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THE HOMOLOGICAL THEORY OF CONTRAVARIANTLY FINITE SUBCATEGORIES: AUSLANDER-BUCHWEITZ CONTEXTS, GORENSTEIN CATEGORIES AND (CO-)STABILIZATION

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1. Introduction

Let $\mathcal{C}$ be an abelian or exact category with enough projectives and let $\mathcal{P}$ be the full subcategory of projective objects of $\mathcal{C}$. We consider the stable category $\mathcal{C}/\mathcal{P}$ modulo projectives, as a left triangulated category [14], [36]. Then there is a triangulated category $\mathcal{S}(\mathcal{C}/\mathcal{P})$ associated to $\mathcal{C}/\mathcal{P}$, which is universal in the following sense. There exists an exact functor $\mathbf{S} : \mathcal{C}/\mathcal{P} \to \mathcal{S}(\mathcal{C}/\mathcal{P})$ such that any exact functor out of $\mathcal{C}/\mathcal{P}$ to a triangulated category has a unique exact factorization through $\mathbf{S}$. The triangulated category $\mathcal{S}(\mathcal{C}/\mathcal{P})$ is called the **stabilization** of $\mathcal{C}/\mathcal{P}$ and the functor $\mathbf{S}$ is called the **stabilization functor**. There is also the dual construction of the **costabilization** $\mathcal{R}(\mathcal{C}/\mathcal{P})$ of $\mathcal{C}/\mathcal{P}$, which is a triangulated category equipped with an exact functor $\mathbf{R} : \mathcal{R}(\mathcal{C}/\mathcal{P}) \to \mathcal{C}/\mathcal{P}$, the **costabilization functor**, such that any exact functor from a triangulated category to $\mathcal{C}/\mathcal{P}$ has a unique exact factorization through $\mathbf{R}$. If $\mathcal{C}$ has enough injectives we can stabilize and costabilize in the above sense the stable category modulo injectives. These constructions have topological origin and make sense for any stable category $\mathcal{C}/\mathcal{X}$, where now $\mathcal{C}$ is an additive category and $\mathcal{X}$ is a contravariantly or covariantly finite subcategory of $\mathcal{C}$ in the sense of Auslander-Smalø [8], assuming that $\mathcal{C}$ satisfies some mild condition. The stabilization construction in our setting is due to Heller [33], see also [24], [44], and later was used by Keller-Vossieck in [36]. For the costabilization construction we refer to the work of Grandis [27].

Our purpose in this paper is to investigate when the stabilization $\mathcal{S}(\mathcal{C}/\mathcal{P})$ or the costabilization $\mathcal{R}(\mathcal{C}/\mathcal{P})$ can be represented as a full (triangulated) subcategory $\mathcal{T}$ of $\mathcal{C}/\mathcal{P}$. In the first case we call the abelian or exact category $\mathcal{C}$, $\mathcal{P}$**−Gorenstein**, and in the second case we call the abelian or exact category $\mathcal{C}$, $\mathcal{P}$**−Co-Gorenstein**. In both cases $\mathcal{T}$ is realized by the stable category $\mathcal{A}/\mathcal{P}$ of a specific resolving subcategory $\mathcal{A}$ of $\mathcal{C}$. In general if $\mathcal{A}$ is any resolving subcategory of $\mathcal{C}$, then our results are dealing with the relations between the coordinates of the triple $(\mathcal{C},\mathcal{A},\mathcal{P})$ in the following three levels. First in the *exact* level, i.e. inside the exact category $\mathcal{C}$, second in the *stable* level, i.e. inside the stable categories $\mathcal{C}/\mathcal{P}$, $\mathcal{A}/\mathcal{P}$, $\mathcal{C}/\mathcal{A}$ and finally in the *derived* level, i.e. inside the derived categories $\mathcal{D}^b(\mathcal{C})$, $\mathcal{D}^b(\mathcal{A})$, $\mathcal{D}^b(\mathcal{P})$ and their Verdier-quotients $\mathcal{D}^b(\mathcal{C})/\mathcal{D}^b(\mathcal{P})$, $\mathcal{D}^b(\mathcal{A})/\mathcal{D}^b(\mathcal{P})$, $\mathcal{D}^b(\mathcal{C})/\mathcal{D}^b(\mathcal{A})$.
We study $\mathcal{P}-$(Co-)Gorenstein categories with respect to the above three levels, with a close view to applications in the module theory of an associative ring or an Artin algebra. The organization of the article is as follows.

In section 2, we study the relative homological algebra induced by a pair $(\mathcal{C},\mathcal{X})$ consisting of an additive category $\mathcal{C}$ and a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{C}$, assuming that any $\mathcal{X}-$epic has kernel in $\mathcal{C}$. Then the stable category $\mathcal{C}/\mathcal{X}$ is left triangulated and we give necessary and sufficient conditions for $\mathcal{C}/\mathcal{X}$ to contain a full triangulated subcategory. This suggests to introduce the concept of an $\mathcal{X}$**−Gorenstein object** of $\mathcal{C}$, which is a natural generalization of a module of zero Gorenstein dimension in the sense of Auslander-Bridger [3]. The full subcategory $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ of $\mathcal{X}-$Gorenstein objects of $\mathcal{C}$ is of central importance in this paper.

In section 3 we recall the construction of the stabilization and costabilization of a left triangulated category from [33], [36], and we compute them in the case of the left triangulated category $\mathcal{C}/\mathcal{X}$ induced by the pair $(\mathcal{C},\mathcal{X})$ mentioned above, in terms of complexes of objects of $\mathcal{X}$. The representation of the stabilization of $\mathcal{C}/\mathcal{X}$ by means of complexes generalizes (and is inspired by) a result of Keller-Vossieck [36]. In this section we introduce the important concept of a **(Co-)Gorenstein** left triangulated category, which will be used in the next sections.

In section 4 we introduce the concept of an $\mathcal{X}$**−Gorenstein category** $\mathcal{C}$, where $\mathcal{C}$ is an exact category and $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{C}$ such that any $\mathcal{X}-$epic in $\mathcal{C}$ is admissible in the sense of Quillen [40]. We prove that if $\mathcal{C}$ is $\mathcal{X}-$Gorenstein, then the stabilization of the left triangulated category $\mathcal{C}/\mathcal{X}$ is realized always by the stable category $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ of the $\mathcal{X}-$Gorenstein objects of $\mathcal{C}$. The category $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ can be thought of as a category of (relative) maximal Cohen-Macaulay objects, and using this category we define the notion of the $\mathcal{X}$**−Gorenstein dimension** of an object of $\mathcal{C}$. Then we prove that $\mathcal{C}$ is $\mathcal{X}-$Gorenstein iff any object of $\mathcal{C}$ has finite $\mathcal{X}-$Gorenstein dimension. In this case the category $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ is contravariantly finite in $\mathcal{C}$ and we give sufficient conditions for the existence of minimal $\mathcal{G}_{\mathcal{X}}(\mathcal{C})-$approximations. The results of this section are related to the fundamental work of Auslander-Buchweitz [4] on maximal Cohen-Macaulay approximations in an abelian category. If $\mathcal{C}$ is $\mathcal{X}-$Gorenstein, then all the results of Auslander-Buchweitz in [4] are valid in $\mathcal{C}$ and conversely if, roughly speaking, the Auslander-Buchweitz theory is true in $\mathcal{C}$, then $\mathcal{C}$ is $\mathcal{X}-$Gorenstein. The crucial points of the Auslander-Buchweitz theory have been recently formulated by M. Hashimoto in the concept of an **Auslander-Buchweitz context** or

AB-context for short, which is a triple of full subcategories $(\mathcal{A}, \mathcal{B}, \mathcal{X})$ of $\mathcal{C}$ satisfying certain properties [30]. We define AB-contexts relative to $\mathcal{X}$ and we prove that the exact category $\mathcal{C}$ is $\mathcal{X}$−Gorenstein iff $\mathcal{X}$ is the base of a relative AB-context. We characterize also when the exact category $\mathcal{C}$ is $\mathcal{X}$−Co-Gorenstein and we prove that in many cases any $\mathcal{X}$−Gorenstein category is $\mathcal{X}$−Co-Gorenstein.

Inspired by the definition of the stable homotopy groups in Algebraic Topology [24], we introduce in section 5 the concepts of **complete $\mathcal{X}$−extension functors** and **complete $\mathcal{X}$−resolutions** of objects of $\mathcal{C}$, for the pair $(\mathcal{C}, \mathcal{X})$ mentioned above. The complete $\mathcal{X}$−extension functors can be regarded as generalized Tate-Vogel cohomology functors and the complete $\mathcal{X}$−resolutions as generalized Tate-Vogel resolutions. The main result of section 5 shows that $\mathcal{C}$ is $\mathcal{X}$−Gorenstein iff any object has a complete $\mathcal{X}$−resolution, and in this case we can compute the complete $\mathcal{X}$−extension functors via complete $\mathcal{X}$−resolutions. These results generalize the results of Gendrich-Gruenberg [25], Cornick-Kropholler [18], Mislin [39] and Avramov-Buchweitz-Martsinkovsky-Reiten [17] concerning complete projective or injective resolutions and complete extension functors.

The theory developed in sections 4, 5, indicate that the concept of a Gorenstein category which is defined using universal properties of stable categories, unifies the concepts: AB-context, global existence of complete resolutions, global existence of complete extension functors, finiteness of Gorenstein-dimension, and appears to be the natural setting for the study of stable phenomena in module theory.

In section 6 we apply our results to module categories. If Λ is an associative ring, then we denote by $\mathrm{Mod}(\Lambda)$, resp. $\mathrm{mod}(\Lambda)$, the category of all, resp. finitely presented, right Λ−modules and by $\mathbf{P}_\Lambda$, resp, $\mathbf{I}_\Lambda$, the full subcategory of projective, resp. injective, modules. Choosing $\mathcal{C} = \mathrm{Mod}(\Lambda)$ and $\mathcal{X} = \mathbf{P}_\Lambda$, or $\mathcal{X} = \mathbf{I}_\Lambda$, we show that $\mathrm{Mod}(\Lambda)$ is $\mathbf{P}_\Lambda$−Gorenstein iff $\mathrm{Mod}(\Lambda)$ is $\mathbf{I}_\Lambda$−Gorenstein iff any projective right module has finite injective dimension and any injective right module has finite projective dimension. We call these rings **right Gorenstein rings**. It is easy to see that QF-rings or rings with finite right global dimension are right Gorenstein. It turns out that a Noetherian right Gorenstein ring is left Gorenstein and this class of rings coincides with the class introduced by Iwanaga [35] and studied by Enochs-Jenda et al., in a long series of papers (see for instance [22], [20]). The classical example of a Gorenstein ring is a local Noetherian ring of finite selfinjective dimension [3]. In case of Artin algebras this class of rings coincides with the class of Gorenstein algebras introduced by Auslander-Reiten [5] using tilting theory and studied also by Happel [32] using derived categories. Our theory has as corollaries the corresponding results of these papers, is valid for all modules not only finitely generated, and can be applied also to (Gorenstein) Orders. D. Happel considered in [32] a certain Verdier quotient $\mathcal{D}_{\mathcal{P}}$ of the bounded derived category $\mathcal{D}^b(\Lambda)$ of an Artin algebra Λ, and he computed this quotient in case Λ is Gorenstein. He says that the computation of $\mathcal{D}_{\mathcal{P}}$ is hard in general. It turns out that $\mathcal{D}_{\mathcal{P}}$ is the stabilization of the stable category $\underline{\mathrm{mod}}(\Lambda)$ modulo projectives and an easy computation of $\mathcal{D}_{\mathcal{P}}$ is available only in case Λ is Gorenstein or equivalently if $\mathrm{Mod}(\Lambda)$ is a $\mathbf{P}_\Lambda$−Gorenstein category. We close the paper applying our previous results to the study of some of the homological conjectures for Artin algebras, and we prove some reductions.

Some of the results of this paper were obtained indepedently by L.L Avramov, R.O. Buchweitz, A. Martsinkovsky and I. Reiten [17]. A general convention used in the paper is that we compose morphisms in a category in the diagrammatic order: the composition of $f : A \to B$ and $g : B \to C$ is denoted by $f \circ g$.

2. Relative Homology and Stable Categories

Troughout this section we fix a pair $(\mathcal{C}, \mathcal{X})$, where $\mathcal{C}$ is an additive category and $\mathcal{X} \subseteq \mathcal{C}$ is a full additive subcategory of $\mathcal{C}$ which is closed under direct summands and isomorphisms. First we recall some notions of relative homological algebra extracted from [8], [15], [23]

A morphism $f : A \to B$ in $\mathcal{C}$ is called *$\mathcal{X}$-epic* if the induced morphism $\mathcal{C}(\mathcal{X}, f) : \mathcal{C}(\mathcal{X}, A) \to \mathcal{C}(\mathcal{X}, B)$ in $\mathcal{A}b$ is epic, and f is called *$\mathcal{X}$-monic* if the induced morphism $\mathcal{C}(f, \mathcal{X}) : \mathcal{C}(B, \mathcal{X}) \to \mathcal{C}(A, \mathcal{X})$ in $\mathcal{A}b$ is epic. A morphism $\chi_A : X_A \to A$ is a *right $\mathcal{X}$-approximation* of A [8], if χ_A is an $\mathcal{X}$-epic and $X_A \in \mathcal{X}$. Dually a morphism $\chi^A : A \to X^A$ is a *left $\mathcal{X}$-approximation* of A if χ^A is an $\mathcal{X}$-monic and $X^A \in \mathcal{X}$. The subcategory $\mathcal{X}$ is *contravariantly finite* (*covariantly finite*) [8], if any object of $\mathcal{C}$ has a right (left) $\mathcal{X}$-approximation. Finally $\mathcal{X}$ is *functorially finite* if $\mathcal{X}$ is covariantly and contravariantly finite in $\mathcal{C}$.

Consider a complex $A^\bullet : \cdots \to A_{i+1} \to A_i \to A_{i-1} \to \cdots$ in $\mathcal{C}$. The complex $A^\bullet$ is called *covariantly $\mathcal{X}$-exact*, if the induced complex $\mathcal{C}(\mathcal{X}, A^\bullet) : \cdots \to \mathcal{C}(\mathcal{X}, A_{i+1}) \to \mathcal{C}(\mathcal{X}, A_i) \to \mathcal{C}(\mathcal{X}, A_{i-1}) \to \cdots$ is exact in $\mathcal{A}b$. Dually the complex $A^\bullet$ is *contravariantly $\mathcal{X}$-exact*, if the induced complex $\mathcal{C}(A^\bullet, \mathcal{X}) : \cdots \to \mathcal{C}(A_{i-1}, \mathcal{X}) \to \mathcal{C}(A_i, \mathcal{X}) \to \mathcal{C}(A_{i+1}, \mathcal{X}) \to \cdots$ is exact in $\mathcal{A}b$. The complex $A^\bullet$ is *functorially $\mathcal{X}$-exact*, if $A^\bullet$ is contravariantly $\mathcal{X}$-exact and covariantly $\mathcal{X}$-exact. Using these notions we can define $\mathcal{X}$-resolutions and $\mathcal{X}$-coresolutions of objects of $\mathcal{C}$. If $A \in \mathcal{C}$, then an *$\mathcal{X}$-resolution* of A is a covariantly $\mathcal{X}$-exact complex $X_A^\bullet : \cdots \to X_1 \to X_0 \to A \to 0$, where $X_n \in \mathcal{X}, \forall n \geq 0$. Then A has *finite contravariant $\mathcal{X}$-dimension* if there exists an $\mathcal{X}$-resolution of A of the form $0 \to X_n \to \cdots \to X_0 \to A \to 0$. In this case we write $\mathcal{X}\text{-dim}A \leq n$. The least such integer n is the *contravariant $\mathcal{X}$-dimension* of A and is denoted by $\mathcal{X}\text{-dim}A$. The *global contravariant $\mathcal{X}$-dimension* of $\mathcal{C}$ is defined by $\mathcal{X}\text{-gl.dim}\mathcal{C} := sup\{\mathcal{X}\text{-dim}A; A \in \mathcal{C}\}$. Dually if $A \in \mathcal{C}$, then an *$\mathcal{X}$-coresolution* of A is a contravariantly $\mathcal{X}$-exact complex $X_\bullet^A : 0 \to A \to X^0 \to X^1 \to \cdots$, where $X^n \in \mathcal{X}, \forall n \geq 0$. The object A has *finite covariant $\mathcal{X}$-dimension* if there exists an $\mathcal{X}$-coresolution of A of the form $0 \to A \to X^0 \to X^1 \to \cdots \to X^n \to 0$. In this case we write $\mathcal{X}\text{-co.dim}A \leq n$. The *covariant $\mathcal{X}$-dimension*, $\mathcal{X}\text{-co.dim}A$, of A is the least such integer n. The *global covariant $\mathcal{X}$-dimension* of $\mathcal{C}$ is defined by $\mathcal{X}\text{-gl.co.dim}\mathcal{C} := sup\{\mathcal{X}\text{-co.dim}A; A \in \mathcal{C}\}$.

The following is a direct consequence of the definitions.

Proposition 2.1. (1) *$\mathcal{X}$ is coreflective in $\mathcal{C}$ iff* $\mathcal{X}\text{-gl.dim}\mathcal{C} = 0$.
(2) *$\mathcal{X}$ is reflective in $\mathcal{C}$ iff* $\mathcal{X}\text{-gl.co.dim}\mathcal{C} = 0$.

We denote by $\mathcal{C}/\mathcal{X}$ the stable category of $\mathcal{C}$ with respect to the subcategory $\mathcal{X}$. We recall that the objects of $\mathcal{C}/\mathcal{X}$ are the objects of $\mathcal{C}$. If A, B are objects of $\mathcal{C}$, then $\mathcal{C}/\mathcal{X}(A, B)$ is the factor group $\mathcal{C}(A, B)/\mathcal{C}_\mathcal{X}(A, B)$, where $\mathcal{C}_\mathcal{X}(A, B)$ is the subgroup of morphisms factorizing through an object of $\mathcal{X}$. If $A \in \mathcal{C}$, then we denote by $\underline{A}$ the same object considered as an object of $\mathcal{C}/\mathcal{X}$, and if $f : A \to B$ is a morphism in $\mathcal{C}$, then we denote by $\underline{f}$ the residue class of f in $\mathcal{C}/\mathcal{X}(A,B)$. Setting $\varpi(A) = \underline{A}$ and $\varpi(f) = \underline{f}$, we obtain the additive *projection* functor $\varpi : \mathcal{C} \to \mathcal{C}/\mathcal{X}$.

2.1. **Contravariant Finiteness.** Suppose that $\mathcal{X}$ is contravariant finite and any $\mathcal{X}$-epic has a kernel in $\mathcal{C}$. Then any object in $\mathcal{C}$ has an $\mathcal{X}$-resolution. Indeed let $\chi_A : X_A \to A$ be a right $\mathcal{X}$-approximation of A, let $k_A^1 : K_A^1 \to X_A$ be the kernel of χ_A, and let $\chi_A^1 : X_A^1 \to K_A^1$ be a right $\mathcal{X}$-approximation of K_A^1. Setting $f_A^1 = \chi_A^1 \circ k_A^1 : X_A^1 \to X_A^0$ and continuing in this way we obtain a complex

$$\mathbf{X}_A^\bullet \quad \cdots \to X_A^{i+1} \xrightarrow{f_A^{i+1}} X_A^i \to \cdots \to X_A^1 \xrightarrow{f_A^1} X_A^0 \xrightarrow{\chi_A} A \to 0$$

which is an $\mathcal{X}$-resolution of A. A *deleted $\mathcal{X}$-resolution* of A is an $\mathcal{X}$-resolution as above with A deleted. The objects K_A^n are called the n^{th} *– $\mathcal{X}$-sysygy* objects of A with respect to the $\mathcal{X}$-resolution $\mathbf{X}_A^\bullet$.

Now let $F : \mathcal{C} \to \mathcal{A}$ and $G : \mathcal{C}^{op} \to \mathcal{A}$ be additive functors with values in an abelian category $\mathcal{A}$. Then as in [15], we can define the *left $\mathcal{X}$-derived functor $\mathcal{L}_n^{\mathcal{X}}F$* of F and the *right $\mathcal{X}$-derived functor $\mathcal{R}_{\mathcal{X}}^nG$* of G:

$$\mathcal{L}_n^{\mathcal{X}}F : \mathcal{C} \to \mathcal{A} \quad \text{and} \quad \mathcal{R}_{\mathcal{X}}^nG : \mathcal{C}^{op} \to \mathcal{A}, \ \ \forall n \geq 0$$

as follows. If $\mathbf{X}_A^\bullet$ is a deleted $\mathcal{X}$-resolution of A, then $\mathcal{L}_n^{\mathcal{X}}F(A) := H_n(F(\mathbf{X}_A^\bullet))$ and $\mathcal{R}_{\mathcal{X}}^nG(A) := H^n(G(\mathbf{X}_A^\bullet))$. Then $\forall B \in \mathcal{C}$, the *contravariant $\mathcal{X}$-extension functors*

$$\underline{\mathcal{E}xt}_{\mathcal{X}}^n(-,B) : \mathcal{C}^{op} \to \mathcal{A}b, \ \ \forall n \geq 0$$

are defined as the right $\mathcal{X}$-derived functor of $\mathcal{C}(-,B)$. Similarly for any object $C \in \mathcal{C}$, we have the left $\mathcal{X}$-derived functors

$$\mathcal{L}_n^{\mathcal{X}}(C,-) : \mathcal{C} \to \mathcal{A}b, \ \ \forall n \geq 0$$

defined as the left $\mathcal{X}$-derived functors of $\mathcal{C}(C,-)$. In particular there are natural morphisms $\phi_{C,-} : \mathcal{L}_0^{\mathcal{X}}(C,-) \to \mathcal{C}(C,-)$ and $\psi_{-,B} : \mathcal{C}(-,B) \to \underline{\mathcal{E}xt}_{\mathcal{X}}^n(-,B)$, $\forall B, C \in \mathcal{C}$. Moreover if $0 \to A_1 \to A_2 \to A_3 \to 0$ is a contravariantly $\mathcal{X}$-exact complex, then $\forall B, C$ in $\mathcal{C}$, we have long exact sequences:

$$0 \to \underline{\mathcal{E}xt}_{\mathcal{X}}^0(A_3,B) \to \underline{\mathcal{E}xt}_{\mathcal{X}}^0(A_2,B) \to \underline{\mathcal{E}xt}_{\mathcal{X}}^0(A_1,B) \to \underline{\mathcal{E}xt}_{\mathcal{X}}^1(A_3,B) \to \cdots \quad (1)$$

$$\cdots \to \mathcal{L}_1^{\mathcal{X}}(C,A_3) \to \mathcal{L}_0^{\mathcal{X}}(C,A_1) \to \mathcal{L}_0^{\mathcal{X}}(C,A_2) \to \mathcal{L}_0^{\mathcal{X}}(C,A_3) \to 0 \quad (2)$$

Remark 2.2. If $\mathcal{X}$ is contravariantly finite and any $\mathcal{X}$-epic has a kernel in $\mathcal{C}$, then the stable category $\mathcal{C}/\mathcal{X}$ has a natural left triangulated structure $(\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$, where $\Omega_{\mathcal{X}} : \mathcal{C}/\mathcal{X} \to \mathcal{C}/\mathcal{X}$ is the loop functor and $\Delta_{\mathcal{X}}$ is the triangulation. By construction $\Omega_{\mathcal{X}}(\underline{A}) = \underline{K}_A^1$, where K_A^1 is the kernel of a right $\mathcal{X}$-approximation of A and the triangulation $\Delta_{\mathcal{X}}$ consists of all diagrams $\Omega_{\mathcal{X}}(\underline{C}) \to \underline{A} \to \underline{B} \to \underline{C}$ which are induced by $\mathcal{X}$-exact sequences $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$, where $g = ker(f)$. See [14] for details. We consider always the stable category $\mathcal{C}/\mathcal{X}$ as a left triangulated category with the described left triangulation.

Lemma 2.3. *For any object $C \in \mathcal{C}$, there are isomorphisms:*

$$\mathcal{L}_n^{\mathcal{X}}(C,-) \xrightarrow{\cong} \mathcal{C}/\mathcal{X}(\underline{C}, \Omega_{\mathcal{X}}^{n+1}(-)), \ \ \forall n \geq 1$$

and an exact sequence, with $Im(\phi_{C,-}) = \mathcal{C}_{\mathcal{X}}(C,-)$:

$$0 \to \mathcal{C}/\mathcal{X}(\underline{C}, \Omega_{\mathcal{X}}(-)) \to \mathcal{L}_0^{\mathcal{X}}(C,-) \xrightarrow{\phi_{C,-}} \mathcal{C}(C,-) \to \mathcal{C}/\mathcal{X}(\underline{C},-) \to 0.$$

Proof. Consider the $\mathcal{X}$−resolution $\mathbf{X}_A^{\bullet}$ of A as above. Then by definition we have $\mathcal{L}_1^{\mathcal{X}}(C,A) = Ker\mathcal{C}(C,f_A^1)/Im\mathcal{C}(C,f_A^2)$. Let $a : C \to X_A^1$ be a morphism with $a \circ f_A^1 = 0$. Then $a \circ \chi_A^1 \circ k_A^1 = 0 \Rightarrow a \circ \chi_A^1 = 0 \Rightarrow \exists! b : C \to K_A^2$ such that $b \circ k_A^2 = a$. Define a morphism $\rho : \mathcal{L}_1^{\mathcal{X}}(C,A) \to \mathcal{C}/\mathcal{X}(\underline{C}, \Omega_{\mathcal{X}}^2(\underline{A}))$ by $\rho(a) = \underline{b}$. It is easy to see that ρ is a well-defined isomorphism. The general case follows by dimension shifting. Now from the exact sequence $\mathcal{C}(C,X_A^1) \xrightarrow{\mathcal{C}(C,f_A^1)} \mathcal{C}(C,X_A^0) \xrightarrow{c} \mathcal{L}_0^{\mathcal{X}}(C,A) \to 0$, there exists a unique morphism $\phi_{C,A} : \mathcal{L}_0^{\mathcal{X}}(C,A) \to \mathcal{C}(C,A)$ such that $c \circ \phi_{C,A} = \mathcal{C}(C,\chi_A)$. The existence of the desired exact sequence follows by a simple diagram-chasing argument in the following diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{C}(C,X_A^1) & \xrightarrow{\mathcal{C}(C,f_A^1)} & \mathcal{C}(C,X_A^0) & \xrightarrow{c} & \mathcal{L}_0^{\mathcal{X}}(C,A) & \longrightarrow & 0 \\
 & & \exists ! w \downarrow & & \| \downarrow & & \phi_{C,A} \downarrow & & \\
0 & \longrightarrow & Ker\mathcal{C}(C,\chi_A) & \longrightarrow & \mathcal{C}(C,X_A^0) & \xrightarrow{\mathcal{C}(C,\chi_A)} & \mathcal{C}(C,A) & &
\end{array}$$

□

2.2. **Covariant Finiteness.** Suppose that $\mathcal{X}$ is covariant finite and any $\mathcal{X}$−monic has a cokernel in $\mathcal{C}$. Then any object in $\mathcal{C}$ has an $\mathcal{X}$−coresolution (see [15]). Let $F : \mathcal{C} \to \mathcal{A}$ and $G : \mathcal{C}^{op} \to \mathcal{A}$ be additive functors with values in an abelian category $\mathcal{A}$. Then as in subsection 2.1, we can define the $\mathcal{X}$−derived functors of F, G:

$$\mathcal{R}_{\mathcal{X}}^n F : \mathcal{C} \to \mathcal{A} \quad \text{and} \quad \mathcal{L}_n^{\mathcal{X}} G : \mathcal{C}^{op} \to \mathcal{A}, \ \ \forall n \geq 0$$

as follows. If $\mathbf{X}_{\bullet}^A$ is a deleted $\mathcal{X}$−coresolution of A, then $\mathcal{R}_{\mathcal{X}}^n F(A) := H^n(F(\mathbf{X}_{\bullet}^A))$ and $\mathcal{L}_n^{\mathcal{X}} G(A) := H_n(G(\mathbf{X}_{\bullet}^A))$. Hence $\forall B \in \mathcal{C}$, the *covariant* $\mathcal{X}$*−extension functors*

$$\overline{\mathcal{E}xt}_{\mathcal{X}}^n(A,-) : \mathcal{C} \to \mathcal{A}b, \ \ \forall n \geq 0$$

are defined as the right $\mathcal{X}$−derived functor of $\mathcal{C}(A,-)$. Similarly for any object $C \in \mathcal{C}$, we have the left $\mathcal{X}$−derived functors of $\mathcal{C}(-,C)$ denoted by

$$\mathcal{L}_n^{\mathcal{X}}(-,C) : \mathcal{C}^{op} \to \mathcal{A}b, \ \ \forall n \geq 0.$$

