bundle on N. If $\mathscr{E}$ is a submodule of $C_c^\infty(M;E)$, the *pull-back module* $p^*(\mathscr{E})$ is the submodule of $C_c^\infty\big(N;p^*(E)\big)$ generated by $f(\xi \circ p)$, with $f \in C_c^\infty(N)$ and $\xi \in \mathscr{E}$.

If N is a submanifold of M, the module $p^*(\mathscr{E})$ is called the *restriction* of $\mathscr{E}$ to N.

(5) A submodule $\mathscr{E}$ of $C_c^\infty(M;E)$ is said to be *locally finitely generated* if there exists an open cover $(U_i)_i$ of $\mathscr{E}$ such that the restriction of $\mathscr{E}$ to each U_i is finitely generated.

1.2. Singular foliations. As usual (cf. [27], [28]), a singular foliation will denote here something more precise than just the partition into leaves.

1.2.1. Definitions and examples.

Definition 1.1. Let M be a smooth manifold. A *foliation* on M is a locally finitely generated submodule of $C_c^\infty(M;TM)$ stable under Lie brackets.

Definition 1.2. Let M, $\mathscr{F}$ be a foliation and $x \in M$. The *tangent space of the leaf* is the image F_x of $\mathscr{F}$ in T_xM. Put $I_x = \{f \in C^\infty(M); f(x) = 0\}$. The *fiber of* $\mathscr{F}$ is the quotient $\mathscr{F}_x = \mathscr{F}/I_x\mathscr{F}$.

The evaluation $\tilde{e}_x : \mathscr{F} \to T_xM$ vanishes on $I_x\mathscr{F}$. We therefore get a surjective homomorphism $e_x : \mathscr{F}_x \to F_x$ by $e_x(\xi + f\eta) = \xi(x)$.

The kernel of $\tilde{e}_x$ is a Lie subalgebra of $\mathscr{F}$ and $I_x\mathscr{F}$ is an ideal in this subalgebra. It follows that $\ker e_x = \ker \tilde{e}_x / I_x\mathscr{F}$ is a Lie algebra $\mathfrak{g}_x$.

Examples 1.3. (1) Recall that a *Lie algebroid* on M is given by a vector bundle A, a Lie bracket $[\,,\,]$ on $C^\infty(M;A)$ together with an *anchor* which is a bundle map $\# : A \to TM$ satisfying the compatibility relations $[\#\xi, \#\eta] = \#[\xi,\eta]$ and $[\xi, f\eta] = f[\xi,\eta] + (\#\xi)(f)\eta$ for all $\xi, \eta \in C^\infty(M;A)$ and $f \in C^\infty(M)$. To a Lie algebroid there is naturally associated a foliation: the image of the anchor $\mathscr{F} = \#\big(C_c^\infty(M;A)\big)$. In particular to any Lie groupoid is associated a foliation.

Note that the anchor maps $I_xC_c(A)$ onto $I_x\mathscr{F}$, and since the quotient $C_c(A)/I_xC_c(A)$ is naturally identified with the fiber A_x we get an onto linear map $A_x \to \mathscr{F}_x$ and the following commutative diagram:

$$\begin{array}{ccc} A_x & & \\ \downarrow & \searrow & \\ \mathscr{F}_x & \xrightarrow{ev} & F_x. \end{array}$$

(2) Recall that a *regular foliation* is a subbundle F of TM whose sections form a Lie algebroid, i.e. $C^\infty(M;F)$ is stable under Lie brackets. The set of sections $C_c^\infty(M;F)$ is a foliation in the above sense. We have $\mathscr{F}_x = F_x$ for every $x \in M$.

Conversely, let us show that if $\mathscr{F}$ is a foliation such that F is a vector bundle then $\mathscr{F} = C_c^\infty(M;F)$. By definition of F, we have $\mathscr{F} \subset C_c^\infty(M;F)$. Let $x \in M$ and let $\xi_1, \dots, \xi_k$ be a basis of F_x. There exist $X_1, \dots, X_k \in \mathscr{F}$ such that $X_i(x) = \xi_i$. It follows that $X_1, \dots, X_k$ form a basis of sections of F in a neighborhood of x. By a compactness argument $C_c^\infty(M;F) \subset \mathscr{F}$.

In particular, in the regular case the distribution F (whence the partition into leaves) determines the foliation. This is no longer the case for singular foliations as the following examples show.

(3) Consider the partition of $\mathbb{R}$ into three leaves: $\mathbb{R}_-^*$, $\{0\}$ and $\mathbb{R}_+^*$. It corresponds to various foliations $\mathscr{F}_k$ with $k > 0$, where $\mathscr{F}_k$ is the module generated by the vector field $x^k \partial/\partial x$. The foliations $\mathscr{F}_k$ are different for all different k.

In this example, one may legitimately consider that $\mathscr{F}_1$ is in some sense the best choice of module defining the partition into leaves.

In some other cases there may be no preferred choice of a foliation (i.e. a module) defining the same partition into leaves as the following examples show:

(4) Consider the partition of $\mathbb{R}$ where the leaves are $\mathbb{R}_+^*$ and $\{x\}$ for every $x \leqq 0$. These are the integral curves of any vector field $f \dfrac{\partial}{\partial x}$ where $f(x)$ vanishes for every $x \leqq 0$. So any module generated by such a vector field defines this foliation. This forces us to impose an a priori choice of such a module. Note that the module of all vector fields vanishing in $\mathbb{R}_-$ is not a foliation, as it is not finitely generated.

(5) Consider the partition of $\mathbb{R}^2$ into two leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. It is given by the action of a Lie group G, where G can be $\mathrm{GL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{R})$ or $\mathbb{C}^*$. The foliation is different for these three actions. The corresponding $\mathscr{F}_x$ are equal to $T_x\mathbb{R}^2$ at each non zero $x \in \mathbb{R}^2$, but $\mathscr{F}_0$ is the Lie algebra $\mathfrak{g}$. To see this we give the following more general result:

Proposition 1.4. *Let E be a (real) vector space and G a Lie subgroup of* $\mathrm{GL}(E)$ *with Lie algebra $\mathfrak{g}$. Consider the (linear) action of G on E, and the foliation $\mathscr{F}$ on E associated with the Lie groupoid $E \rtimes G$, with Lie algebroid $E \rtimes \mathfrak{g}$. Then $\mathscr{F}_0 = \mathfrak{g}$, i.e. the map $p : A_0 = \mathfrak{g} \to \mathscr{F}_0$ described in example* 1.3(1) *is injective.*

Proof. Let $\tilde{\mathscr{F}} \subset C_c^\infty(E; TE)$ be the set of vector fields on E vanishing at 0, and $J^1 : \tilde{\mathscr{F}} \to \mathscr{L}(E)$ be the map which associates to a vector field its 1-jet, i.e. its derivative at 0. This restricts to $I_0\mathscr{F} \subset I_0\tilde{\mathscr{F}}$. It therefore induces a map $\mathscr{F}_0 \to \mathscr{L}(E)$. On the other hand, $J^1 \circ p : \mathfrak{g} \to \mathscr{L}(E)$ is the inclusion of $\mathfrak{g}$ into the Lie algebra of $\mathrm{GL}(E)$. It follows that p is injective. □

Let us gather in the following proposition a few easy (and well known) facts about the fibers and tangent spaces of the leaves.

Proposition 1.5. *Let M, $\mathscr{F}$ be a foliation and $x \in M$.*

(a) *Let $X_1, \dots, X_k \in \mathscr{F}$ whose images in $\mathscr{F}_x$ form a basis of $\mathscr{F}_x$. There is a neighborhood U of x in M such that $\mathscr{F}$ restricted to U is generated by $X_1, \dots, X_k$.*

(b) *The dimension of F_x is lower semi-continuous and the dimension of $\mathscr{F}_x$ is upper semi-continuous.*

(c) *The set $U = \{x \in M; e_x : \mathscr{F}_x \to F_x$ is bijective$\}$ is the set of continuity of $x \mapsto \dim F_x$. It is open and dense in M. The restriction of F to U is a subbundle of TU; the restriction of the foliation $\mathscr{F}$ to U is the set of sections of this subbundle. It is a regular foliation.*

Proof. Since $\mathscr{F}$ is locally finitely generated there exist $Y_1, \dots, Y_N$ which generate $\mathscr{F}$ in a neighborhood V of x.

(a) Since the images of $X_1, \dots, X_k$ form a basis of $\mathscr{F}_x$, there exist $a_{\ell,i} \in \mathbb{C}$ for $1 \leqq i \leqq N$ and $1 \leqq \ell \leqq k$ such that $Y_i - \sum_{\ell=1}^{k} a_{\ell,i} X_\ell \in I_x \mathscr{F}$. It follows that there exist functions $\alpha_{i,j} \in C^\infty(M)$ for $1 \leqq i, j \leqq N$ such that $\alpha_{i,j}(x) = 0$ and for all i we have $Y_i - \sum_{\ell=1}^{k} a_{\ell,i} X_\ell = \sum_{i=1}^{n} \alpha_{j,i} Y_j$ in a neighborhood of x. This can be written as $\sum_{j=1}^{N} \beta_{j,i} Y_j = \sum_{\ell=1}^{k} a_{\ell,i} X_\ell$ for all $1 \leqq i \leqq N$, where $\beta_{i,j} = -\alpha_{i,j}$ if $i \neq j$ and $\beta_{i,i} = 1 - \alpha_{i,i}$.

For $y \in M$, let B_y denote the matrix with entries $\beta_{i,j}(y)$. Since B_x is the identity matrix, for y in a neighborhood U of x, the matrix $B(y)$ is invertible. Write $\big(B(y)\big)^{-1} = \big(\gamma_{i,j}(y)\big)$. We find on U, $Y_i = \sum_{\ell=1}^{k} c_{\ell,i} X_\ell$, where $c_{\ell,i} = \sum a_{\ell,j} \gamma_{j,i}$.

(b) For $y \in V$, F_y is the image of $\mathbb{R}^N$ through the map $\varphi_y : (t_1, \dots, t_N) \mapsto \sum t_i Y_i(y)$. Since $y \mapsto \varphi_y$ is continuous, the rank of F_y is lower semi-continuous.

By (a), it follows that on a neighborhood U of x, $\mathscr{F}$ is generated by $\dim \mathscr{F}_x$ elements, and therefore, for $y \in U$, $\dim \mathscr{F}_y \leqq \dim \mathscr{F}_x$. In other words, the dimension of $\mathscr{F}_x$ is upper semi-continuous.

(c) The function $y \mapsto \dim \mathscr{F}_y - \dim F_y$ is nonnegative (since e_y is surjective) and upper semi-continuous by (b). It follows that $U = \{y; \dim \mathscr{F}_y - \dim F_y < 1\}$ is open.

Let V be the (open) set where $\dim F_x$ is continuous, i.e. has a local minimum. Note that on U the functions $x \mapsto F_x$ and $x \mapsto \mathscr{F}_x$ coincide and are therefore continuous. Therefore $U \subset V$. Now F restricted to V is a vector bundle, and by example 1.3(2), $\mathscr{F}$ restricted to V is equal to $C_c^\infty(V; F)$, whence for every $y \in V$ we have $\mathscr{F}_y = F_y$, thus $U = V$. $\square$

1.2.2. Groups of diffeomorphisms. Let M, $\mathscr{F}$ be a foliation. Let $g : M \to N$ be a diffeomorphism. Associated to g is an isomorphism of modules $g_* : C_c^\infty(M; TM) \to C_c^\infty(N; TN)$. The image of $g_*(\mathscr{F})$ of $\mathscr{F}$ is a foliation of N. Let us denote by $\mathscr{F}'$ this foliation. We obviously have $F'_{g(x)} = dg_x(F_x)$. Also $g_*(I_x \mathscr{F}_x) = I_{g(x)} \mathscr{F}'$, therefore $\mathscr{F}'_{g(x)} \simeq \mathscr{F}_x$.

There are several groups of diffeomorphisms of M to be considered:

(1) The group $\mathrm{Aut}(M, \mathscr{F})$ of diffeomorphisms of M preserving the foliation, i.e. those diffeomorphisms g such that $g_*(\mathscr{F}) = \mathscr{F}$.

(2) The group $\exp \mathscr{F}$ generated by $\exp X$ with X in $\mathscr{F}$.

The following fact is fundamental. This is in some sense the main ingredient in the Frobenius theorem:

Proposition 1.6. *We have* $\exp \mathscr{F} \subset \mathrm{Aut}(M, \mathscr{F})$. *It is actually a normal subgroup in* $\mathrm{Aut}(M, \mathscr{F})$.

Proof. Let $X \in \mathscr{F}$; we have to show that $\exp X \in \mathrm{Aut}(M, \mathscr{F})$. Replacing M by a neighborhood of the support of X, we may assume that $\mathscr{F}$ is finitely generated. Take $Y_1, \ldots, Y_n$ to be global sections of $\mathscr{F}$ generating $\mathscr{F}$. Since $[X, Y_i] \in \mathscr{F}$, there exist functions $\alpha_{i,j} \in C_c^\infty(M)$ such that $[X, Y_i] = \sum_j \alpha_{j,i} Y_j$.

Denote by L the linear mapping of $C^\infty(M)^n$ given by $L(f_1, \ldots, f_n) = (g_1, \ldots, g_n)$, where $g_i = X(f_i) + \sum_j \alpha_{i,j} f_j$.

Let $S : C^\infty(M)^n \to \mathscr{F}$ be the map $(f_1, \ldots, f_n) \mapsto \sum f_i Y_i$; since $L_X \circ S = S \circ L$, we find $\exp X \circ S = S \circ \exp L$. Therefore, $\exp X(\mathscr{F})$, which is the image of $\exp X \circ S$, is contained in the image of S, i.e. it is contained in $\mathscr{F}$.

Furthermore, if $g \in \mathrm{Aut}(M, \mathscr{F})$, we find $g \circ \exp X \circ g^{-1} = \exp(g_* X) \in \exp \mathscr{F}$. □

Definition 1.7. The *leaves* are the orbits of the action of the group $\exp \mathscr{F}$.

Remarks 1.8. (1) For every x, y in the same leaf, there exists (by definition of the leaves) a diffeomorphism $g \in \exp \mathscr{F}$ such that $g(x) = y$. Since $g \in \mathrm{Aut}(M, \mathscr{F})$ it follows that the dimension of F_x and of $\mathscr{F}_x$ is constant along the leaves.

(2) Another group of diffeomorphisms of $\mathscr{F}$ is the subgroup of $\mathrm{Aut}(M, \mathscr{F})$ consisting of those $g \in \mathrm{Aut}(M, \mathscr{F})$ preserving the leaves, i.e. such that $g(x)$ and x are in the same leaf for all $x \in M$.

It will also be useful to consider *local diffeomorphisms* preserving the foliation. Let U, V be open subsets in M. A local diffeomorphism $\varphi : U \to V$ is said to preserve the foliation if $\varphi_*(\mathscr{F}_U) = \mathscr{F}_V$, i.e. if the image through the diffeomorphism φ of the restriction of $\mathscr{F}$ to U is the restriction of $\mathscr{F}$ to V. The local diffeomorphisms preserving the foliation form a pseudogroup. Associated to it is the *étale groupoid of germs of local diffeomorphisms preserving the foliation*.

1.2.3. Transverse maps and pull back foliations.

Definition 1.9. Let M, N be two manifolds, $\varphi : N \to M$ be a smooth map and $\mathscr{F}$ be a foliation on M.

(a) Denote by $\varphi^{-1}(\mathscr{F})$ the submodule $\varphi^{-1}(\mathscr{F}) = \{X \in C_c^\infty(N; TN); d\varphi(X) \in \varphi^*(\mathscr{F})\}$ of $C_c^\infty(N; TN)$.

(b) We say that φ is *transverse* to $\mathscr{F}$ if the natural map $\varphi^*(\mathscr{F}) \oplus C_c^\infty(N; TN) \to C_c^\infty\big(N; \varphi^*(TM)\big)$ (defined by $(\xi, \eta) \mapsto \xi + d\varphi(\eta)$) is onto.

Of course, a submersion is transverse to any foliation.

Proposition 1.10. *Let* M, N *be two manifolds,* $\varphi : N \to M$ *be a smooth map and* $\mathscr{F}$ *be a foliation on* M.

(a) *The $C^\infty(N)$-module $\varphi^{-1}(\mathscr{F})$ is stable under Lie brackets.*

(b) *If φ is transverse to $\mathscr{F}$, the $C^\infty(N)$-module $\varphi^{-1}(\mathscr{F})$ is locally finitely generated. It is a foliation.*

Proof. (a) Take $X, X' \in f^{-1}(\mathscr{F})$ and write

$$dp(X) = \sum f_i Y_i \circ p \quad \text{and} \quad dp(X') = \sum f_i' Y_i' \circ p.$$

We have

$$dp([X, X']) = \sum f_i f_j' [Y_i, Y_j'] \circ p + \sum X(f_j') Y_j' \circ p - \sum X'(f_i) Y_i \circ p.$$

(b) By restricting to small open subsets of N and M, we may assume that the tangent bundles of M and N are trivial and $\mathscr{F}$ is finitely generated. By projectiveness, there is a section $s : C_c^\infty\big(N; p^*(TM)\big) \to p^*(\mathscr{F}) \oplus C_c^\infty(N; TN)$. Then, the fibered product

$$p^{-1}(\mathscr{F}) = p^*(\mathscr{F}) \times_{C_c^\infty(N; p^*(TM))} C_c^\infty(N; TN)$$

identifies with the quotient $\big(p^*(\mathscr{F}) \oplus C_c^\infty(N; TN)\big)/s\big(C_c^\infty\big(N; p^*(TM)\big)\big)$ and is therefore finitely generated. □

Let us note the following obvious facts about pull-back foliations:

Proposition 1.11. *Let M, N be two manifolds, $\mathscr{F}$ be a foliation on M and $\varphi : N \to M$ be a smooth map transverse to $\mathscr{F}$. Denote by $\mathscr{F}_N$ the pull back foliation.*

(a) *For all $x \in N$, we have $(\mathscr{F}_N)_x = \{\xi \in T_x N; d\varphi_x(\xi) \in F_x\}$.*

(b) *Let P be a manifold and $\psi : P \to N$ be a smooth map. The map ψ is transverse to $\mathscr{F}_N$ if and only if $\varphi \circ \psi$ is transverse to M and we have $(\varphi \circ \psi)^{-1}(\mathscr{F}) = \psi^{-1}(\mathscr{F}_N)$.* □

1.3. The leaves and the longitudinal smooth structure. We now recall the manifold structure of the leaves.

1.3.1. The local picture. Let us recall the following well-known fact:

Proposition 1.12. *Let $(M, \mathscr{F})$ be a foliated manifold and let $x \in M$. Put $k = \dim F_x$ and $q = \dim T_x M - \dim F_x$.*

(a) *There exist an open neighborhood W of x in M, a foliated manifold $(V, \mathscr{F}_V)$ of dimension q and a submersion $\varphi : W \to V$ with connected fibers such that $\mathscr{F}_W = \varphi^{-1}(\mathscr{F}_V)$ where $\mathscr{F}_W$ is the restriction of $\mathscr{F}$ to W.*

(b) *Moreover, the tangent space of the leaf of $(V, \mathscr{F}_V)$ at the point $\varphi(x)$ is* 0, *we have* $\ker(d\varphi)_x = F_x$ *and each fiber of φ is contained in a leaf of $(M, \mathscr{F})$.*

Proof. (a) We proceed by induction in k noting that if $k = 0$, there is nothing to be proved: just take $\varphi : M \to M$ to be the identity.

Let $X \in \mathscr{F}$ be a vector field such that $X(x) \neq 0$. Denote by $\varphi_t = \exp(tX)$ the corresponding one-parameter group. Let V_0 be a locally closed submanifold of M containing x and such that $T_x V_0 \oplus \mathbb{R} X(x) = T_x M$.

The map $h : \mathbb{R} \times V_0 \to M$ given by $h(t, v) = \varphi_t(v)$ is smooth and we have $h(0, x) = x$ and $(dh)_0(s, \xi) = sX(x) + \xi$ for every $s \in \mathbb{R}$ and $\xi \in T_x V_0$. In particular it is bijective, and it follows that its restriction $\psi : I \times V \to W$ to an open neighborhood of $(0, x)$ is a diffeomorphism into an open subset W of M. Let $\iota : V \to M$ be the inclusion and $\varphi : W \to V$ be the composition of $\psi^{-1} : W \to I \times V$ with the projection $I \times V \to V$. Denote by $\mathcal{F}_W$ the restriction of $\mathcal{F}$ to W. Note that V is transverse to $\mathcal{F}$ and put $\mathcal{F}_V = \iota^{-1}(\mathcal{F})$.

Writing $W \simeq I \times V$, every vector field $Y \in C_c^\infty(W; TW)$ decomposes uniquely as $Y = fX + Z$, where Z is in the V direction, i.e. $Z(s, v) = \varphi_s\big(Z_s'(v)\big)$ with $Z_s' \in C_c^\infty(V; TV)$. Since $fX \in \mathcal{F}_W$ and $\varphi_s \in \mathrm{Aut}(\mathcal{F})$, it follows that $Y \in \mathcal{F}_W$ if and only if $Z_s' \in \mathcal{F}_V$ for all $s \in I$; in other words, $\mathcal{F}_W = \varphi^{-1}(V)$. Now, $(F_V)_x = F_x \cap T_x V$ has dimension $k - 1$. We may now apply the induction hypothesis to V.

(b) follows from proposition 1.11. To see the last statement, note that every vector field Y on W tangent to the fibers of $\varphi : W \to V$ satisfies $d\varphi(Y) = 0$, whence $Y \in \mathcal{F}_W = \varphi^{-1}(\mathcal{F}_V)$. $\square$

1.3.2. Leaves and the smooth structure along the leaves. Let M, $\mathcal{F}$ be a foliation. We now describe the topology and smooth structure of a leaf. We actually consider the collection of leaves as a single manifold with underlying set M, i.e. a smooth structure on M for which the leaves are open smooth manifolds (of different dimensions).

Definition 1.13. Let N be a manifold and $f : N \to M$ be a smooth map.

- We say that f is *leafwise* if $f^{-1}(\mathcal{F}) = C_c^\infty(N; TN)$, i.e. if $df\big(C_c^\infty(N; TN)\big) \subset f^*(\mathcal{F})$.

- We say that f is *longitudinally étale* if it is leafwise and for all $x \in N$, the differential $(df)_x$ is an isomorphism $T_x N \to F_{f(x)}$.

- A *longitudinal chart* is a locally closed submanifold U of M such that the inclusion $i : U \to M$ is longitudinally étale.

Proposition 1.14. (a) *There is a new smooth structure on M called the* leafwise structure, *such that a map $f : N \to M$ (where N is a manifold) is smooth (resp. étale) for this structure if and only if it is smooth and leafwise (resp. longitudinally étale).*

(b) *The leaves are the connected components of this structure.*

Proof. (a) We prove that longitudinal charts form an atlas of M.

Let $x \in M$. Let W, V, φ be as in proposition 1.12. Put $U = \varphi^{-1}\big(\{\varphi(x)\}\big)$.

We show that:

(i) U is a longitudinal chart.

(ii) Given a leafwise smooth map $g : N \to M$, the set $g^{-1}(U)$ is open in N.

Since $\mathscr{F}_W = \varphi^{-1}(\mathscr{F}_V)$, it follows that $g : N \to W$ is leafwise if and only if $\varphi \circ g$ is. Since the restriction of φ to U is constant, it is leafwise whence the inclusion $U \to M$ is leafwise. Furthermore, for each $y \in U$, since it is in the same leaf as x, it follows that $\dim F_y = \dim F_x$, and since $T_y U \subset F_y$, the inclusion $U \to M$ is longitudinally étale. This establishes (i).

Replacing N by its open subset $g^{-1}(W)$, we may assume that $g(N) \subset W$. We may now replace W by V and g by $\varphi \circ g$. Therefore (ii) is equivalent to proving: assume $F_x = 0$ and $g : N \to M$ is leafwise; then $g^{-1}(\{x\})$ is open. We may further assume that N is connected and $g^{-1}(\{x\}) \neq \emptyset$. We then prove g is constant.

Let then $y, z \in N$ with $g(y) = x$. Let y_t be a smooth path in N joining y to z. By definition, $(dg)_{y_t}\big(y'(t)\big) \in F_{y(t)}$; whence there exists a smooth path $X_t \in \mathscr{F}$ such that $X_t(y_t) = (dg)_{y_t}\big(y'(t)\big)$. Now $x_t = g(y_t)$ as well as the constant path satisfy a differential equation $x_t' = X_t(x_t)$, and by uniqueness in Cauchy-Lipschitz theorem they are equal.

Let us now show that longitudinal charts form an atlas. By (i) they cover M. If U_1 and U_2 are connected longitudinal charts and $x \in U_1 \cap U_2$, let U be as above. It follows by (ii) that $U \cap U_i$ is open in U_i, but since they are submanifolds with the same dimension, $U \cap U_i$ is open in U, whence $U_1 \cap U_2 \cap U$ is a neighborhood of x in U_i. It follows that $U_1 \cap U_2$ is open in both U_1 and U_2. This shows that the longitudinal charts form an atlas.

Now it is quite easy to characterize smooth maps for this new structure.

(b) We denote by N the space M endowed with the new structure.

The identity $N \to M$ is leafwise. Let $x \in M$, let W, V, φ be as in proposition 1.12, and put $U = \varphi^{-1}\big(\{\varphi(x)\}\big)$. By (a), U is open in N, and by proposition 1.12 it is contained in the leaf L_x of x. In particular L_x is a neighborhood of x in N. This means that the leaves are open in N. Therefore, they contain the connected components of N.

On the other hand, let $X_1, \dots, X_m \in \mathscr{F}$ and $x \in M$. The map $\mathbb{R} \to M$ defined by $f(t) = t \mapsto (\exp tX_1) \circ (\exp tX_2) \circ \cdots \circ (\exp tX_m)(x)$ is leafwise, therefore f is continuous with values in N. It follows that $\exp \mathscr{F}$ fixes the connected components of N, i.e. the leaves are connected. □

Remarks 1.15. (1) Note that from the proof above it follows that if $f : N \to M$ satisfies $(df)_x(T_x N) \subset F_{f(x)}$, then it is leafwise.

(2) Let $x \in M$. The tangent space at x for the leafwise structure is F_x. In other words, F_x is the tangent space to the leaf through x.

Remark 1.16. The foliation $\mathscr{F}$, when restricted to a leaf gives rise to a transitive algebroid. Indeed, when M is endowed with the leaf topology $(\mathscr{F}_x)_{x \in M}$ is a vector bundle and the Lie bracket on the sections of this bundle are well defined. More precisely (and locally) take $x \in M$ and let V, $\mathscr{F}_V$, W and φ be as in proposition 1.12. Put $P = \varphi^{-1}\big(\{\varphi(x)\}\big)$. Let $\mathscr{F}_W = C_c^\infty(W)\mathscr{F}$ be the restriction of $\mathscr{F}$ to W and put $A_P = \mathscr{F}_W / I_P \mathscr{F}_W$ where $I_P = \{f \in C_c(W); f_{|P} = 0\}$. It is easily seen to be a Lie algebroid over P which is a neighborhood of x for the leaf topology.

2. Bi-submersions and bi-sections

Throughout this section, we fix a foliation $(M, \mathscr{F})$.

2.1. Bi-submersions.

Definition 2.1. Let $(N, \mathscr{F}_N)$ be another foliated manifold.

(a) A *bi-submersion* between $(M, \mathscr{F})$ and $(N, \mathscr{F}_N)$ is a smooth manifold U endowed with two smooth maps $s : U \to M$ $t : U \to N$ which are submersions and satisfying:

(i) $s^{-1}(\mathscr{F}) = t^{-1}(\mathscr{F}_N)$,

(ii) $s^{-1}(\mathscr{F}) = C_c^\infty(U; \ker ds) + C_c^\infty(U; \ker dt)$.

(b) A *morphism* (resp. a *local morphism*) of bi-submersions (U, t_U, s_U) and (V, t_V, s_V) (between $(M, \mathscr{F})$ and $(N, \mathscr{F}_N)$) is a smooth mapping $f : U \to V$ (resp. $f : U' \to V$ where U' is an open subset of U) such that for all $u \in U$ (resp. $u \in U'$) we have $s_V(f(u)) = s_U(u)$ and $t_V(f(u)) = t_U(u)$.

(c) A bi-submersion of $(M; \mathscr{F})$ is a bi-submersion between $(M; \mathscr{F})$ and $(M; \mathscr{F})$.

(d) A bi-submersion (U, t, s) of $(M; \mathscr{F})$ is said to be *leave-preserving* if for every $u \in U$, $s(u)$ and $t(u)$ are in the same leaf.