In particular there are natural morphisms $\phi_{-,C} : \mathcal{L}_n^{\mathcal{X}}(-,C) \to \mathcal{C}(-,C)$ and $\psi_{A,-} : \mathcal{C}(A,-) \to \overline{\mathcal{E}xt}_{\mathcal{X}}^n(A,-)$, $\forall A, C \in \mathcal{C}$. Moreover if $0 \to B^1 \to B^2 \to B^3 \to 0$ is a covariantly $\mathcal{X}$−exact complex, then $\forall A, C \in \mathcal{C}$, there are long exact sequences:

$$0 \to \overline{\mathcal{E}xt}_{\mathcal{X}}^0(A,B^1) \to \overline{\mathcal{E}xt}_{\mathcal{X}}^0(A,B^2) \to \overline{\mathcal{E}xt}_{\mathcal{X}}^0(A,B^3) \to \overline{\mathcal{E}xt}_{\mathcal{X}}^1(A,B^1) \to \cdots \quad (3)$$

$$\cdots \to \mathcal{L}_1^{\mathcal{X}}(B^1,C) \to \mathcal{L}_0^{\mathcal{X}}(B^3,C) \to \mathcal{L}_0^{\mathcal{X}}(B^2,C) \to \mathcal{L}_0^{\mathcal{X}}(B^1,C) \to 0 \quad (4)$$

By [14], under the above assumptions the stable category $\mathcal{C}/\mathcal{X}$ has a natural right triangulated structure $(\mathcal{C}/\mathcal{X}, \Sigma_{\mathcal{X}}, \nabla_{\mathcal{X}})$, where $\Sigma_{\mathcal{X}} : \mathcal{C}/\mathcal{X} \to \mathcal{C}/\mathcal{X}$ is the loop functor and $\nabla_{\mathcal{X}}$ is the triangulation, see [14] for details. The dual of Lemma 2.3 also holds:

Lemma 2.4. *For any object $C \in \mathcal{C}$, there are isomorphisms:*

$$\mathcal{L}_n^{\mathcal{X}}(-,C) \xrightarrow{\cong} \mathcal{C}/\mathcal{X}(\Sigma_{\mathcal{X}}^{n+1}(-), \underline{C}), \ \ \forall n \geq 1$$

and an exact sequence, with $Im(\phi_{-,C}) = \mathcal{C}_{\mathcal{X}}(-,C)$*:*

$$0 \to \mathcal{C}/\mathcal{X}(\Sigma_{\mathcal{X}}(-),\underline{C}) \to \mathcal{L}_0^{\mathcal{X}}(-,C) \xrightarrow{\phi_{-,C}} \mathcal{C}(-,C) \to \mathcal{C}/\mathcal{X}(-,\underline{C}) \to 0.$$

2.3. **Functorial Finiteness.** Suppose now that $\mathcal{X}$ is a functorially finite subcategory of $\mathcal{C}$, any $\mathcal{X}$−epic has a kernel and any $\mathcal{X}$−monic has a cokernel in $\mathcal{C}$. Then by the above observations, the stable category $\mathcal{C}/\mathcal{X}$ admits a left traingulated structure $(\mathcal{C}/\mathcal{X},\Omega_{\mathcal{X}},\Delta_{\mathcal{X}})$ and a right triangulated structure $(\mathcal{C}/\mathcal{X},\Sigma_{\mathcal{X}},\nabla_{\mathcal{X}})$. Moreover any object of $\mathcal{C}$ admits an $\mathcal{X}$−resolution and an $\mathcal{X}$−coresolution, and for any objects $A, C \in \mathcal{C}$ the derived functors $\mathcal{L}_n^{\mathcal{X}}(-,C), \mathcal{L}_n^{\mathcal{X}}(C,-), \underline{\mathcal{E}xt}_{\mathcal{X}}^n(-,A), \overline{\mathcal{E}xt}_{\mathcal{X}}^n(A,-)$ are defined, $\forall n \geq 0$. We note that in general the left and right triangulated structures in the stable category $\mathcal{C}/\mathcal{X}$ are not the same and $\mathcal{C}/\mathcal{X}$ is not necessarily triangulated. Indeed if Λ is an Artin algebra, then the full subcategory $\mathcal{P}_\Lambda$ of finitely generated projective right Λ−modules, satisfies all the above assumptions. Hence the stable category $\underline{\mathrm{mod}}(\Lambda)$ modulo projectives is left and right triangulated. But $\underline{\mathrm{mod}}(\Lambda)$ is triangulated iff Λ is selfinjective. However we have the following.

Proposition 2.5. *The pair $(\Sigma_{\mathcal{X}},\Omega_{\mathcal{X}})$ is an adjoint pair in the stable category $\mathcal{C}/\mathcal{X}$. Hence there exist a natural isomorphism:*

$$\mathcal{C}/\mathcal{X}[\Sigma_{\mathcal{X}}(?),-] \xrightarrow{\cong} \mathcal{C}/\mathcal{X}[?,\Omega_{\mathcal{X}}(-)].$$

Proof. Let A be an arbitrary object of $\mathcal{C}$, and consider the $\mathcal{X}$−exact sequence $0 \to K_A^1 \xrightarrow{k_A^1} X_A^0 \xrightarrow{\chi_A} A$, where χ_A is a right $\mathcal{X}$−approximation of A and $k_A^1 = ker(\chi_A)$. Let $\chi^{K_A^1} : K_A^1 \to X^{K_A^1}$ be a left $\mathcal{X}$−approximation of A with cokernel $l_1^{K_A^1} : X^{K_A^1} \to L_1^{K_A^1}$. Then in $\mathcal{C}/\mathcal{X}$ we have $\Omega_{\mathcal{X}}(\underline{A}) = \underline{K}_A^1$ and $\Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}(\underline{A}) = \underline{L}_1^{K_A^1}$. Since $\chi^{K_A^1}$ is a left $\mathcal{X}$−approximation, there exists a morphism $m : X^{K_A^1} \to X_A^0$ such that $\chi^{K_A^1} \circ m = k_A^1$ and a commutative diagram:

$$\begin{tikzcd}
 & K_A^1 \arrow[r, "\chi^{K_A^1}"] \arrow[d, "\parallel"'] & X^{K_A^1} \arrow[r, "l_1^{K_A^1}"] \arrow[d, "\exists m"'] & L_1^{K_A^1} \arrow[r] \arrow[d, "\exists! e"'] & 0 \\
0 \arrow[r] & K_A^1 \arrow[r, "k_A^1"] & X_A^0 \arrow[r, "\chi_A"] & A. &
\end{tikzcd}$$

In this way we obtain a morphism $\varepsilon_{\underline{A}} := \underline{e} : \Sigma_{\mathcal{X}}\Omega_{\mathcal{X}}(\underline{A}) \to \underline{A}$ in $\mathcal{C}/\mathcal{X}$. Dually we construct a morphism $\delta_{\underline{A}} : \underline{A} \to \Omega_{\mathcal{X}}\Sigma_{\mathcal{X}}(\underline{A})$ in $\mathcal{C}/\mathcal{X}$. We leave to the reader to check that ε is the counit and δ is the unit of an adjoint pair $(\Sigma_{\mathcal{X}},\Omega_{\mathcal{X}})$ in $\mathcal{C}/\mathcal{X}$. □

Let $\Omega_{\mathcal{X}}^n(\mathcal{C})$, $\Sigma_{\mathcal{X}}^n(\mathcal{C})$ be the full additive subcategories of $\mathcal{C}$ generated by $\mathcal{X}$ and the n^{th}−syzygies, n^{th}−cosyzygies, with respect to $\mathcal{X}$−resolutions, $\mathcal{X}$−coresolutions of objects of $\mathcal{C}$. The next two results are direct consequences of Proposition 2.5.

Corollary 2.6. (1) *$\Omega_{\mathcal{X}}^n(\mathcal{C}/\mathcal{X})$ is a reflective subcategory of $\mathcal{C}/\mathcal{X}$ and $\Sigma_{\mathcal{X}}^n(\mathcal{C}/\mathcal{X})$ is a coreflective subcategory of $\mathcal{C}/\mathcal{X}$, $\forall n \geq 0$.*

(2) *$\Omega_{\mathcal{X}}^n(\mathcal{C})$ is a covariantly finite subcategory of $\mathcal{C}$ and $\Sigma_{\mathcal{X}}^n(\mathcal{C}/\mathcal{X})$ is a contravariantly finite subcategory of $\mathcal{C}/\mathcal{X}$, $\forall n \geq 0$.*

Corollary 2.7. *$\forall A, B \in \mathcal{C}$ there are isomorphisms:*

$$\mathcal{L}_n^{\mathcal{X}}(-,B)(A) \xrightarrow{\cong} \mathcal{L}_n^{\mathcal{X}}(A,-)(B), \ \forall n \geq 1.$$

The next result gives some sufficient conditions, for the coincidence of the contravariant and covariant $\mathcal{X}$−extension functors. The proof is the same as in the classical case, using a simple spectral sequence argument.

Proposition 2.8. *If* $\overline{\mathcal{E}xt}^n_{\mathcal{X}}(\mathcal{X}, B) = \underline{\mathcal{E}xt}^n_{\mathcal{X}}(A, \mathcal{X}) = 0, \forall n \geq 1$ *and the canonical morphisms* $\psi_{A,\mathcal{X}} : \mathcal{C}(A, \mathcal{X}) \to \underline{\mathcal{E}xt}^0_{\mathcal{X}}(A, \mathcal{X}), \psi_{\mathcal{X},B} : \mathcal{C}(\mathcal{X}, B) \to \overline{\mathcal{E}xt}^0_{\mathcal{X}}(\mathcal{X}, B)$ *are invertible, then:* $\overline{\mathcal{E}xt}^n_{\mathcal{X}}(A, -)(B) \cong \underline{\mathcal{E}xt}^n_{\mathcal{X}}(-, B)(A)$, $\forall n \geq 0$.

2.4. **Stable Triangulated Categories.** We close this section studying when a stable category is triangulated. For the theory of (left or right) triangulated categories and exact functors we refer to [14], [31], [36], [45].

Definition 2.9. The category $\mathcal{X}$ is called an $\mathcal{X}-$*cogenerator* of $\mathcal{C}$ if for any $A \in \mathcal{C}$, there exists an $\mathcal{X}-$epic $f : X \to B$ with $X \in \mathcal{X}$ such that $\underline{A} \cong \underline{Ker(f)}$ in $\mathcal{C}/\mathcal{X}$.

Dually $\mathcal{X}$ is called an $\mathcal{X}-$*generator* of $\mathcal{C}$ if for any $B \in \mathcal{C}$, there exists an $\mathcal{X}-$monic $f : A \to X$ with $X \in \mathcal{X}$ such that $\underline{B} \cong \underline{Coker(f)}$ in $\mathcal{C}/\mathcal{X}$.

If $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$epic has a kernel, the functors $\underline{\mathcal{E}xt}^n_{\mathcal{X}}(-, A)$ are defined, and if $\mathcal{X}$ is covariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$monic has a cokernel, the functors $\overline{\mathcal{E}xt}^n_{\mathcal{X}}(A, -)$ are defined, for any object A of $\mathcal{C}$. In these cases we define the left $\mathcal{X}-$orthogonal subcategory ${}^\perp\mathcal{X}$ and the right $\mathcal{X}-$orthogonal subcategory $\mathcal{X}^\perp$ of $\mathcal{X}$ as follows:

$$ {}^\perp\mathcal{X} = \{A \in \mathcal{C} : \ \underline{\mathcal{E}xt}^n_{\mathcal{X}}(A, \mathcal{X}) = 0, \ \forall n \geq 1 \text{ and } \psi_{A,\mathcal{X}} : \mathcal{C}(A, \mathcal{X}) \xrightarrow{\cong} \underline{\mathcal{E}xt}^0_{\mathcal{X}}(A, \mathcal{X})\}, $$

$$ \mathcal{X}^\perp = \{B \in \mathcal{C} : \ \overline{\mathcal{E}xt}^n_{\mathcal{X}}(\mathcal{X}, B) = 0, \ \forall n \geq 1 \text{ and } \psi_{\mathcal{X},B} : \mathcal{C}(\mathcal{X}, B) \xrightarrow{\cong} \overline{\mathcal{E}xt}^0_{\mathcal{X}}(\mathcal{X}, B)\}. $$

Definition 2.10. Let $\mathcal{A}, \mathcal{B}, \mathcal{X}$ be full subcategories of $\mathcal{C}$.

(α) If any $\mathcal{X}-$epic has a kernel in $\mathcal{C}$, then $\mathcal{A}$ is called $\mathcal{X}-$*resolving*, if $\mathcal{X} \subseteq \mathcal{A}$ and $\mathcal{A}$ is closed under kernels of $\mathcal{X}-$epics. Moreover if $A \xrightarrow{g} B \xrightarrow{f} C$ is a diagram in $\mathcal{C}$, where f is $\mathcal{X}-$epic, $g = ker(f)$ and $A, C \in \mathcal{A}$, then $B \in \mathcal{A}$.

(β) If any $\mathcal{X}-$monic has a cokernel in $\mathcal{C}$, then $\mathcal{B}$ is called $\mathcal{X}-$*coresolving*, if $\mathcal{X} \subseteq \mathcal{B}$ and $\mathcal{B}$ is closed under cokernels of $\mathcal{X}-$monics. Moreover if $A \xrightarrow{f} B \xrightarrow{g} C$ is a diagram in $\mathcal{C}$, where f is $\mathcal{X}-$monic, $g = coker(f)$ and $A, C \in \mathcal{B}$, then $B \in \mathcal{B}$.

If $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$epic in $\mathcal{C}$ has a kernel, then for any $\mathcal{X}-$resolving subcategory $\mathcal{A}$, the stable category $\mathcal{A}/\mathcal{X}$ is a full left triangulated subcategory of $\mathcal{C}/\mathcal{X}$. Dually if $\mathcal{X}$ is covariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$monic in $\mathcal{C}$ has a cokernel, then for any $\mathcal{X}-$coresolving subcategory $\mathcal{B}$, the stable category $\mathcal{B}/\mathcal{X}$ is a full right triangulated subcategory of $\mathcal{C}/\mathcal{X}$ [14]. For the notions of an exact category, admissible epic, monic, short exact sequence, we refer to [40].

Theorem 2.11. *Let $\mathcal{X}$ be a full subcategory of an exact category $\mathcal{C}$.*

1. *Suppose that $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$epic is an admissible epic. If $\mathcal{A}$ is an $\mathcal{X}-$resolving subcategory of $\mathcal{C}$, the following are equivalent:*
   (α) *$\mathcal{A}/\mathcal{X}$ is a triangulated subcategory of $(\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$.*
   (β) *$\mathcal{A} \subseteq {}^\perp\mathcal{X}$ and $\mathcal{X}$ is an $\mathcal{X}-$cogenerator of $\mathcal{A}$.*
2. *Suppose that $\mathcal{X}$ is covariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$monic is an admissible monic. If $\mathcal{B}$ is an $\mathcal{X}-$coresolving subcategory of $\mathcal{C}$, the following are equivalent:*
   (α) *$\mathcal{B}/\mathcal{X}$ is a triangulated subcategory of $(\mathcal{C}/\mathcal{X}, \Sigma_{\mathcal{X}}, \nabla_{\mathcal{X}})$.*

(β) $\mathcal{B} \subseteq \mathcal{X}^{\perp}$ *and* $\mathcal{X}$ *is an* $\mathcal{X}-$*generator of* $\mathcal{B}$.

(3) *If the assumptions in* (1), (2) *are true for* $\mathcal{A} = \mathcal{B}$, *then the triangulated structures* $(\mathcal{A}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$ *and* $(\mathcal{A}/\mathcal{X}, \Sigma_{\mathcal{X}}, \nabla_{\mathcal{X}})$ *on* $\mathcal{A}/\mathcal{X}$ *coincide:* $\Omega_{\mathcal{X}} = \Sigma_{\mathcal{X}}^{-1}$ *and*

$$\Omega_{\mathcal{X}}(\underline{C}) \xrightarrow{h} \underline{A} \xrightarrow{g} \underline{B} \xrightarrow{f} \underline{C} \in \Delta_{\mathcal{X}} \Leftrightarrow \underline{A} \xrightarrow{g} \underline{B} \xrightarrow{f} \underline{C} \xrightarrow{\Sigma_{\mathcal{X}}(\underline{h})} \Sigma_{\mathcal{X}}(\underline{A}) \in \nabla_{\mathcal{X}}.$$

Proof. (1) By the above remarks $\mathcal{C}/\mathcal{X}$ carries a left triangulated structure and $\mathcal{A}/\mathcal{X}$ is a full left triangulated subcategory of $\mathcal{C}/\mathcal{X}$. An easy modification of Theorem 3.3 of [1] in our setting, shows that the loop functor $\Omega_{\mathcal{X}}$ is fully faithful in $\mathcal{A}/\mathcal{X}$ iff $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$. Trivially $\Omega_{\mathcal{X}}$ is surjective on objects iff $\mathcal{X}$ is an $\mathcal{X}-$cogenerator of $\mathcal{A}$. (2) Follows by duality, and (3) follows from (1), (2) and Proposition 2.5. □

Suppose again that $\mathcal{X}$ is a full subcategory of $\mathcal{C}$. We define some classes of objects in $\mathcal{C}$, which will play an important role in the next sections. First let $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ be the full subcategory of $\mathcal{C}$ of all objects of finite contravariant $\mathcal{X}-$dimension.

Definition 2.12. (1) $C \in \mathcal{C}$ is called **$\mathcal{X}-$stable** if C has a functorially $\mathcal{X}-$exact resolution.

(2) $C \in \mathcal{C}$ is called **$\mathcal{X} - n-$torsion free object**, $n \geq 0$, if there exists a functorially $\mathcal{X}-$exact sequence $0 \to C \to X^0 \to \cdots \to X^n$, with $X^i \in \mathcal{X}$, $0 \leq i \leq n$.

(3) $C \in \mathcal{C}$ is called **$\mathcal{X}-$torsion free**, if C is $\mathcal{X} - n-$torsion free, $\forall n \geq 0$, i.e. if C has a functorially $\mathcal{X}-$exact coresolution.

(4) $C \in \mathcal{C}$ is called an **$\mathcal{X}-$Gorenstein object** if C is $\mathcal{X}-$stable and $\mathcal{X}-$torsion free. Equivalently C has a functorially $\mathcal{X}-$exact resolution and coresolution.

We denote by $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ the full subcategory of $\mathcal{C}$ consisting of all $\mathcal{X}-$Gorenstein objects. By definition $\mathcal{X}$ is a functorially finite subcategory of $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$. Observe that if $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$epic has a kernel, then C is $\mathcal{X}-$stable iff $\underline{\mathcal{E}xt}^i_{\mathcal{X}}(C, \mathcal{X}) = 0, \forall i \geq 1$. Hence if any $\mathcal{X}-$epic is epic, then C is $\mathcal{X}-$stable iff $C \in {}^{\perp}\mathcal{X}$. The category of arbitrary $\mathcal{X}-$syzygy objects is defined by $\Omega^{\infty}_{\mathcal{X}}(\mathcal{C}) = \bigcap_{n \geq 1} \Omega^n_{\mathcal{X}}(\mathcal{C})$. Dually if $\mathcal{X}$ is covariantly finite in $\mathcal{C}$ and any $\mathcal{X}-$monic has a cokernel, then C is $\mathcal{X}-$torsion free iff $\overline{\mathcal{E}xt}^i_{\mathcal{X}}(\mathcal{X}, C) = 0, \forall i \geq 1$. Hence if any $\mathcal{X}-$monic is monic, then C is $\mathcal{X}-$torsion free iff $C \in \mathcal{X}^{\perp}$. The category of arbitrary $\mathcal{X}-$cosyzygy objects $\Sigma^{\infty}_{\mathcal{X}}(\mathcal{C})$ is defined similarly. The final result of this section shows that in many cases, the category $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ of $\mathcal{X}-$Gorenstein objects is the largest $\mathcal{X}-$resolving subcategory of $\mathcal{C}$, such that the stable category $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ is a full triangulated subcategory of $\mathcal{C}/\mathcal{X}$. First we recall [33] that an exact category $\mathcal{E}$ is called *Frobenius*, if $\mathcal{E}$ has enough projectives and injectives and the projectives coincide with the injectives.

Proposition 2.13. *Let* $\mathcal{C}$ *be an exact category and* $\mathcal{X}$ *a full subcategory of* $\mathcal{C}$.

(1) *If* $\mathcal{X}$ *is contravariantly finite, any* $\mathcal{X}-$*epic is an admissible epic and any left* $\mathcal{X}-$*approximation of an* $\mathcal{X}$*-Gorenstein object is an admissible monic, then* $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ *is an* $\mathcal{X}-$*resolving Frobenius exact subcategory of* $\mathcal{C}$ *and the stable category* $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ *is a full triangulated subcategory of* $(\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$.

Moreover if $\mathcal{A}$ *is a subcategory of* $\mathcal{C}$, *such that* $\mathcal{A}/\mathcal{X}$ *is a triangulated subcategory of* $\mathcal{C}/\mathcal{X}$, *then* $\mathcal{A}$ *is* $\mathcal{X}-$*resolving and* $\mathcal{A} \subseteq \mathcal{G}_{\mathcal{X}}(\mathcal{C})$.

(2) *If* $\mathcal{X}$ *is covariantly finite, any* $\mathcal{X}-$*monic is an admissible monic and any right* $\mathcal{X}-$*approximation of an* $\mathcal{X}$*-Gorenstein object is an admissible epic, then* $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ *is*

an $\mathcal{X}$−coresolving Frobenius exact subcategory of $\mathcal{C}$ and the stable category $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ is a full triangulated subcategory of $(\mathcal{C}/\mathcal{X}, \Sigma_{\mathcal{X}}, \nabla_{\mathcal{X}})$.

Moreover if $\mathcal{B}$ is a subcategory of $\mathcal{C}$, such that $\mathcal{B}/\mathcal{X}$ is a triangulated subcategory of $\mathcal{C}/\mathcal{X}$, then $\mathcal{B}$ is $\mathcal{X}$−coresolving and $\mathcal{B} \subseteq \mathcal{G}_{\mathcal{X}}(\mathcal{C})$.

Proof. (1) Since any $\mathcal{X}$−epic is admissible epic, it has a kernel in $\mathcal{C}$ so $\mathcal{C}/\mathcal{X}$ is left triangulated. Since any $\mathcal{X}$−epic is epic, we have $\mathcal{C}(-,B) = \underline{\mathcal{E}xt}^0_{\mathcal{X}}(-,B)$. If A is $\mathcal{X}$−Gorenstein, then consider a covariantly $\mathcal{X}$−exact $\mathcal{X}$−coresolution $0 \to A \xrightarrow{\alpha_0} X^0 \xrightarrow{\alpha_1} X^1 \to \cdots$ of A. Since this coresolution is contravariantly $\mathcal{X}$−exact, α_0 is a left $\mathcal{X}$−approximation, so by hypothesis there exists an admisssible exact sequence $0 \to A \xrightarrow{\alpha_0} X^0 \xrightarrow{\kappa} A^1 \to 0$, and a factorization $\alpha_1 = \kappa \circ \lambda$, where $\lambda : A^1 \to X^1$. Similarly λ is a left $\mathcal{X}$−approximation of A^1, so it is admissible monic. Inductively we see easily that the objects A^n are $\mathcal{X}$−Gorenstein and the above $\mathcal{X}$−coresolution of A is a Yoneda composition of admissible and functorially $\mathcal{X}$−exact sequences $0 \to A^n \to X^n \to A^{n+1} \to 0$, $A^0 = A$. Suppose now that $(\dagger) : A \xrightarrow{g} B \xrightarrow{f} C$ is a sequence in $\mathcal{C}$ with $A, C \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$, f is an $\mathcal{X}$−epic and $g = ker(f)$, so $(\dagger)$ is admissible and covariantly $\mathcal{X}$−exact. Then we have the long exact sequence: $0 \to \mathcal{C}(C,X) \to \mathcal{C}(B,X) \to \mathcal{C}(A,X) \to \underline{\mathcal{E}xt}^1_{\mathcal{X}}(C,X) \to \cdots$ which implies that $B \in {}^{\perp}\mathcal{X}$ and $(\dagger)$ is contravariantly $\mathcal{X}$−exact, since A, C are $\mathcal{X}$−Gorenstein. Moreover using that $A, C \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$, we can construct by standard arguments a covariantly $\mathcal{X}$−exact $\mathcal{X}$−coresolution of B, from the covariantly $\mathcal{X}$−exact $\mathcal{X}$−coresolutions of A, C. This shows that $B \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. Let $(\dagger) : A \xrightarrow{g} B \xrightarrow{f} C$ be a sequence in $\mathcal{C}$ with $B, C \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$, f is an $\mathcal{X}$−epic and $g = ker(f)$, so $(\dagger)$ is admissible and covariantly $\mathcal{X}$−exact. As above we see that $A \in {}^{\perp}\mathcal{X}$ and $(\dagger)$ is contravariantly $\mathcal{X}$−exact. Consider the functorially $\mathcal{X}$−exact admissible sequence $0 \to B \xrightarrow{\alpha} X^0 \xrightarrow{\beta} B^1 \to 0$ which starts a covariantly $\mathcal{X}$−exact $\mathcal{X}$−coresolution of B. Then we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "\|"'] & B \arrow[r, "f"] \arrow[d, "\alpha"'] & C \arrow[r] \arrow[d, "\delta"'] & 0 \\
0 \arrow[r] & A \arrow[r, "\phi"] & X^0 \arrow[r, "\psi"] & D \arrow[r] & 0
\end{tikzcd}$$

and an admissible covariantly $\mathcal{X}$−exact sequence $0 \to C \xrightarrow{\delta} D \xrightarrow{\epsilon} B^1 \to 0$. Since $C, B^1 \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$, we have that $D \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. Since, as easily seen, the sequence $0 \to A \xrightarrow{\phi} X^0 \xrightarrow{\psi} D \to 0$ is functorially $\mathcal{X}$−exact and D is an $\mathcal{X}$−Gorenstein object, we infer that A is an $\mathcal{X}$−Gorenstein object. Hence $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ is an $\mathcal{X}$−resolving subcategory of $\mathcal{C}$. We leave to the reader to show that $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ is actually a Frobenius subcategory of $\mathcal{C}$. By Theorem 2.11, $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ is a full triangulated subcategory of $\mathcal{C}/\mathcal{X}$, since by definition $\mathcal{X}$ is an $\mathcal{X}$−cogenerator of $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ and $\mathcal{G}_{\mathcal{X}}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{X}$.

If $\mathcal{A} \subseteq \mathcal{C}$ is such that $\mathcal{A}/\mathcal{X}$ is a triangulated subcategory of $\mathcal{C}/\mathcal{X}$, then it is not difficult to see that $\mathcal{A}$ is $\mathcal{X}$−resolving using that the inclusion $\mathcal{A}/\mathcal{X} \hookrightarrow \mathcal{C}/\mathcal{X}$ is exact. Then by Theorem 2.11, $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$, so any object of $\mathcal{A}$ has a contravariantly $\mathcal{X}$−exact resolution. Since $\mathcal{X}$ is a cogenerator of $\mathcal{A}$, any object of $\mathcal{A}$ has a covariantly $\mathcal{X}$−exact coresolution. We conclude that $\mathcal{A} \subseteq \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. The proof of part (2) is dual. □

3. Stable Categories and (Co)Stabilization

In this section we associate to a fixed left triangulated category $\mathcal{C}$ two triangulated categories: $\mathcal{S}(\mathcal{C})$, the stabilization of $\mathcal{C}$ and $\mathcal{R}(\mathcal{C})$, the costabilization of $\mathcal{C}$. The category $\mathcal{S}(\mathcal{C})$ is the universal triangulated category for exact functors starting at $\mathcal{C}$ and $\mathcal{R}(\mathcal{C})$ is the universal triangulated category for exact functors ending at $\mathcal{C}$. The existence of $\mathcal{S}(\mathcal{C})$ is a result of Heller [33], see also [24], [36], and the existence of $\mathcal{R}(\mathcal{C})$ is due to Grandis [27]. Both existence results were inspired by well-known constructions in Algebraic Topology [38], and they are very useful tools for the study of stable categories. We define a *looped* category to be a pair $(\mathcal{A},\Omega)$ where $\mathcal{A}$ is an additive category and $\Omega : \mathcal{A} \to \mathcal{A}$ is an additive functor. If $(\mathcal{A},\Omega),(\mathcal{B},\Sigma)$ are looped categories, then a *stable* functor $(\mathcal{A},\Omega) \to (\mathcal{B},\Sigma)$ is a pair (F,ϕ) where $F : \mathcal{A} \to \mathcal{B}$ is an additive functor and $\phi : \Sigma F \xrightarrow{\cong} F\Omega$ is a natural isomorphism.

3.1. Stabilization.

Let $\mathcal{C} = (\mathcal{C},\Omega,\Delta)$ be a left triangulated category.

Definition 3.1. The **stabilization** of $\mathcal{C}$ is a pair $(\mathbf{S},\mathcal{S}(\mathcal{C}))$, where $\mathcal{S}(\mathcal{C})$ is a triangulated category and $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$ is an exact functor, the **stabilization functor**, such that for any exact functor $F : \mathcal{C} \to \mathcal{D}$ to a triangulated category $\mathcal{D}$, there exists a unique exact functor $F^* : \mathcal{S}(\mathcal{C}) \to \mathcal{D}$ such that: $F^*\mathbf{S} = F$.

We recall the construction of $\mathcal{S}(\mathcal{C})$ from [33], which consists of formally inverting the endofunctor Ω. An object of $\mathcal{S}(\mathcal{C})$ is a pair (A,n) where $A \in \mathcal{C}$ and $n \in \mathbb{Z}$. If $n,m \in \mathbb{Z}$, then we consider the directed set $I_{n,m} = \{k \in \mathbb{Z} : k \geq n, k \geq m\}$. The space of morphisms between $(A,n),(B,m) \in \mathcal{S}(\mathcal{C})$ is defined by

$$\mathcal{S}(\mathcal{C})[(A,n),(B,m)] = \varinjlim_{k\in I_{n,m}} \mathcal{C}(\Omega^{k-n}(A),\Omega^{k-m}(B)).$$

Then $\mathcal{S}(\mathcal{C})$ is an additive category and there exists an equivalence $\tilde{\Omega} : \mathcal{S}(\mathcal{C}) \to \mathcal{S}(\mathcal{C})$ defined as follows: $\tilde{\Omega}(A,n) = (A,n-1)$ and if $\tilde{f} : (A,n) \to (B,m)$ then choose a representative $f_k : \Omega^{k-n}(A) \to \Omega^{k-m}(B)$ where $k \in I_{n,m}$ and define $\tilde{\Omega}(f)$ to be the class of f_{k-1} in $\mathcal{S}(\mathcal{C})[(A,n-1),(B,m-1)]$. The inverse of $\tilde{\Omega}$ is defined by $\tilde{\Omega}^{-1}(A,n) = (A,n+1)$. There exists a natural additive functor $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$ defined as follows: $\mathbf{S}(A) = (A,0)$, and if $f : A \to B$ is a morphism in $\mathcal{C}$ then $\mathbf{S}(f)$ is defined by the zero-representative of f. The functor $\mathbf{S}$ is a stable functor, i.e. there exists a natural isomorphism $\omega : \tilde{\Omega}\mathbf{S}(?) = (?,-1) \cong (\Omega(?),0) = \mathbf{S}\Omega(?)$, and the pair $(\mathbf{S},\mathcal{S}(\mathcal{C}))$ has the following universal property. If $(\mathcal{D},\Sigma)$ is a looped category with Σ a self equivalence of $\mathcal{D}$, and $F : \mathcal{C} \to \mathcal{D}$ is a stable functor, then there exists a unique stable functor $F^* : \mathcal{S}(\mathcal{C}) \to \mathcal{D}$ such that $F^*\mathbf{S} = F$. Indeed this follows directly by defining $F^*(A,n) = \Sigma^{-n}F(A)$.

Using the functor $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$ and the triangulation Δ of $\mathcal{C}$, we define a triangulation $\tilde{\Delta}$ of the pair $(\mathcal{S}(\mathcal{C}),\tilde{\Omega})$ as follows. A diagram $\tilde{\Omega}(C,l) \to (A,n) \to (B,m) \to (C,l)$ belongs to $\tilde{\Delta}$ if there exists $k \in 2\mathbb{Z}$ and a triangle of representatives $\Omega(\Omega^{k-l}(C)) \to \Omega^{k-n}(A) \to \Omega^{k-m}(B) \to \Omega^{k-l}(C)$ in $\mathcal{C}$. Then the triple $(\mathcal{S}(\mathcal{C}),\tilde{\Omega},\tilde{\Delta})$ is a triangulated category and has the required universal property of definition 3.1. In fact the functor $\mathbf{S}$ is exact and if $F : \mathcal{C} \to \mathcal{D}$ is an exact functor to a triangulated category $\mathcal{D}$, then the functor F^* defined above, is the unique exact functor which extends F through the stabilization functor $\mathbf{S}$.

Remark 3.2. In case $\mathcal{C}$ has a right triangulated structure $(\mathcal{C}, \Sigma, \nabla)$, then the above construction works, producing the stabilization $\mathcal{S}(\mathcal{C})$ of $\mathcal{C}$ which also is triangulated and satisfies the same universal property. The stabilization has the same objects as before but the space of morphisms is defined by the formula

$$\mathcal{S}(\mathcal{C})[(A,n),(B,m)] = \varinjlim_{k\in J_{n,m}} \mathcal{C}(\Sigma^{k+n}(A), \Sigma^{k+m}(B))$$

where $J_{n,m} = \{k \in \mathbb{Z} : k+n, k+m \geq 0\}$. In this case we invert the suspension functor Σ to obtain the equivalence $\tilde{\Sigma}$, where $\tilde{\Sigma}(A,n) = (A, n+1)$. As above the triangulation ∇ of $\mathcal{C}$ induces a triangulation $\tilde{\nabla}$ in the stabilization.

The following is a direct consequence of the construction.

Corollary 3.3. (1) *$\mathcal{C}$ is triangulated iff $\mathbf{S} : \mathcal{C} \xrightarrow{\approx} \mathcal{S}(\mathcal{C})$ is a triangle equivalence.*
(2) *$\mathcal{S}(\mathcal{C}) = 0$ iff $\forall A \in \mathcal{C}$, there exists $n_A \geq 0$, such that: $\Omega^{n_A}(A) = 0$.*
(3) *Consider a morphism $\tilde{f} : (A,n) \to (B,m)$ in $\mathcal{S}(\mathcal{C})$. Then $\tilde{f} = 0$ iff there exists a representative $f_k : \Omega^{k-n}(A) \to \Omega^{k-m}(B)$ of $\tilde{f}$ and $l \geq k$, such that: $\Omega^{l-k}(f_k) = 0$. Also $\tilde{f}$ is an isomorphism iff there exists a representative $f_k : \Omega^{k-n}(A) \to \Omega^{k-m}(B)$ of $\tilde{f}$ and $l \geq k$, such that: $\Omega^{l-k}(f_k)$ is an isomorphism.*
(4) *If $\mathcal{C}, \mathcal{D}$ are triangle equivalent, then $\mathcal{S}(\mathcal{C}), \mathcal{S}(\mathcal{D})$ are triangle equivalent.*

The next result is useful studying when an exact functor $\mathcal{C} \to \mathcal{D}$ to a triangulated category $\mathcal{D}$ extends to a triangle equivalence $F^* : \mathcal{S}(\mathcal{C}) \to \mathcal{D}$. For a proof see [44].

Proposition 3.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor to a triangulated category $\mathcal{D}$ with translation functor Σ, and let $F^* : \mathcal{S}(\mathcal{C}) \to \mathcal{D}$ be the unique exact functor extending F through the stabilization functor. Then:*

(1) *F^* is faithful iff for any morphism $f : A \to B$ in $\mathcal{C}$ such that $F(f) = 0$, there exists $k \geq 0$ such that: $\Omega^k(f) = 0$.*

(2) *F^* is full iff for any morphism $g : F(A) \to F(B)$ in $\mathcal{D}$, there exists $k \geq 0$ and a morphism $f : \Omega^k(A) \to \Omega^k(B)$, such that: $\Sigma^k(g) = F(f)$.*

(3) *F^* is surjective on objects iff for any object $D \in \mathcal{D}$, there exists $k \geq 0$ and an object $A \in \mathcal{C}$, such that: $\Sigma^k(D) = F(A)$.*

Corollary 3.5. *The functor $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$ is faithful (full, resp. dense) iff Ω is faithful (full, resp. dense). In case Ω is fully faithful then $\mathcal{S}(\mathcal{C})$ is the smallest triangulated category containing $\mathcal{C}$ as a full left triangulated subcategory.*

Example 3.6. Let $\mathcal{U}$ be an additive category and let $\mathcal{K}(\mathcal{U})$ be the (triangulated) homotopy category of complexes over $\mathcal{U}$. Let $\mathcal{K}^{0]}(\mathcal{U})$, $\mathcal{K}^{b,0]}(\mathcal{U})$ be the full subcategories of negative, negative and bounded below complexes respectively. These categories are right triangulated subcategories of $\mathcal{K}(\mathcal{U})$. A simple application of Proposition 3.4 shows that $\mathcal{S}(\mathcal{K}^{0]}(\mathcal{U})) = \mathcal{K}^-(\mathcal{U})$ and $\mathcal{S}(\mathcal{K}^{b,0]}(\mathcal{U})) = \mathcal{K}^b(\mathcal{U})$. Similar remarks are applied to the derived category, in case $\mathcal{U}$ is abelian or exact.

The next result shows that the Grothendieck group [15] is invariant under stabilization.

Proposition 3.7. *The stabilization functor $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$ induces an isomorphism*

$$\mathrm{K}_0(\mathbf{S}) : \mathrm{K}_0(\mathcal{C}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{S}(\mathcal{C})).$$

Proof. Since $\mathbf{S}$ is exact, we have the induced morphism $\mathrm{K}_0(\mathbf{S})$ which is defined by $\mathrm{K}_0(\mathbf{S})[A] = [(A,0)]$. Consider the function $\psi : Ob(\mathcal{S}(\mathcal{C})) \to \mathrm{K}_0(\mathcal{C})$ defined by $\psi(A,n) = (-1)^n[A]$. If $(A,n) \cong (B,m)$ then there exists $k \in I_{n,m}$ and an isomorphism $\Omega^{k-n}(A) \cong \Omega^{k-m}(B)$. Then in $\mathrm{K}_0(\mathcal{C})$ we have $(-1)^{k-n}[A] = [\Omega^{k-n}(A)] = [\Omega^{k-m}(B)] = (-1)^{k-m}[B]$, hence $\psi(A,n) = \psi(B,m)$. If $\tilde{\Omega}(C,l) \to (A,n) \to (B,m) \to (C,l)$ is a triangle in $\mathcal{S}(\mathcal{C})$, then by definition there exists a triangle $\Omega(\Omega^{k-l}(C)) \to \Omega^{k-n}(A) \to \Omega^{k-m}(B) \to \Omega^{k-l}(C)$ in $\mathcal{C}$. Then in $\mathrm{K}_0(\mathcal{C})$ we have $(-1)^{k-m}[B] = (-1)^{k-n}[A] + (-1)^{k-l}[C]$. This implies that $\psi(B,m) = \psi(A,n) + \psi(C,l)$. Hence there exists a unique group homomorphism $\phi : \mathrm{K}_0(\mathcal{S}(\mathcal{C})) \to \mathrm{K}_0(\mathcal{C})$, such that $\phi([A,n]) = \psi([(A,n)]) = (-1)^n[A]$. If A is an object of $\mathcal{C}$, then $\phi\mathrm{K}_0(\mathbf{S})([A]) = \phi([(A,0)]) = [A]$ and if (A,n) is an object of $\mathcal{S}(\mathcal{C})$, then $\mathrm{K}_0(\mathbf{S})\phi([(A,n)]) = \mathrm{K}_0(\mathbf{S})((-1)^n[A]) = (-1)^n[(A,0)]$. But $(-1)^n[(A,0)] = (-1)^{-n}[(A,0)] = [\tilde{\Omega}^{-n}(A,0)] = [(A,n)]$. So $\mathrm{K}_0(\mathbf{S})\phi([(A,n)]) = [(A,n)]$. This shows that $\mathrm{K}_0(\mathbf{S})$ is an isomorphism with inverse ϕ. □

Consider now an additive category $\mathcal{C}$ and a full additive contravariantly finite subcategory $\mathcal{X} \subseteq \mathcal{C}$, closed under direct summands and suppose throughout that any $\mathcal{X}$-epic has a kernel in $\mathcal{C}$. Our purpose is to compute the stabilization $\mathcal{S}(\mathcal{C}/\mathcal{X})$ of the left triangulated category $\mathcal{C}/\mathcal{X} = (\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$. Let $\mathcal{K}(\mathcal{X})$ be the unbounded homotopy category of complexes over $\mathcal{X}$ and let $\mathcal{K}^-(\mathcal{X})$ be the full subcategory consisting of bounded above complexes, where a complex $X^\bullet : \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots$ is bounded above if $X^k = 0$ for all sufficiently large $k > 0$. We call a complex $X^\bullet$ *n-acyclic*, if the morphism d^{n-1} admits a factorization $X^{n-1} \xrightarrow{\epsilon^{n-1}} Ker(d^n) \xrightarrow{ker(d^n)} X^n$, where ϵ^{n-1} is $\mathcal{X}$-epic. The complex $X^\bullet$ is called *acyclic*, if it is n-acyclic, for any $n \in \mathbb{Z}$. Let $\mathcal{K}^{-,b}(\mathcal{X})$ be the full subcategory of $\mathcal{K}^-(\mathcal{X})$ consisting of all complexes which are acyclic almost everywhere, i.e. except of a finite number of degrees. Then we have exact inclusions of triangulated categories $\mathcal{K}^b(\mathcal{X}) \hookrightarrow \mathcal{K}^{-,b}(\mathcal{X}) \hookrightarrow \mathcal{K}^-(\mathcal{X}) \hookrightarrow \mathcal{K}(\mathcal{X})$, where $\mathcal{K}^b(\mathcal{X})$ is the bounded homotopy category of complexes over $\mathcal{X}$. Obviously $\mathcal{K}^b(\mathcal{X})$ is closed under direct summands in $\mathcal{K}^{-,b}(\mathcal{X})$, hence the Verdier quotient $\mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^b(\mathcal{X})$ is defined and it is a triangulated category.

The following result generalizes (and is inspired by) a result of Keller-Vossieck (see [36], where the next result is proved for an exact category with enough injectives objects).

Theorem 3.8. *There exists a triangle equivalence*

$$\mathcal{S}(\mathcal{C}/\mathcal{X}) \cong \mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^b(\mathcal{X}).$$

Moreover $\mathrm{Ker}\mathbf{S} = \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ *and* $\mathcal{S}(\mathcal{C}/\mathcal{X}) = 0$ *iff* $\forall C \in \mathcal{C}$: $\mathcal{X}$-$\dim C < \infty$, *i.e.* $\mathcal{C} = \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$. *Finally* $\mathrm{Ker}\mathbf{S} = 0$ *iff* $\mathcal{X} = \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ *iff* $sup\{\mathcal{X}\text{-}\dim C : C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})\} = 0$.

Proof. Define a functor $F : \mathcal{C} \to \mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^b(\mathcal{X})$ as follows. If A is an object of $\mathcal{C}$, let $\mathbf{X}^\bullet_A$ be a deleted $\mathcal{X}$-resolution of A as in section 2. Hence $\mathbf{X}^\bullet_A : \cdots \to X^{i+1}_A \xrightarrow{f^{i+1}_A} X^i_A \to \cdots \to X^1_A \xrightarrow{f^1_A} X^0_A \to 0$, where X^i is in degree $-i$. We set $F(A) = Q(\mathbf{X}^\bullet_A)$, where $Q : \mathcal{K}^{-,b}(\mathcal{X}) \to \mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^b(\mathcal{X})$ is the quotient functor. Since any two $\mathcal{X}$-resolutions are homotopy equivalent, the functor F is well-defined. Obviously $F(\mathcal{X}) = 0$. Moreover if $C \xrightarrow{g} B \xrightarrow{f} A$ is a contravariantly $\mathcal{X}$-exact sequence in $\mathcal{C}$, then there exists a sequence $0 \to \mathbf{X}^\bullet_C \to \mathbf{X}^\bullet_B \to \mathbf{X}^\bullet_A \to 0$ of complexes which is split short exact in each degree. Hence we obtain a triangle $\mathbf{X}^\bullet_A[-1] \to \mathbf{X}^\bullet_C \to \mathbf{X}^\bullet_B \to \mathbf{X}^\bullet_A$ in $\mathcal{K}^{-,b}(\mathcal{X})$. Applying the exact quotient functor Q, we see that the contravariantly $\mathcal{X}-$exact sequence $C \to B \to A$ in $\mathcal{C}$ induces a triangle $F(A)[-1] \to F(C) \to F(B) \to F(A)$ in $\mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^b(\mathcal{X})$. By Theorem 2.2 of [12], there exists a unique exact functor $G : \mathcal{C}/\mathcal{X} \to \mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^b(\mathcal{X})$, such that $G\varpi = F$, where $\varpi : \mathcal{C} \to \mathcal{C}/\mathcal{X}$ is the projection functor. By the universal property of the stabilization $\mathcal{S}(\mathcal{C}/\mathcal{X})$, there exists a unique exact functor $G^* : \mathcal{S}(\mathcal{C}/\mathcal{X}) \to \mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^-(\mathcal{X})$, such that $G^*\mathbf{S} = G$. We claim that G^* is an equivalence. This is easy to see applying Proposition 3.4 and using the definition of the functor G, and the construction of the quotient $\mathcal{K}^{-,b}(\mathcal{X})/\mathcal{K}^b(\mathcal{X})$ in [45]. The last part is trivial. □

Corollary 3.9. *Let $\mathcal{C}$ be an abelian (or exact) category with enough projectives, and let $\mathcal{P}$ be the full subcategory of projective objects of $\mathcal{C}$.*

(1) *There exists a triangle equivalence*

$$\mathcal{S}(\mathcal{C}/\mathcal{P}) \approx \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P}).$$

$\mathcal{S}(\mathcal{C}/\mathcal{P}) = 0$ iff $\mathcal{D}^b(\mathcal{C}) = \mathcal{K}^b(\mathcal{P})$ iff any object of $\mathcal{C}$ has finite projective dimension.

(2) *There exists an isomorphism: $\mathrm{K}_0(\mathcal{C}/\mathcal{P}) \cong \mathrm{K}_0(\mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P}))$ such that the canonical morphisms $\mathrm{K}_0(\mathcal{P}, \oplus) \to \mathrm{K}_0(\mathcal{K}^b(\mathcal{P}))$ and $\mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{D}^b(\mathcal{C}))$ are embedded in the exact commutative diagram, where $c_{\mathcal{C}}, c^\bullet_{\mathcal{C}}$ are the Cartan morphisms:*

$$\begin{tikzcd}
Ker(c_{\mathcal{C}}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{K}_0(\mathcal{P},\oplus) \arrow[r, "c_{\mathcal{C}}"] \arrow[d, "\cong"'] & \mathrm{K}_0(\mathcal{C}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{K}_0(\mathcal{C}/\mathcal{P}) \arrow[r] \arrow[d, "\cong"'] & 0 \\
Ker(c^\bullet_{\mathcal{C}}) \arrow[r] & \mathrm{K}_0(\mathcal{K}^b(\mathcal{P})) \arrow[r, "c^\bullet_{\mathcal{C}}"] & \mathrm{K}_0(\mathcal{D}^b(\mathcal{C})) \arrow[r] & \mathrm{K}_0(\mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P})) \arrow[r] & 0
\end{tikzcd}$$

(3) *$\mathcal{C}$ is Frobenius iff there exists a triangle equivalence:*

$$\mathcal{C}/\mathcal{P} \approx \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P}).$$

Proof. (1) Follows from Thorem 3.8, since if $\mathcal{X} = \mathcal{P}$, then $\mathcal{K}^{-,b}(\mathcal{X}) = \mathcal{D}^b(\mathcal{C})$.

(2) By a well known result of Grothendieck, the central square of the above diagram commutes and the middle vertical arrows are invertible. Since the cokernel of $c_{\mathcal{C}}$ is the stable Grothendieck of $\mathcal{C}$ modulo projectives [15], the result follows from Proposition 3.7, [16] and part (1).

(3) It is well-known that if $\mathcal{C}$ is Frobenius, then $\mathcal{C}/\mathcal{P}$ is triangulated (see [31] or Theorem 2.11). Hence in this case $\mathcal{C}/\mathcal{P}$ is triangle equivalent to its stabilization, and the result follows from (1). If $\mathcal{C}/\mathcal{P}$ is triangle equivalent to $\mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P})$, then $\mathcal{C}/\mathcal{P}$ is triangulated. Then by section 2, we know that $\mathcal{C} = {}^\perp\mathcal{P}$ and $\mathcal{P}$ is a $\mathcal{P}-$cogenerator of $\mathcal{C}$. The fact that $\mathcal{C} = {}^\perp\mathcal{P}$ implies that any projective is injective. Since $\mathcal{P}$ is a $\mathcal{P}-$cogenerator of $\mathcal{C}$, for any object of $\mathcal{C}$, there exists a short exact sequence $0 \to C \to P \to D \to 0$ with $P \in \mathcal{P}$. In particular $\mathcal{C}$ has enough injectives and trivially any injective is projective. Hence $\mathcal{C}$ is Frobenius. □

3.2. **Costabilization.** Let $\mathcal{C} = (\mathcal{C}, \Omega, \Delta)$ be a left triangulated category.

Definition 3.10. The **costabilization** of $\mathcal{C}$ is a pair $(\mathbf{R}, \mathcal{R}(\mathcal{C}))$, where $\mathcal{R}(\mathcal{C})$ is a triangulated category and $\mathbf{R} : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ is an exact functor, the **costabilization**

functor, such that for any exact functor $F : \mathcal{D} \to \mathcal{C}$ from a triangulated category $\mathcal{D}$, there exists a unique exact functor $F^* : \mathcal{D} \to \mathcal{R}(\mathcal{C})$ such that: $\mathbf{R}F^* = F$.

We recall the construction of the pair $(\mathbf{R}, \mathcal{R}(\mathcal{C}))$ from [27], which consists of constructing formal Ω-spectra. An object of $\mathcal{R}(\mathcal{C})$ is a family (A_n, α_n) where $n \in \mathbb{Z}$ and $\alpha_n : A_n \xrightarrow{\cong} \Omega(A_{n+1})$ is an isomorphism $\forall n \in \mathbb{Z}$. A morphism $f_\bullet : (A_n, \alpha_n) \to (B_n, \beta_n)$ is a family $f_\bullet = (f_n)$, where $f_n : A_n \to B_n$ is a morphism in $\mathcal{C}$ such that the following diagram commutes, $\forall n \in \mathbb{Z}$:

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & B_n \\ {\scriptstyle \alpha_n}\downarrow & & \downarrow{\scriptstyle \beta_n} \\ \Omega(A_{n+1}) & \xrightarrow{\Omega(f_{n+1})} & \Omega(B_{n+1}). \end{array}$$

Then $\mathcal{R}(\mathcal{C})$ is an additive category and defining $\hat{\Omega}(A_n, \alpha_n) = (B_n, \beta_n)$ where $B_n = A_{n-1}$ and $\beta_n = \alpha_{n-1}$ and for a morphism $f_\bullet : (A_n, \alpha_n) \to (B_n, \beta_n)$ in $\mathcal{R}(\mathcal{C})$, $\hat{\Omega}(f_\bullet) = g_\bullet$, where $g_n = f_{n-1}$, we obtain an equivalence $\hat{\Omega} : \mathcal{R}(\mathcal{C}) \to \mathcal{R}(\mathcal{C})$. Hence the pair $(\mathcal{R}(\mathcal{C}), \hat{\Omega})$ is a looped category. Now define an additive functor $\mathbf{R} : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ by $\mathbf{R}(A_n, \alpha_n) = A_0$ and $\mathbf{R}(f_\bullet) = f_0$. The functor $\mathbf{R}$ is stable since $\mathbf{R}\hat{\Omega}(A_n, \alpha_n) = A_{-1} \cong \Omega(A_0) = \Omega\mathbf{R}(A_n, \alpha_n)$, by the isomorphism α_{-1}. The pair $(\mathbf{R}, \mathcal{R}(\mathcal{C}))$ has the following universal property. If $(\mathcal{D}, \Sigma)$ is a looped category where Σ is a self-equivalence of $\mathcal{D}$, and $F : \mathcal{D} \to \mathcal{C}$ is a stable functor, then there exists a unique stable functor $F^* : \mathcal{D} \to \mathcal{R}(\mathcal{C})$ with $\mathbf{R}F^* = F$. Indeed define $F^*(D) = (D_n, d_n)$ where $D_n = F\Sigma^{-n}(D)$ and d_n is the isomorphism $D_n = F\Sigma^{-n}(D) = F\Sigma\Sigma^{-n-1}(D) \cong \Omega F\Sigma^{-n-1}(D) = \Omega(D_{n+1})$. Then $\mathbf{R}F^*(D) = D_0 = F(D)$, and F^* is obviously the unique stable functor lifting F through the costabilization functor $\mathbf{R}$.

Using the functor $\mathbf{R} : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ and the triangulation Δ of $\mathcal{C}$, we define a triangulation $\hat{\Delta}$ in the looped category $(\mathcal{R}(\mathcal{C}), \hat{\Omega})$ as follows. A diagram $\hat{\Omega}(C_n, \gamma_n) \xrightarrow{(h_n)} (A_n, \alpha_n) \xrightarrow{(g_n)} (B_n, \beta_n) \xrightarrow{(f_n)} (C_n, \gamma_n)$ belongs to $\hat{\Delta}$ if for any $n \in \mathbb{Z}$ there are triangles $\Omega(C_n) \xrightarrow{\gamma_{n-1}^{-1} \circ h_n} A_n \xrightarrow{g_n} B_n \xrightarrow{f_n} C_n$ in Δ. Then the triple $(\mathcal{R}(\mathcal{C}), \hat{\Omega}, \hat{\Delta})$ is a triangulated category, the functor $\mathbf{R} : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ is exact and if $\mathcal{D}$ is a triangulated category with translation functor Σ, and $F : \mathcal{D} \to \mathcal{C}$ is an exact functor, then the functor F^* defined above is the unique exact functor lifting F through $\mathbf{R}$. In case $\mathcal{C}$ has a right triangulation, then the above construction with the necessary modifications also works, producing the costabilization of $\mathcal{C}$ which also is triangulated and satisfies the same universal property. We leave to the reader to state and prove the analogous results of Corollaries 3.3, 3.5 and Proposition 3.4.

Consider now the pair $(\mathcal{C}, \mathcal{X})$ where $\mathcal{C}$ is an additive category and $\mathcal{X} \subseteq \mathcal{C}$ is a full additive contravariantly finite subcategory of $\mathcal{C}$, closed under direct summands and suppose that any $\mathcal{X}$-epic has a kernel in $\mathcal{C}$. Our purpose is to compute the costabilization $\mathcal{R}(\mathcal{C}/\mathcal{X})$ of the left triangulated category $\mathcal{C}/\mathcal{X} = (\mathcal{C}/\mathcal{X}, \Omega_{\mathcal{X}}, \Delta_{\mathcal{X}})$. Let $\mathcal{K}(\mathcal{X})$ be the unbounded homotopy category over $\mathcal{X}$ and let $\mathcal{K}_{Ac}(\mathcal{X})$ be the full subcategory of acyclic complexes, as defined in Subsection 3.1. Clearly $\mathcal{K}_{Ac}(\mathcal{X})$ is a full triangulated subcategory of $\mathcal{K}(\mathcal{X})$.