The next proposition motivates the definition of bi-submersions.

Proposition 2.2. *Let G be a Lie groupoid over M and assume that $\mathscr{F}$ is the foliation on M defined by the Lie algebroid AG (example 1.3.1). Then (G, t, s) is a bi-submersion of $(M, \mathscr{F})$.*

Proof. The Lie algebroid is the restriction $\ker ds|_M$ to $M = G^{(0)}$ of the bundle $\ker ds$. Recall that $\mathscr{F} = dt(C_c^\infty(M; A))$. In the same way, $\mathscr{F} = ds(C_c^\infty(M; \ker dt|_M))$.

We just have to show that $t^{-1}(\mathscr{F}) = C_c^\infty(G; \ker ds) + C_c^\infty(G; \ker dt)$. Using the isomorphism $x \mapsto x^{-1}$, the equality $s^{-1}(\mathscr{F}) = C_c^\infty(G; \ker ds) + C_c^\infty(G; \ker dt)$ will follow.

For $a \in M$, we denote $G_a = \{x \in G; s(x) = a\}$. Let $x \in G$. The map $y \mapsto yx^{-1}$ is a diffeomorphism between $G_{s(x)}$ and $G_{t(x)}$, and identifies $(\ker ds)_x$ with $A_{t(x)}$. In this way, we obtain a bundle isomorphism $\ker ds \to t^*(A)$. It follows that $t^*(\mathscr{F}) = dt(t^*A) = dt(\ker ds)$. The proposition follows. $\square$

More generally, when the foliation is defined by an algebroid A then any *local Lie groupoid* integrating A gives also a bi-submersion.

Let us come back to the general case. We will use the rather obvious:

Lemma 2.3. *Let (U, t, s) be a bi-submersion of $(M, \mathscr{F})$ and $p : W \to U$ be a submersion. Then $(W, t \circ p, s \circ p)$ is a bi-submersion.*

Proof. The statement is local and we may therefore assume $W = U \times V$ where V is a manifold and p is the first projection, in which case the lemma is quite obvious. □

Proposition 2.4 (Inverse and composition of bi-submersions). *Let (U, t_U, s_U) and (V, t_V, s_V) be bi-submersions.*

(a) *(U, s_U, t_U) is a bi-submersion.*

(b) *Put $W = U \times_{s_U, t_V} V = \{(u, v) \in U \times V; s_U(u) = t_V(v)\}$, and let*

$$s_W : (u, v) \mapsto s_V(v) \quad \textit{and} \quad t_W : (u, v) \mapsto t_U(u).$$

The triple (W, t_W, s_W) is a bi-submersion.

The bi-submersions (U, s_U, t_U) and (W, t_W, s_W) constructed in this proposition are respectively called the *inverse* of (U, t_U, s_U) and the *composition* of (U, t_U, s_U) and (V, t_V, s_V). They are respectively denoted by U^{-1} and $U \circ V$.

Proof. (a) is obvious.

(b) Denote by $\alpha : W \to M$ the map $(u, v) \mapsto s_U(u) = t_V(v)$. Note that $p : (u, v) \mapsto u$ and $q : (u, v) \mapsto v$ are submersions. It follows by lemma 2.3 that (W, t_W, α) and (W, α, s_W) are bi-submersions. In particular $t_W^{-1}(\mathcal{F}) = \alpha^{-1}(\mathcal{F}) = s_W^{-1}(\mathcal{F})$, whence $C_c^\infty(W; \ker dt_W) + C_c^\infty(W; \ker ds_W) \subset t_W^{-1}(\mathcal{F})$. Note that $\ker d\alpha = \ker dp \oplus \ker dq$. Therefore

$$\begin{aligned} t_W^{-1}(\mathcal{F}) &= C_c^\infty(W; \ker dt_W) + C_c^\infty(W; \ker d\alpha) \\ &= C_c^\infty(W; \ker dt_W) + C_c^\infty(W; \ker dp) + C_c^\infty(W; \ker dq) \\ &= C_c^\infty(W; \ker dt_W) + C_c^\infty(W; \ker dq) \quad \text{since } \ker dp \subset \ker dt_W \\ &\subset C_c^\infty(W; \ker dt_W) + C_c^\infty(W; \ker ds_W) \end{aligned}$$

since $\ker dp \subset \ker dt_W$. Thus $t_W^{-1}(\mathcal{F}) = C_c^\infty(W; \ker dt_W) + C_c^\infty(W; \ker ds_W) = s_W^{-1}(\mathcal{F})$. □

Remark 2.5. We may actually define a notion of bi-transverse map, more general than that of bi-submersion. This notion is not used here, but could be of some help. Let $(N, \mathcal{F}_N)$ be another foliated manifold.

(a) A *bi-transverse map* between $(M, \mathcal{F})$ and $(N, \mathcal{F}_N)$ is a smooth manifold U endowed with two smooth maps $s : U \to M$ transverse to $\mathcal{F}$ and $t : U \to N$ transverse to $\mathcal{F}_N$ satisfying:

(i) $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}_N)$,

(ii) $s^{-1}(\mathcal{F}) = C_c^\infty(U; \ker ds) + C_c^\infty(U; \ker dt)$.

(b) A *morphism* (resp. a *local morphism*) of bi-transverse maps (U, t_U, s_U) and (V, t_V, s_V) is a smooth mapping $f : U \to V$ (resp. $f : U' \to V$ where U' is an open subset of U) such that for all $u \in U$ (resp. $u \in U'$) we have $s_V(f(u)) = s_U(u)$ and $t_V(f(u)) = t_U(u)$.

2.2. Bisections. The notion of bisections will allow us to analyze bi-submersions.

Definition 2.6. Let (U, t, s) be a bi-submersion of $(M, \mathcal{F})$.

(a) A *bisection* of (U, t, s) is a locally closed submanifold V of U such that the restrictions of both s and t to V are diffeomorphisms from V onto open subsets of M.

(b) The *local diffeomorphism associated to a bisection* V is $\varphi_V = t_V \circ s_V^{-1}$ of M where $s_V : V \to s(V)$ and $t_V : V \to t(V)$ are the restrictions of s and t to V.

(c) Let $u \in U$ and φ a local diffeomorphism of M. We will say that φ is carried by (U, t, s) at u if there exists a bisection V such that $u \in V$ and whose associated local diffeomorphism coincides with φ in a neighborhood of u.

Proposition 2.7. *Let (U, t, s) be a bi-submersion of a foliation $(M, \mathcal{F})$ and $u \in U$. There exists a bisection V of (U, t, s) containing u.*

Proof. The subspaces $\ker(ds)_u$ and $\ker(dt)_u$ of T_uM have the same dimension. Take a locally closed submanifold V_0 of M such that T_uV_0 supplements both $\ker(ds)_u$ and $\ker(dt)_u$ in T_uM. One can then take V to be a small neighborhood of u in V_0—using the local diffeomorphism theorem. □

The following results are rather obvious. We omit their proofs.

Proposition 2.8. *Let (U, t_U, s_U) and (V, t_V, s_V) be bi-submersions.*

(a) *Let $u \in U$ and φ a local diffeomorphism of M carried by (U, t_U, s_U) at u. Then φ^{-1} is carried by the inverse (U, s_U, t_U) of (U, t_U, s_U) at u.*

(b) *Let $u \in U$ and $v \in V$ be such that $s_U(u) = t_V(v)$ and let φ, ψ local diffeomorphisms of M carried respectively by (U, t_U, s_U) at u and by (V, t_V, s_V) at v. Then $\varphi \circ \psi$ is carried by the composition (W, t_W, s_W) of (U, t_U, s_U) and (V, t_V, s_V) at (u, v).* □

Proposition 2.9. *If $\varphi : U \to V$ is a local diffeomorphism carried by a bi-submersion then $\varphi_*(\mathcal{F}_U) = \mathcal{F}_V$.* □

2.3. Bi-submersions near the identity; comparison of bi-submersions. The following result is crucial:

Proposition 2.10. *Let $x \in M$. Let $X_1, \dots, X_n \in \mathcal{F}$ be vector fields whose images form a basis of $\mathcal{F}_x$. For $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, put $\varphi_y = \exp(\sum y_i X_i) \in \exp \mathcal{F}$. Put $\mathcal{W}_0 = \mathbb{R}^n \times M$, $s_0(y, x) = x$ and $t_0(y, x) = \varphi_y(x)$.*

(a) *There is a neighborhood $\mathcal{W}$ of $(0, x)$ in $\mathcal{W}_0$ such that $(\mathcal{W}, t, s)$ is a bi-submersion where s and t are the restrictions of s_0 and t_0.*

(b) *Let* (V, t_V, s_V) *be a bi-submersion and* $v \in V$. *Assume that* $s(v) = x$ *and that the identity of* M *is carried by* (V, t_V, s_V) *at* v. *There exists an open neighborhood* V' *of* v *in* V *and a submersion* $g : V' \to \mathscr{W}$ *which is a morphism of bi-submersions and* $g(v) = (0, x)$.

Proof. (a) Consider the vector field $Z : (y, x) \mapsto (0, \sum y_i X_i)$ on $\mathscr{W}_0$. Since $Z \in s_0^{-1}(\mathscr{F})$, it follows that the diffeomorphism $\varphi = \exp Z$ fixes the foliation $s_0^{-1}(\mathscr{F})$ (proposition 1.6). Put also $\alpha : (y, x) \mapsto (-y, x)$ and $\kappa = \alpha \circ \varphi$. These diffeomorphisms also fix the foliation $s_0^{-1}(\mathscr{F})$. Now $\kappa^2 = \mathrm{id}$ and $s_0 \circ \kappa = t_0$. It follows that $s_0^{-1}(\mathscr{F}) = t_0^{-1}\mathscr{F}$.

In particular $C_c^\infty(\mathbb{R}^n \times M; \ker dt_0) \subset s_0^{-1}(\mathscr{F})$ hence

$$C_c^\infty(\mathbb{R}^n \times M; \ker ds_0) + C_c^\infty(\mathbb{R}^n \times M; \ker dt_0) \subset s_0^{-1}(\mathscr{F}).$$

Now, since $\mathscr{F}$ is spanned by the X_i's near x, there exists a smooth function $h = (h_{i,j})$ defined in a neighborhood W_0 of $(0, x)$ in $\mathbb{R}^n \times M$ with values in the space of $n \times n$ matrices such that for $(y, u) \in W$ and $z \in \mathbb{R}^n$ we have: $(dt_0)_{(y,u)}(z, 0) = \sum z_i h_{i,j}(y, u) X_j$, and $h_{i,j}(0, x) = \delta_{i,j}$. Taking a smaller neighborhood W, we may assume that $h(y, u)$ is invertible. In this neighborhood, $(dt_W)\big(C_c^\infty(W; \ker ds_W)\big) = t_N^*\mathscr{F}$, whence

$$t_W^{-1}(\mathscr{F}) \subset C_c^\infty(W; \ker ds_W) + C_c^\infty(W; \ker dt_W).$$

(b) Replacing V by an open subset containing v, we may assume that $s_V(V) \subset s(\mathscr{W})$ and that the bundles $\ker dt_V$ and $\ker ds_V$ are trivial. Since $t_V^{-1}(\mathscr{F}) = C_c^\infty(V; \ker ds_V) + C_c^\infty(V; \ker dt_V)$, the map $dt : C_c^\infty(V; \ker ds_V) \to t_V^*(\mathscr{F})$ is onto, and there exist $Y_1, \dots, Y_n \in C^\infty(V; \ker ds_V)$ such that $dt_V(Y_i) = X_i$. Since $X_i(x)$ form a basis of $\mathscr{F}_x$, the $Y_i(v)$ are independent. Replacing V by an open neighborhood of v, we may assume that the Y_i's are independent everywhere on V. Let $Z_{n+1}, \dots, Z_k$ be sections of $\ker ds$ such that $(Y_1, \dots, Y_n, Z_{n+1}, \dots, Z_k)$ is a basis of $\ker ds_V$. Since $dt_V(Z_i) \in s_V^*(\mathscr{F})$ which is spanned by the Y_i's, we may subtract a combination of the Y_i's so to obtain a new basis $(Y_1, \dots, Y_n, Y_{n+1}, \dots, Y_k)$ satisfying $t_V^*(Y_i) = X_i$ if $i \leqq n$ and $t_V^*(Y_i) = 0$ if $i > n$. For $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ small enough we denote by ψ_y the (partially defined) diffeomorphism $\exp(\sum y_i Y_i)$ of V.

Let $U_0 \subset V$ be a bisection through v representing the identity, i.e. such that s_V and t_V coincide on U_0. We identify U_0 with an open subset of M via this map. There exists an open neighborhood U of v in U_0 and an open ball B in $\mathbb{R}^k$ such that $h : (y, u) \mapsto \psi_y(u)$ is a diffeomorphism of $U \times B$ into an open neighborhood V' of v. Let $p : \mathbb{R}^k \to \mathbb{R}^n$ be the projection to n first coordinates. The map $p \circ h^{-1} : V' \to \mathscr{W}$ is the desired morphism. It is a submersion. □

Corollary 2.11. *Let* (U, t_U, s_U) *and* (V, t_V, s_V) *be bi-submersions and let* $u \in U$, $v \in V$ *be such that* $s_U(u) = s_V(v)$.

(a) *If the identity local diffeomorphism is carried by* U *at* u *and by* V *at* v, *there exists an open neighborhood* U' *of* u *in* U *and a morphism* $f : U' \to V$ *such that* $f(u) = v$.

(b) *If there is a local diffeomorphism carried both by* U *at* u *and by* V *at* v, *there exists an open neighborhood* U' *of* u *in* U *and a morphism* $f : U' \to V$ *such that* $f(u) = v$.

(c) *If there is a morphism of bi-submersions $g : V \to U$ such that $g(v) = u$, there exists an open neighborhood U' of u in U and a morphism $f : U' \to V$ such that $f(u) = v$.*

Proof. (a) Put $x = s_U(u)$ and let $\mathscr{W}$ be as in the proposition 2.10. By this proposition there are open neighborhoods $U' \subset U$ of u and $V' \subset V$ of v, and submersions $g : U' \to \mathscr{W}$ and $h : V' \to \mathscr{W}$ which are morphisms and $g(u) = h(v) = (0, x)$. Let h_1 be a local section of h such that $h'(0, x) = v$. Up to reducing U', we may assume that the range of g is contained in the domain of h_1. Put then $f = h_1 \circ g$.

(b) Let φ be a local diffeomorphism carried both by U at u and by V at v. Up to reducing U and V, we may assume that $t_U(U)$ and $t_V(V)$ are contained in the range of φ. Then $(U, \varphi^{-1} \circ t_U, s_U)$ and $(V, \varphi^{-1} \circ t_V, s_V)$ are bi-submersions carrying respectively at u and v the identity local diffeomorphism. We may therefore apply (a).

(c) Let $V_0 \subset V$ be a bi-section through v and φ be the local diffeomorphism associated with V_0. Obviously $g(V_0)$ is a bi-section and defines the same local diffeomorphism. The conclusion follows from (b). □

3. The holonomy groupoid

3.1. Atlases and groupoids.

Definition 3.1. Let $\mathscr{U} = (U_i, t_i, s_i)_{i \in I}$ be a family of bi-submersions.

(a) A bi-submersion (U, t, s) is said to be *adapted to $\mathscr{U}$ at $u \in U$* if there exists an open subset $U' \subset U$ containing u, an element $i \in I$ and a morphism of bi-submersions $U' \to U_i$.

(b) A bi-submersion (U, t, s) is said to be *adapted* to $\mathscr{U}$ if for all $u \in U$, (U, t, s) is adapted to $\mathscr{U}$ at $u \in U$.

(c) We say that $(U_i, t_i, s_i)_{i \in I}$ is an *atlas* if:

(i) $\bigcup_{i \in I} s_i(U_i) = M$.

(ii) The inverse of every element in $\mathscr{U}$ is adapted to $\mathscr{U}$.

(iii) The composition of any two elements of $\mathscr{U}$ is adapted to $\mathscr{U}$.

(d) Let $\mathscr{U} = (U_i, t_i, s_i)_{i \in I}$ and $\mathscr{V} = (V_j, t_j, s_j)_{j \in J}$ be two atlases. We say that $\mathscr{U}$ is *adapted* to $\mathscr{V}$ if every element of $\mathscr{U}$ is adapted to $\mathscr{V}$. We say that $\mathscr{U}$ and $\mathscr{V}$ are *equivalent* if they are adapted to each other.

Proposition 3.2 (Groupoid of an atlas). *Let $\mathscr{U} = (U_i, t_i, s_i)_{i \in I}$ be an atlas.*

(a) *The disjoint union $\coprod_{i \in I} U_i$ is endowed with an equivalence relation $\sim$ given by: $U_i \ni u \sim v \in U_j$ if there exists a local morphism from U_i to U_j mapping u to v.*

Let $G = G_{\mathscr{U}}$ denote the quotient of this equivalence relation. Denote by $Q = (q_i)_{i \in I} : \coprod_{i \in I} U_i \to G$ the quotient map.

(b) *There are maps $t, s : G \to M$ such that $s \circ q_i = s_i$ and $t \circ q_i = t_i$.*

(c) *For every bi-submersion (U, t_U, s_U) adapted to $\mathscr{U}$, there exists a map $q_U : U \to G$ such that, for every local morphism $f : U' \subset U \to U_i$ and every $u \in U'$, we have $q_U(u) = q_i(f(u))$.*

(d) *There is a groupoid structure on G with set of objects M, source and target maps s and t defined above and such that $q_i(u)q_j(v) = q_{U_i \circ U_j}(u, v)$.*

Proof. It follows from corollary 2.11(c) that $\sim$ is an equivalence relation. Assertions (b) and (c) are obvious.

Let (U, t_U, s_U) and (V, t_V, s_V) be bi-submersions adapted to $\mathscr{U}$, $u \in U$ and $v \in V$. It follows from corollary 2.11 that $q_U(u) = q_V(v)$ if and only if there exists a local diffeomorphism carried both by U at u and by V at v. Let $u_i \in U_i$ and $u_j \in U_j$ be such that $s_i(u_i) = t_j(u_j)$. If $q_U(u) = q_i(u_i)$ and $q_V(v) = q_j(u_j)$, it follows from proposition 2.8 that $(u, v) \in U \circ V$ and $(u_i, u_j) \in U_i \circ U_j$ carry the same local diffeomorphism, whence $q_{U \circ V}(u, v) = q_{U_i \circ U_j}(u_i, u_j)$. Assertion (d) now follows. □

Remarks 3.3. (a) Let $\mathscr{U}$ and $\mathscr{V}$ be two atlases. If $\mathscr{U}$ is adapted to $\mathscr{V}$ there is a natural morphism of groupoids $G_{\mathscr{U}} \to G_{\mathscr{V}}$. This homomorphism is injective. It is surjective if and only if $\mathscr{U}$ and $\mathscr{V}$ are equivalent.

(b) We may also consider a bi-submersion (U, t, s) defined on a *not necessarily Hausdorff* manifold U. Since every point in U admits a Hausdorff neighborhood, we still get a map $q_U : U \to G$.

Examples 3.4. There are several choices of non equivalent atlases that can be constructed.

(1) The *full holonomy atlas*. Take all possible bi-submersions.[1)] Of course, any bi-submersion is adapted to this atlas and the corresponding groupoid is the biggest possible.

(2) The *leaf preserving atlas*. Take all leave preserving bi-submersions, i.e. those bi-submersions (U, t, s) such that for all $u \in U$, $s(u)$ and $t(u)$ are in the same leaf. It is immediately seen to be an atlas.

(3) Take a cover of M by s-connected bi-submersions given by proposition 2.10. Let $\mathscr{U}$ be the atlas generated by those. This atlas is adapted to any other atlas and the corresponding groupoid is the smallest possible. We say that $\mathscr{U}$ is a *path holonomy atlas*. The associated maximal atlas will be called the path holonomy atlas.

(4) Assume that the foliation $\mathscr{F}$ is associated with a (not necessarily Hausdorff) Lie groupoid (G, t, s). Then (G, t, s) is a bi-submersion (proposition 2.2). It is actually an atlas.

1) To make them a set, take bi-submersions defined on open subsets of $\mathbb{R}^N$ for all N.

Indeed, G^{-1} is isomorphic to G via $x \mapsto x^{-1}$ and $G \circ G = G^{(2)}$ is adapted to G via the composition map $G^{(2)} \to G$ which is a morphism of bi-submersions.

If G is s-connected, this atlas is equivalent to the path holonomy atlas. Indeed, consider the Lie algebroid AG associated to G. Let U be an open neighborhood of a point $x \in M$ and choose a base of sections $\xi_1, \dots, \xi_n$ of A_U. Then the exponential map gives rise to a diffeomorphism $U \times B^n \to W$, where B^n is an open ball at zero in $\mathbb{R}^n$ and W is an open subset at $1_x \in G$. This map is $(x, \theta_1, \dots, \theta_n) \mapsto \exp(\sum \theta_i \xi_i)(1_x)$. (For an account of the exponential map, as well as this diffeomorphism see [19].)

The restrictions of the source and target maps of G make W a bi-submersion. On the other hand $U \times B^n$ is a bi-submersion with source the first projection and target the map $(x, \theta_1, \dots, \theta_n) \mapsto \exp\big(\sum \theta_i \alpha(\xi_i)\big)(x)$, where α is the anchor map. It is then straightforward to check that the diffeomorphism above is actually an isomorphism of bi-submersions.

Since G is s-connected, it is generated by W, and it follows that the atlas $\{(G, t, s)\}$ is equivalent to the atlas generated by $(U_i \times B^n, s, t)_{i \in I}$, i.e. the path holonomy atlas. The path holonomy groupoid is therefore a quotient of G.

Definition 3.5. The *holonomy groupoid* is the groupoid associated with the path holonomy atlas.

Remark 3.6. Let $x \in M$ and $g \in \exp \mathscr{F}$. There is a bi-submersion carrying g at x adapted to the path holonomy atlas. In other words, we get a surjective morphism of groupoids $\exp \mathscr{F} \ltimes M \to G$. If $g \in \exp(I_x \mathscr{F})$, then the image of (g, x) is equal to that of (id, x).

Examples 3.7. (1) Let us illustrate in a simple case how bad the topology of the holonomy groupoid is when the foliation is not almost regular. Consider the foliation of $\mathbb{R}^2$ with two leaves $\{0\}$ and $\mathbb{R}^2 \backslash \{0\}$ given by the action of $\mathrm{SL}(2, \mathbb{R})$. Recall that the groupoid $\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$ is, as a set $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}^2$, that we have $s(g, u) = u$ and $t(g, u) = gu$ for $u \in \mathbb{R}^2$ and $g \in \mathrm{SL}(2, \mathbb{R})$. It is a bi-submersion and an atlas for our foliation; this atlas is equivalent to the path holonomy atlas since it is s-connected. The path holonomy groupoid G is therefore a quotient of this groupoid.

Let us prove that $G = \big(\mathrm{SL}(2, \mathbb{R}) \times \{0\}\big) \cup (\mathbb{R}^2 \backslash \{0\}) \times (\mathbb{R}^2 \backslash \{0\})$. Note that $(\mathbb{R}^2 \backslash \{0\}) \times (\mathbb{R}^2 \backslash \{0\})$ is a bi-submersion. We therefore just have to show that the map $\big(\mathrm{SL}(2, \mathbb{R}) \times \{0\}\big) \to G$ is injective.

Let $g \in \mathrm{SL}(2, \mathbb{R})$ and let V be a bi-section through $(g, 0)$. In particular, $s : V \to \mathbb{R}^2$ is a local diffeomorphism, so that V is the graph of a smooth map $\varphi : B \to \mathrm{SL}(2, \mathbb{R})$, where B is a neighborhood of 0 in $\mathbb{R}^2$, such that $\varphi(0) = g$. The partial diffeomorphism defined by V is then $u \mapsto \varphi(u)u$ and its derivative at 0 is g. In particular, two distinct elements g, g' in $\mathrm{SL}(2, \mathbb{R})$ don't give rise to the same bisection. Therefore $(g, 0)$ and $(g', 0)$ are different.

Let now $x \in \mathbb{R}^2 \backslash \{0\}$ and $g \in \mathrm{SL}(2, \mathbb{R})$ such that $gx = x$. The sequence $\left(\dfrac{x}{n}, \dfrac{x}{n}\right) \in G$ converges in G to $(g, 0)$. In other words, this sequence converges to every point of the stabilizer of x: the set of its limits is a whole real line!

(2) More generally, let E be a finite dimensional real vector space and Γ a closed connected subgroup of $\mathrm{GL}(E)$. Consider the group Γ acting on the space E and the associated Lie groupoid $E \rtimes \Gamma$. It defines a foliation $\mathscr{F}$ on E. The holonomy groupoid G is again a quotient of this groupoid. As above, one sees that $(g, 0) \sim (g', 0) \Leftrightarrow g = g'$ (for $g, g' \in \Gamma$).

3.2. The "quasi-regular" case. The holonomy groupoid was first defined by Winkelnkemper in the regular case (cf. [30]) and generalized to singular cases by various authors. In particular, a construction suggested by Pradines and Bigonnet [24], [2] was carefully analyzed, and its precise range of applicability found, by Claire Debord [9], [10]. Their construction holds for foliations which are locally projective i.e. when $\mathscr{F}$ is a Lie algebroid (cf. [9], [10]). It follows that such a Lie algebroid is integrable (a result given also by Crainic and Fernandes in [7]). One can summarize their result in the following way:

Proposition 3.8 (cf. [30], [24], [1], [2], [9], [10]). *Assume that $\mathscr{F}$ is a Lie algebroid. Then there exists an s-connected Lie groupoid G with Lie algebroid $\mathscr{F}$ which is minimal in the sense that if H is an s-connected Lie groupoid with Lie algebroid $\mathscr{F}$, there is a surjective étale homomorphism of groupoids $H \to G$ which is the identity at the Lie algebroid level. Moreover, G is a quasi-graphoid.* □

Recall, cf. [1], [10] that a *quasi-graphoid* is a Lie groupoid G such that if $V \subset G$ is a locally closed submanifold such that s and t coincide on V and $s : V \to M$ is étale, then $V \subset G^{(0)}$.

It is now almost immediate that the holonomy groupoid given here generalizes the holonomy groupoid of Winkelnkemper, Pradines-Bigonnet and Debord.

Proposition 3.9. *Let G be an s-connected quasi-graphoid and let $\mathscr{F}$ be the associated foliation. Then the holonomy groupoid of $\mathscr{F}$ is canonically isomorphic to G.*

Proof. By proposition 2.2 and example 3.4.4, it follows that G is a bi-submersion associated with $\mathscr{F}$, and an atlas equivalent to the path holonomy atlas $\mathscr{U}$ (since G is s-connected). We thus get a surjective morphism $G \to G_{\mathscr{U}}$. This morphism is injective by the definition of quasi-graphoids. □

Putting together proposition 3.8 and proposition 3.9, we find:

Corollary 3.10. *In the regular case, our groupoid coincides with the one defined by Winkelnkemper; in the quasi-regular case, it coincides with the one defined by Pradines-Bigonnet and Debord.* □

3.3. The longitudinal smooth structure of the holonomy groupoid. The topology of the holonomy groupoid, as well as all groupoids associated with other atlases, is usually quite bad. Let us discuss the smoothness issue.

Fix an atlas $\mathscr{U}$ and let G be the associated groupoid. For every bi-submersion (U, t, s) adapted to $\mathscr{U}$, let $\Gamma_U \subset U$ be the set of $u \in U$ such that $\dim T_u U = \dim M + \dim \mathscr{F}_{s(u)}$. It is an open subset of U when U is endowed with the smooth structure along the leaves of the foliation $t^{-1}(\mathscr{F}) = s^{-1}(\mathscr{F})$.

Proposition 3.11. (a) *For every $x \in G$, there is a bi-submersion (U, t, s) adapted to $\mathcal{U}$ such that $x \in q_U(\Gamma_U)$.*

(b) *Let (U, t, s) and (U', t', s') be two bi-submersions and let $f : U \to U'$ be a morphism of bi-submersions. Let $u \in U$. If $u \in \Gamma_U$, then $(df)_u$ is injective; if $f(u) \in \Gamma_{U'}$, then $(df)_u$ is surjective.*

Proof. (a) Let $x \in G$. Let (W, t, s) be a bi-submersion adapted to $\mathcal{U}$ and $w \in W$ and such that $x = q_W(w)$. Let $A \subset W$ be a bi-section through w. Let $g : s(A) \to t(A)$ be the partial diffeomorphism of M defined by A. By proposition 2.10 there exists a bi-submersion (U_0, t_0, s_0) and $u_0 \in U_0$ such that $\dim U_0 = \dim \mathcal{F}_{s(u)} + \dim M$, $s_0(u_0) = s(u)$ and carrying the identity through u_0. Put then $U = \{x \in U_0;\ t_0(u) \in s(A)\}$ and let s be the restriction of s_0 to U and t be the map $u \mapsto g\bigl(t_0(u)\bigr)$. Obviously (U, t, s) is a bi-submersion which carries g at u_0. It follows that (U, t, s) is adapted to $\mathcal{U}$ at u_0 and $q_U(u_0) = q_W(w) = x$. It is obvious that $u_0 \in \Gamma_U$.