Theorem 3.11. *There exists a triangle equivalence:*

$$\mathcal{K}_{Ac}(\mathcal{X}) \approx \mathcal{R}(\mathcal{C}/\mathcal{X}).$$

Moreover $\mathrm{Im}(\mathbf{R}) = \Omega^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$. *Hence:* $\mathrm{Im}(\mathbf{R}) = 0$ *iff* $\mathcal{R}(\mathcal{C}/\mathcal{X}) = 0$ *iff* $\mathcal{X} = \Omega^{\infty}_{\mathcal{X}}(\mathcal{C})$.

Proof. Define a functor $F : \mathcal{K}_{Ac}(\mathcal{X}) \to \mathcal{R}(\mathcal{C}/\mathcal{X})$ as follows. Let $X^{\bullet}$ be an acyclic complex of $\mathcal{X}$-objects. Then by definition, $\forall n \in \mathbb{Z}$, there are sequences $A_{n-1} \xrightarrow{\mu_{n-1}} X^n \xrightarrow{\varepsilon_n} A_n$, where ε_n is $\mathcal{X}$-epic and $\mu_{n-1} = ker(\varepsilon_n)$, such that the differential $d^n = \varepsilon_n \circ \mu_n : X^n \to X^{n+1}$. We set $F(X^{\bullet}) = (\underline{A}_n, \underline{\alpha}_n)$, where $\underline{\alpha}_n : \underline{A}_n \cong \Omega_{\mathcal{X}}(\underline{A}_{n+1})$ is the natural identification. Let $f_{\bullet} : X^{\bullet} \to Y^{\bullet}$ be a morphism, and let $B_{n-1} \xrightarrow{\nu_{n-1}} Y^n \xrightarrow{\zeta_n} B_n$ be sequences as above, where ζ_n is $\mathcal{X}$-epic and $\nu_{n-1} = ker(\zeta_n)$, such that the differential $d^n = \zeta_n \circ \nu_n : Y^n \to Y^{n+1}$. Since the morphisms μ_{t-1}, ν_{t-1} are monics, we have that $\mu_{t-1} : A_{t-1} \to X^t$ is the kernel of $d^t_{X^{\bullet}} : X^t \to X^{t+1}$ and similarly $\nu_{t-1} : B_{t-1} \to Y^{t+1}$ is the kernel of $d^t_{Y^{\bullet}} : Y^t \to Y^{t+1}$. Hence there exists a unique morphism $\rho_{t-1} : A_{t-1} \to B_{t-1}$ with $\rho_{t-1} \circ \nu_{t-1} = \mu_{t-1} \circ f_t$. The family of morphisms $\rho_t : A_t \to B_t$ has obviously the property $\rho_t \circ \nu_t = \mu_t \circ f_{t+1}$ and $f_t \circ \zeta_t = \varepsilon_t \circ \rho_t$, $\forall t \in \mathbb{Z}$. This means that $\underline{\alpha}_t \circ \Omega_{\mathcal{X}}(\underline{\rho}_{t+1}) = \underline{\rho}_t \circ \underline{\beta}_t$, $\forall t \in \mathbb{Z}$. Hence the family $\rho_{\bullet} : (\underline{A}_t, \underline{\alpha}_t) \to (\underline{B}_t, \underline{\beta}_t)$ is a morphism in $\mathcal{R}(\mathcal{C}/\mathcal{X})$. We set $F(f_{\bullet}) = \rho_{\bullet}$. It is easy to see that in this way we obtain an exact functor $F : \mathcal{K}_{Ac}(\mathcal{X}) \to \mathcal{R}(\mathcal{C}/\mathcal{X})$. We leave to the reader the easy proof that F is an equivalence. The proof of the last assertion is trivial. □

Corollary 3.12. *Let $\mathcal{C}$ be an exact category with enough projectives, and let $\mathcal{P}$ be the full subcategory of projective objects. Then there is a triangle equivalence*

$$\mathcal{R}(\mathcal{C}/\mathcal{P}) \approx \mathcal{K}_{Ac}(\mathcal{P})$$

and $\mathcal{R}(\mathcal{C}/\mathcal{P}) = 0$ *iff* $\mathcal{P} = \Omega^{\infty}(\mathcal{C})$, *i.e. the only arbitrary syzygy objects of $\mathcal{C}$ are the projectives. In particular $\mathcal{C}$ is Frobenius iff there exists a triangle equivalence*

$$\mathcal{C}/\mathcal{P} \approx \mathcal{K}_{Ac}(\mathcal{P}).$$

We leave to the reader to state and prove the dual results concerning (co-)stabilizations of right triangulated stable categories $\mathcal{C}/\mathcal{X}$, induced by covariantly finite subcategories $\mathcal{X}$ in $\mathcal{C}$, such that any $\mathcal{X}$-monic has a cokernel in $\mathcal{C}$. For example stable categories modulo injectives of exact categories with enough injectives.

3.3. **(Co-)Gorenstein Left Triangulated Categories.** Throughout this subsection $\mathcal{C}$ will denote a left triangulated category $(\mathcal{C}, \Omega, \Delta)$. Our purpose here is to examine when the (co-)stabilization of $\mathcal{C}$ can be realized as a full subcategory of $\mathcal{C}$. We shall obtain more complete results in the next section when the left triangulated category $\mathcal{C}$ is a stable category. We denote as always by $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$ the stabilization functor and by $\mathbf{R} : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ the costabilization functor.

Definition 3.13. (1) The left triangulated category $\mathcal{C}$ is called **Gorenstein** if there exists a full left triangulated subcategory $\mathcal{V}$ of $\mathcal{C}$, such that the composite functor $\mathbf{S}i_{\mathcal{V}} : \mathcal{V} \xrightarrow{i_{\mathcal{V}}} \mathcal{C} \xrightarrow{\mathbf{S}} \mathcal{S}(\mathcal{C})$ is a triangle equivalence, where $i_{\mathcal{V}} : \mathcal{V} \hookrightarrow \mathcal{C}$ is the inclusion. In this case we say that $\mathcal{V}$ *realizes the stabilization* of $\mathcal{C}$.

(2) The left triangulated category $\mathcal{C}$ is called **Co-Gorenstein** if there exists a full left triangulated subcategory $\mathcal{U}$ of $\mathcal{C}$, such that the inclusion $i_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{C}$ is the costabilization functor. In this case we say that $\mathcal{U}$ *realizes the costabilization* of $\mathcal{C}$.

The easy proof of the next Lemma is left to the reader (use Proposition 3.4).

Lemma 3.14. *If $i_{\mathcal{V}} : \mathcal{V} \hookrightarrow \mathcal{C}$ is a left triangulated subcategory of $\mathcal{C}$, then the loop functor $\Omega : \mathcal{V} \to \mathcal{V}$ is fully faithful $\Leftrightarrow$ the functor $\mathbf{S}i_{\mathcal{V}} : \mathcal{V} \to \mathcal{S}(\mathcal{C})$ is fully faithful.*

Theorem 3.15. *The following are equivalent:*

(1) *$\mathcal{C}$ is Gorenstein.*
(2) *There exists a full triangulated subcategory $\mathcal{V}$ of $\mathcal{C}$, such that: $\forall C \in \mathcal{C}$ and $n \in \mathbb{Z}$, there exists $t \geq n, 0$ with $\Omega^{t-n}(C) \in \mathcal{V}$.*
(3) *There exists a full triangulated subcategory $\mathcal{V}$ of $\mathcal{C}$, such that: $\forall C \in \mathcal{C}$, there exists $t \geq 0$ with $\Omega^t(C) \in \mathcal{V}$.*

In this case the triangulated category $\mathcal{V}$ is uniquely determined up to a triangle equivalence and realizes the stabilization of $\mathcal{C}$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathcal{C}$ is Gorenstein, and let $i_{\mathcal{V}} : \mathcal{V} \hookrightarrow \mathcal{C}$ be a full triangulated subcategory of $\mathcal{C}$ realizing the stabilization. Let C be in $\mathcal{C}$ and $n \in \mathbb{Z}$. Consider the object $(C, n) \in \mathcal{S}(\mathcal{C})$. Since $\mathbf{S}i_{\mathcal{V}}$ is dense, there exists an object $A \in \mathcal{V}$ and an isomorphism $\tilde{f} : (C, n) \to \mathbf{S}i_{\mathcal{V}}(A) = (A, 0)$, with inverse $\tilde{g} : (A, 0) \to (C, n)$. Choose representatives $f_k : \Omega^{k-n}(C) \to \Omega^k(A)$ of $\tilde{f}$ with $k \geq 0, n$ and $g_l : \Omega^l(A) \to \Omega^{l-n}(C)$ of $\tilde{g}$ with $l \geq 0, n$. Analyzing the relations $\tilde{g} \circ \tilde{f} = 1_{(C,n)}$ and $\tilde{f} \circ \tilde{g} = 1_{(A,0)}$ and choosing $t \geq k, l$, we see that $\Omega^{t-k}(f_k) : \Omega^{t-n}(C) \to \Omega^t(A)$ is invertible with inverse $\Omega^{t-l}(g_l)$. Since $\Omega^t(A) \in \mathcal{V}$, the assertion (2) follows. The direction (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Since $\mathcal{V}$ is a triangulated subcategory, by the above Lemma the exact functor $\mathbf{S}i_{\mathcal{V}} : \mathcal{V} \to \mathcal{S}(\mathcal{C})$ is fully faithful. Let (C, n) be an arbitrary object of $\mathcal{S}(\mathcal{C})$. Then by hypothesis there exists $t \geq 0$ such that $\Omega^t(C) := A \in \mathcal{V}$. Applying the functor $\mathbf{S}$, we have $\mathbf{S}(\Omega^t(C)) = \mathbf{S}(A) \Rightarrow \mathbf{S}(C) = \mathbf{S}\Omega^{-t}(A)$. Since $A \in \mathcal{V}$ and $\mathcal{V}$ is triangulated, we can write $\Omega^{-t}(A) = \Omega^r(B)$ with $B \in \mathcal{V}$ and $r \geq 0, n$. Then $(C, n) = \tilde{\Omega}^{-n}\mathbf{S}(C) = \tilde{\Omega}^{-n}\mathbf{S}(\Omega^{-t}(A)) = \tilde{\Omega}^{-n}\mathbf{S}(\Omega^r(B)) = \tilde{\Omega}^{-n}\tilde{\Omega}^r\mathbf{S}(B) = \tilde{\Omega}^{r-n}\mathbf{S}(B) = \mathbf{S}\Omega^{r-n}(B)$. Since B is in $\mathcal{V}$, this shows that $\mathbf{S}i_{\mathcal{V}}$ is dense. □

We denote the kernel of the stabilization functor $\mathbf{S} : \mathcal{C} \to \mathcal{S}(\mathcal{C})$, by

$$\mathcal{P}^{\infty}(\mathcal{C}) = \mathrm{Ker}\mathbf{S} = \{C \in \mathcal{C} : \exists n \geq 0 : \Omega^n(C) = 0\}.$$

If $\mathcal{C}$ is Gorenstein, we call the uniquely determined triangulated subcategory of $\mathcal{C}$ realizing the stabilization, the category of **maximal Cohen-Macaulay objects** (or maximal CM-objects for short) of $\mathcal{C}$, and we denote it by $\mathrm{CM}(\mathcal{C})$.

Proposition 3.16. *If the left triangulated category $\mathcal{C}$ is Gorenstein, then*

(1) *The stabilization functor $\mathbf{S} : \mathcal{C} \to \mathrm{CM}(\mathcal{C})$ is given as follows:*

$$\forall C \in \mathcal{C} : \ \mathbf{S}(C) = \Omega^{-t}\Omega^t(C) \ \text{ where } t \geq 0 \text{ is such that}: \ \Omega^t(C) \in \mathrm{CM}(\mathcal{C}).$$

(2) *$\mathcal{P}^{\infty}(\mathcal{C})$ is a full left triangulated subcategory of $\mathcal{C}$, $\mathcal{P}^{\infty}(\mathcal{C}) \bigcap \mathrm{CM}(\mathcal{C}) = 0$ and the inclusions $\mathcal{P}^{\infty}(\mathcal{C}) \hookrightarrow \mathcal{C}$ and $\mathrm{CM}(\mathcal{C}) \hookrightarrow \mathcal{C}$ induce isomorphisms:*

$$\mathrm{K}_0(\mathcal{P}^{\infty}(\mathcal{C})) = 0 \ \text{ and } \ \mathrm{K}_0(\mathrm{CM}(\mathcal{C})) \cong \mathrm{K}_0(\mathcal{C}).$$

Proof. (1) By Theorem 3.15, $\forall C \in \mathcal{C}$ there exists $t \geq 0$, such that $\Omega^t(C) \in \mathrm{CM}(\mathcal{C})$. Suppose also that $\Omega^s(C) \in \mathrm{CM}(\mathcal{C})$ with $s \geq 0$ and assume without restriction that

$s = t + r$. Then $\Omega^{-s}\Omega^{s}(C) = \Omega^{-t-r}\Omega^{t+r}(C) = \Omega^{-t}\Omega^{-r}\Omega^{r}\Omega^{t}(C) = \Omega^{-t}\Omega^{t}(C)$, i.e. the object $\Omega^{-t}\Omega^{t}(C) \in \mathrm{CM}(\mathcal{C})$ is uniquely determined. We set $\mathbf{S}'(C) = \Omega^{-t}\Omega^{t}(C)$. If $f : C \to D$ is a morphism in $\mathcal{C}$, define $\mathbf{S}'(f) = \Omega^{-t}\Omega^{t}(f)$. A similar argument as above shows that in this way we obtain a functor $\mathbf{S}' : \mathcal{C} \to \mathrm{CM}(\mathcal{C})$ which clearly is exact. If $F : \mathcal{C} \to \mathcal{D}$ is an exact functor to a triangulated category $\mathcal{D}$ with translation functor Σ, then define $F^* : \mathrm{CM}(\mathcal{C}) \to \mathcal{D}$ by $F^* = F|_{\mathrm{CM}(\mathcal{C})}$. Trivially F^* is exact and $\forall C \in \mathcal{C}$ with $\Omega^{t}(C) \in \mathrm{CM}(\mathcal{C})$, we have $F^*\mathbf{S}'(C) = F^*\Omega^{-t}\Omega^{t}(C) = \Sigma^{-t}F^*\Omega^{t}(C) = \Sigma^{-t}F\Omega^{t}(C) = \Sigma^{-t}\Sigma^{t}F(C) = F(C)$. It is easy to see that these identifications are natural, so we have $F^*\mathbf{S}' = F$. If $G : \mathrm{CM}(\mathcal{C}) \to \mathcal{D}$ is another exact functor with $G\mathbf{S}' = F$, then $\forall A \in \mathrm{CM}(\mathcal{C})$: $G(A) = G\mathbf{S}'(A) = F(A) = F^*(A)$. Hence $G = F^*$. This shows that $\mathbf{S}' = \mathbf{S}$ is the stabilization functor. (2) Let $C \in \mathcal{P}^{\infty}(\mathcal{C}) \bigcap \mathrm{CM}(\mathcal{C})$. Since $C \in \mathcal{P}^{\infty}(\mathcal{C})$ we have $\mathbf{S}(C) = 0$ and since $C \in \mathrm{CM}(\mathcal{C})$ we have $\mathbf{S}(C) = C$, so $C = 0$. All other assertions are trivial, noting that $\mathcal{S}(\mathcal{P}^{\infty}(\mathcal{C})) = 0$, since the loop functor Ω in $\mathcal{P}^{\infty}(\mathcal{C})$ is locally nilpotent. □

We turn now our attention to Co-Gorenstein left triangulated categories. We consider the following full subcategory of $\mathcal{C}$:

$$\Omega^{\infty}(\mathcal{C}) = \bigcap_{n \geq 1} \Omega^{n}(\mathcal{C}) = \{C \in \mathcal{C} \mid \exists \{C_n\}_{n \geq 0} \subseteq \mathcal{C} : C_n = \Omega(C_{n+1}), \forall n \geq 0, C_0 = C\}.$$

If $\mathcal{C}$ is Co-Gorenstein, we call the uniquely determined triangulated subcategory of $\mathcal{C}$ realizing the costabilization, the category of **maximal Co-Cohen-Macaulay objects** (or maximal Co-CM-objects for short) of $\mathcal{C}$, and we denote it by $\mathrm{CoCM}(\mathcal{C})$.

Theorem 3.17. *The following are equivalent.*

(1) *$\mathcal{C}$ is Co-Gorenstein.*
(2) *$\Omega^{\infty}(\mathcal{C})$ is a triangulated subcategory of $\mathcal{C}$.*
(3) *There exists a triangulated subcategory $\mathcal{U}$ of $\mathcal{C}$ with the property: $\Omega^{\infty}(\mathcal{C}) \subseteq \mathcal{U}$.*

In this case $\mathcal{U} = \mathrm{CoCM}(\mathcal{C}) = \Omega^{\infty}(\mathcal{C})$, there exists a triangle equivalence $\mathcal{R}(\mathcal{C}) \approx \Omega^{\infty}(\mathcal{C})$ and the inclusion $\Omega^{\infty}(\mathcal{C}) \hookrightarrow \mathcal{C}$ is the costabilization functor.

Proof. (1) $\Rightarrow$ (3) If $\mathcal{C}$ is Co-Gorenstein, then there exists a triangulated subcategory $\mathcal{U}$ of $\mathcal{C}$ such that the inclusion $i : \mathcal{U} \hookrightarrow \mathcal{C}$ is the costabilization functor. Since $\mathcal{U}$ is triangulated, we have $\mathcal{U} \subseteq \Omega^{\infty}(\mathcal{C})$ and since the strict image of the costabilization functor i is $\Omega^{\infty}(\mathcal{C})$, we conclude that $\Omega^{\infty}(\mathcal{C}) = \mathcal{U}$. (3) $\Rightarrow$ (2) Trivial. (2) $\Rightarrow$ (1) To prove (1) it suffices to show that the costabilization functor $\mathbf{R} : \mathcal{R}(\mathcal{C}) \to \mathcal{C}$ is fully faithful, since then $\mathbf{R}$ induces an equivalence: $\mathcal{R}(\mathcal{C}) \approx \Omega^{\infty}(\mathcal{C})$. If $f_\bullet : (A_n, \alpha_n) \to (B_n, \beta_n)$ is a morphism in $\mathcal{R}(\mathcal{C})$ with $\mathbf{R}(f_\bullet) = 0$, then $\Omega^{n}(f_n) = 0$, $\forall n \geq 0$ and $f_n = 0$, $\forall n \leq 0$. Since $A_n, B_n \in \Omega^{\infty}(\mathcal{C})$, and since the latter is triangulated, we have $f_n = 0$, $\forall n \in \mathbb{Z}$. Hence $f_\bullet = 0$ and this shows that $\mathbf{R}$ is faithful. A similar argument shows that $\mathbf{R}$ is full, using that $\mathrm{Im}\mathbf{R} = \Omega^{\infty}(\mathcal{C})$ is triangulated. □

The next result, which is a consequence of Theorems 3.15, 3.17, shows that jn some cases the stabilization and the costabilization of $\mathcal{C}$ coincide.

Corollary 3.18. *Suppose that $\mathcal{C}$ contains a full triangulated subcategory $\mathcal{V}$ enjoying the property: $\exists d \geq 0$ such that $\Omega^{d}(\mathcal{C}) \subseteq \mathcal{V}$. Then $\mathcal{C}$ is Gorenstein and Co-*

Gorenstein, there are triangle equivalences

$$\mathcal{R}(\mathcal{C}) \xrightarrow{\approx} \Omega^{\infty}(\mathcal{C}) \xrightarrow{\approx} \mathrm{CoCM}(\mathcal{C}) \xrightarrow{\approx} \mathcal{V} \xrightarrow{\approx} \mathrm{CM}(\mathcal{C}) \xrightarrow{\approx} \mathcal{S}(\mathcal{C}),$$

the functor $\Omega^{-d}\Omega^{d} : \mathcal{C} \to \mathcal{V}$ is the stabilization functor and the inclusion $i_{\mathcal{V}} : \mathcal{V} \hookrightarrow \mathcal{C}$ is the costabilization functor.

We leave to the reader the formulation of the dual concepts and results concerning (Co-)Gorenstein right triangulated categories.

4. Gorenstein Categories and Auslander-Buchweitz Contexts

Throughout this section we assume that $\mathcal{C}$ is an exact category and $\mathcal{X}$ is a full contravariantly finite additive subcategory of $\mathcal{C}$ which is closed under isomorphisms, direct summands, and such that any $\mathcal{X}$−epic is an admissible epic.

4.1. Stabilization and $\mathcal{X}$−Gorenstein Exact categories.

Definition 4.1. The exact category $\mathcal{C}$ is called **$\mathcal{X}$−Gorenstein** if the stable left triangulated category $\mathcal{C}/\mathcal{X}$ is Gorenstein.

By section 2, we know that if $\mathcal{C}/\mathcal{X}$ is Gorenstein, then the triangulated subcategory of $\mathcal{C}/\mathcal{X}$ realizing the stabilization is the stable category of an $\mathcal{X}$−resolving subcategory $\mathcal{A}$ of $\mathcal{C}$. So we fix an $\mathcal{X}$−resolving subcategory $\mathcal{A}$ of $\mathcal{C}$, and let $i_{\mathcal{A}} : \mathcal{A}/\mathcal{X} \hookrightarrow \mathcal{C}/\mathcal{X}$ be the inclusion functor. Then we know that $\mathcal{A}/\mathcal{X}$ is left triangulated and the inclusion functor $i_{\mathcal{A}}$ is exact. Let $\widehat{\mathcal{A}}$ be the full subcategory of $\mathcal{C}$ consisting of all objects having finite $\mathcal{X}$−resolutions by objects of $\mathcal{A}$. Hence an object $C \in \mathcal{C}$ is in $\widehat{\mathcal{A}}$ if there exists an $\mathcal{X}$−exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to C \to 0$, where $A_i \in \mathcal{A}$, $\forall i \geq 0$. Let $\mathbf{S} : \mathcal{C}/\mathcal{X} \to \mathcal{S}(\mathcal{C}/\mathcal{X})$ be the stabilization functor.

Theorem 4.2. (1) *The following are equivalent.*

- (α) *The exact functor $\mathbf{S}i_{\mathcal{A}} : \mathcal{A}/\mathcal{X} \to \mathcal{S}(\mathcal{C}/\mathcal{X})$ is fully faithful.*
- (β) *The loop functor $\Omega_{\mathcal{X}} : \mathcal{A}/\mathcal{X} \to \mathcal{A}/\mathcal{X}$ is fully faithful.*
- (γ) $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$.

(2) *The following are equivalent.*

- (α) *The functor $\mathbf{S}i_{\mathcal{A}} : \mathcal{A}/\mathcal{X} \to \mathcal{S}(\mathcal{C}/\mathcal{X})$ is a triangle equivalence.*
- (β) (i) $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$.
  - (ii) *$\mathcal{X}$ is an $\mathcal{X}$−cogenerator of $\mathcal{A}$.*
  - (iii) *$\forall C \in \mathcal{C}$ and $\forall n \in \mathbb{Z}$, $\exists t \geq 0$ with $t \geq n$, such that: $\Omega_{\mathcal{X}}^{t-n}(\underline{C}) \in \mathcal{A}/\mathcal{X}$.*
- (γ) (i) $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$.
  - (ii) *$\mathcal{X}$ is an $\mathcal{X}$−cogenerator of $\mathcal{A}$.*
  - (iii) $\mathcal{C} = \widehat{\mathcal{A}}$.

Proof. Part (1) and the direction (α) $\Leftrightarrow$ (β) of part (2) are consequences of Lemma 3.14, Theorem 3.15 and the results of section 2. (2)(β) $\Rightarrow$ (γ) If $C \in \mathcal{C}$, then by Theorem 3.15, there exists $t \geq 0$ such that $\Omega_{\mathcal{X}}^{t}(\underline{C}) = \underline{A} \in \mathcal{A}/\mathcal{X}$. Hence there exists an $\mathcal{X}$−exact sequence $0 \to A \to X_{t-1} \to \cdots \to X_0 \to C \to 0$ with $A \in \mathcal{A}$ and $X_i \in \mathcal{X} \subseteq \mathcal{A}, \forall i = 0, \ldots, t-1$. This implies that $C \in \widehat{\mathcal{A}}$, and consequently $\mathcal{C} = \widehat{\mathcal{A}}$.

(γ) $\Rightarrow$ (α) By Theorem 3.15, it suffices to show that for any $C \in \mathcal{C}$ there exists $t \geq 0$ such that $\Omega_{\mathcal{X}}^{t}(\underline{C}) \in \mathcal{A}/\mathcal{X}$. Since $\widehat{\mathcal{A}} = \mathcal{C}$, there exists an $\mathcal{X}$−exact sequence $0 \to A_t \to A_{t-1} \to \cdots \to A_1 \to A_0 \to C \to 0$, where $A_i \in \mathcal{A}, \forall i \geq 0$. Consider the $\mathcal{X}$-exact sequence $0 \to A_t \to A_{t-1} \to M_{t-1} \to 0$, and let $\Omega_{\mathcal{X}}(\underline{M}_{t-1}) \to \underline{A}_t \to \underline{A}_{t-1} \to \underline{M}_{t-1}$ be the induced triangle in $\mathcal{C}/\mathcal{X}$. Since $\mathcal{A}$ is $\mathcal{X}$-resolving, imbedding the morphism $\underline{A}_t \to \underline{A}_{t-1}$ in a triangle in $\mathcal{A}/\mathcal{X}$, it follows that $\Omega_{\mathcal{X}}(\underline{M}_{t-1}) \in \mathcal{A}/\mathcal{X}$. Continuing in this way we see that $\Omega^k_{\mathcal{X}}(\underline{M}_{t-k}) \in \mathcal{A}/\mathcal{X}$, and finally that $\Omega^t_{\mathcal{X}}(\underline{C}) := \underline{A} \in \mathcal{A}/\mathcal{X}$. □

Let $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ of $\mathcal{C}$ be the full subcategory of $\mathcal{C}$ of all objects with finite contravariant $\mathcal{X}$-dimension. Obviously $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ is an $\mathcal{X}$-resolving subcategory of $\mathcal{C}$ and we know that the induced stable category $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} = \mathrm{Ker}(\mathbf{S})$. So in the notation of Subsection 3.3, $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} = \mathcal{P}^{\infty}(\mathcal{C}/\mathcal{X})$. If $\mathcal{C}$ is $\mathcal{X}$-Gorenstein, then we know that the stabilization of $\mathcal{C}/\mathcal{X}$ is realized by the full subcategory $\mathrm{CM}(\mathcal{C}/\mathcal{X})$. Hence there exists an $\mathcal{X}$-resolving subcategory $\mathrm{CM}(\mathcal{C})$ of $\mathcal{C}$ such that $\mathrm{CM}(\mathcal{C}/\mathcal{X}) = \mathrm{CM}(\mathcal{C})/\mathcal{X}$.

Theorem 4.3. *If $\mathcal{C}$ is $\mathcal{X}$-Gorenstein, then the following are true.*

(1) *The stabilization functor $\mathbf{S} : \mathcal{C}/\mathcal{X} \to \mathrm{CM}(\mathcal{C}/\mathcal{X})$ is the coreflection of the category $\mathrm{CM}(\mathcal{C}/\mathcal{X})$ in $\mathcal{C}/\mathcal{X}$ and is given as follows:*

$$\mathbf{S}(\underline{C}) = \Omega^{-t}_{\mathcal{X}}\Omega^t_{\mathcal{X}}(\underline{C}), \quad \text{where } t \geq 0 \text{ is such that } \Omega^t_{\mathcal{X}}(\underline{C}) \in \mathrm{CM}(\mathcal{C}/\mathcal{X}).$$

(2) *For any $C \in \mathcal{C}$, there exists an $\mathcal{X}$-exact sequence*

$$(\mathbf{A}_C) \quad 0 \to P_C \xrightarrow{p_C} A_C \xrightarrow{\alpha_C} C \to 0 \ \text{ with } \ P_C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \ \text{ and } \ A_C \in \mathrm{CM}(\mathcal{C}).$$

$\mathrm{CM}(\mathcal{C})$ is a contravariantly finite subcategory of $\mathcal{C}$. The morphism $\alpha_C : A_C \to C$ gives a right $\mathrm{CM}(\mathcal{C})$-approximation of C.