(b) Since s and s' are submersions and $s' \circ f = s$, df_u is injective or surjective if and only if its restriction $(df_{|\ker ds})_u : \ker(ds)_u \to \ker(ds')_{f(u)}$ is. Consider the composition

$$\ker(ds)_u \xrightarrow{(df_{|\ker ds})_u} \ker(ds')_{f(u)} \xrightarrow{t'_*} \mathcal{F}_{s(u)}.$$

By definition of bi-submersions the maps t'_* and $t_* = t'_* \circ (df)_u$ are onto; if $u \in \Gamma_U$, then $t_* : \ker(ds)_u \to \mathcal{F}_{s(u)}$ is an isomorphism; if $f(u) \in \Gamma_{U'}$, then $t'_* : \ker(ds')_{f(u)} \to \mathcal{F}_{s(u)}$ is an isomorphism. The conclusion follows. □

It follows from 3.11(b) that the restriction of f to a neighborhood of $\Gamma_U \cap f^{-1}(\Gamma_{U'})$ is étale. This restriction preserves the foliation, and is therefore étale also with respect to this structure. Now the Γ_U are open in U with respect to the longitudinal structure; they are manifolds. The groupoid G is obtained by gluing them through local diffeomorphisms.

Let us record for further use the following immediate consequence of proposition 3.11.

Corollary 3.12. *Let (U, t, s) and (U', t', s') be two bi-submersions, let $f : U \to U'$ be a morphism of bi-submersions. Let $u \in U$. There exist neighborhoods V of u in U, V' of $f(u)$ in U', a bi-submersion (U'', t'', s'') and a morphism $p : V' \to U''$ which is a submersion, such that $f(V) \subset V'$ and the map $v \mapsto p\bigl(f(v)\bigr)$ is also a submersion.*

Proof. Just take a bi-submersion (U'', t'', s'') and $u'' \in \Gamma_{U''}$ such that $q_U(u) = q_{U''}(u'')$. By definition of the groupoid, there exists a morphism p of bi-submersions from a neighborhood of $f(u)$ to U'', it follows from proposition 3.11(b) that up to restricting to small neighborhoods, one may assume that p and $p \circ f$ are submersions. □

Remark 3.13. In many cases, the groupoid $G_{\mathcal{U}}$ is a manifold when looked at longitudinally. We observe that the necessary and sufficient condition for it to be a manifold is the following: We need to ensure that for every $x \in M$ there exists a bi-submersion (U, t, s) in the path holonomy atlas and a $u \in U$ which carries the identity at x and has an open neighborhood $U_u \subseteq U$ with respect to the leaf topology such that the quotient map $U_u \to G_{\mathscr{U}}$ is injective. Under this condition the s (t)-fibers of $G_{\mathscr{U}}$ are smooth manifolds (of dimension equal to the dimension of $\mathscr{F}_x$) and the quotient map $q_U : U \to G_{\mathscr{U}}$ is a submersion for the leaf smooth structure. Note that this condition does not depend on the atlas $\mathscr{U}$.

If this condition is satisfied, we say that the groupoid $G_{\mathscr{U}}$ is *longitudinally smooth.*

Let us note that if $G_{\mathscr{U}}$ is longitudinally smooth, then its restriction to each leaf is a Lie groupoid (with respect to the smooth structure of the leaf). The Lie algebroid of this groupoid is then the transitive Lie algebroid considered in remark 1.16. So a necessary condition is that this algebroid be integrable. Since this algebroid is transitive, a necessary and sufficient condition is the one given by Crainic and Fernandes in [7] following Mackenzie ([17]), namely that a given subgroup of the center of the Lie algebra $\mathfrak{g}_x (= \ker(\mathscr{F}_x \to F_x))$ be discrete.

We therefore come up with the two following questions:

(1) Could it happen that the above Mackenzie-Crainic-Fernandes obstruction always vanishes for the algebroid associated with a foliation?

(2) If this obstruction vanishes, does this imply that our groupoid is longitudinally smooth?

In case the groupoid $G_{\mathscr{U}}$ is not longitudinally smooth, we are led to give the following definition:

Definition 3.14. A *holonomy pair* for a foliation $(M, \mathscr{F})$ is a pair $(\mathscr{U}, G)$ where $\mathscr{U}$ is an atlas of bi-submersions and G is a groupoid over M which is a Lie groupoid for the smooth longitudinal structure, together with a surjective groupoid morphism $\alpha : G_{\mathscr{U}} \to G$ such that the maps $\alpha \circ q_U$ are leafwise submersions for each $U \in \mathscr{U}$.

We will describe in section 4 how to construct reduced C^*-algebras associated with a given holonomy pair. In case $(\mathscr{U}, G_{\mathscr{U}})$ is not a holonomy pair for some atlas $\mathscr{U}$ we can always replace $G_{\mathscr{U}}$ with the groupoid $R_{\mathscr{F}}$ defined naturally by the equivalence relation of "belonging in the same leaf" (or leaves that are related by $\mathscr{U}$—if $\mathscr{U}$ is not leaf preserving). This groupoid is not smooth in general, but it is always leafwise smooth: it is the disjoint union of $L \times L$ where L is a leaf (or the disjoint union of $\mathscr{U}$-equivalent leaves).

This will allow us to deal with the cases where the atlas $\mathscr{U}$ does not satisfy the condition we mentioned above.

4. The C^*-algebra of the foliation

We will construct a convolution $*$-algebra on the holonomy groupoid G. We then construct a family of regular representations, which will yield the reduced C^*-algebra of the foliation. Moreover, we construct an L^1-norm, and therefore a full C^*-algebra of the foliation.

4.1. Integration along the fibers of a submersion. Let E be a vector bundle over a space X and $\alpha \in \mathbb{R}$. We denote by $\Omega^\alpha E$ the bundle of α-densities on E. Let $\varphi : M \to N$ be a submersion, and let E be a vector bundle on N. Integration along the fibers gives rise to a linear map $\varphi_! : C_c\big(M; \Omega^1(\ker d\varphi) \otimes \varphi^* E\big) \to C_c(N; E)$ defined by $\varphi_!(f) : x \mapsto \int\limits_{\varphi^{-1}(x)} f \ (\in E_x)$.

The following result is elementary:

Lemma 4.1. *With the above notation, if $\varphi : M \to N$ is a surjective submersion, then the map $\varphi_! : C_c\big(M; \Omega^1(\ker d\varphi) \otimes \varphi^* E\big) \to C_c(N; E)$ is surjective.*

Proof. This is obvious if $M = \mathbb{R}^k \times N$. The general case follows using partitions of the identity. □

4.2. Half densities on bi-submersions. Let (U, t_U, s_U) be a bi-submersion. Consider the bundle $\ker dt_U \times \ker ds_U$ over U. Let $\Omega^{1/2} U$ denote the bundle of (complex) half densities of this bundle.

Recall that the inverse bi-submersion U^{-1} of U is (U, s_U, t_U). Note that $\Omega^{1/2} U^{-1}$ is canonically isomorphic to $\Omega^{1/2} U$.

Let (V, t_V, s_V) be another bi-submersion. Recall that the composition $U \circ V$ is the fibered product $U \times_{s_U, t_V} V$. Denote by $p : U \circ V \to U$ and $q : U \circ V \to V$ the projections. By definition of the fibered product, we have identifications: $\ker dp \sim q^* \ker dt_V$ and $\ker dq \sim p^* \ker ds_U$. Furthermore, since $t_{U \circ V} = t_U \circ p$ and $s_{U \circ V} = s_V \circ q$, we have exact sequences

$$0 \to \ker dp \to \ker dt_{U \circ V} \to p^*(\ker dt_U) \to 0 \quad \text{and}$$

$$0 \to \ker dq \to \ker ds_{U \circ V} \to q^*(\ker ds_V) \to 0.$$

In this way, we obtain a canonical isomorphism $\Omega^{1/2}(U \circ V) \simeq p^*(\Omega^{1/2} U) \otimes q^*(\Omega^{1/2} V)$.

Definition 4.2. Let (U, t_U, s_U) and (V, t_V, s_V) be bi-submersions.

(a) Denote by U^{-1} the inverse bi-submersion and $\kappa : U \to U^{-1}$ the (identity) isomorphism. For $f \in C_c^\infty(U; \Omega^{1/2} U)$ we set $f^* = \bar{f} \circ \kappa^{-1} \in C_c^\infty(U^{-1}; \Omega^{1/2} U^{-1})$—via the identification $\Omega^{1/2} U^{-1} \sim \Omega^{1/2} U$.

(b) Denote by $U \circ V$ the composition. For $f \in C_c^\infty(U; \Omega^{1/2} U)$ and $g \in C_c(V; \Omega^{1/2} V)$ we set $f \otimes g : (u, v) \mapsto f(u) \otimes g(v) \in C_c^\infty(U \circ V; \Omega^{1/2} U \circ V)$—via the identification $\Omega^{1/2}(U \circ V)_{(u,v)} \simeq (\Omega^{1/2} U)_u \otimes (\Omega^{1/2} V)_v$.

Let (U, t_U, s_U) and (V, t_V, s_V) be bi-submersions and let $p : U \to V$ be a submersion which is a morphism of bi-submersions. We have exact sequences

$$0 \to \ker dp \to \ker ds_U \to p^*(\ker ds_V) \to 0 \quad \text{and}$$

$$0 \to \ker dp \to \ker dt_U \to p^*(\ker dt_V) \to 0$$

of vector bundles over U, whence a canonical isomorphism

$$\Omega^{1/2}(U) \simeq \Omega^{1/2}\big(\big(\ker dp \oplus p^*(\ker dt_U)\big) \oplus \big(\ker dp \oplus p^*(\ker ds_U)\big)\big)$$
$$\simeq (\Omega^1 \ker dp) \otimes p^*(\Omega^{1/2} V).$$

Integration along the fibers then gives a map $p_! : C_c^\infty(U; \Omega^{1/2} U) \to C_c^\infty(V; \Omega^{1/2} V)$.

4.3. The $*$-algebra of an atlas. Let us now fix an atlas $\mathscr{U} = (U_i, t_i, s_i)_{i \in I}$. Denote by U the disjoint union $\coprod_{i \in I} U_i$, and let $t_U, s_U : U \to M$ be the submersions whose restrictions to U_i are t_i and s_i respectively. Then (U, t_U, s_U) is a bi-submersion, and $C_c^\infty(U; \Omega^{1/2} U) = \bigoplus_{i \in I} C_c^\infty(U_i; \Omega^{1/2} U_i)$.

Lemma 4.3. *Let (V, t_V, s_V) be a bi-submersion adapted to $\mathscr{U}$.*

(a) *Let $v \in V$. Then there exist a bi-submersion (W, t_W, s_W) and submersions $p : W \to U$ and $q : W \to V$ which are morphisms of bi-submersions such that $v \in q(W)$.*

(b) *Let $f \in C_c^\infty(V; \Omega^{1/2} V)$. Then there exist a bi-submersion (W, t_W, s_W), submersions $p : W \to U$ and $q : W \to V$ which are morphisms of bi-submersions and $g \in C_c^\infty(W; \Omega^{1/2} W)$ such that $q_!(g) = f$.*

Proof. (a) Let h be the local diffeomorphism carried by V at v. Since V is adapted to $\mathscr{U}$, there exists $u \in U$ such that h is also carried by U at u. By proposition 3.11, there exist neighborhoods V' of $v \in V$ and U' of u in U a bi-submersion (N, t_N, s_N) and morphisms $\alpha : U' \to N$ and $\beta : V' \to N$ such that $\alpha(u) = \beta(v) \in \Gamma_N$. Taking smaller neighborhoods, we may further assume that α and β are submersions. Let W be the fibered product $U' \times_N V'$.

(b) Using (a), we find bi-submersions (W_j, t_j, s_j) and submersions which are morphisms of bi-submersions $p_j : W_j \to U$ and $q_j : W_j \to V$ such that $\bigcup q_j(W_j)$ contains the support of f. Put $W = \coprod W_j$. It follows from Lemma 4.1 that f is of the form $q_!(g)$. □

Our algebra $\mathscr{A}_\mathscr{U}$ is the quotient $\mathscr{A}_\mathscr{U} = C_c(U; \Omega^{1/2} U)/\mathscr{I}$ of

$$C_c(U; \Omega^{1/2} U) = \bigoplus_{i \in I} C_c(U_i; \Omega^{1/2} U_i)$$

by the subspace $\mathscr{I}$ spanned by the $p_!(f)$ where $p : W \to U$ is a submersion and $f \in C_c(W; \Omega^{1/2} W)$ is such that there exists a morphism $q : W \to V$ of bi-submersions which is a submersion and such that $q_!(f) = 0$.

From lemma 4.3, it follows:

Proposition 4.4. *To every bi-submersion V adapted to $\mathscr{U}$ one associates a linear map $Q_V : C_c(V; \Omega^{1/2} V) \to \mathscr{A}_\mathscr{U}$. The maps Q_V are characterized by the following properties*:

(i) *If $(V, t_V, s_V) = (U_i, t_i, s_i)$, Q_V is the quotient map*

$$C_c(U_i, \Omega^{1/2} U_i) \subset \bigoplus_{j \in I} C_c(U_j; \Omega^{1/2} U_j) \to \mathscr{A}_\mathscr{U} = \bigoplus_{i \in I} C_c(U_i; \Omega^{1/2} U_i)/\mathscr{I}.$$

(ii) *For every morphism* $p : W \to V$ *of bi-submersions which is a submersion, we have* $Q_W = Q_V \circ p_!$.