(3) *Let $f : A_C \to A_C$ be a morphism such that $f \circ \alpha_C = \alpha_C$. Then $\underline{f}$ is an isomorphism. All objects of $\mathcal{C}$ have minimal right $\mathrm{CM}(\mathcal{C})$-approximations if $\mathrm{CM}(\mathcal{C})$ is a Krull-Schmidt category or if the ideal $\mathcal{J}_{\mathcal{X}}(\mathrm{CM}(\mathcal{C}))$ of morphisms in $\mathrm{CM}(\mathcal{C})$ factoring through $\mathcal{X}$, is contained in the Jacobson radical of $\mathrm{CM}(\mathcal{C})$.*

(4) *$\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \cap \mathrm{CM}(\mathcal{C}) = \mathcal{X}$. Moreover $\forall C \in \mathcal{C} : C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \Leftrightarrow A_C \in \mathcal{X}$.*

(5) *$\forall A \in \mathrm{CM}(\mathcal{C})$, $\forall C \in \mathcal{C}$ we have: $\underline{\mathcal{E}xt}^i_{\mathcal{X}}(A, C) \cong \mathcal{A}/\mathcal{X}(\Omega^i_{\mathcal{X}}(\underline{A}), \underline{A}_C), \forall i \geq 1$. Moreover: $\mathcal{C}/\mathcal{X}[\mathrm{CM}(\mathcal{C}/\mathcal{X}), \mathcal{P}^{\infty}(\mathcal{C}/\mathcal{X})] = 0$.*

(6) *$\mathrm{CM}(\mathcal{C}) = {}^{\perp}\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) = {}^{\perp}\mathcal{X}$.*

(7) *For any $C \in \mathcal{C}$, there exists an $\mathcal{X}$-exact sequence*

$$(\mathbf{P}^C) \quad 0 \to C \xrightarrow{p^C} P^C \xrightarrow{\alpha^C} A^C \to 0 \ \text{ with } \ P^C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \ \text{ and } \ A^C \in \mathrm{CM}(\mathcal{C}).$$

$\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) = \mathrm{CM}(\mathcal{C})^{\perp}$ is a covariantly finite subcategory of $\mathcal{C}$. The left $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$-approximation of C is given by the morphism $p^C : C \to P^C$.

(8) *Let $f : P^C \to P^C$ be a morphism such that $p^C \circ f = p^C$. Then $\underline{f}$ is a monomorphism and $\Omega^n_{\mathcal{X}}(\underline{f})$ is an isomorphism $\forall n \geq 1$. All objects of $\mathcal{C}$ have minimal left $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$-approximations if $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ is a Krull-Schmidt category or if the ideal $\mathcal{J}_{\mathcal{X}}(\mathrm{CM}(\mathcal{C}))$ of morphisms in $\mathrm{CM}(\mathcal{C})$ factoring through $\mathcal{X}$ is contained in the Jacobson radical $\mathcal{J}ac(\mathrm{CM}(\mathcal{C}))$ of $\mathrm{CM}(\mathcal{C})$.*

(9) *The stable category $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ is a reflective left triangulated subcategory of $\mathcal{C}/\mathcal{X}$ and the reflection is given by the functor*

$$\mathbf{T} : \mathcal{C}/\mathcal{X} \to \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}, \quad \text{defined by } \ \mathbf{T}(\underline{C}) = \underline{P}^C.$$

(10) *The category $\mathcal{C}/\mathcal{X}$ admits a "direct sum decomposition" $\mathcal{C}/\mathcal{X} = \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} \oplus \mathrm{CM}(\mathcal{C}/\mathcal{X})$ in the sense that the sequences of categories and functors*

$$0 \to \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} \xrightarrow{j} \mathcal{C}/\mathcal{X} \xrightarrow{\mathbf{S}} \mathrm{CM}(\mathcal{C}/\mathcal{X}) \to 0$$

$$0 \to \mathrm{CM}(\mathcal{C}/\mathcal{X}) \xrightarrow{i} \mathcal{C}/\mathcal{X} \xrightarrow{\mathbf{T}} \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} \to 0$$

satisfy the relations: $\mathbf{S}j = 0$, $\mathbf{T}i = 0$, $\mathbf{S}i = \mathrm{Id}_{\mathrm{CM}(\mathcal{C}/\mathcal{X})}$, $\mathbf{T}j = \mathrm{Id}_{\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}}$. *Moreover there exists an equivalence of categories:*

$$\mathcal{C}/\mathcal{X}/\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} \approx \mathrm{CM}(\mathcal{C}/\mathcal{X})$$

where the first category indicates localization of the left triangulated category $\mathcal{C}/\mathcal{X}$ with respect to the (eppaise) subcategory $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$.

(11) *The pair $[\mathrm{CM}(\mathcal{C}/\mathcal{X}), \mathcal{P}^{\infty}(\mathcal{C}/\mathcal{X})]$ is a torsion theory in $\mathcal{C}/\mathcal{X}$:*

$$\mathrm{CM}(\mathcal{C}/\mathcal{X})^{\perp} := \{\underline{C} \in \mathcal{C}/\mathcal{X} : \mathcal{C}/\mathcal{X}(\underline{A}, \underline{C}) = 0, \forall \underline{A} \in \mathrm{CM}(\mathcal{C}/\mathcal{X})\} = \mathcal{P}^{\infty}(\mathcal{C}/\mathcal{X}),$$

$$^{\perp}\mathcal{P}^{\infty}(\mathcal{C}/\mathcal{X}) := \{\underline{C} \in \mathcal{C}/\mathcal{X} : \mathcal{C}/\mathcal{X}(\underline{C}, \underline{P}) = 0, \forall \underline{P} \in \mathcal{P}^{\infty}(\mathcal{C}/\mathcal{X})\} = \mathrm{CM}(\mathcal{C}/\mathcal{X}).$$

In particular there exists a triangle in $\mathcal{C}/\mathcal{X}$: $\Omega_{\mathcal{X}}\mathbf{T} \to \mathbf{S} \to \mathrm{Id}_{\mathcal{C}/\mathcal{X}} \to \mathbf{T}$.

(12) *Consider the relative Grothendieck groups $\mathrm{K}_0(\mathcal{C}, \mathcal{X})$ and $\mathrm{K}_0(\mathrm{CM}(\mathcal{C}), \mathcal{X})$ and the stable Grothendieck groups $\mathrm{K}_0(\mathcal{C}/\mathcal{X})$ and $\mathrm{K}_0(\mathrm{CM}(\mathcal{C}/\mathcal{X}))$ as defined in* [15]. *Then there are isomorphisms $\mathrm{K}_0(\mathrm{CM}(\mathcal{C}), \mathcal{X}) \cong \mathrm{K}_0(\mathcal{C}, \mathcal{X})$ and $\mathrm{K}_0(\mathrm{CM}(\mathcal{C}/\mathcal{X})) \cong \mathrm{K}_0(\mathcal{C}/\mathcal{X})$ and an exact commutative diagram* (†):

$$\begin{array}{ccccccc}
\mathrm{K}_0(\mathcal{X}, \oplus) & \longrightarrow & \mathrm{K}_0(\mathrm{CM}(\mathcal{C}), \mathcal{X}) & \longrightarrow & \mathrm{K}_0(\mathrm{CM}(\mathcal{C}/\mathcal{X})) & \longrightarrow & 0 \\
\| \downarrow & & \cong \downarrow & & \cong \downarrow & & \\
\mathrm{K}_0(\mathcal{X}, \oplus) & \longrightarrow & \mathrm{K}_0(\mathcal{C}, \mathcal{X}) & \longrightarrow & \mathrm{K}_0(\mathcal{C}/\mathcal{X}) & \longrightarrow & 0.
\end{array}$$

Proof. (1) By Proposition 3.16 it suffices to show that the functor $\mathbf{S}$ is the coreflection of $\mathrm{CM}(\mathcal{C}/\mathcal{X})$. If $C \in \mathcal{C}$, then by Theorem 4.1, there exists $t \geq 0$ such that $\Omega^t_{\mathcal{X}}(\underline{C}) := \underline{A}$ with $A \in \mathrm{CM}(\mathcal{C})$. Hence there exists an $\mathcal{X}$−resolution (α) : $0 \to A \to X^{t-1}_C \to X^{t-2}_C \to \cdots \to X^0_C \to C \to 0$ of C in $\mathcal{C}$. Since A belongs to $\mathrm{CM}(\mathcal{C})$ and $\mathrm{CM}(\mathcal{C}/\mathcal{X})$ is triangulated, $\underline{A}$ is an arbitrary $\mathcal{X}$−syzygy object. Hence there exists an $\mathcal{X}$−exact sequence (β) : $0 \to A \to X^{t-1} \to X^{t-2} \to \cdots \to X^0 \to A_C \to 0$ in $\mathrm{CM}(\mathcal{C})$ and $\underline{A_C} = \Omega^{-t}_{\mathcal{X}}(\underline{A}) = \Omega^{-t}_{\mathcal{X}}\Omega^t_{\mathcal{X}}(\underline{C})$. Consider the $\mathcal{X}$−exact sequences (γ) : $0 \to A \to X^{t-1}_C \to K^{t-1}_C \to 0$ in $\mathcal{C}$ and (δ) : $0 \to A \to X^{t-1} \to A^{t-1} \to 0$ in $\mathrm{CM}(\mathcal{C})$. Since A^{t-1} by construction is in $\mathrm{CM}(\mathcal{C})$ and $\mathrm{CM}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{X}$, the push-out of the above admissible sequence along the morphism $A \to X^{t-1}_C$ splits, and this induces a morphism $(\delta) \to (\gamma)$ of short exact sequences. Continuing in this way we obtain finally a morphism of $\mathcal{X}$−resolutions $(\beta) \to (\alpha)$, in particular we obtain a morphism $\alpha_C : A_C \to C$. It is easy to see that $\underline{\alpha}_C$ is independent of the above construction and induces a natural morphism $\underline{\alpha} : i_{\mathcal{A}}\mathbf{S} \to \mathrm{Id}_{\mathcal{C}/\mathcal{X}}$. Since by construction $\Omega^t_{\mathcal{X}}(\underline{\alpha}_C) = 1_{\Omega^t_{\mathcal{X}}(\underline{C})}$, trivially $\underline{\alpha}_C$ gives the coreflection of $\underline{C}$ in $\mathrm{CM}(\mathcal{C}/\mathcal{X})$. Hence $\mathrm{CM}(\mathcal{C}/\mathcal{X})$ is a coreflective (triangulated) subcategory of $\mathcal{C}/\mathcal{X}$ with coreflector the stabilization functor $\mathbf{S}$.

(2) We use the notation of part (1). Adding a right $\mathcal{X}$−approximation X_C to A_C if necessary and using that any $\mathcal{X}$−epic is admissible, we can assume that $\alpha_C : A_C \to C$ is an (admissible $\mathcal{X}$−)epic. Hence the $\mathcal{X}$−exact sequence $0 \to$

$P_C \to A_C \to C \to 0$ is defined. This sequence induces a triangle $\Omega_{\mathcal{X}}(\underline{C}) \to \underline{P}_C \to \underline{A}_C \to \underline{C}$ in $\mathcal{C}/\mathcal{X}$ and a triangle $\tilde{\Omega}_{\mathcal{X}}\mathbf{S}(\underline{C}) \to \mathbf{S}(\underline{P}_C) \to \mathbf{S}(\underline{A}_C) \to \mathbf{S}(\underline{C})$ in $\mathcal{S}(\mathcal{C}/\mathcal{X})$. But obviously the stabilization functor $\mathbf{S}$ induces an isomorphism $\mathbf{S}(\underline{\alpha}_C) : \underline{A}_C = \mathbf{S}(\underline{A}_C) \cong \mathbf{S}(\underline{C})$. Hence $\mathbf{S}(\underline{P}_C) = 0$ which is equivalent to P_C being of finite contravariant $\mathcal{X}$-dimension, i.e. $P_C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$. It remains to show that $\alpha_C : A_C \to C$ is a right $\mathrm{CM}(\mathcal{C})$-approximation, but this follows directly from (1).

(3) Let $f : A_C \to A_C$ be a morphism such that $f \circ \alpha_C = \alpha_C$. Then $\underline{f} \circ \underline{\alpha}_C = \underline{\alpha}_C$ in $\mathcal{C}/\mathcal{X}$. Applying the stabilization functor to this relation we see directly that $\mathbf{S}(\underline{f})$ is an isomorphism. Since $\underline{A}_C$ is in $\mathrm{CM}(\mathcal{C}/\mathcal{X})$, we have that $\underline{f}$ is an isomorphism. If the ideal $\mathcal{J}_{\mathcal{X}}(\mathrm{CM}(\mathcal{C}))$ is contained in $\mathcal{J}ac(\mathrm{CM}(\mathcal{C}))$ then the projection functor $\mathrm{CM}(\mathcal{C}) \to \mathrm{CM}(\mathcal{C}/\mathcal{X})$ reflects isomorphisms. Hence f is an isomorphism and any object of $\mathcal{C}$ has a minimal right $\mathrm{CM}(\mathcal{C})$-approximation. If $\mathrm{CM}(\mathcal{C})$ is a Krull-Schmidt category, then the proof of Lemma 2.6 of [1] can be applied, showing that any object of $\mathcal{C}$ has a minimal right $\mathrm{CM}(\mathcal{C})$-approximation.

(4) That $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \cap \mathrm{CM}(\mathcal{C}) = \mathcal{X}$, follows from Proposition 3.16. Suppose now that $C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$. Then the sequence $0 \to P_C \to A_C \to C \to 0$, induces a triangle $\Omega_{\mathcal{X}}(\underline{C}) \to \underline{P}_C \to \underline{A}_C \to \underline{C} \to$ in $\mathcal{C}/\mathcal{X}$ and then a triangle $\tilde{\Omega}_{\mathcal{X}}\mathbf{S}(\underline{C}) \to \mathbf{S}(\underline{P}_C) \to \mathbf{S}(\underline{A}_C) \to \mathbf{S}(\underline{C})$ in $\mathcal{S}(\mathcal{C}/\mathcal{X})$. But since P_C, C have finite $\mathcal{X}$-dimension, we have $\mathbf{S}(\underline{P}_C) = \mathbf{S}(\underline{C}) = 0$. Hence $\mathbf{S}(\underline{A}_C) = 0$, so A_C is in $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \cap \mathrm{CM}(\mathcal{C}) = \mathcal{X}$. Conversely if $A_C \in \mathcal{X}$, then obviously $C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$.

(5) Consider objects $A \in \mathrm{CM}(\mathcal{C})$ and $C \in \mathcal{C}$, and let $0 \to \Omega_{\mathcal{X}}(A) \to X_A \to A \to 0$ be a right $\mathcal{X}$-approximation of A. Applying the functor $\mathcal{C}(-, C)$ to this sequence and using that $\mathrm{CM}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{X}$, it is easy to see that there exists an isomorphism $\underline{\mathcal{E}xt}^1_{\mathcal{X}}(A, C) \cong \mathcal{C}/\mathcal{X}(\Omega_{\mathcal{X}}(\underline{A}), \underline{C})$. Consider now the triangle $\Omega_{\mathcal{X}}(\underline{C}) \to \underline{P}_C \to \underline{A}_C \to \underline{C}$ in $\mathcal{C}/\mathcal{X}$ induced by the sequence $(\mathbf{A}_C)$ in (2). Since by (1), $\mathrm{CM}(\mathcal{C}/\mathcal{X})$ is coreflective in $\mathcal{C}/\mathcal{X}$ and $\Omega_{\mathcal{X}}(\underline{A})$ is in $\mathrm{CM}(\mathcal{C}/\mathcal{X})$, it follows that we have an isomorphism $\mathrm{CM}(\mathcal{C}/\mathcal{X})(\Omega_{\mathcal{X}}(\underline{A}), \underline{A}_C) \to \mathcal{C}/\mathcal{X}(\Omega_{\mathcal{X}}(\underline{A}), \underline{C})$. Hence $\underline{\mathcal{E}xt}^1_{\mathcal{X}}(A, C) \cong \mathcal{C}/\mathcal{X}(\Omega_{\mathcal{X}}(\underline{A}), \underline{C}) \cong \mathrm{CM}(\mathcal{C}/\mathcal{X})(\Omega_{\mathcal{X}}(\underline{A}), \underline{A}_C)$. The general case follows by dimension shifting since $\mathrm{CM}(\mathcal{C})$ is $\mathcal{X}$-resolving.

If $\underline{f} : \underline{A} \to \underline{P}$ is a morphism in $\mathcal{C}/\mathcal{X}$ with $A \in \mathrm{CM}(\mathcal{C})$ and $P \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$, then from the right $\mathrm{CM}(\mathcal{C})$-approximation sequence $(\mathbf{A}_P)$: $0 \to P_P \to A_P \to P \to 0$, we see that f factors through A_P. But by (4), $A_P \in \mathcal{X}$. Hence $\underline{f} = 0$ in $\mathcal{C}/\mathcal{X}$.

(6) By hypothesis on $\mathrm{CM}(\mathcal{C})$ and parts (3), (4) we have $\mathrm{CM}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{X}$. By part (2), if $C \in {}^{\perp}\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$, then $C \in \mathrm{CM}(\mathcal{C})$ as a direct summand of A_C. Hence $\mathrm{CM}(\mathcal{C}) = {}^{\perp}\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ and it remains to show that ${}^{\perp}\mathcal{X} \subseteq {}^{\perp}\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$. So let $C \in {}^{\perp}\mathcal{X}$. We first prove that any morphism $f : C \to P$ with $P \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ factors through $\mathcal{X}$. If $P \in \mathcal{X}$, this is trivial. If $\mathcal{X}\text{-}\dim P = 1$ and $0 \to X_1 \to X_0 \to P \to 0$ is a $\mathcal{X}$-resolution, then the pull-back sequence along the morphism f splits since $C \in {}^{\perp}\mathcal{X}$. An easy induction argument shows our claim. Now as in the proof of (4), we have $\underline{\mathcal{E}xt}^1_{\mathcal{X}}(C, P) \cong \mathcal{C}/\mathcal{X}(\Omega_{\mathcal{X}}(\underline{C}), \underline{P}) = 0$, since $P \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ and $\Omega_{\mathcal{X}}(C) \in {}^{\perp}\mathcal{X}$. By dimension shifting we conclude that $C \in {}^{\perp}\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$.

(7) Consider the $\mathcal{X}$-exact sequence $(\mathbf{A}_C)$ of (2) and let $0 \to A_C \xrightarrow{\kappa} X \xrightarrow{\lambda} A^C \to 0$ be an $\mathcal{X}$-exact sequence in $\mathrm{CM}(\mathcal{C})$ with $X \in \mathcal{X}$, which exists since $A_C \in \mathrm{CM}(\mathcal{C})$. Since the morphisms $p_C : P_C \to A_C$ and $\kappa : A_C \to X$ are admissible monics, their composition $p_C \circ \kappa : P_C \to X$ is admissible monic. Let $0 \to P_C \to X \to P^C \to 0$ be the induced admissible short exact sequence in $\mathcal{C}$. Then we have a pull-back

diagram, which defines the object P^C, and a push-out diagram:

$$\begin{array}{ccccc} P_C & \xrightarrow{p_C} & A_C & \xrightarrow{\alpha_C} & C \\ \| \downarrow & & \kappa \downarrow & & p^C \downarrow \\ P_C & \xrightarrow{p_C \circ \kappa} & X & \xrightarrow{\nu} & P^C \end{array} \qquad \begin{array}{ccccc} A^C & \xrightarrow{\kappa} & X & \xrightarrow{\lambda} & A^C \\ \alpha_C \downarrow & & \nu \downarrow & & \| \downarrow \\ C & \xrightarrow{p^C} & P^C & \xrightarrow{\alpha^C} & A^C. \end{array}$$

Since the admissible sequence $0 \to A_C \to X \to A^C \to 0$ is $\mathcal{X}$-exact, from the above diagram it follows directly that the admissible sequence $(\mathbf{P}^C)$: $0 \to C \xrightarrow{p^C} P^C \xrightarrow{\alpha^C} A^C \to 0$ is $\mathcal{X}$-exact, hence induces a triangle $\tilde{\Omega}\mathbf{S}(\underline{A}^C) \to \mathbf{S}(\underline{C}) \to \mathbf{S}(\underline{P}^C) \to \mathbf{S}(\underline{A}^C)$ in $\mathrm{CM}(\mathcal{C}/\mathcal{X})$. But from (2) we have $\underline{A}^C = \Omega_{\mathcal{X}}^{-1}(\underline{A}_C) = \Omega_{\mathcal{X}}^{-t-1}\Omega_{\mathcal{X}}^{t}(\underline{C})$. Hence $\tilde{\Omega}\mathbf{S}(\underline{A}^C) = \tilde{\Omega}\mathbf{S}(\Omega_{\mathcal{X}}^{-t-1}\Omega_{\mathcal{X}}^{t}(\underline{C})) = \tilde{\Omega}\tilde{\Omega}_{\mathcal{X}}^{-t-1}\mathbf{S}(\Omega_{\mathcal{X}}^{t}(\underline{C})) = \mathbf{S}(\underline{C})$. Obviously the morphism $\tilde{\Omega}\mathbf{S}(\underline{A}^C) \to \mathbf{S}(\underline{C})$ in the above triangle is the identity, hence $\mathbf{S}(\underline{P}^C) = 0$ and then P^C belongs to $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$. Now let $f : C \to Q$ be a morphism with $Q \in \mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$ and consider the push-out sequence $0 \to Q \to M \to A^C \to 0$ of $(\mathbf{P}^C)$ along the morphism f. Trivially the push-out sequence is also admissible and $\mathcal{X}$-exact, hence is split by (6). It follows that the morphism $p^C : C \to P^C$ is the left $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$-approximation of C, since then f factors through p^C.

Finally by (6), $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C}) \subseteq \mathrm{CM}(\mathcal{C})^{\perp}$. If $C \in \mathrm{CM}(\mathcal{C})^{\perp}$, then the $\mathcal{X}$-exact sequence $(\mathbf{P}^C)$ splits, and $C \in \mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$ as a direct summand of P^C. So $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C}) = \mathrm{CM}(\mathcal{C})^{\perp}$.

(8) Let $f : P^C \to P^C$ be a morphism such that $p^C \circ f = p^C$. Then from the sequence $(\mathbf{P}^C)$ of (7), there exists a unique morphism $g : A^C \to A^C$ such that $\alpha^C \circ g = f \circ \alpha^C$. Since $A^C \in \mathrm{CM}(\mathcal{C})$, as in (3) we have that $\underline{g}$ is an isomorphism. Then the triple of morphisms $(1_{\underline{C}}, \underline{f}, \underline{g})$ is an endomorphism of the induced triangle in $\mathcal{C}/\mathcal{X}$ of the sequence $(\mathbf{P}^C)$. Applying Yoneda's lemma we conclude directly that $\Omega_{\mathcal{X}}(\underline{f})$ is an isomorphism and $\underline{f}$ is a monomorphism. If the ideal $\mathcal{J}_{\mathcal{X}}(\mathrm{CM}(\mathcal{C}))$ is contained in $\mathcal{J}ac(\mathrm{CM}(\mathcal{C}))$ then the projection functor $\mathrm{CM}(\mathcal{C}) \to \mathrm{CM}(\mathcal{C}/\mathcal{X})$ reflects isomorphisms. Hence g is an isomorphism and this implies that f is an isomorphism. Hence any object of $\mathcal{C}$ has a minimal left $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$-approximation. If $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$ is Krull-Schmidt, then as in (3) we can apply the method of proof of Lemma 2.6 of [1], to conclude that any object of $\mathcal{C}$ has a minimal left $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$-approximation.

(9) It suffices to show that $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$ is reflective in $\mathcal{C}/\mathcal{X}$. For any $C \in \mathcal{C}$, we claim that the object P^C of part (7) is uniquely determined in $\mathcal{C}/\mathcal{X}$. Indeed if $0 \to C \xrightarrow{p^C} P^C \xrightarrow{\alpha^C} A^C \to 0$ and $0 \to C \xrightarrow{q^C} Q^C \xrightarrow{\beta^C} B^C \to 0$ are left $\mathcal{P}_{\mathcal{X}}^{\infty}(\mathcal{C})$-approximations of C with $A^C, B^C \in \mathrm{CM}(\mathcal{C})$, then we have the following commutative diagrams

$$\begin{array}{ccccc} C & \xrightarrow{p^C} & P^C & \xrightarrow{\alpha^C} & A^C \\ \| \downarrow & & f \downarrow & & \exists\rho \downarrow \\ C & \xrightarrow{q^C} & Q^C & \xrightarrow{\beta^C} & B^C \end{array} \qquad \begin{array}{ccccc} C & \xrightarrow{q^C} & Q^C & \xrightarrow{\beta^C} & B^C \\ \| \downarrow & & g \downarrow & & \exists\sigma \downarrow \\ C & \xrightarrow{p^C} & P^C & \xrightarrow{\alpha^C} & A^C. \end{array}$$

It is easy to see then that there are morphisms $k : A^C \to P^C$ and $m : B^C \to Q^C$ such that: $1_{P^C} - f \circ g = \alpha^C \circ k$ and $1_{Q^C} - g \circ f = \beta^C \circ m$. But from part (5) we have that $\underline{k} = 0$ and $\underline{m} = 0$ in $\mathcal{C}/\mathcal{X}$. Hence $\underline{f}$ and $\underline{g}$ are isomorphisms in $\mathcal{C}/\mathcal{X}$. We set $\mathbf{T}(\underline{C}) := \underline{P}^C$. If $Q \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$ and $\underline{f} : \underline{C} \to \underline{Q}$ is a morphism, then by part (7) there exists a morphism $g : P^C \to Q$ with $p^C \circ g = f$. Then $\underline{p}^C \circ \underline{g} = \underline{f}$. If $\underline{h} : \underline{P}^C \to \underline{Q}$ is another morphism with $\underline{p}^C \circ \underline{h} = \underline{f}$, then $p^C \circ (g - h) = k \circ m$ where $k : C \to X$, $m : X \to Q$ and $X \in \mathcal{X}$. Let $t : P^C \to X$ be a morphism with $p^C \circ t = k$. Then $p^C \circ (g-h) = p^C \circ t \circ m$, hence the morphism $g - h - t \circ m = \alpha^C \circ z$ for a unique morphism $z : A^C \to Q$. Since Q has finite $\mathcal{X}$-dimension, $\underline{z} = 0$. Hence $\underline{g} - \underline{h} = \underline{t} \circ \underline{m} = 0$ and $\underline{g} = \underline{h}$. So $\underline{p}^C : \underline{C} \to \mathbf{T}(\underline{C})$ is the reflection of $\underline{C}$ in $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$.

(10) The first part is a consequence of (9) and (1). Let T be the class of morphisms $\underline{f} : \underline{C} \to \underline{D}$ in $\mathcal{C}/\mathcal{X}$, such that in a triangle $\Omega_{\mathcal{X}}(\underline{D}) \to \underline{P} \to \underline{C} \xrightarrow{\underline{f}} \underline{D}$, the object $\underline{P} \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$. Obviously the functor $\mathbf{S} : \mathcal{C}/\mathcal{X} \to \mathrm{CM}(\mathcal{C}/\mathcal{X})$ sends the class T to isomorphisms. If $F : \mathcal{C}/\mathcal{X} \to \mathcal{D}$ is a functor with the same property, then since the morphism $\underline{\alpha}_C : \underline{A}_C \to \underline{C}$ of the sequence $(\mathbf{A}_C)$ belongs to T, $F(\underline{\alpha}_C)$ is an isomorphism. Define a functor $F^* : \mathrm{CM}(\mathcal{C}/\mathcal{X}) \to \mathcal{D}$ by $F^* = F|_{\mathrm{CM}(\mathcal{C}/\mathcal{X})}$. Then $\forall \underline{C} \in \mathcal{C}/\mathcal{X}$ we have $F^*\mathbf{S}(\underline{C}) = F^*(\underline{A}_C) = F(\underline{A}_C) \cong F(\underline{C})$. Hence $F^*\mathbf{S} = F$. Trivially F^* is the unique functor with this property. This implies that $\mathrm{CM}(\mathcal{C}/\mathcal{X})$ is equivalent to the localization $\mathcal{C}/\mathcal{X}/\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} = \mathcal{C}/\mathcal{X}[\mathrm{T}^{-1}]$.

(11) Follows easily from part (5) and parts (2), (9).