Proof. With the notation of lemma 4.3(b), we define $Q_V(f)$ to be the class of $p_!(g)$. This map is well defined, since if we are given two bi-submersions W and W' with morphisms $p : W \to U$, $q : W \to V$, $p' : W' \to U$ and $q' : W' \to V$ which are submersions and elements $g \in C_c(W, \Omega^{1/2}W)$ and $g' \in C_c(W', \Omega^{1/2}W')$ such that $q_!(g) = q'_!(g')$, we let W'' to be the disjoint union of W and W' and g'' to agree with g on W and with $-g'$ in W'. With obvious notations, we have $q''_!(g'') = q_!(g) - q'_!(g') = 0$, hence $p_!(g) - p'_!(g') = p''_!(g'') = 0$. It is obvious that this construction satisfies (i) and (ii) and is characterized by these properties. □

Proposition 4.5. *There is a well defined structure of* $*$*-algebra on* $\mathcal{A}_{\mathcal{U}}$ *such that, if V and W are bi-submersions adapted to* $\mathcal{U}$, $f \in C_c(V; \Omega^{1/2}V)$ *and* $g \in C_c(W; \Omega^{1/2}W)$, *with the notation given in definition* 4.2, *we have*

$$\big(Q_V(f)\big)^* = (Q_{V^{-1}})(f^*) \quad \text{and} \quad Q_V(f)Q_W(g) = Q_{V\circ W}(f \otimes g).$$

Proof. If $p : V \to V'$ is a submersion which is a morphism of bi-submersions, and if $f \in C_c(V; \Omega^{1/2}V)$ satisfies $p_!(f) = 0$, then $(p \times \mathrm{id}_W)_!(f \otimes g) = 0$, where $(p \times \mathrm{id}_W) : V \circ W \to V' \circ W$ is the map $(v, w) \mapsto \big(p(v), w\big)$. Together with a similar property, given a submersion $p : W \to W'$ which is a morphism of bi-submersions, this shows that the product is well defined. □

Definition 4.6. The $*$-algebra of the atlas $\mathcal{U}$ is $\mathcal{A}_{\mathcal{U}}$ endowed with these operations. The $*$-algebra $\mathcal{A}(\mathcal{F})$ of the foliation is $\mathcal{A}_{\mathcal{U}}$ in the case $\mathcal{U}$ is the path holonomy atlas.

4.4. The L^1-norm and the full C^*-algebra. Let (U, t, s) be a bi-submersion, $u \in U$ and put $x = t(u)$, and $y = s(u)$. The image by the differential dt_u of $\ker ds$ is F_y and the image by the differential ds_u of $\ker dt$ is F_x. We therefore have exact sequences of vector spaces

$$0 \to (\ker ds)_u \cap (\ker dt)_u \to (\ker ds)_u \to F_y \to 0$$

and

$$0 \to (\ker ds)_u \cap (\ker dt)_u \to (\ker dt)_u \to F_x \to 0.$$

We therefore get isomorphisms

$$\Omega^{1/2}(\ker ds)_u \simeq \Omega^{1/2}\big((\ker ds)_u \cap (\ker dt)_u\big) \otimes \Omega^{1/2}F_y$$

and

$$\Omega^{1/2}(\ker dt)_u \simeq \Omega^{1/2}\big((\ker ds)_u \cap (\ker dt)_u\big) \otimes \Omega^{1/2}F_x.$$

Choose a Riemannian metric on M. It gives a Euclidean metric on F_x and F_y and whence trivializes the one dimensional vector spaces $\Omega^{1/2}F_x$ and $\Omega^{1/2}F_y$. We thus get an isomorphism $\rho_u^U : \Omega^{1/2}(\ker ds)_u \to \Omega^{1/2}(\ker dt)_u$.

Let f be a nonnegative measurable section of the bundle

$$\Omega^{1/2}U = \Omega^{1/2}\ker dt \otimes \Omega^{1/2}\ker ds.$$

Using ρ^U, we obtain nonnegative sections $\rho_*^U(f) = (1\otimes\rho^U)(f)$ of the bundle $\Omega^1 \ker dt$ and $(\rho^U)_*^{-1}(f) = \big((\rho^U)^{-1}\otimes 1\big)(f)$ of the bundle $\Omega^1\ker ds$; we may then integrate these functions along the fibers of t and s respectively and obtain functions $t_!\big(\rho_*^U(f)\big)$ and $s_!\big((\rho^U)_*^{-1}f\big)$ defined on M (with values in $[0,+\infty]$).

Lemma 4.7. *Thus defined the families ρ^U and $(\rho^U)^{-1}$ are measurable and bounded over compact subsets of U.*

Note that ρ^U is a section of the one dimensional bundle $\mathrm{Hom}\big(\Omega^{1/2}(\ker ds_U),\Omega^{1/2}(\ker dt_U)\big)$, which explains the above statement.

Proof. The family ρ^U is continuous on the locally closed subsets of U on which the dimension of $F_{s(u)}$ is constant; it is therefore measurable.

We just have to show that ρ^U is bounded in the neighborhood of every point $u\in U$. Let $u\in U$. Let θ be a local diffeomorphism of M carried by U. There exists a local diffeomorphism $\varphi: U'\to U''$, where U' and U'' are neighborhoods of u in U such that $\varphi(u) = u$, $t_U\circ\varphi = \theta\circ s_U$ and $s_U\circ\varphi = \theta^{-1}\circ t_U$. The map $d\varphi$ induces a smooth isomorphism of vector bundles $\ker ds_U$ and $\ker dt_U$. We just have to compare ρ^U with the isomorphism $\psi: \mathrm{Hom}\big(\Omega^{1/2}(\ker ds_U),\Omega^{1/2}(\ker dt_U)\big)$ induced by $d\varphi$.

Let $v\in U'$; put $x = s_U(v)$ and $E_v = \ker(ds_U)_v\cap\ker(dt_U)_v$. The map $d\varphi_v$ fixes E_v and induces an automorphism k_v of the vector space E_v; the differential $(d\theta)_x$ induces an isomorphism ℓ_v between the Euclidean vector spaces F_x and $F_{t_U(v)}$. The composition $(\rho^U)_v^{-1}\circ\psi_v$ is the multiplication by the square root of the product $|\det(k_v)|.\det|\ell_v|$.

Choose a Riemannian metric g_U on U, and let K be a compact subspace of U'. Since $d\varphi$ and $d\theta$ are continuous, they are bounded on K and $s_U(K)$ respectively. It follows that they multiply the norm of a vector by a ratio bounded by above and below. This gives the desired estimates for $|\det(k_v)|.\det|\ell_v|$ when $v\in K$. □

Definition 4.8. Let $\mathscr{U} = (U_i, t_i, s_i)_{i\in I}$ be an atlas.

(a) Let (V, t_V, s_V) be a bi-submersion adapted to $\mathscr{U}$. The L^1-norm associated with ρ^V is the map which to $f\in C_c(V;\Omega^{1/2}V)$ associates

$$\|f\|_{1,\rho^V} = \max\big(\sup\{t_!(\rho_*|f|)(x), x\in M\}, \sup\{s_!(\rho_*^{-1}|f|)(x), x\in M\}\big).$$

(b) In the case $(U,t,s) = \coprod_{i\in I}(U_i,t_i,s_i)$, we obtain the L^1-norm $\|f\|_{1,\rho^{\mathscr{U}}} = \|f\|_{1,\rho^U}$ for $f\in C_c(U;\Omega^{1/2}U) = \bigoplus_{i\in I} C_c(U_i;\Omega^{1/2}U_i)$.

(c) We note $\|\ \|_{1,\rho}$ the quotient norm in $\mathscr{A}_{\mathscr{U}} = \bigoplus_{i\in I} C_c(U_i;\Omega^{1/2}U_i)/\mathscr{I}$.

It follows from lemma 4.7 that the quantity $\|f\|_{1,\rho^V}$ is well defined and finite for every $f \in C_c(V;\Omega^{1/2}V)$. The map $f \mapsto \|f\|_{1,\rho^V}$ is obviously a norm, whence $\| \ \|_{1,\rho}$ is a semi-norm.

Proposition 4.9. *The semi-norm* $\| \ \|_{1,\rho}$ *is a* $*$*-algebra norm, i.e. satisfies* $\|f \star g\|_{1,\rho} \leqq \|f\|_{1,\rho}\|g\|_{1,\rho}$ *and* $\|f^*\|_{1,\rho} = \|f\|_{1,\rho}$ *(for every* $f, g \in \mathcal{A}_{\mathcal{U}}$*).*

Proof. Note that in lemma 4.1, one may show that one can choose g so that $p_!(g) = f$ and $p_!(|g|) = |f|$. It follows that for every bi-submersion V adapted to $\mathcal{U}$ the map $Q_V : C_c(V;\Omega^{1/2}V) \to \mathcal{A}_{\mathcal{U}}$ is norm reducing. Now, it follows from the Fubini theorem that, if V, W are bi-submersions, for every $f \in C_c(V;\Omega^{1/2}V)$ and every $g \in C_c(W;\Omega^{1/2}W)$, we have $\|f \otimes g\|_{1,\rho^{V\circ W}} \leqq \|f\|_{1,\rho^V}\|g\|_{1,\rho^W}$. The proposition follows. □

Definition 4.10. (a) We denote by $L^1_\rho(M,\mathcal{U},\rho)$ the Hausdorff completion of $\mathcal{A}_{\mathcal{U}}$ with respect to the L^1-norm associated with ρ. When $\mathcal{U}$ is the path holonomy atlas, we write $L^1(M,\mathcal{F})$ instead of $L^1_\rho(M,\mathcal{U})$.

(b) The *full* C^**-algebra* of $\mathcal{U}$ is the enveloping C^*-algebra $C^*(M,\mathcal{U})$ of $L^1_\rho(M,\mathcal{U})$. Equivalently, it is the Hausdorff completion of $\mathcal{A}_{\mathcal{U}}$ by the norm $\|(Q_U)(f)\| = \sup_\pi\{\|\pi(f)\|\}$ for every $*$-representation π of $\mathcal{A}_{\mathcal{U}}$ on a Hilbert space $\mathcal{H}$ bounded in the sense $\|\pi(Q_U(f))\| \leqq \|f\|_1$. When $\mathcal{U}$ is the path holonomy atlas, we write $C^*(M,\mathcal{F})$ instead of $C^*(M,\mathcal{U})$.

4.5. Regular representations associated with holonomy pairs and the reduced C^*-algebra. In order to define the reduced C^*-algebra of the foliation, we have to decide which set of representations we consider as being the regular ones. In fact, there are several possible choices:

- If the groupoid $G_{\mathcal{U}}$ is longitudinally smooth, we can just consider the corresponding regular representations.

- In the general case, we may replace $G_{\mathcal{U}}$ by a suitable ergodic decomposition leafwise.

- We could also decide to consider the representations on square integrable functions on the leaves, or just consider the (dense set of) leaves without holonomy.

In order to give a flexible solution, covering many different choices, we fix a holonomy pair $(\mathcal{U}, G)$. In particular, we have a groupoid homomorphism $\alpha : G_{\mathcal{U}} \to G$. If U is a bi-submersion adapted to $\mathcal{U}$, we put $\alpha_U = \alpha \circ q_U : U \to G$.

Fix $x \in M$. The set $G_x = \{\gamma \in G; s(\gamma) = x\}$ is a smooth manifold by definition of a holonomy pair. We denote by $L^2(G_x)$ the completion of the space $C_c^\infty\big(G_x;\Omega^{1/2}_{\mathbb{C}}\ker(ds)_x\big)$. This is a Hilbert space, with inner product $\langle f, g\rangle = \int_{G_x} \bar{f} \otimes g$ (we can integrate $\bar{f} \otimes g$ as a section of $\Omega^1\big(\ker(ds)_x\big)$).

Proposition 4.11. *There is a* $*$*-representation* $\pi_x : \mathcal{A}_{\mathcal{U}} \to \mathcal{L}\big(L^2(G_x)\big)$ *satisfying*

$$\left[\pi_x\big(Q_U(f)\big)(\xi)\right](\gamma) = \int_{t_U(u)=t(\gamma)} f(u)\xi\big(\alpha_U(u)^{-1}\gamma\big)$$

for every adapted bi-submersion (U, t_U, s_U), *every* $f \in C_c^\infty(U; \Omega^{1/2}U)$ *and* $\xi \in L^2(G_x)$. *Let* ρ *be a positive isomorphism* $\rho : \Omega^{1/2} \ker ds_U \to \Omega^{1/2} \ker dt_U$ *associated with a Riemannian metric as above. Then* π_x *extends to a representation of* $L^1(M, \mathscr{U}; \rho)$.

Proof. To see that π_x is well defined, one just needs to check that if $p : U \to V$ is a morphism of bi-submersions which is a submersion then

$$\pi_x\big(Q_U(f)\big)(\xi) = \pi_x\big(Q_V\big(p_!(f)\big)\big)(\xi),$$

which is easy.

To show that π_x extends to $L^1(M, \mathscr{U}; \rho)$, we have to show that for every $f \in C_c^\infty(U; \Omega^{1/2}U)$ and $\xi \in C_c^\infty(G_x; \Omega^{1/2}G_x)$, we have $\left\|\pi_x\big(Q_U(f)\big)(\xi)\right\|_2 \leqq \|f\|_{1,\rho^U}\|\xi\|_2$.

Endow U and G with the leafwise smooth structures. Then α_U becomes a smooth submersion. For $u, v \in G_x$, put then $F(u,v) = (\alpha_U)_!(f)(u^{-1}v)$. Then F is a section of the bundle of half densities on $G_x \times G_x$. The inequality follows then from the following classical fact (applied to $Z = G_x$): Let Z be a smooth manifold. Fix a positive density section μ on Z. We thus identify half density sections on Z and $Z \times Z$ with functions. Let F be a smooth function on $Z \times Z$. Put $\|F\|_1 = \max\left(\sup_x \int |F(x,y)|\, d\mu(y), \sup_y \int |F(x,y)|\, d\mu(x)\right)$. If $\|F\|_1 < \infty$, then the map $\xi \mapsto F \star \xi$ given by $(F \star \xi)(x) = \int F(x,y)\xi(y)\, d\mu(y)$ is a well defined bounded map from $L^2(Z)$ into itself with norm $\leqq \|F\|_1$.

Finally, the fact π_x is a $*$-representation follows from the Fubini theorem. □

Definition 4.12. Let $(\mathscr{U}, G)$ be a holonomy pair. We define the *reduced norm* in $\mathscr{A}_\mathscr{U}$ by $\|f\|_r = \sup_{x \in M} \|\pi_x(f)\|$. The *reduced* C^*-*algebra* $C_r^*(G)$ of G is the Hausdorff-completion of $\mathscr{A}_\mathscr{U}$ by the reduced norm. If $\mathscr{U}$ is the path holonomy atlas and $G_\mathscr{U}$ is longitudinally smooth, we write $C_r^*(M, \mathscr{F})$ instead of $C_r^*(G_\mathscr{U})$.