(12) The diagram (†) is obviously commutative. By the results of section 3, the morphism $\mathrm{K}_0(\mathrm{CM}(\mathcal{C}/\mathcal{X})) \to \mathrm{K}_0(\mathcal{C}/\mathcal{X})$ is an isomorphism, since $\mathrm{CM}(\mathcal{C}/\mathcal{X})$ is the stabilization of $\mathcal{C}/\mathcal{X}$. Hence the middle morphism of (†) is an epimorphism, which by [40] is actually an isomorphism, since $\widehat{\mathrm{CM}(\mathcal{C})} = \mathcal{C}$. □

Many of the consequences of the above Theorem are identical with the theory developed by Auslander-Buchweitz in [4], in case $\mathcal{C}$ is abelian. The crucial points of the Auslander-Buchweitz theory have been formulated by Hashimoto (see [30] or Subsection 4.3 below), in the concept of an Auslander-Buchweitz context in an abelian category. In our relative setting this concept can be formulated as follows.

Definition 4.4. Let $\mathcal{C}$ be an exact category and consider full additive subcategories $\mathcal{A}, \mathcal{B}, \mathcal{X}$ of $\mathcal{C}$. The triple $(\mathcal{A}, \mathcal{B}, \mathcal{X})$ is called an $\mathcal{X}$**−Auslander-Buchweitz context** (or $\mathcal{X}$*−AB-context*), if the following conditions are true:

- (α) $\mathcal{A}$ is $\mathcal{X}$−resolving.
- (β) $\mathcal{B}$ is closed under extensions of $\mathcal{X}$−exact admissible sequences, direct summands and cokernels of $\mathcal{X}$−monics.
- (γ) $\mathcal{A} \cap \mathcal{B} = \mathcal{X}$ is an $\mathcal{X}$−cogenerator of $\mathcal{A}$, and any $\mathcal{X}$−exact admissible short exact sequence $0 \to X \to C \to A \to 0$ with $X \in \mathcal{X}$ and $A \in \mathcal{A}$, splits.
- (δ) $\widehat{\mathcal{A}} = \mathcal{C}$.

Next we characterize the exact categories which admit an $\mathcal{X}$−AB-context.

Theorem 4.5. *The following are equivalent.*

1. *The subcategory $\mathcal{X}$ is part of an $\mathcal{X}$−Auslander-Buchweitz context $(\mathcal{A}, \mathcal{B}, \mathcal{X})$.*
2. *The category $\mathcal{C}$ is $\mathcal{X}$−Gorenstein.*

In this case: $\mathcal{A} = \mathcal{G}_{\mathcal{X}}(\mathcal{C}) = {}^{\perp}\mathcal{X} = {}^{\perp}\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) = \mathrm{CM}(\mathcal{C})$ *and* $\mathcal{B} = \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$.

Proof. (2) $\Rightarrow$ (1) is the content of Theorem 4.3. Conversely let $(\mathcal{A}, \mathcal{B}, \mathcal{X})$ be an $\mathcal{X}-$Auslander-Buchweitz context. Then $\widehat{\mathcal{A}} = \mathcal{C}$, and $\mathcal{X}$ is a cogenerator of $\mathcal{A}$. Since any $\mathcal{X}-$exact admissible sequence of the form $0 \to X \to C \to A \to 0$ with $X \in \mathcal{X}$ and $A \in \mathcal{A}$, splits, and since $\mathcal{A}$ is $\mathcal{X}-$resolving, we have directly that $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$. By Theorem 4.2, the subcategory $\mathcal{A}$ realizes the stabilization, so $\mathcal{C}$ is $\mathcal{X}-$Gorenstein.

It remains to show that $\mathcal{A} = \mathcal{G}_{\mathcal{X}}(\mathcal{C})$ and $\mathcal{B} = \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$. Let $A \in \mathcal{A}$. Then the $\mathcal{X}-$resolution of A is by definition contravariantly $\mathcal{X}-$exact and it is covariantly $\mathcal{X}-$exact since $\mathcal{A} = {}^{\perp}\mathcal{X}$. Since $\mathcal{X}$ is a cogenerator of $\mathcal{A}$, there exists an admissible contravariantly $\mathcal{X}-$exact sequence $0 \to A \to X^0 \to B \to 0$ with $B \in \mathcal{A}$. Since $B \in \mathcal{A}$ and $\mathcal{A} = {}^{\perp}\mathcal{X}$, this sequence is also contravariantly $\mathcal{X}-$exact. Continuing in this way we obtain a contravariantly $\mathcal{X}-$exact $\mathcal{X}-$coresolution of A. Hence A has a covariantly $\mathcal{X}-$exact $\mathcal{X}-$resolution and a contravariantly $\mathcal{X}-$exact $\mathcal{X}-$coresolution, i.e. $A \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. Conversely if $C \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$, then since the $\mathcal{X}-$resolution of C is contravariantly $\mathcal{X}-$exact, we have $C \in {}^{\perp}\mathcal{X} = \mathcal{A}$. Hence $\mathcal{A} = \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. Since $\mathcal{X} \subseteq \mathcal{B}$, by property (β) of definition 4.4 it follows easily that $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}) \subseteq \mathcal{B}$. Let $C \in \mathcal{B}$ and let $0 \to C \to P^C \to A^C \to 0$ be the left $\mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})-$approximation of C. By $(\beta), (\gamma)$ of definition 4.4, we have $A^C \in \mathcal{B} \cap \mathcal{A} = \mathcal{X}$, so the above sequence splits since it is $\mathcal{X}-$exact. Then $C \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$, since $\mathcal{B}$ is closed under direct summands. □

Hence for any $\mathcal{X}-$Gorenstein category $\mathcal{C}$, we always have: $\mathcal{G}_{\mathcal{X}}(\mathcal{C}) = \mathrm{CM}(\mathcal{C})$.

Definition 4.6. Let $\mathcal{A}$ be an $\mathcal{X}-$resolving subcategory of $\mathcal{C}$ and $C \in \mathcal{C}$. Then C *has finite $\mathcal{A}-$resolution dimension* iff there exists an $\mathcal{X}-$exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to C \to 0$ with $A_k \in \mathcal{A}$, $\forall k = 0, 1, ..., n$. In this case the least such integer n is called the *$\mathcal{A}-$resolution dimension* of C and is denoted by $\mathcal{A}-\mathrm{res.dim}C$. Otherwise we define $\mathcal{A}-\mathrm{res.dim}C = \infty$. The *global $\mathcal{A}-$resolution dimension* of $\mathcal{C}$ is defined by $\mathcal{A}-\mathrm{gl.res.dim}\mathcal{C} = \sup\{\mathcal{A}-\mathrm{res.dim}C; C \in \mathcal{C}\}$.

It is not difficult to see, using that $\mathcal{A}$ is $\mathcal{X}-$resolving, that $\mathcal{A}-\mathrm{res.dim}C$ is well-defined. Obviously $\widehat{\mathcal{A}} = \mathcal{C}$ iff any object of $\mathcal{C}$ has finite $\mathcal{A}-$resolution dimension.

Corollary 4.7. *Suppose that any left $\mathcal{X}-$approximation of an $\mathcal{X}-$Gorenstein object of $\mathcal{C}$ is admissible monic. Then the following are equivalent:*

(1) *The category $\mathcal{C}$ is $\mathcal{X}-$Gorenstein.*

(2) *Any object of $\mathcal{C}$ has finite $\mathcal{G}_{\mathcal{X}}(\mathcal{C})-$resolution dimension, i.e. $\mathcal{C} = \widehat{\mathcal{G}_{\mathcal{X}}(\mathcal{C})}$.*

Proof. (1) $\Rightarrow$ (2) Follows from Theorems 4.3 and 4.5. By hypothesis and Proposition 2.13, $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ is $\mathcal{X}-$resolving and the stable category $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ is triangulated. Then condition (2) implies that $\mathcal{C}$ is $\mathcal{X}-$Gorenstein by Theorem 4.2. □

Corollary 4.8. *Suppose that $\mathcal{X}$ is an $\mathcal{X}-$cogenerator of ${}^{\perp}\mathcal{X}$ and for any object $C \in \mathcal{C}$, there exists $d \geq 1$ such that $\underline{\mathcal{E}xt}^i_{\mathcal{X}}(C, \mathcal{X}) = 0, \forall i \geq d$.*

Then the triple $({}^{\perp}\mathcal{X}, \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}), \mathcal{X})$ is an $\mathcal{X}-$Auslander-Buchweitz context.

Proof. The last assumption implies that for any $C \in \mathcal{C}$ there exists $d \geq 0$ such that $\Omega^d_{\mathcal{X}}(C) \in {}^{\perp}\mathcal{X}$. Hence $\mathcal{C} = \widehat{{}^{\perp}\mathcal{X}}$ and then the triple $({}^{\perp}\mathcal{X}, \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C}), \mathcal{X})$ is an $\mathcal{X}-$Auslander-Buchweitz context by Theorem 4.2. □

4.2. **Costabilization and $\mathcal{X}-$Co-Gorenstein Exact categories.** We turn now our attention to the representation of the costabilization $\mathcal{R}(\mathcal{C}/\mathcal{X})$ of $\mathcal{C}/\mathcal{X}$ as a full subcategory of $\mathcal{C}/\mathcal{X}$. Let again $\mathcal{A}$ be an $\mathcal{X}$−resolving subcategory of $\mathcal{C}$, and consider the costabilization functor $\mathbf{R} : \mathcal{R}(\mathcal{C}/\mathcal{X}) \to \mathcal{C}/\mathcal{X}$. By the description of $\mathcal{R}(\mathcal{C}/\mathcal{X})$ in Theorem 3.11, it follows that the essential image $\mathrm{Im}(\mathbf{R})$ of $\mathbf{R}$ is the full subcategory $\Omega^{\infty}_{\mathcal{X}}(\mathcal{C}/\mathcal{X}) = \Omega^{\infty}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ of $\mathcal{C}/\mathcal{X}$ induced by all objects which are arbitrary $\mathcal{X}$−syzygies. Hence $\mathcal{A}/\mathcal{X} \subseteq \mathrm{Im}(\mathbf{R})$ iff $\mathcal{A} \subseteq \Omega^{\infty}_{\mathcal{X}}(\mathcal{C})$.

Definition 4.9. The exact category $\mathcal{C}$ is called **$\mathcal{X}$−Co-Gorenstein** if the left triangulated category $\mathcal{C}/\mathcal{X}$ is Co-Gorenstein.

Theorem 4.10. *The following are equivalent.*

(1) *$\mathcal{C}$ is $\mathcal{X}$−Co-Gorenstein.*

(2) $\Omega^{\infty}_{\mathcal{X}}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{X}$.

(3) *There exist an $\mathcal{X}$−resolving subcategory $\mathcal{A}$ of $\mathcal{C}$ satisfying the following:*

  (α) $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$.

  (β) *$\mathcal{X}$ is an $\mathcal{X}$−cogenerator of $\mathcal{A}$.*

  (γ) $\Omega^{\infty}_{\mathcal{X}}(\mathcal{C}) \subseteq \mathcal{A}$.

In one of the above equivalent statements is true, then $\mathcal{A} = \Omega^{\infty}_{\mathcal{X}}(\mathcal{C})$ and if the left $\mathcal{X}$−approximation of any $\mathcal{X}$−Gorenstein object is admissible monic, then $\mathcal{A} = \Omega^{\infty}_{\mathcal{X}}(\mathcal{C}) = \mathcal{G}_{\mathcal{X}}(\mathcal{C})$ and $\mathcal{A}/\mathcal{X} = \mathrm{CoCM}(\mathcal{C}/\mathcal{X})$.

Proof. The proof is a direct consequence of Theorem 3.17 and section 2. If $\mathcal{C}$ is $\mathcal{X}$−Co-Gorenstein, then by (3) it follows trivially that $\mathcal{A} \subseteq \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. If $A \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$, then by hypothesis and section 2, $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$ is a triangulated subcategory of $\mathcal{C}/\mathcal{X}$. Hence the exact inclusion $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} \hookrightarrow \mathcal{C}/\mathcal{X}$, factors uniquely through the inclusion $\mathcal{A}/\mathcal{X} \hookrightarrow \mathcal{C}/\mathcal{X}$. But then obviously $\mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X} \subseteq \mathcal{A}/\mathcal{X}$. Hence $\mathcal{A} = \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. □

Our next result which is a consequence of Corollary 3.18, shows that in some cases any $\mathcal{X}$−Gorenstein exact category is $\mathcal{X}$−Co-Gorenstein. For instance this is true if $\mathcal{C}$ is $\mathcal{X}$−Gorenstein with $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$−gl.res.dim$\mathcal{C} < \infty$.

Corollary 4.11. *Suppose that $\mathcal{X}$ is an $\mathcal{X}$−cogenerator of $\mathcal{A}$ and $\mathcal{A} \subseteq {}^{\perp}\mathcal{X}$. If $\Omega^{d}_{\mathcal{X}}(\mathcal{C}/\mathcal{X}) \subseteq \mathcal{A}/\mathcal{X}$ for some $d \geq 0$, then $\mathcal{C}$ is $\mathcal{X}$−Gorenstein and $\mathcal{X}$−Co-Gorenstein and there are triangle equivalences*

$$\mathcal{R}(\mathcal{C}/\mathcal{X}) \xrightarrow{\approx} \mathcal{A}/\mathcal{X} \xrightarrow{\approx} \mathcal{S}(\mathcal{C}/\mathcal{X}),$$

the functor $\Omega^{-d}_{\mathcal{X}}\Omega^{d}_{\mathcal{X}} : \mathcal{C}/\mathcal{X} \to \mathcal{A}/\mathcal{X}$ is the stabilization functor and the inclusion $i_{\mathcal{A}} : \mathcal{A}/\mathcal{X} \hookrightarrow \mathcal{C}/\mathcal{X}$ is the costabilization functor.

Remark 4.12. All the above results are true in case we deal with the dual situation of relative injectives, i.e. when $\mathcal{X}$ is a covariantly finite subcategory of the exact category $\mathcal{C}$ and any $\mathcal{X}$−monic is an admissible monic. Then the stable category $\mathcal{C}/\mathcal{X}$ is right triangulated and its stabilization $\mathcal{S}(\mathcal{C}/\mathcal{X})$ (costabilization $\mathcal{R}(\mathcal{C}/\mathcal{X})$) is defined. Then $\mathcal{C}$ is called $\mathcal{X}$−(Co-)Gorenstein if the category $\mathcal{C}/\mathcal{X}$ is (Co-)Gorenstein as a right triangulated category. We entrust to the reader the definition of an $\mathcal{X}$−Auslander-Buchweitz context in this case, which however we call an *dual* $\mathcal{X}$−Auslander-Buchweitz context.

4.3. Gorenstein and Co-Gorenstein Abelian Categories.

We assume in this subsection that $\mathcal{C}$ is an abelian category. Then a resolving subcategory of $\mathcal{C}$ is a full additive subcategory of $\mathcal{C}$ which is closed under extensions, kernels of epics and contains the projective objects. Dually a coresolving subcategory of $\mathcal{C}$ is a full additive subcategory of $\mathcal{C}$ which is closed under extensions, cokernels of monics and contains the injective objects. We denote always by $\mathcal{P}$ ($\mathcal{I}$) the full subcategories of projective (injective) objects and by $\mathcal{P}^{\infty}$ ($\mathcal{I}^{\infty}$) the full subcategory of objects with finite projective (injective) dimension. We call $\mathcal{P}-$cogenerators simply cogenerators.

An Auslander-Buchweitz context as defined in [30], is a triple $(\mathcal{A}, \mathcal{B}, \mathcal{X})$ of full subcategories of $\mathcal{C}$, such that the following conditions are true:

(i) $\mathcal{A}$ is closed under extensions, direct summands and kernels of epics. $\mathcal{B}$ is closed under extensions, direct summands and cokernels of monics.

(ii) $\mathcal{X} = \mathcal{A} \cap \mathcal{B}$ is a cogenerator of $\mathcal{A}$ with the property

$$\mathcal{X} \subseteq \mathcal{A}^{\perp} = \{C \in \mathcal{C} : \mathcal{E}xt^{i}(\mathcal{A}, C) = 0, \ \ \forall i \geq 1\}.$$

(iii) $\mathcal{C} = \widehat{\mathcal{A}}$, i.e. any object of $\mathcal{C}$ has a finite resolution by objects of $\mathcal{A}$.

We refer to [4], [30] for examples of Auslander-Buchweitz contexts, in Commutative Algebra, Algebraic Geometry and Ring Theory. Given an Auslander-Buchweitz context $(\mathcal{A}, \mathcal{B}, \mathcal{X})$ in the abelian category $\mathcal{C}$, we are interested in describing the stable category $\mathcal{C}/\mathcal{X}$, when the latter is left triangulated. This happens if $\mathcal{X}$ is in addition contravariantly finite. So the results of Subsection 4.1 give a clear picture of this situation. Now we restrict ourselves in the case when the abelian category $\mathcal{C}$ has enough projectives (resp. injectives), studying when $\mathcal{P}$ (resp. $\mathcal{I}$) is part of an Auslander-Buchweitz context $(\mathcal{A}, \mathcal{B}, \mathcal{P})$ (resp. dual Auslander-Buchweitz context $(\mathcal{A}, \mathcal{B}, \mathcal{I})$) in $\mathcal{C}$. We shall see that in this case our results are more complete. So from now on suppose that $\mathcal{C}$ has enough projectives. In case $\mathcal{C}$ has enough injectives we entrust to the reader to formulate the dual definitions of an $\mathcal{I}-$Gorenstein category, $\mathcal{B}-$gl.cores.dim$\mathcal{C}$ for a coresolving subcategory of $\mathcal{B}$ of $\mathcal{C}$, and of a dual Auslander-Buchweitz context. The dual results using injectives are also true.

Corollary 4.13. *The following are equivalent.*

(1) *The subcategory $\mathcal{P}$ is part of an Auslander-Buchweitz context $(\mathcal{A}, \mathcal{B}, \mathcal{P})$.*
(2) *The triple $({}^{\perp}\mathcal{P}, \mathcal{P}^{\infty}, \mathcal{P})$ is an Auslander-Buchweitz context.*
(3) *The category $\mathcal{C}$ is $\mathcal{P}-$Gorenstein.*
(4) *The natural functor ${}^{\perp}\mathcal{P}/\mathcal{P} \to \mathcal{D}^{b}(\mathcal{C})/\mathcal{K}^{b}(\mathcal{P})$ is a triangle equivalence.*
(5) *Any object of $\mathcal{C}$ has finite $\mathcal{P}-$Gorenstein resolution dimension.*
(6) *The natural functor $\mathcal{D}^{b}(\mathcal{G}_{\mathcal{P}}(\mathcal{C})) \to \mathcal{D}^{b}(\mathcal{C})$ is a triangle equivalence.*

In this case $\mathcal{A} = {}^{\perp}\mathcal{P} = {}^{\perp}\mathcal{P}^{\infty} = \mathcal{G}_{\mathcal{P}}(\mathcal{C})$ and $\mathcal{B} = \mathcal{P}^{\infty}$.

Proof. By our previous results, the statements (1) to (5) are equivalent. So it suffice to show that (3) is equivalent to (6). We view $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$ as an exact subcategory of $\mathcal{C}$ with enough projectives. If one of the statements (1) to (5) is true, then we know from Corollary 4.7 that $\widehat{\mathcal{G}_{\mathcal{P}}(\mathcal{C})} = \mathcal{C}$. Then by [28] we have that the natural functor $\mathcal{D}^{b}(\mathcal{G}_{\mathcal{P}}(\mathcal{C})) \to \mathcal{D}^{b}(\mathcal{C})$ is a triangle equivalence. Conversely if the above functor is a triangle equivalence, then the Verdier-quotients $\mathcal{D}^{b}(\mathcal{G}_{\mathcal{P}}(\mathcal{C}))/\mathcal{K}^{b}(\mathcal{P})$ and $\mathcal{D}^{b}(\mathcal{C})/\mathcal{K}^{b}(\mathcal{P}) = \mathcal{S}(\mathcal{C}/\mathcal{P})$ are triangle equivalent. Since $\mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P}$ is triangulated, by section 3, $\mathcal{D}^{b}(\mathcal{G}_{\mathcal{P}}(\mathcal{C}))/\mathcal{K}^{b}(\mathcal{P}) = \mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P}$. Hence the canonical functor $\mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P} \to \mathcal{S}(\mathcal{C}/\mathcal{P})$ is a triangle equivalence. By Theorem 4.2, $\mathcal{C}$ is $\mathcal{P}-$Gorenstein. □

Corollary 4.14. *Let $\mathcal{A}$, $\mathcal{B}$ be $\mathcal{P}-$Gorenstein abelian (or exact) categories with enough projectives. If there exists an equivalence $\mathcal{G}_{\mathcal{P}}(\mathcal{A}) \approx \mathcal{G}_{\mathcal{P}}(\mathcal{B})$, then there exists a triangle equivalence: $\mathcal{D}^b(\mathcal{A}) \approx \mathcal{D}^b(\mathcal{B})$.*

Corollary 4.15. *The following are equivalent.*

(1) *There exists an Auslander-Buchweitz context $(\mathcal{C}, \mathcal{B}, \mathcal{P})$.*
(2) *The category $\mathcal{C}$ is Frobenius.*
(3) *The natural functor $\mathcal{C}/\mathcal{P} \to \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P})$ is a triangle equivalence.*
(4) *The natural functor $\mathcal{K}_{Ac}(\mathcal{P}) \to \mathcal{C}/\mathcal{P}$ is a triangle equivalence.*
(5) *Any object of $\mathcal{C}$ is $\mathcal{P}-$Gorenstein.*

In this case $\mathcal{B} = \mathcal{P}$.

Theorem 4.16. *Let $\mathcal{C}$ be an abelian category with exact products and coproducts and with enough projectives and injectives. Then the following are equivalent.*

(1) *$\mathcal{C}$ is $\mathcal{P}-$Gorenstein.*
(2) *$\mathcal{C}$ is $\mathcal{I}-$Gorenstein.*
(3) *$({}^{\perp}\mathcal{P}, \mathcal{P}^{\infty}, \mathcal{P})$ is an Auslander-Buchweitz context.*
(4) *$(\mathcal{I}^{\perp}, \mathcal{I}^{\infty}, \mathcal{I})$ is a dual Auslander-Buchweitz context.*
(5) *$d := sup\{\text{p.d.}I : I \in \mathcal{I}\} = sup\{\text{i.d.}P : P \in \mathcal{P}\} < \infty$.*
(6) *$\mathcal{P}^{\infty} = \mathcal{I}^{\infty}$.*
(7) *$\mathcal{K}^b(\mathcal{P}) = \mathcal{K}^b(\mathcal{I})$ as full subcategories of $\mathcal{D}^b(\mathcal{C})$.*
(8) *The natural functor ${}^{\perp}\mathcal{P}/\mathcal{P} \to \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P})$ is a triangle equivalence.*
(9) *The natural functor $\mathcal{I}^{\perp}/\mathcal{I} \to \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{I})$ is a triangle equivalence.*
(10) *Any object of $\mathcal{C}$ has finite $\mathcal{P}-$Gorenstein resolution dimension.*
(11) *Any object of $\mathcal{C}$ has finite $\mathcal{I}-$Gorenstein resolution dimension.*
(12) *The natural functor $\mathcal{D}^b(\mathcal{G}_{\mathcal{P}}(\mathcal{C})) \to \mathcal{D}^b(\mathcal{C})$ is a triangle equivalence.*
(13) *The natural functor $\mathcal{D}^b(\mathcal{G}_{\mathcal{I}}(\mathcal{C})) \to \mathcal{D}^b(\mathcal{C})$ is a triangle equivalence.*

If one of the above equivalent conditions is true, then we have the following:

(α) *$\mathcal{C}$ is $\mathcal{P}-$Co-Gorenstein and $\mathcal{I}-$Co-Gorenstein and*

$$\mathcal{G}_{\mathcal{P}}(\mathcal{C})-\text{gl.res.dim}\mathcal{C} = \mathcal{G}_{\mathcal{I}}(\mathcal{C})-\text{gl.cores.dim}\mathcal{C} = d.$$

(β) *$\mathcal{P}^{\infty} = \mathcal{I}^{\infty}$ is functorially finite in $\mathcal{C}$, $\mathcal{G}_{\mathcal{P}}(\mathcal{C}) = \Omega^d(\mathcal{C})$ is contravariantly finite in $\mathcal{C}$, $\mathcal{G}_{\mathcal{I}}(\mathcal{C}) = \Sigma^d(\mathcal{C})$ is covariantly finite in $\mathcal{C}$, $\mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P}$ is coreflective in $\mathcal{C}/\mathcal{P}$ and $\mathcal{G}_{\mathcal{I}}(\mathcal{C})/\mathcal{I}$ is reflective in $\mathcal{C}/\mathcal{I}$.*

(γ) *If $\mathcal{P}$ is covariantly finite in $\mathcal{C}$, then $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$ is functorially finite in $\mathcal{C}$ and $\mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P}$ is reflective in $\mathcal{C}/\mathcal{P}$. If $\mathcal{I}$ is contravariantly finite in $\mathcal{C}$, then $\mathcal{G}_{\mathcal{I}}(\mathcal{C})$ is functorially finite in $\mathcal{C}$ and $\mathcal{G}_{\mathcal{I}}(\mathcal{C})/\mathcal{I}$ is coreflective in $\mathcal{C}/\mathcal{I}$.*

(δ) *There exist triangle equivalences:*

$$\mathcal{K}_{Ac}(\mathcal{P}) \approx \mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P} \approx {}^{\perp}\mathcal{P}/\mathcal{P} \approx {}^{\perp}(\mathcal{P}^{\infty})/\mathcal{P} \approx \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P}) \approx$$
$$\approx \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{I}) \approx (\mathcal{I}^{\infty})^{\perp}/\mathcal{I} \approx \mathcal{I}^{\perp}/\mathcal{I} \approx \mathcal{G}_{\mathcal{I}}(\mathcal{C})/\mathcal{I} \approx \mathcal{K}_{Ac}(\mathcal{I}).$$

(ϵ) *The costabilization functors are the inclusions*

$$\mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P} \hookrightarrow \mathcal{C}/\mathcal{P} \quad \text{and} \quad \mathcal{G}_{\mathcal{I}}(\mathcal{C})/\mathcal{I} \hookrightarrow \mathcal{C}/\mathcal{I}$$

and the stabilization functors are given by

$$\Omega^{-d}\Omega^d : \mathcal{C}/\mathcal{P} \to \mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P} \quad \text{and} \quad \Sigma^{-d}\Sigma^d : \mathcal{C}/\mathcal{I} \to \mathcal{G}_{\mathcal{I}}(\mathcal{C})/\mathcal{I}.$$

(ζ) $\mathcal{C}$ *is Frobenius iff* $\mathcal{G}_{\mathcal{P}}(\mathcal{C}) = \mathcal{G}_{\mathcal{I}}(\mathcal{C})$.

Proof. By our previous results we have that (1) is equivalent to (3), (8), (10) and (12). By duality (2) is equivalent to (4), (9), (11) and (13). By the results of [25], [18] we have (5) $\Leftrightarrow$ (6). Also it is easy to see that (7) is equivalent to (6). So it suffices to show the equivalence (1) $\Leftrightarrow$ (5). Then (2) $\Leftrightarrow$ (5) will follow by duality. We set $d = sup\{\text{p.d.}I : I \in \mathcal{I}\}$ and $\delta = sup\{\text{i.d.}P : P \in \mathcal{P}\}$.

(1) $\Rightarrow$ (5) Suppose that $\mathcal{C}$ is $\mathcal{P}-$Gorenstein. By Theorem 4.2 we have that $\forall C \in \mathcal{C}$ there exists $r_C \geq 0$ such that $\Omega^{r_C}(C) \in {}^{\perp}\mathcal{P}$. Hence $\forall C \in \mathcal{C} : \mathcal{E}xt^i(\Omega^{r_C}(C), \mathcal{P}) = 0, \forall i \geq 1$ or equivalently $\mathcal{E}xt^{i+r_C}(C, \mathcal{P}) = 0, \forall i \geq 1$. This implies that any projective object has finite injective dimension, i.e. $\mathcal{P} \subseteq \mathcal{I}^{\infty}$. If I is an injective object, then the left $\mathcal{P}^{\infty}-$approximation ($\mathbf{P}^{\mathbf{I}}$) of Theorem 4.3 splits, hence I has finite pojective dimension as a direct summand of P^I, so $\mathcal{I} \subseteq \mathcal{P}^{\infty}$. By the arguments of [25], [18] it follows that $d = \delta < \infty$.

(5) $\Rightarrow$ (1) Since $\delta = d < \infty$, for any $C \in \mathcal{C}$ we have $\Omega^d(C) \in {}^{\perp}\mathcal{P}$, hence $\Omega^d(\mathcal{C}) = {}^{\perp}\mathcal{P}$. Let $A \in {}^{\perp}\mathcal{P}$ and consider an injective resolution $0 \to A \to I^0 \to I^1 \to \cdots$ of A, with corresponding cosyzygies $B^n = \Sigma^n(A), \forall n \geq 1$. The exact sequence $0 \to A \to I^0 \to B^1 \to 0$ induces a triangle $\Omega(\underline{B}^1) \to \underline{A} \to \underline{I}^0 \to \underline{B}^1$ in $\mathcal{C}/\mathcal{P}$. Applying the stabilization functor $\mathbf{S}$ to this triangle and using that $\mathcal{I} \subseteq \mathcal{P}^{\infty}$, we have $\mathbf{S}(\underline{A}) \cong \mathbf{S}\Omega(\underline{B}^1) = \mathbf{S}\Omega(\underline{\Sigma(A)})$. Inductively we have $\mathbf{S}(\underline{A}) \cong \mathbf{S}\Omega^k(\underline{B}^k) = \mathbf{S}\Omega^k(\underline{\Sigma^k(A)})$, $\forall k \geq 1$, in particular $\mathbf{S}(\underline{A}) \cong \mathbf{S}\Omega^{d+1}(\underline{B}^{d+1}) = \mathbf{S}\Omega^{d+1}(\underline{\Sigma^{d+1}(A)})$. Since $\underline{\Omega^{d+1}(\Sigma^{d+1}(A))} \in {}^{\perp}\mathcal{P}/\mathcal{P}$ and since the stabilization functor $\mathbf{S}$ is fully faithful restricted to ${}^{\perp}\mathcal{P}/\mathcal{P}$, we have $\underline{A} \cong \Omega^{d+1}(\underline{\Sigma^{d+1}(A)})$. Setting $A' = \Omega^d\Sigma^{d+1}(A) \in {}^{\perp}\mathcal{P}$ we see that $A' \in {}^{\perp}\mathcal{P}$ and $\Omega(\underline{A}') = \underline{A}$. Hence $\mathcal{P}$ is a cogenerator of ${}^{\perp}\mathcal{P}$. Since $\Omega^d(\mathcal{C}) = {}^{\perp}\mathcal{P}$, by Corollary 4.11 we have that $\mathcal{C}$ is $\mathcal{P}-$Gorenstein, $\mathcal{P}-$Co-Gorenstein and $\mathcal{G}_{\mathcal{P}}(\mathcal{C})-$gl.res.dim$\mathcal{C} = d < \infty$.

If one of the equivalent conditions (1), (13) is true, then as the above proof shows, we have $\Omega^d(\mathcal{C}) = {}^{\perp}\mathcal{P}$ and $\Sigma^d(\mathcal{C}) = \mathcal{I}^{\perp}$. Hence by Corollary 4.11 and its dual, $\mathcal{C}$ is $\mathcal{P}-$Co-Gorenstein and $\mathcal{I}-$Co-Gorenstein. Also the above proof shows that $\mathcal{P} \subseteq \mathcal{I}^{\infty}$ and $\mathcal{I} \subseteq \mathcal{P}^{\infty}$, hence trivially $\mathcal{P}^{\infty} = \mathcal{I}^{\infty}$ and by the Theorem 4.3 and its dual we see that these categories are functorially finite. Since $\mathcal{C}$ is $\mathcal{P}-$Gorenstein, by Theorem 4.2 the category $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$ is contravariantly finite in $\mathcal{C}$ and $\mathcal{G}_{\mathcal{P}}(\mathcal{C})/\mathcal{P}$ is coreflective in $\mathcal{C}/\mathcal{P}$. Dually $\mathcal{G}_{\mathcal{I}}(\mathcal{C})$ is covariantly finite in $\mathcal{C}$ and $\mathcal{G}_{\mathcal{I}}(\mathcal{C})/\mathcal{I}$ is reflective in $\mathcal{C}/\mathcal{I}$. If $\mathcal{P}$ is covariantly finite in $\mathcal{C}$, then by section 2, we have that $\Omega^d(\mathcal{C})$ is covariantly finite. Since $\mathcal{G}_{\mathcal{P}}(\mathcal{C}) = {}^{\perp}\mathcal{P} = \Omega^d(\mathcal{C})$, we conclude that $\mathcal{G}_{\mathcal{P}}(\mathcal{C})$ is functorially finite. Dually if $\mathcal{I}$ is contravariantly finite, then $\mathcal{G}_{\mathcal{I}}(\mathcal{C})$ is functorially finite. The other assertions in parts $(\alpha), (\beta), (\gamma), (\delta), (\epsilon)$ are consequences of our previous results.

Finally if $\mathcal{G}_{\mathcal{P}}(\mathcal{C}) = \mathcal{G}_{\mathcal{I}}(\mathcal{C})$, and if I is an injective object, the I is $\mathcal{P}-$Gorenstein. But since I also has finite projective dimension it is projective by Theorem 4.3. Hence any projective is injective. By duality any injective is projective and $\mathcal{C}$ is Frobenius. Conversely if $\mathcal{C}$ is Frobenius, then obviously $\mathcal{G}_{\mathcal{P}}(\mathcal{C}) = \mathcal{C} = \mathcal{G}_{\mathcal{I}}(\mathcal{C})$. □

The following are consequences of Theorem 4.10 and Corollary 4.8.

Corollary 4.17. *The abelian category* $\mathcal{C}$ *with enough projectives (injectives) is* $\mathcal{P}-$*Co-Gorenstein (* $\mathcal{I}-$*Co-Gorenstein) iff* $\Omega^{\infty}(\mathcal{C}) \subseteq {}^{\perp}\mathcal{P}$ ($\Sigma^{\infty}(\mathcal{C}) \subseteq \mathcal{I}^{\perp}$).

Corollary 4.18. *Let $\mathcal{C}$ be an abelian category with enough projectives and enough injectives and let $\mathcal{A}$, resp. $\mathcal{B}$, be a resolving, resp. coresolving, subcategory of $\mathcal{C}$.*

(1) *Suppose that $\mathcal{P}$ is a cogenerator of $\mathcal{A}$ and $\mathcal{A} \subseteq {}^{\perp}\mathcal{P}$. If there exists $d \geq 0$, such that $\Omega^d(\mathcal{C}) \subseteq \mathcal{A}$, then there are triangle equivalences*
$$\mathcal{K}_{Ac}(\mathcal{P}) \approx \mathcal{A}/\mathcal{P} \approx \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{P}),$$
the functor $\Omega^{-d}\Omega^d : \mathcal{C}/\mathcal{P} \longrightarrow \mathcal{A}/\mathcal{P}$ is the stabilization functor and the inclusion $i_{\mathcal{A}} : \mathcal{A}/\mathcal{P} \hookrightarrow \mathcal{C}/\mathcal{P}$ is the costabilization functor.

(2) *Suppose that $\mathcal{I}$ is generator of $\mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{I}^{\perp}$. If there exists $d \geq 0$, such that $\Sigma^d(\mathcal{C}) \subseteq \mathcal{B}$, then there are triangle equivalences*
$$\mathcal{K}_{Ac}(\mathcal{I}) \approx \mathcal{B}/\mathcal{I} \approx \mathcal{D}^b(\mathcal{C})/\mathcal{K}^b(\mathcal{I}),$$
the functor $\Sigma^{-d}\Sigma^d : \mathcal{C}/\mathcal{I} \longrightarrow \mathcal{B}/\mathcal{I}$ is the stabilization functor and the inclusion $i_{\mathcal{B}} : \mathcal{B}/\mathcal{I} \hookrightarrow \mathcal{C}/\mathcal{I}$ is the costabilization functor.

(3) *Suppose that $\mathcal{C}$ in addition has exact products and coproducts and $\mathcal{A}$ satisfies the conditions in (1) or $\mathcal{B}$ satisfies the conditions in (2). Then $\mathcal{C}$ is $\mathcal{P}-$Gorenstein, $\mathcal{P}-$Co-Gorenstein, $\mathcal{I}-$Gorenstein, $\mathcal{I}-$Co-Gorenstein.*

Remark 4.19. If in the definition 4.4 of an $\mathcal{X}-$Auslander-Buchweitz context we remove condition (δ) that $\mathcal{C} = \widehat{\mathcal{A}}$ and we add the condition $\mathcal{B} \subseteq \widehat{\mathcal{A}}$, then the triple $(\mathcal{A}, \mathcal{B}, \mathcal{X})$ is called a *weak* $\mathcal{X}-$Auslander-Buchweitz context, see [30]. All the results of this section are true for weak $\mathcal{X}-$Auslander-Buchweitz contexts in $\mathcal{C}$, but now $\mathcal{C}$ has to be replaced everywhere by $\widehat{\mathcal{A}}$.

5. Complete Resolutions and Complete (Co-)Homological Functors

5.1. **Complete Resolutions and Complete Extension Functors.** Troughout this section we fix a pair $(\mathcal{C}, \mathcal{X})$, where $\mathcal{C}$ is an additive category and $\mathcal{X} \subseteq \mathcal{C}$ is a full contravariantly finite additive subcategory of $\mathcal{C}$ which is closed under direct summands, such that any $\mathcal{X}-$epic has a kernel. Then $\mathcal{C}/\mathcal{X}$ is left triangulated.

Definition 5.1. The **complete $\mathcal{X}-$extension bifunctors** of $\mathcal{C}$ are defined by
$$\widehat{\operatorname{Ext}}^n_{\mathcal{X}}(-,-) \ : \ \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{A}b,$$
$$\widehat{\operatorname{Ext}}^n_{\mathcal{X}}(A,B) = \operatorname{Hom}_{\mathcal{S}(\mathcal{C}/\mathcal{X})}[(\underline{A},-n),(\underline{B},0)], \ \forall n \in \mathbb{Z}.$$

If $\mathbf{S} : \mathcal{C}/\mathcal{X} \longrightarrow \mathcal{S}(\mathcal{C}/\mathcal{X})$ is the stabilization functor, then since $(\underline{A},-n) = \tilde{\Omega}^n_{\mathcal{X}}(\underline{A},0) = \tilde{\Omega}^n_{\mathcal{X}}\mathbf{S}(\underline{A})$, it follows that
$$\widehat{\operatorname{Ext}}^n_{\mathcal{X}}(A,B) = \operatorname{Hom}_{\mathcal{S}(\mathcal{C}/\mathcal{X})}[\tilde{\Omega}^n_{\mathcal{X}}\mathbf{S}(\underline{A}),\mathbf{S}(\underline{B})] = \operatorname{Hom}_{\mathcal{S}(\mathcal{C}/\mathcal{X})}[\mathbf{S}(\underline{A}),\tilde{\Omega}^{-n}_{\mathcal{X}}\mathbf{S}(\underline{B})] \cong$$
$$\cong \varinjlim_{k,k+n\geq 0} \mathcal{C}/\mathcal{X}[\Omega^{k+n}_{\mathcal{X}}(\underline{A}),\Omega^k_{\mathcal{X}}(\underline{B})].$$

The above definition is inspired by the definition of the stable homotopy groups of spheres and CW-complexes in algebraic topology, see for instance [24], [38]. We shall see in the next section that the complete $\mathcal{X}-$Extension Bifunctors can be regarded as generalized Tate-Vogel cohomology functors.

Remark 5.2. In case $\mathcal{X}$ is a covariantly finite subcategory of $\mathcal{C}$ and any $\mathcal{X}-$monic has a cokernel in $\mathcal{C}$, so the stable category $\mathcal{C}/\mathcal{X}$ is right triangulated, then we can define the *complete $\mathcal{X}$-Extension Bifunctors* as follows:

$$\widehat{\mathrm{Ext}}_n^{\mathcal{X}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{A}b,$$

$$\widehat{\mathrm{Ext}}_n^{\mathcal{X}}(A,B) = \mathrm{Hom}_{\mathcal{S}(\mathcal{C}/\mathcal{X})}[(\underline{A},0),(\underline{B},n)] = \mathcal{S}(\mathcal{C}/\mathcal{X})[\mathrm{S}(\underline{A}),\tilde{\Sigma}^n_{\mathcal{X}}\mathrm{S}(\underline{B})] =$$

$$= \varinjlim_{k,k+n\geq 0} \mathcal{C}/\mathcal{X}(\Sigma^k_{\mathcal{X}}(\underline{A}),\Sigma^{k+n}_{\mathcal{X}}(\underline{B})), \ \forall n \in \mathbb{Z},$$

where $\mathcal{S}(\mathcal{C}/\mathcal{X})$ is the stabilization of $\mathcal{C}/\mathcal{X}$ as a *right* triangulated category, i.e. inverting the suspension functor $\Sigma_{\mathcal{X}}$ to an automorphism $\tilde{\Sigma}_{\mathcal{X}}$ of $\mathcal{S}(\mathcal{C}/\mathcal{X})$. Observe that in case $\mathcal{X}$ is functorially finite, any $\mathcal{X}$-monic has a cokernel and any $\mathcal{X}$-epic has a kernel in $\mathcal{C}$, then the above complete $\mathcal{X}$-extension bifunctors are different, since in general the stabilizations of the stable category, first as a left triangulated category and second as a right triangulated category are not equivalent.

Remark 5.3. (1) From the above description we see that we can define Yoneda products. Indeed if $\tilde{\alpha} \in \widehat{\mathrm{Ext}}^n_{\mathcal{X}}(A,B)$ and $\tilde{\beta} \in \widehat{\mathrm{Ext}}^m_{\mathcal{X}}(B,C)$, then $\tilde{\alpha} : \tilde{\Omega}^n_{\mathcal{X}}\mathrm{S}(\underline{A}) \to \mathrm{S}(\underline{B})$ and $\tilde{\beta} : \tilde{\Omega}^m_{\mathcal{X}}\mathrm{S}(\underline{B}) \to \mathrm{S}(\underline{C})$. Then define the Yoneda product $\tilde{\alpha} \odot \tilde{\beta}$ as the composition $\tilde{\Omega}^m(\tilde{\alpha}) \circ \tilde{\beta} \in \widehat{\mathrm{Ext}}^{n+m}_{\mathcal{X}}(A,C)$. In this way for any object $A \in \mathcal{C}$, we obtain a $\mathbb{Z}$-graded ring $\widehat{\mathrm{Ext}}^*_{\mathcal{X}}(A,A)$.

(2) $\widehat{\mathrm{Ext}}^n_{\mathcal{X}}(\Omega^r_{\mathcal{X}}(A),\Omega^s_{\mathcal{X}}(B)) = \widehat{\mathrm{Ext}}^{n+r-s}_{\mathcal{X}}(A,B), \ \forall n \in \mathbb{Z}, \ \forall r,s \geq 0.$

(3) If $\mathcal{C}/\mathcal{X}$ is triangulated, then since $\mathcal{C}/\mathcal{X}$ is triangle equivalent to $\mathcal{S}(\mathcal{C}/\mathcal{X})$:

$$\widehat{\mathrm{Ext}}^n_{\mathcal{X}}(A,B) = \mathrm{Hom}_{\mathcal{C}/\mathcal{X}}[\Omega^n(\underline{A}),\underline{B}] \cong \mathrm{Hom}_{\mathcal{C}/\mathcal{X}}[\underline{A},\Omega^{-n}(\underline{B})], \ \forall n \in \mathbb{Z}.$$

The next result shows in particular that the complete $\mathcal{X}$-cohomology of $\mathcal{C}$ is non-trivial only if $\mathcal{X}$-gl.dim$\mathcal{C} = \infty$.

Proposition 5.4. (1) *If A or $B \in \mathcal{P}^{\infty}_{\mathcal{X}}(\mathcal{C})$, then $\widehat{\mathrm{Ext}}^*_{\mathcal{X}}(A,B) = 0$. In particular if $\mathcal{X}$-gl.dim$\mathcal{C} < \infty$, then $\widehat{\mathrm{Ext}}^*_{\mathcal{X}}(-,-) = 0$.*

(2) *Let $A \xrightarrow{g} B \xrightarrow{f} C$ be a sequence in $\mathcal{C}$ with f an $\mathcal{X}$-epic and $g = ker(f)$. Then for any $D \in \mathcal{C}$, there are long exact sequences:*

$$\cdots \to \widehat{\mathrm{Ext}}^1_{\mathcal{X}}(A,D) \to \widehat{\mathrm{Ext}}^0_{\mathcal{X}}(C,D) \to \widehat{\mathrm{Ext}}^0_{\mathcal{X}}(B,D) \to \widehat{\mathrm{Ext}}^0_{\mathcal{X}}(A,D) \to$$

$$\to \widehat{\mathrm{Ext}}^{-1}_{\mathcal{X}}(C,D) \to \widehat{\mathrm{Ext}}^{-1}_{\mathcal{X}}(B,D) \to \widehat{\mathrm{Ext}}^{-1}_{\mathcal{X}}(A,D) \to \widehat{\mathrm{Ext}}^{-2}_{\mathcal{X}}(C,D) \to \cdots \qquad (\alpha)$$

$$\cdots \to \widehat{\mathrm{Ext}}^{-1}_{\mathcal{X}}(D,C) \to \widehat{\mathrm{Ext}}^0_{\mathcal{X}}(D,A) \to \widehat{\mathrm{Ext}}^0_{\mathcal{X}}(D,B) \to \widehat{\mathrm{Ext}}^0_{\mathcal{X}}(D,C)$$

$$\to \widehat{\mathrm{Ext}}^1_{\mathcal{X}}(D,A) \to \widehat{\mathrm{Ext}}^1_{\mathcal{X}}(D,B) \to \widehat{\mathrm{Ext}}^1_{\mathcal{X}}(D,C) \to \widehat{\mathrm{Ext}}^2_{\mathcal{X}}(D,A) \to \cdots \qquad (\beta)$$

Proof. (2) The sequence $A \to B \to C$ induces a triangle $\Omega_{\mathcal{X}}(\underline{C}) \to \underline{A} \to \underline{B} \to \underline{C}$ in $\mathcal{C}/\mathcal{X}$ by the definition of the triangulation $\Delta_{\mathcal{X}}$ in [14]. Applying the stabilization functor $\mathrm{S} : \mathcal{C}/\mathcal{X} \to \mathcal{S}(\mathcal{C}/\mathcal{X})$ we have a triangle $(\underline{C},-1) \to (\underline{A},0) \to (\underline{B},0) \to (\underline{C},0)$ in $\mathcal{S}(\mathcal{C}/\mathcal{X})$. Since $\mathcal{S}(\mathcal{C}/\mathcal{X})$ is triangulated, applying to this triangle the cohomological functor $\mathrm{Hom}_{\mathcal{S}(\mathcal{C}/\mathcal{X})}[?,(\underline{D},0)]$ we get the long exact sequence (α) and applying the homological functor $\mathrm{Hom}_{\mathcal{S}(\mathcal{C}/\mathcal{X})}[(\underline{D},0),?]$ we get the long exact sequence (β). Part (1) follows directly from the definition. □

Our purpose here is to compute the complete $\mathcal{X}$-extension bifunctors by using suitable resolutions. To simplify things, we suppose throughout that $\mathcal{C}$ is an exact category, any $\mathcal{X}$-epic is an admissible epic and any left $\mathcal{X}$-approximation (which always exist) of an $\mathcal{X}$-Gorenstein object, is an admissible monic.

Definition 5.5. A **complete $\mathcal{X}$-resolution** of $A \in \mathcal{C}$ is a functorially $\mathcal{X}$-exact complex

$$\mathbf{X}_c^\bullet(A) \qquad \cdots \to X^{-n} \to \cdots \to X^{-1} \to X^0 \to X^1 \to \cdots \to X^n \to \cdots$$

for which there exists $t := t_A \in \mathbb{Z}$, such that the complex $\cdots \to X^{t-2} \to X^{t-1} \to X^t \to 0$ coincides with a part $\cdots \to X_A^s \to X_A^{s-1} \to X_A^s$ of an $\mathcal{X}$-resolution $\cdots \to X_A^s \to X_A^{s-1} \to \cdots \to X_A^1 \to X_A^0 \to A \to 0$ of A.

The following is a direct consequence of the definition.

Lemma 5.6. *An object A has a complete $\mathcal{X}$-resolution iff there exists $t \geq 0$ such that $\Omega_{\mathcal{X}}^t(A)$ is an $\mathcal{X}$-Gorenstein object or equivalently if A has finite $\mathcal{X}$-Gorenstein resolution dimension, i.e. $A \in \widehat{\mathcal{G}_{\mathcal{X}}(\mathcal{C})}$.*

In particular any $\mathcal{X}$-Gorenstein object A has a complete $\mathcal{X}$-resolution (choose $t = t_A = 0$ in definition 5.5). For simplicity we set $\mathcal{G}(\mathcal{C}/\mathcal{X}) := \mathcal{G}_{\mathcal{X}}(\mathcal{C})/\mathcal{X}$.

Theorem 5.7. *The following are equivalent:*

(1) *The category $\mathcal{C}$ is $\mathcal{X}$-Gorenstein.*
(2) *Any object of $\mathcal{C}$ has a complete $\mathcal{X}$-resolution.*

Proof. (1) $\Rightarrow$ (2) If $\mathcal{C}$ is $\mathcal{X}$-Gorenstein, then by section 4, we know that there exists $d \geq 0$, such that $\Omega_{\mathcal{X}}^d(A)$ is an $\mathcal{X}$-Gorenstein object. Since $\Omega_{\mathcal{X}}^d(A)$ is an arbitrary $\mathcal{X}$-syzygy object, there exists a functorially $\mathcal{X}$- exact complex $0 \to \Omega_{\mathcal{X}}^d(A) \to X^0 \to X^1 \to \cdots$. Composing the complex $0 \to X^0 \to X^1 \to \cdots$ with a deleted $\mathcal{X}$-resolution $\cdots \to X^{-2} \to X^{-1} \to 0$ of $\Omega_{\mathcal{X}}^d(A)$, since $\Omega_{\mathcal{X}}^d(A) \in {}^{\perp}\mathcal{X}$, we obtain a functorially $\mathcal{X}$-exact complex, which obviously is a complete $\mathcal{X}$-resolution of A.

(2) $\Rightarrow$ (1) Suppose that any object C of $\mathcal{C}$ has a complete $\mathcal{X}$-resolution $\mathbf{X}_c^\bullet(C)$. Then by the above Lemma, there exists $d = d_C \geq 0$, such that $\Omega_{\mathcal{X}}^d(C)$ is an $\mathcal{X}$-Gorenstein object. By Corollary 4.7, we have that $\mathcal{C}$ is $\mathcal{X}$-Gorenstein. □

Corollary 5.8. *Suppose that $\mathcal{C}$ is $\mathcal{X}$-Gorenstein. Then $\forall A, B \in \mathcal{C}$:*

$$\widehat{\mathrm{Ext}}_{\mathcal{X}}^n(A,B) \cong \mathcal{G}(\mathcal{C}/\mathcal{X})[\Omega_{\mathcal{X}}^{n-t+r}\Omega_{\mathcal{X}}^t(\underline{A}), \Omega_{\mathcal{X}}^r(\underline{B})], \quad \forall n \in \mathbb{Z},$$

where $t, r \geq 0$ are such that: $\Omega_{\mathcal{X}}^t(\underline{A}), \Omega_{\mathcal{X}}^r(\underline{B}) \in \mathcal{G}(\mathcal{C}/\mathcal{X})$.

Proof. This follows from the description of the stabilization of the $\mathcal{X}$-Gorenstein category $\mathcal{C}$ as the stable category $\mathcal{G}(\mathcal{C}/\mathcal{X})$ in section 4. □

Corollary 5.9. *Suppose that there exists $d \geq 0$, such that $\Omega_{\mathcal{X}}^d(\mathcal{C}) \subseteq \mathcal{G}_{\mathcal{X}}(\mathcal{C})$. Then the complete $\mathcal{X}$-extension functors are given by:*

$$\widehat{\mathrm{Ext}}_{\mathcal{X}}^n(-,-) \cong \mathcal{G}(\mathcal{C}/\mathcal{X})[\Omega_{\mathcal{X}}^n\Omega_{\mathcal{X}}^d(-), \Omega_{\mathcal{X}}^d(-)], \quad \forall n \in \mathbb{Z}.$$

5.2. **Complete (Co-)Homological Functors.** Our purpose in this subsection is to define under some assumptions, another sequence of (co-)homological functors, using resolutions of objects. Consider an object A in $\mathcal{C}$, let

$$\mathbf{X}^{\bullet}_A \quad \cdots \to X^{i+1}_A \xrightarrow{f^{i+1}_A} X^i_A \to \cdots \to X^1_A \xrightarrow{f^1_A} X^0_A \xrightarrow{\chi_A} A \to 0$$

be an $\mathcal{X}$-resolution of A and let

$$\mathbf{X}^A_{\bullet} \quad 0 \to A \xrightarrow{\chi^A} X^A_0 \xrightarrow{f^A_1} X^A_1 \to \cdots \to X^A_i \xrightarrow{f^A_{i+1}} X^A_{i+1} \to \cdots$$

be an $\mathcal{X}$-coresolution of A. The complex

$$\mathbf{X}^{\bullet}(A) \qquad \cdots \to X^1_A \xrightarrow{f^1_A} X^0_A \xrightarrow{\vartheta_A} X^A_0 \xrightarrow{f^A_1} X^A_1 \to \cdots$$

where $\vartheta_A := \chi_A \circ \chi^A$, is called an $\mathcal{X}$*-biresolution* of A. Since we consider all our complexes as cohomological, X^A_0 is in degree 0 and X^0_A is in degree -1.

Let A, B be two objects in $\mathcal{C}$ having $\mathcal{X}$-biresolutions. We define $\mathcal{X}$-homology groups $\widehat{\mathrm{H}}^{\mathcal{X}}_n(A,B)$ and $\mathcal{X}$-cohomology groups $\widehat{\mathrm{H}}^n_{\mathcal{X}}(A,B)$, as follows:

$$\widehat{\mathrm{H}}^{\mathcal{X}}_n(A,B) = \mathrm{H}^n(A, \mathbf{X}^{\bullet}(B)) \quad \text{and} \quad \widehat{\mathrm{H}}^n_{\mathcal{X}}(A,B) = \mathrm{H}^n(\mathbf{X}^{\bullet}(A), B).$$

Let $0 \to A \to B \to C \to 0$ be a functorially $\mathcal{X}$-exact sequence in $\mathcal{C}$ of objects having $\mathcal{X}$-biresolutions. By section 2, $\forall D \in \mathcal{C}$, the following sequences are exact:

$$\cdots \to \widehat{\mathrm{H}}^{\mathcal{X}}_n(D,A) \to \widehat{\mathrm{H}}^{\mathcal{X}}_n(D,B) \to \widehat{\mathrm{H}}^{\mathcal{X}}_n(D,C) \to \widehat{\mathrm{H}}^{\mathcal{X}}_{n-1}(D,A) \to \cdots$$

$$\cdots \to \widehat{\mathrm{H}}^n_{\mathcal{X}}(C,D) \to \widehat{\mathrm{H}}^n_{\mathcal{X}}(B,D) \to \widehat{\mathrm{H}}^n_{\mathcal{X}}(A,D) \to \widehat{\mathrm{H}}^{n+1}_{\mathcal{X}}(C,D) \to \cdots$$

Our aim is to study the relationship between the complete $\mathcal{X}$-(co)homology functors $\widehat{\mathrm{H}}^{\mathcal{X}}_n(A,B)$, $\widehat{\mathrm{H}}^n_{\mathcal{X}}(A,B)$ and the complete $\mathcal{X}$-extensions functors $\widehat{\mathrm{Ext}}^n_{\mathcal{X}}(A,B)$, and $\widehat{\mathrm{Ext}}^{\mathcal{X}}_n(A,B)$, $n \in \mathbb{Z}$, when all these functors make sense.