5. Representations of $C^*(M, \mathscr{F})$

In this section, we extend to the (highly singular) groupoid associated with an atlas $\mathscr{U}$ of bi-submersions the following result given by Renault in [26] (see also [12]): given a locally compact groupoid $G \rightrightarrows M$ there is a correspondence between the representations of $C^*(G)$ (on a Hilbert space) and the representations of the groupoid G (on a Hilbert bundle). This will clarify that the L^1-norm we gave earlier provides a good estimate, although defined up to the choice of a Riemannian metric. As an obvious consequence, $C^*(M, \mathscr{F})$ does not depend on the choice of the Riemannian metric.

Let us fix again an atlas of bi-submersions $\mathscr{U}$ for the foliation $\mathscr{F}$ on M (not necessarily the path holonomy atlas). We denote $G = G_\mathscr{U}$ the associated groupoid. If U is a bi-submersion adapted to $\mathscr{U}$, we denote by $q_U : U \to G_\mathscr{U}$ the quotient map.

Let $\mathscr{A}_{\mathscr{U}}$ be the associated $*$-algebra. The full C^*-algebra will be denoted $C^*(\mathscr{A}_{\mathscr{U}})$.

In order to avoid uninteresting technicalities, we consider only representations on separable Hilbert spaces; we assume that the atlas $\mathscr{U} = (U_i)_{i \in I}$ is countably generated, i.e. that I is countable and each U_i is σ-compact.

5.1. Representations of $G_{\mathscr{U}}$ and their integration. We begin with a few considerations about measures.

Let $p : N \to M$ be a submersion and μ a measure on M. Any positive Borel section λ^p of the bundle $\Omega^1(\ker dp)$ may be thought of as a family of positive Borel measures λ^p_x on $p^{-1}(\{x\})$. Such a section defines a measure $\mu \circ \lambda^p$ on N by

$$\mu \circ \lambda^p(f) = \int_M \left(\int_{p^{-1}(\{x\})} f(y)\, d\lambda^p_x(y) \right) d\mu(x)$$

for every $f \in C_c(N)$.

Definition 5.1. The measure μ on M is *quasi-invariant* if for all $(U, t_U, s_U) \in \mathscr{U}$ the measures $\mu \circ \lambda^{t_U}$ and $\mu \circ \lambda^{s_U}$ are equivalent.

In other words, there exists a positive locally $\mu \circ \lambda^{t_U}$-integrable function D^U such that $\mu \circ \lambda^{s_U} = D^U \cdot \mu \circ \lambda^{t_U}$.

Let us show that this Radon-Nikodym derivative D gives rise to a homomorphism $G_{\mathscr{U}} \to \mathbb{R}^*_+$ if we make suitable choices of the densities λ^{t_U} and λ^{s_U}.

We fix a Riemannian metric on M. We saw in 4.4 that it induces a Borel isomorphism $\rho^U : \Omega^1(\ker ds_U) \to \Omega^1(\ker dt_U)$ for every bi-submersion $(U, t_U, s_U) \in \mathscr{U}$. Choose a section $\lambda^{s_U} \in C_c\big(U; \Omega^1(\ker ds_U)\big)$ and take the corresponding $\lambda^{t_U} = \rho^U(\lambda^{s_U})$ which is a positive Borel section, bounded over compact sets of the bundle $\Omega^1(\ker dt_U)$. Let $\mu \circ \lambda^{s_U}$ and $\mu \circ \lambda^{t_U}$ be the associated measures of U.

(1) The function D^U depends on the chosen Riemannian metric but does not depend on the choice of λ^{s_U}. This follows from the linearity of the map ρ^U applied to the product of λ^{s_U} with a positive continuous function.

(2) The maps D^U may be replaced by a map $D : G_{\mathscr{U}} \to \mathbb{R}^+$ defined by $D\big(q_U(u)\big) = D^U(u)$ for any bi-submersion of the atlas $\mathscr{U}$. More precisely:

Let $\varphi : U \to V$ be a morphism of bi-submersions. Then $D^U = D^V \circ \phi$ almost everywhere.

Indeed, the statement is local: we may replace U by a small open neighborhood of a point u. Thanks to corollary 3.12, we may assume that φ is a submersion. Finally, we may assume that $U = L \times V$ where L is a manifold and φ is the projection. We may then take $\lambda^{s_U} = \lambda^L \times \lambda^{s_V}$ where λ^L is a 1 density on L; then $\lambda^{t_U} = \lambda^L \times \lambda^{t_V}$, and the statement follows.

(3) To show that D is a *homomorphism* we need to examine the behavior of the D^Us with respect to the composition of bi-submersions. For two bi-submersions (U_1, t_1, s_1) and (U_2, t_2, s_2) in $\mathscr{U}$ denote $(U_1 \circ U_2, t, s)$ their composition. Then $s^{-1}(x) = U_1 \times_{s_1, t_1} (U_2)_x$ for every $x \in M$. Notice that $\ker(ds)_{(u_1, u_2)}$ contains $\ker(ds_1)_{u_1} \times \{0\}$ and the quotient is isomorphic to $\ker(ds_2)_{u_2}$. So we obtain a canonical isomorphism

$$\Omega^1\big(\ker(ds)_{(u_1,u_2)}\big) = \Omega^1\big(\ker(ds_1)_{u_1}\big) \otimes \Omega^1\big(\ker(ds_2)_{u_2}\big).$$

Likewise, we find

$$\Omega^1\big(\ker(dt)_{(u_1,u_2)}\big) = \Omega^1\big(\ker(dt_1)_{u_1}\big) \otimes \Omega^1\big(\ker(dt_2)_{u_2}\big).$$

One checks that, up to these isomorphisms, we have an equality $\rho_{U_1 \circ U_2} = \rho_{U_1} \otimes \rho_{U_2}$. An easy computation gives then

$$D^{U_1 \circ U_2} = D^{U_1} \circ p_1 \cdot D^{U_2} \circ p_2,$$

where p_1, p_2 are the projections of $U_1 \circ U_2$ to U_1 and U_2 respectively.

Due to the last two observations we may think of D as being a homomorphism $D : G_{\mathscr{U}} \to \mathbb{R}^+$.

In the same way, we define representations of $G_{\mathscr{U}}$ as being 'measurable homomorphisms from $G_{\mathscr{U}}$ to the unitary group'. The precise definition is the following:

Definition 5.2. A *representation* of $G_{\mathscr{U}}$ is a triple (μ, H, π) where:

(a) μ is a quasi-invariant measure on M.

(b) $H = (H_x)_{x \in M}$ is a measurable (with respect to μ) field of Hilbert spaces over M.

(c) For every bi-submersion (U, t, s) adapted to $\mathscr{U}$, π^U is a measurable (with respect to $\mu \circ \lambda$) section of the field of unitaries $\pi_u^U : H_{s(u)} \to H_{s(u)}$.

Moreover, we assume:

(1) π is 'defined on $G_{\mathscr{U}}$':

If $f : U \to V$ is a morphism of bi-submersions, for almost all $u \in U$ we have $\pi_u^U = \pi_{f(u)}^V$.

(2) π is a homomorphism:

If U and V are bi-submersions adapted to $\mathscr{U}$, we have $\pi_{(u,v)}^{U \circ V} = \pi_u^U \pi_v^V$ for almost all $(u, v) \in U \circ V$.

Any representation (μ, H, π) where μ is quasi-invariant gives rise to a representation of $C^*(M, \mathscr{F})$ on the space $\mathscr{H} = \int_M^{\oplus} H_x \, d\mu(x)$ of sections of the Hilbert bundle H:

For every bi-submersion U adapted to $\mathscr{U}$, define $\hat{\pi}_U : C_c(U, \Omega^{1/2}U) \to B(\mathscr{H})$ by putting

$$\hat{\pi}_U(f)(\xi)(x) = \int_{U^x} (1 \otimes \rho^U)\big(f(u)\big)\pi_u^U\big(\xi\big(s_U(u)\big)\big)D^U(u)^{1/2}$$

μ-a.e. for every $f \in C_c\big(U, \Omega^{1/2}(U)\big)$, $\xi \in \mathscr{H}$ and $x \in M$.

One checks that we thus define a representation of $C^*(M;\mathscr{F})$ in a few steps:

(1) If $\varphi : U \to V$ is a morphism of bi-submersions which is a submersion, for every $f \in C_c\big(U, \Omega^{1/2}(U)\big)$, we obviously have $\hat{\pi}_U(f) = \hat{\pi}_V\big(\varphi_!(f)\big)$.

(2) Let (U_1, t_1, s_1) and (U_2, t_2, s_2) be two bi-submersions in $\mathscr{U}$, and denote $(U_1 \circ U_2, t, s)$ their composition. For every $f_1 \in C_c\big(U_1, \Omega^{1/2}(U_1)\big)$ and $f_2 \in C_c\big(U_2, \Omega^{1/2}(U_2)\big)$, we have $\hat{\pi}_{U_1}(f_1)\hat{\pi}_{U_2}(f_2) = \hat{\pi}_{U_1 \circ U_2}(f_1 \otimes f_2)$.

(3) Let (U, t, s) be a bisubmersion in $\mathscr{U}$, and let $U^{-1} = (U, s, t)$ be its inverse. Denote by $\kappa : U \to U^{-1}$ the identity. We have $\hat{\pi}_{U^{-1}}(\bar{f} \circ \kappa) = \big(\hat{\pi}_U(f)\big)^*$.

It follows from (1) that there is a linear map $\hat{\pi} : \mathscr{A}_{\mathscr{U}} \to \mathscr{L}(\mathscr{H})$ such that for every U adapted to $\mathscr{U}$ we have $\hat{\pi}_U = \hat{\pi} \circ Q_U$; it then follows from (2) and (3) that the map $\hat{\pi}$ is a $*$-homomorphism.

(4) The proof of [26] can be adapted to show that $\hat{\pi}$ is continuous with respect to the L^1-norm (cf. definition 4.8), and therefore defines a representation $\hat{\pi}$ of $C^*(M, \mathscr{U})$ on $\mathscr{H}$.

5.2. Desintegration of representations. We now show that every representation of $C^*(M, \mathscr{U})$ is of the above form i.e. it is the integrated form of a representation of the groupoid $G_{\mathscr{U}}$. Actually, the proof given by J. Renault in [26] can be easily adapted to our case. We give here an alternate route, based on Hilbert C^*-modules.

Let us first note that $C_0(M)$ sits in the multipliers of $C^*(M, \mathscr{U})$: if (U, t_U, s_U) is a bi-submersion adapted to $\mathscr{U}$, for $f \in C_c(U; \Omega^{1/2})$ and $h \in C_0(M)$, we just put $h.Q_U(f) = Q_U\big((h \circ t)f\big)$ and $Q_U(f).h = Q_U\big(f(h \circ s)\big)$. It follows that every non degenerate representation of $C^*(M, \mathscr{U})$ on a separable Hilbert space $\mathscr{H}$ gives rise to a representation of $C_0(M)$ and therefore a measure (class) μ on M and a measurable field of Hilbert spaces $(H_x)_{x \in M}$ such that $\mathscr{H} = \int^{\oplus} H_x \, d\mu(x)$ on which $C_0(M)$ is naturally represented. We are going to show that μ is quasi-invariant and $G_{\mathscr{U}}$ is represented on the Hilbert field $(H_x)_{x \in M}$.

Recall (cf. e.g. [6]) that along with a submersion $p : N \to M$ is naturally associated a Hilbert $C_0(N) - C_0(M)$-bimodule $\mathscr{E}_p$ which is the continuous family of Hilbert spaces $\big(L^2\big(p^{-1}(\{x\})\big)\big)_{x \in M}$, on which $C_0(N)$ is represented by multiplication. More specifically, $\mathscr{E}_p$ is the completion of $C_c(N; \Omega^{1/2} \ker dp)$ with respect to the $C_0(M)$ valued scalar product given by the formula $\langle \xi, \eta \rangle(x) = \int_{p^{-1}(\{x\})} \bar{\xi}\eta$. The operation of $C_0(N)$ by multiplication obviously extends to $\mathscr{E}_p$.

Now, if $p : N \to M$ is a submersion, we put $\tilde{\mathscr{E}}_p = \mathscr{E}_p \otimes_{C_0(M)} C^*(M, \mathscr{U})$. A typical element of $\tilde{\mathscr{E}}_p$ is the class of a section $C_c(N; \Omega^{1/2} \ker dp) \otimes C_c(U; \Omega^{1/2})$ where

U is a bi-submersion adapted to U; we therefore associate elements of $\tilde{\mathscr{E}}_p$ to every $\varphi \in C_c\big(N \times U; \Omega^{1/2}(\ker dp \times \ker dt_U \times \ker ds_U)\big)$; it is easily seen that the image of such a φ only depends on its restriction to $N \times_{p,t_U} U$. Now, $N \times_{p,t_U} U$ is easily seen to be a bi-submersion between $(M, \mathscr{F})$ and $(N, p^{-1}\mathscr{F})$. It is in fact quite easy to prove:

Proposition 5.3. *For every bi-submersion (V, t_V, s_V) between $(M, \mathscr{F})$ and $(N, p^{-1}\mathscr{F})$ such that $(V, p \circ t_V, s_V)$ is adapted to $\mathscr{U}$, we have a map $Q_V : C_c(V, \Omega^{1/2} V) \to \tilde{\mathscr{E}}_p$. The images of Q_V span a dense subspace of $\tilde{\mathscr{E}}_p$. If V, W are two bi-submersions as above between $(M, \mathscr{F})$ and $(N, p^{-1}\mathscr{F})$, $f \in C_c(V, \Omega^{1/2} V)$, $g \in C_c(W, \Omega^{1/2} W)$ and $h \in C_0(N)$, we have the formulae*

$$(1) \quad hQ_V(f) = Q_V\big((h \circ t_V)f\big) \quad \text{and} \quad \langle Q_V(f), Q_W(g)\rangle = Q_{V^{-1}\circ W}(\bar{f} \otimes g).$$

Let us give a few explanations on the statement of this proposition:

- $\Omega^{1/2} V$ denotes the bundle $\Omega^{1/2}(\ker dt_V \times \ker ds_V)$.
- $V^{-1} \circ W = V \times_{t_V, t_W} W$; it is a bi-submersion of $(M, \mathscr{F})$.
- We identify $\mathscr{A}_{\mathscr{U}}$ with its image in $C^*(M, \mathscr{U})$, thus $Q_{V^{-1}\circ W}(\bar{f} \otimes g) \in C^*(M, \mathscr{U})$.

Proof. One checks easily formulae (1) for bi-submersions of the form $N \times_{p,t_U} U$ and f, g of the form $f_1 \otimes f_2$ where U is a bi-submersion of $(M, \mathscr{F})$, $f_1 \in C_c(N, \Omega^{1/2})$, $f_2 \in C_c(U, \Omega^{1/2})$. In general, V is covered by open subsets of this form, and the conclusion follows using partitions of the identity. □

Remarks and notation 5.4. Let $p, q : N \to M$ be two submersions such that $p^{-1}\mathscr{F} = q^{-1}\mathscr{F}$; assume moreover that for every bi-submersion (V, t_V, s_V) of $(M, \mathscr{F})$ between $(M, \mathscr{F})$ and $(N, p^{-1}\mathscr{F})$, $(V, p \circ t_V, s_V)$ is adapted to $\mathscr{U}$ if and only if $(V, q \circ t_V, s_V)$ is adapted to $\mathscr{U}$. It follows from proposition 5.3 that there is a canonical isomorphism between $\sigma_p^q \in \mathscr{L}(\tilde{\mathscr{E}}_p, \tilde{\mathscr{E}}_q)$.

(a) Obviously, $\sigma_p^p = \mathrm{id}$, $\sigma_q^p = (\sigma_p^q)^{-1}$; furthermore, if $r : N \to M$ is a third submersion satisfying the same properties, we have $\sigma_p^r = \sigma_q^r \sigma_p^q$.

(b) Let (U, t_U, s_U) be a bi-submersion adapted with $\mathscr{U}$. To simplify notation, we will just denote by $\sigma^U \in \mathscr{L}(\tilde{\mathscr{E}}_{s_U}, \tilde{\mathscr{E}}_{t_U})$ the unitary operator $\sigma_{s_U}^{t_U}$.

Now let $\varpi : C^*(M, \mathscr{U}) \to \mathscr{L}(\mathscr{H})$ be a representation of $C^*(M, \mathscr{U})$. Since $C_0(M)$ sits in the multiplier algebra of $C^*(M, \mathscr{U})$, the representation ϖ gives rise to a representation of $C_0(M)$. We thus get a measure (class) μ on M together with a measurable family of Hilbert spaces $(H_x)_{x \in M}$. If $p : N \to M$ is a submersion, write $\mathscr{E}_p \otimes_{C_0(M)} \mathscr{H} = \int_N^{\oplus} H_{p(y)}\, d\mu \circ \lambda(y)$: this gives the representations of $C_0(N)$ on $\mathscr{E}_p \otimes_{C_0(M)} \mathscr{H}$. Let (U, t_U, s_U) be a bi-submersion adapted to $\mathscr{U}$. The representations of $C_0(U)$ on $\mathscr{E}_{s_U} \otimes_{C_0(M)} \mathscr{H} = \tilde{\mathscr{E}}_{s_U} \otimes_{C^*(M,\mathscr{U})} \mathscr{H}$ and $\mathscr{E}_{t_U} \otimes_{C_0(M)} \mathscr{H} = \tilde{\mathscr{E}}_{t_U} \otimes_{C^*(M,\mathscr{U})} \mathscr{H}$ are isomorphic through the unitary operator $\alpha^U = \sigma^U \otimes_{C^*(M,\mathscr{U})} \mathrm{id}_{\mathscr{H}}$. It follows that μ is quasi-invariant; furthermore the operator $\sigma^U \otimes_{C^*(M,\mathscr{U})} \mathrm{id}_{\mathscr{H}}$ yields a measurable family $(\pi_u^U)_{u \in U}$ where $\pi_u^U : H_{s_U} \to H_{t_U}$ is a unitary operator.

One sees easily that:

(1) If $f : U \to V$ is a morphism of bi-submersions, $\pi_u^U = \pi_{f(u)}^V$ (almost everywhere):

(a) This is obvious if U is an open subset of V.

(b) This is also easy if $f : U = L \times V \to V$ is the projection.

(c) From these facts the formula is established if f is a submersion.

(d) The general case follows from corollary 3.12.

(2) Let (U, t_U, s_U) and (V, t_V, s_V) be bi-submersions. Let (W, t_W, s_W) be the bi-submersion $U \circ V$. We have $\pi_{(u,v)}^W = \pi_u^U \pi_v^V$ (almost everywhere in W).

Indeed, let $\alpha : W \to M$ be the map $(u, v) \mapsto s_U(u) = t_V(v)$. The map $(u, v) \mapsto u$ is a morphism of bi-submersions between (W, t_W, α) and U and the map $(u, v) \mapsto v$ is a morphism of bi-submersions between (W, α, s_W) and V. The equality follows from 5.4.

It follows from the above that (μ, H, π) is a representation of the groupoid $G_{\mathscr{U}}$ in the sense of definition 5.2. Finally, one checks that ϖ is the representation $\hat{\pi}$ associated with π.

6. Further developments

We end with a couple of remarks that we will expand elsewhere:

6.1. The tangent groupoid, longitudinal pseudo-differential operators and analytic index.

6.1.1. The cotangent space. The K-theory class of a symbol should take place in the K-theory of the total space of a cotangent bundle. The cotangent bundle of the foliation is the "bundle" $(\mathscr{F}_x^*)_{x \in M}$. Let us discuss this space.

Definition 6.1. Every $\xi \in \mathscr{F}_x^*$ with $x \in M$ defines a linear functional on $\mathscr{F}$ as a composition $\xi \circ e_x : \mathscr{F} \to \mathscr{F}_x \to \mathbb{R}$. Let $\mathscr{F}^*$ be the union of $\mathscr{F}_x^*$ endowed with the topology of pointwise convergence on $\mathscr{F}$. It is a locally compact space.

6.1.2. The tangent groupoid. Let $\lambda : M \times \mathbb{R} \to \mathbb{R}$ be the second projection. Consider the foliation $\mathscr{F} \times \{0\}$ of $M \times \mathbb{R}$. Denote by $T\mathscr{F}$ the foliation $\{\lambda X;\ X \in \mathscr{F} \times \{0\}\}$.

The groupoid of this foliation is $\bigcup_{x \in M} \mathscr{F}_x \times \{0\} \cup G \times \mathbb{R}^*$. It is called the *tangent groupoid*. Its C^*-algebra contains as an ideal $C_0(\mathbb{R}^*) \otimes C^*(M; \mathscr{F})$ with quotient $C_0(\mathscr{F}^*)$.

This tangent groupoid allows to construct a map $K\big(C_0(\mathscr{F}^*)\big) \to K\big(C^*(M; \mathscr{F})\big)$. This map is the *analytic index*.

6.1.3. Pseudo-differential calculus along the foliation. One can also define the longitudinal pseudo-differential operators associated with a foliation. These are unbounded multipliers of the C^*-algebra.

The differential operators are very easily defined: They are generated by vector fields in $\mathscr{F}$.

Let (U,t,s) be a bi-submersion and $V \subset U$ an identity bisection. Denote by N the normal bundle to V in U and let a be a (classical) symbol on N^*. Let χ be a smooth function on U supported on a tubular neighborhood of V in U and let $\phi : U \to N$ be an inverse of the exponential map (defined on the neighborhood of V). A pseudodifferential kernel on U is a (generalized) function $k_a : x \mapsto \int a(x,\xi) \exp\big(i\phi(u)\xi\big)\chi(u)\, d\xi$ (the integral is an oscillatory integral, taken over the vector space N_x^*).

As in the case of foliations and Lie groupoids (cf. [3], [18], [20]), we show:

- The kernel k_a defines a multiplier of $\mathscr{A}(\mathscr{F})$.

- Those multipliers when U runs over an atlas, V covers M and a runs over the classical symbols form an algebra. This algebra only depends on the class of the atlas.

- The algebra of pseudodifferential operators is filtered by the order of a. The class of k_a only depends up to lower order on the restriction of the principal part of a on $\mathscr{F}$.

- Negative order pseudodifferential operators are elements of the C^*-algebra (both full and reduced) of the foliation. Zero order pseudodifferential operators define bounded multipliers of the C^*-algebra of the foliation.

We have an exact sequence of C^*-algebras

$$0 \to C^*(M,\mathscr{F}) \to \Psi^*(M,\mathscr{F}) \to C_0(S^*\mathscr{F}) \to 0$$

where $\Psi^*(M,\mathscr{F})$ denotes the closure of the algebra of zero order pseudodifferential operators.

- One can also take coefficients on a smooth vector bundle over M.

- Elliptic operators of positive order (i.e. operators whose principal symbol is invertible when restricted to $\mathscr{F}$) give rise to regular quasi-invertible operators (cf. [29]).

6.1.4. The analytic index. Elliptic (pseudo)-differential operators have an index which is an element of $K_0\big(C^*(M,\mathscr{F})\big)$. One easily sees that this index equals the one given by the 'tangent groupoid' and the corresponding exact sequence of C^*-algebras.

6.2. Generalization to a 'continuous family' case. As it was explained by Paterson in [22], one doesn't really need a Lie groupoid in order to have a nice pseudodifferential calculus: the only thing which matters is the fact that the s and t fibers are smooth manifolds. Examples of longitudinally smooth groupoids naturally appear also in the case of stratified

Lie groupoids recently studied in a fundamental paper by Fernandez, Ortega and Ratiu ([13]).

In the same way, we wish to consider singular foliations on locally compact spaces, which are only smooth in the leaf direction. Actually, one way to define those foliations is to start with an atlas of bi-submersions. Indeed, in the smooth case, an atlas of bi-submersions associated with $\mathscr{F}$, obviously defines the foliation $\mathscr{F}$.

We may then define the holonomy groupoid, the C^*-algebra exactly as above, together with a suitable pseudodifferential calculus.

References

[1] *B. Bigonnet*, Holonomie et graphe de certains feuilletages avec singularités, preprint.

[2] *B. Bigonnet*, *J. Pradines*, Graphe d'un feuilletage singulier, C. R. Acad. Sci. Paris **300** (13) (1985), 439–442.

[3] *A. Connes*, Sur la théorie non commutative de l'intégration, in: Algèbres d'opérateurs, Springer Lect. Notes Math. **725** (1979), 19–143.

[4] *A. Connes*, A survey of foliations and operator algebras, in: Operator Algebras and Applications, Proc. Symp. Pure Math. A.M.S. **38** part I (1982), 521–628.

[5] *A. Connes*, Noncommutative Geometry, Academic Press, 1984.

[6] *A. Connes* and *G. Skandalis*, The Longitudinal Index Theorem for Foliations, Publ. Res. Inst. Math. Sci. **20**, No. 6 (1984), 1139–1183.

[7] *Marius Crainic* and *Rui Loja Fernandes*, Integrability of Lie brackets, Ann. Math. **157** (2003), 575–620.

[8] *Marius Crainic* and *Rui Loja Fernandes*, Integrability of Poisson brackets, J. Diff. Geom. **66** (2004), 71–137.

[9] *C. Debord*, Local integration of Lie algebroids, Banach Center Publ. **54** (2001), 21–33.

[10] *C. Debord*, Holonomy groupoids of singular foliations, J. Diff. Geom. **58** (2001), 467–500.

[11] *C. Ehresmann*, Catégories et structures, Dunod, Paris 1965.

[12] *Th. Fack* and *G. Skandalis*, Sur les représentations et idéaux de la C^*-algèbre d'un feuilletage, J. Op. Th. **8** (1982), 95–129.

[13] *R. L. Fernandes*, *J.-P. Ortega* and *T. S. Ratiu*, The momentum map in Poisson geometry, preprint, arXiv:0705.0562.

[14] *M. Khoshkam*, *G. Skandalis*, Regular representation of groupoid C^*-algebras and applications to inverse semigroups, J. reine angew. Math. **546** (2002), 47–72.

[15] *N. P. Landsman*, Mathematical Topics Between Classical and Quantum Mechanics, Springer, New York 1988.

[16] *N. P. Landsman* and *B. Ramazan*, Quantization of Poisson algebras associated to Lie algebroids, Contemp. Math. **282** (2001), 159–192.

[17] *K. Mackenzie*, Lie groupoids and Lie algebroids in differential geometry, London Math. Soc. Lect. Note Ser. **124**, Cambridge University Press, Cambridge 1987.

[18] *B. Monthubert* and *F. Pierrot*, Indice analytique et groupoïdes de Lie, C. R. Acad. Sci. Paris (I) **325** (1997) no. 2, 193–198.

[19] *V. Nistor*, Groupoids and the integration of Lie algebroids, J. Math. Soc. Japan **52** (2000), 847–868.

[20] *V. Nistor*, *A. Weinstein* and *P. Xu*, Pseudodifferential operators on differential groupoids, Pac. J. Math. **189** (1999) no. 1, 117–152.

[21] *J.-P. Ortega* and *T. S. Ratiu*, Momentum maps and Hamiltonian reduction, Progr. Math. **222**, Birkhäuser Boston, Inc., Boston, MA, 2004.

[22] *A. Paterson*, Continuous family groupoids, Homol. Homotopy Appl. **2** (2000), 89–104.

[23] *J. Pradines*, Lie groupoids and generalised atlases, Cent. Eur. J. Math. **2** (2004) no. 5, 624–662.

[24] *J. Pradines*, How to define the differentiable graph of a singular foliation, C. Top. Geom. Diff. Cat. **XXVI**(4) (1985), 339–381.

[25] *J. Pradines*, Feuilletages: holonomie et graphes locaux, C. R. Acad. Sci. Paris (I) **298**, no. 13 (1984), 297–300.

[26] *J. N. Renault*, A groupoid approach to C^*-algebras, Springer Lect. Notes Math. **793** (1979).

[27] *P. Stefan*, Accessible sets, orbits, and foliations with singularities, Proc. London Math. Soc. (3) **29** (1974), 699–713.

[28] *H. J. Sussmann*, Orbits of families of vector fields and integrability of distributions, Trans. A. M. S. **180** (1973), 171–188.
[29] *S. Vassout*, Unbounded pseudodifferential calculus on Lie groupoids, J. Funct. Anal. **236** (2006), 161–200.
[30] *H. E. Winkelnkemper*, The graph of a foliation, Ann. Glob. Anal. Geom. **1** (3) (1983), 51–75.

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