So let $\mathcal{X}$ be functorially finite in $\mathcal{C}$, any $\mathcal{X}$-epic has a kernel and any $\mathcal{X}$-monic has a cokernel in $\mathcal{C}$. As always $\Sigma_{\mathcal{X}}$ is the left adjoint of $\Omega_{\mathcal{X}}$ in $\mathcal{C}/\mathcal{X}$ as in section 2.

Proposition 5.10. *For any $A, B \in \mathcal{C}$, we have the following*

(1) $\widehat{\mathrm{H}}^{\mathcal{X}}_{-n}(A,B) = \widehat{\mathrm{H}}^{-n}_{\mathcal{X}}(A,B) = \mathcal{C}/\mathcal{X}(\underline{A}, \Omega^n(\underline{B})) = \mathcal{C}/\mathcal{X}(\Sigma^n(\underline{A}), \underline{B})$, $\forall n \geq 2$. *If the right $\mathcal{X}$-approximation of A is epic and the left $\mathcal{X}$-approximation of B is monic, then the above identifications are true,* $\forall n \geq 0$.

(2) *If* $\underline{\mathcal{E}xt}^n_{\mathcal{X}}(A,\mathcal{X}) = \overline{\mathcal{E}xt}^n_{\mathcal{X}}(\mathcal{X},B) = 0, \forall n \geq 1$, *then* $\widehat{\mathrm{H}}^{\mathcal{X}}_n(A,B) = \widehat{\mathrm{H}}^n_{\mathcal{X}}(A,B) = \underline{\mathcal{E}xt}^n_{\mathcal{X}}(A,B) = \overline{\mathcal{E}xt}^n_{\mathcal{X}}(A,B)$, $\forall n \geq 1$.

(3) *If* $\mathcal{C}(A,B) = \underline{\mathcal{E}xt}^0_{\mathcal{X}}(A,B) = \overline{\mathcal{E}xt}^0_{\mathcal{X}}(A,B)$ *and* $\underline{\mathcal{E}xt}^n_{\mathcal{X}}(A,\mathcal{X}) = \overline{\mathcal{E}xt}^n_{\mathcal{X}}(\mathcal{X},B) = 0, \forall n \geq 1$, *then* $\widehat{\mathrm{H}}^n_{\mathcal{X}}(A,-)(B) \cong \widehat{\mathrm{H}}^{\mathcal{X}}_n(-,B)(A), \forall n \in \mathbb{Z}$.

Proof. Parts (1), (2) are consequences of Proposition 2.5 and Proposition 2.8. Part (3) follows from (1), (2) and the definitions. □

The following result is a direct consequence of the definitions.

Proposition 5.11. *The following are equivalent.*

(1) $A \in \mathcal{G}_{\mathcal{X}}(\mathcal{C})$, *i.e. A is an $\mathcal{X}$-Gorenstein object.*

(2) *Any $\mathcal{X}$-coresolution of A is covariantly $\mathcal{X}$-exact and any $\mathcal{X}$-resolution of A is contravariantly $\mathcal{X}$-exact.*

(3) $\widehat{\mathrm{H}}^{\mathcal{X}}_n(\mathcal{X},A) = \widehat{\mathrm{H}}^n_{\mathcal{X}}(A,\mathcal{X}) = 0$, $\forall n \in \mathbb{Z}$.

(4) $A \in {}^{\perp}\mathcal{X} \cap \mathcal{X}^{\perp}$.

Let A, B be objects of $\mathcal{C}$ and consider the complex $\mathcal{C}(\mathbf{X}^{\bullet}(A), B)$. By diagram chasing, there exists the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathcal{C}/\mathcal{X}(\Sigma_{\mathcal{X}}(\underline{A}),\underline{B}) & \longrightarrow & \mathcal{L}_0^{\mathcal{X}}(A,B) & \longrightarrow & \mathcal{C}(A,B) & \longrightarrow & \mathcal{C}/\mathcal{X}(\underline{A},\underline{B}) \\
{\scriptstyle\mu}\downarrow & & \|\downarrow & & \downarrow & & {\scriptstyle\nu}\downarrow \\
\widehat{\mathrm{H}}_{\mathcal{X}}^{-1}(A,B) & \longrightarrow & \mathcal{L}_0^{\mathcal{X}}(A,B) & \longrightarrow & \underline{\mathcal{E}xt}_{\mathcal{X}}^0(A,B) & \longrightarrow & \widehat{\mathrm{H}}_{\mathcal{X}}^0(A,B).
\end{array}$$

From the above diagram we see that that there exist natural morphisms

$\mu : \mathcal{C}/\mathcal{X}(\Sigma_{\mathcal{X}}(-),\underline{B}) \cong \mathcal{C}/\mathcal{X}(-,\Omega_{\mathcal{X}}(\underline{B})) \to \widehat{\mathrm{H}}_{\mathcal{X}}^{-1}(-,B)$, $\nu : \mathcal{C}/\mathcal{X}(-,\underline{B}) \to \widehat{\mathrm{H}}_{\mathcal{X}}^0(-,B)$

with μ monic. Hence the morphisms $\mu_A : \mathcal{C}/\mathcal{X}(\underline{A},\Omega_{\mathcal{X}}(\underline{B})) \to \widehat{\mathrm{H}}_{\mathcal{X}}^{-1}(A,B)$ and $\nu_A : \mathcal{C}/\mathcal{X}(\underline{A},\underline{B}) \to \widehat{\mathrm{H}}_{\mathcal{X}}^0(A,B)$ are isomorphisms iff $\mathcal{C}(A,B) \cong \underline{\mathcal{E}xt}_{\mathcal{X}}^0(A,B)$, which obviously happens if the right $\mathcal{X}$–approximation of A is epic. Suppose that the right $\mathcal{X}$–approximations of A and any of its $\mathcal{X}$–syzygies K_A^n are epics. From the exact sequence $0 \to \mathcal{C}(A,B) \to \mathcal{C}(X_A^0,B) \to \mathcal{C}(K_A^1,B) \to \underline{\mathcal{E}xt}_{\mathcal{X}}^1(A,B) \to 0$, we see directly that there exists an epic $\underline{\mathcal{E}xt}_{\mathcal{X}}^1(A,B) \to \mathcal{C}/\mathcal{X}(\Omega_{\mathcal{X}}(\underline{A}),\underline{B})$. By dimension shifting we obtain in this way epics $\varepsilon_{A,B}^n : \underline{\mathcal{E}xt}_{\mathcal{X}}^n(A,B) \to \mathcal{C}/\mathcal{X}(\Omega_{\mathcal{X}}^n(\underline{A}),\underline{B}), \forall n \geq 1$.

Dually consider the complex $\mathcal{C}(A,\mathbf{X}^\bullet(B))$. Then we have the following commutative diagram, with exact rows

$$\begin{array}{ccccccc}
\mathcal{C}/\mathcal{X}(\underline{A},\Omega_{\mathcal{X}}(\underline{B})) & \longrightarrow & \mathcal{L}_0^{\mathcal{X}}(A,B) & \longrightarrow & \mathcal{C}(A,B) & \longrightarrow & \mathcal{C}/\mathcal{X}(\underline{A},\underline{B}) \\
{\scriptstyle\xi}\downarrow & & \|\downarrow & & \downarrow & & {\scriptstyle\zeta}\downarrow \\
\widehat{\mathrm{H}}_{-1}^{\mathcal{X}}(A,B) & \longrightarrow & \mathcal{L}_0^{\mathcal{X}}(A,B) & \longrightarrow & \overline{\mathcal{E}xt}_{\mathcal{X}}^0(A,B) & \longrightarrow & \widehat{\mathrm{H}}_0^{\mathcal{X}}(A,B)
\end{array}$$

and natural morphisms $\xi : \mathcal{C}/\mathcal{X}(\underline{A},\Omega_{\mathcal{X}}(-)) \to \widehat{\mathrm{H}}_{-1}^{\mathcal{X}}(A,-)$ and $\zeta : \mathcal{C}/\mathcal{X}(\underline{A},-) \to \widehat{\mathrm{H}}_0^{\mathcal{X}}(A,-)$, with ξ monic. Hence the morphisms $\xi_B : \mathcal{C}/\mathcal{X}(\underline{A},\Omega_{\mathcal{X}}(\underline{B})) \to \widehat{\mathrm{H}}_{-1}^{\mathcal{X}}(A,B)$ and $\zeta_B : \mathcal{C}/\mathcal{X}(\underline{A},\Omega_{\mathcal{X}}(\underline{B})) \to \widehat{\mathrm{H}}_0^{\mathcal{X}}(A,B)$ are isomorphisms iff $\mathcal{C}(A,B) \cong \overline{\mathcal{E}xt}_{\mathcal{X}}^0(A,B)$, which obviously happens if the left $\mathcal{X}$–approximation of B is monic. If the left $\mathcal{X}$–approximations of B and any of its $\mathcal{X}$–cosyzygies L_n^B are monics, then we have epimorphisms $\overline{\mathcal{E}xt}_{\mathcal{X}}^n(A,B) \to \mathcal{C}/\mathcal{X}(\underline{A},\Sigma_{\mathcal{X}}^n(\underline{B})), \forall n \geq 1$.

Suppose that any $\mathcal{X}$–epic is epic. Then there is a natural morphism

$$\sigma_{-,B}^n \ : \ \widehat{\mathrm{H}}_{\mathcal{X}}^n(-,B) \to \widehat{\mathrm{Ext}}_{\mathcal{X}}^n(-,B), \ \ \forall B \in \mathcal{C}, \ \ \forall n \in \mathbb{Z}.$$

Indeed in this case $\widehat{\mathrm{H}}_{\mathcal{X}}^{-n}(A,B) = \mathcal{C}/\mathcal{X}(\Sigma_{\mathcal{X}}^n(\underline{A}),\underline{B}) \cong \mathcal{C}/\mathcal{X}(\underline{A},\Omega^n(\underline{B})), \forall n \geq 0, \forall A \in \mathcal{C}$. Hence $\forall n \geq 0$ we have the natural morphism:

$$\sigma_{-,B}^{-n} : \widehat{\mathrm{H}}_{\mathcal{X}}^{-n}(-,B) \xrightarrow{\cong} \mathcal{C}/\mathcal{X}(-,\Omega^n(\underline{B})) \xrightarrow{\mathrm{S}} \mathcal{S}(\mathcal{C}/\mathcal{X})[\mathrm{S}(-),\tilde{\Omega}_{\mathcal{X}}^n\mathrm{S}(\underline{B})] = \widehat{\mathrm{Ext}}_{\mathcal{X}}^{-n}(-,B),$$

and for $n \geq 1$, we have the morphisms:

$$\sigma_{-,B}^n \ : \ \widehat{\mathrm{H}}_{\mathcal{X}}^n(-,B) = \underline{\mathcal{E}xt}_{\mathcal{X}}^n(-,B) \xrightarrow{\varepsilon_{-,B}^n} \mathcal{C}/\mathcal{X}(\Omega_{\mathcal{X}}^n(-),\underline{B}) \xrightarrow{\mathrm{S}}$$

$$\xrightarrow{\mathrm{S}} \mathcal{S}(\mathcal{C}/\mathcal{X})[\tilde{\Omega}_{\mathcal{X}}^n\mathrm{S}(-),\mathrm{S}(\underline{B})] = \widehat{\mathrm{Ext}}_{\mathcal{X}}^n(-,B).$$

Lemma 5.12. *Let $\mathcal{C}$ be an exact category and $\mathcal{X}$ be a contravariantly finite subcategory such that any $\mathcal{X}$–epic is admissible epic. Then the morphism*

$$\mathrm{S}_{A,B} : \mathcal{C}/\mathcal{X}(\underline{A},\underline{B}) \to \mathcal{S}(\mathcal{C}/\mathcal{X})[\mathrm{S}(\underline{A}),\mathrm{S}(\underline{B})]$$

is an isomorphism, $\forall A \in {}^{\perp}\mathcal{X}$ and $\forall B \in \mathcal{C}$.

Proof. Let $\underline{f} : \underline{A} \to \underline{B}$ be a morphism with $\mathbf{S}(\underline{f}) = 0$. Then there exists $t \geq 0$ such that $\Omega^t_{\mathcal{X}}(\underline{f}) = 0$. Choose $\mathcal{X}$-resolutions $\mathbf{X}^\bullet_A$ and $\mathbf{X}^\bullet_B$ of A and B, with corresponding $\mathcal{X}$-syzygies K^*_A, K^*_B. We denote by $k^n_f : K^n_A \to K^n_B$ the induced by f morphisms, such that $\underline{k}^n_f = \Omega^n_{\mathcal{X}}(\underline{f})$. Since $\Omega^t_{\mathcal{X}}(\underline{f}) = 0$, the morphism k^t_f factors through X^t_B. But since $A \in {}^\perp\mathcal{X}$, taking the push-out of the $\mathcal{X}$-exact admissible sequence $0 \to K^t_A \to X^{t-1}_A \to K^{t-1}_A \to 0$ along the morphism $K^t_A \to X^t_B$ we see that there exists a morphism $X^{t-1}_A \to X^t_B$ such that composing this morphism with $K^t_A \to X^{t-1}_A$ we obtain the morphism $K^t_A \to X^t_B$. This implies that in the diagram

$$\begin{tikzcd}
0 \arrow[r] & K^t_A \arrow[r] \arrow[d, "k^t_f"'] & X^{t-1}_A \arrow[r] \arrow[d, "x^{t-1}_f"'] & K^{t-1}_A \arrow[r] \arrow[d, "k^{t-1}_f"'] & 0 \\
0 \arrow[r] & K^t_B \arrow[r] & X^{t-1}_B \arrow[r] & K^{t-1}_B \arrow[r] & 0
\end{tikzcd}$$

the morphism k^t_f factors through X^{t-1}_A. But then the morphism k^{t-1}_f factors through X^{t-1}_B. Hence $\Omega^{t-1}_{\mathcal{X}}(\underline{f}) = \underline{k}^{t-1}_f = 0$. Continuing in this way we have finally that $\underline{f} = 0$. Hence $\mathbf{S}_{A,B}$ is a monomorphism. Suppose now that $\tilde{f} : \mathbf{S}(\underline{A}) \to \mathbf{S}(\underline{B})$ is a morphism in $\mathcal{S}(\mathcal{C}/\mathcal{X})$ and choose a representative $\underline{f}_s : \Omega^s_{\mathcal{X}}(\underline{A}) \to \Omega^s_{\mathcal{X}}(\underline{B})$ of $\tilde{f}$, where $s \geq 0$. Let $g_s : K^s_A \to K^s_B$ be a morphism such that $\underline{g}_s = \underline{f}_s$. Arguing as above and using that $A \in {}^\perp\mathcal{X}$, we see that there are morphisms $\underline{g}_t : K^t_A \to K^t_B$, $\forall t \leq s$ making the above diagram commutative. We set $g = g_0 : A \to B$. Using Corollary 3.3 it follows that $\mathbf{S}(\underline{g}) = \tilde{f}$. Hence $\mathbf{S}_{A,B}$ is an epimorphism. □

The next result shows that the complete $\mathcal{X}$-Extensions functors $\widehat{\mathrm{Ext}}^*_{\mathcal{X}}(A, B)$ defined using the contravariant finiteness of $\mathcal{X}$ and the stabilization functor $\mathbf{S} : \mathcal{C}/\mathcal{X} \to \mathcal{S}(\mathcal{C}/\mathcal{X})$, can be computed via resolutions if A has a complete $\mathcal{X}$-resolution or equivalently if A has finite $\mathcal{X}$-Gorenstein resolution dimension.

Theorem 5.13. *Let $\mathcal{C}$ be an exact category and $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$, such that any $\mathcal{X}$-epic is an admissible epic and the left $\mathcal{X}$-approximation of any $\mathcal{X}$-Gorenstein object of $\mathcal{C}$ is admissible monic. Then for any $B \in \mathcal{C}$ the following natural morphism is defined and is an isomorphism:*

$$\sigma^*_{-,B} \ : \ \widehat{\mathrm{H}}^*_{\mathcal{X}}(-, B) \to \widehat{\mathrm{Ext}}^*_{\mathcal{X}}(-, B) : \ \widehat{\mathcal{G}_{\mathcal{X}}(\mathcal{C})}^{op} \to \mathcal{A}b.$$

Proof. Since any object of finite $\mathcal{X}$-Gorenstein resolution dimension has a complete $\mathcal{X}$-resolution, the morphism $\sigma^*_{A,B}$ is defined, $\forall B \in \mathcal{C}$. Suppose first that A is $\mathcal{X}$-Gorenstein. Then we have seen in section 4 that the morphisms $\epsilon_{A,B}$ above are isomorphisms. Hence to prove that the morphisms $\sigma^*_{A,B}$ are isomorphisms for A an $\mathcal{X}$-Gorenstein object, it suffices to prove that the morphisms $\mathbf{S}_{A,B} : \mathcal{C}/\mathcal{X}(\underline{A}, \underline{B}) \to \mathcal{S}(\mathcal{C}/\mathcal{X})[\mathbf{S}(\underline{A}), \mathbf{S}(\underline{B})]$ are isomorphisms, for all $B \in \mathcal{C}$. But this follows from the above Lemma, since $\mathcal{G}_{\mathcal{X}}(\mathcal{C}) \subseteq {}^\perp\mathcal{X}$. Now if $A \in \widehat{\mathcal{G}_{\mathcal{X}}(\mathcal{C})}$ has a complete $\mathcal{X}$-resolution then $\Omega^t_{\mathcal{X}}(A)$ is in $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$ for some $t \geq 0$. By dimension shifting we conclude that the morphism $\sigma^*_{A,B}$ is an isomorphism, since the functors $\widehat{\mathrm{Ext}}^*_{\mathcal{X}}(-, B)$ and $\widehat{\mathrm{H}}^*_{\mathcal{X}}(-, B)$ are both cohomological with respect to $\mathcal{X}$-exact admissible short exact sequences. □

Suppose that any $\mathcal{X}$-monic is monic. Then dually there is a natural morphism

$$\tau^{A,-}_n \ : \ \widehat{\mathrm{H}}^{\mathcal{X}}_n(A, -) \to \widehat{\mathrm{Ext}}^{\mathcal{X}}_n(A, -), \ \ \forall A \in \mathcal{C}, \ \ \forall n \in \mathbb{Z}$$

but here the stabilization used in the definition of the complete $\mathcal{X}$–cohomology refers to the stabilization of $\mathcal{C}/\mathcal{X}$ as a *right* triangulated category, i.e. inverting the functor $\Sigma_{\mathcal{X}}$. The next result, which we state without proof, shows that the complete $\mathcal{X}$–Extensions functors $\widehat{\mathrm{Ext}}_*^{\mathcal{X}}(A, B)$ defined by the covariant finiteness of $\mathcal{X}$, can be computed via resolutions if B has a complete $\mathcal{X}$–resolution or equivalently A has finite $\mathcal{X}$–Gorenstein coresolution dimension. The full subcategory of $\mathcal{C}$ consisting of all objects with finite $\mathcal{X}$–exact coresolutions by objects of $\mathcal{G}_{\mathcal{X}}(\mathcal{C})$, i.e. the dual of $\widehat{\mathcal{G}_{\mathcal{X}}(\mathcal{C})}$, is denoted by $\widetilde{\mathcal{G}_{\mathcal{X}}(\mathcal{C})}$.

Theorem 5.14. *Let $\mathcal{C}$ be an exact category and $\mathcal{X}$ be a covariantly finite subcategory of $\mathcal{C}$, such that any $\mathcal{X}$–monic is admissible monic and the right $\mathcal{X}$–approximation of any $\mathcal{X}$–Gorenstein object of $\mathcal{C}$ is admissible epic. Then $\forall A \in \mathcal{C}$, the following natural morphism is defined and is an isomorphism:*

$$\tau_{A,-}^{*} \ : \ \widehat{\mathrm{H}}_*^{\mathcal{X}}(A,-) \longrightarrow \widehat{\mathrm{Ext}}_*^{\mathcal{X}}(A,-) \ : \ \widetilde{\mathcal{G}_{\mathcal{X}}(\mathcal{C})} \longrightarrow \mathcal{A}b.$$

6. (Co-)Gorenstein Rings: Applications to Ring Theory

In this section we apply the results of the previous sections to the familiar setting of module categories. We fix throughout an associative ring Λ. We denote by $\mathrm{Mod}(\Lambda)$ the category of right Λ–modules and by $\mathbf{P}_\Lambda$, resp. $\mathbf{I}_\Lambda$, the full subcategory of projective, resp. injective, modules. The category of finitely presented right Λ–modules is denoted by $\mathrm{mod}(\Lambda)$, and its full subcategory of projective, resp. injective, modules is denoted by $\mathcal{P}_\Lambda$, resp. $\mathcal{I}_\Lambda$. The induced stable categories are denoted by: $\mathrm{Mod}(\Lambda)/\mathbf{P}_\Lambda = \underline{\mathrm{Mod}}(\Lambda), \mathrm{Mod}(\Lambda)/\mathbf{I}_\Lambda = \overline{\mathrm{Mod}}(\Lambda)$, $\mathrm{mod}(\Lambda)/\mathcal{P}_\Lambda = \underline{\mathrm{mod}}(\Lambda)$ and $\mathrm{mod}(\Lambda)/\mathcal{I}_\Lambda = \overline{\mathrm{mod}}(\Lambda)$.

Throughout we choose $\mathcal{C}$ to be the module category $\mathrm{Mod}(\Lambda)$, resp. mod–Λ, and $\mathcal{X}$ to be one of the categories $\mathbf{P}_\Lambda$, $\mathbf{I}_\Lambda$, resp. $\mathcal{P}_\Lambda$, $\mathcal{I}_\Lambda$. We use the terminology and notations of the previous sections applied to the above choices.

6.1. Homologically finite Subcategories.

Proposition 6.1. *Let Λ be an arbitrary ring.*

(1) *$\mathbf{P}_\Lambda$ is covariantly finite $\Leftrightarrow$ Λ is left coherent and right perfect. In this case the loop functor $\Omega : \underline{\mathrm{Mod}}(\Lambda) \to \underline{\mathrm{Mod}}(\Lambda)$ admits a left adjoint $\Sigma_{\mathbf{P}}$.*

(2) *$\mathbf{I}_\Lambda$ is contravariantly finite $\Leftrightarrow$ Λ is right Noetherian. In this case the suspension functor $\Sigma : \overline{\mathrm{Mod}}(\Lambda) \to \overline{\mathrm{Mod}}(\Lambda)$ admits a right adjoint $\Omega_{\mathbf{I}}$.*

(3) *${}_\Lambda\mathbf{P}$ is covariantly finite and $\mathbf{I}_\Lambda$ is contravariantly finite $\Leftrightarrow$ Λ is right Artinian.*

(4) *$\mathcal{P}_\Lambda$ is covariantly finite $\Leftrightarrow$ Λ is left coherent. If Λ is coherent then we have an adjoint pair $(\Omega, \Sigma_{\mathcal{P}})$ defined on the left triangulated category $\underline{\mathrm{mod}}(\Lambda)$, and an adjoint pair $(\Omega, \Sigma_{\mathcal{P}})$ defined on the left triangulated category $\underline{\mathrm{mod}}(\Lambda^{op})$.*

Proof. It is well known [12], [19] that Λ is left coherent and right perfect iff $\mathbf{P}_\Lambda$ is covariantly finite, that Λ is left coherent iff $\mathcal{P}_\Lambda$ is covariantly finite and finally that Λ right Noetherian iff $\mathbf{I}_\Lambda$ is contravariantly finite. Since Λ is right Artinian iff Λ is right Noetherian and left perfect, (3) is a consequence of (1), (2). If Λ is coherent, then $\mathrm{mod}(\Lambda)$, $\mathrm{mod}(\Lambda^{op})$ are abelian with projectives, and $\mathcal{P}_\Lambda$, ${}_\Lambda\mathcal{P}$ are covariantly finite in them. The remaining assertions follow from section 2. $\square$

By the above result, if Λ is an Artinian ring then all the subcategories $\mathbf{P}_\Lambda$, $\mathbf{I}_\Lambda$, ${}_\Lambda\mathbf{P}$ and $\mathbf{I}_\Lambda$ are functorially finite. For $n \geq 1$, let $\Omega^n(\mathrm{Mod}(\Lambda))$ be the full additive subcategory of $\mathrm{Mod}(\Lambda)$ generated by $\mathbf{P}_\Lambda$ and the n^{th}–syzygy modules, and let $\Omega^\infty(\mathrm{Mod}(\Lambda)) = \bigcap_{n\geq 1} \Omega^n(\mathrm{Mod}(\Lambda))$ be the full subcategory of arbitrary syzygy modules. The categories $\Sigma^n(\mathrm{Mod}(\Lambda))$ and $\Sigma^\infty(\mathrm{Mod}(\Lambda))$ are defined similarly.

Proposition 6.2. (1) *If Λ is left coherent and right perfect then $\Omega^n(\mathrm{Mod}(\Lambda))$ is covariantly finite in $\mathrm{Mod}(\Lambda)$ and $\Omega^n(\underline{\mathrm{Mod}}(\Lambda))$ is reflective in $\underline{\mathrm{Mod}}(\Lambda)$, $\forall n \geq 1$.*

The category $\Omega^\infty(\mathrm{Mod}(\Lambda))$ is a covariantly finite subcategory of $\mathrm{Mod}(\Lambda)$ and the category $\Omega^\infty(\underline{\mathrm{Mod}}(\Lambda))$ is a reflective subcategory of $\underline{\mathrm{Mod}}(\Lambda)$.

(2) *If Λ is coherent then the categories $\Omega^n(\mathrm{mod}(\Lambda))$, $\Omega^n(\mathrm{mod}(\Lambda^{op}))$ are covariantly finite in $\mathrm{mod}(\Lambda)$, $\mathrm{mod}(\Lambda^{op})$ and the categories $\Omega^n(\underline{\mathrm{mod}}(\Lambda))$, $\Omega^n(\underline{\mathrm{mod}}(\Lambda^{op}))$ are reflective in $\underline{\mathrm{mod}}(\Lambda)$, $\underline{\mathrm{mod}}(\Lambda^{op})$ respectively, $\forall n \geq 1$.*

Proof. Let $\Sigma_{\mathbf{P}}$ be the left adjoint of Ω in the stable category, which exists by Proposition 6.1. Let A be a right Λ–module, and let

$$A \xrightarrow{\pi_0^A} P_0^A \xrightarrow{f_1^A} P_1^A \to \cdots \to P_n^A \xrightarrow{f_{n+1}^A} P_{n+1}^A \to \cdots$$

be a $\mathbf{P}_\Lambda$–coresolution, which is a result of the composition of the exact sequences $A \xrightarrow{\pi_0^A} P_0^A \to L_1^A$, $L_1^A \xrightarrow{\pi_1^A} P_1^A \to L_2^A$ and so on, where π_n^A are left $\mathbf{P}_\Lambda$–approximations and $L_n^A = Coker(\pi_{n-1}^A)$. Then in $\underline{\mathrm{Mod}}(\Lambda)$, we have $\Sigma_{\mathbf{P}}^n(\underline{A}) = \underline{L}_n^A$ and $\Omega\Sigma_{\mathbf{P}}^n(\underline{A}) = \underline{Ker(\pi_{n-1}^A)}$. We set for simplicity $\Sigma_{\mathbf{P}}^n(A) = L_n^A$. Then $\Omega\Sigma_{\mathbf{P}}^n(A) = Ker(\pi_{n-1}^A)$, and we have the following exact commutative diagrams, $\forall n \geq 0$:

$$\begin{array}{ccccccc}
 & & \Sigma_{\mathbf{P}}^n(A) & \longrightarrow & P_n^A & \longrightarrow & \Sigma_{\mathbf{P}}^{n+1}(A) & \longrightarrow & 0 \\
 & & {\scriptstyle \exists\mu_n}\downarrow & & \Vert\downarrow & & \Vert\downarrow & & \\
0 & \longrightarrow & \Omega\Sigma_{\mathbf{P}}^{n+1}(A) & \longrightarrow & P_n^A & \longrightarrow & \Sigma_{\mathbf{P}}^{n+1}(A) & &
\end{array}$$

Hence we obtain morphisms $\underline{\mu}_n : \Sigma_{\mathbf{P}}^n(\underline{A}) \to \Omega\Sigma_{\mathbf{P}}^{n+1}(\underline{A})$, $\forall n \geq 0$. This system of morphisms induces a tower

$$\underline{A} \xrightarrow{\underline{\mu}_0} \Omega\Sigma_{\mathbf{P}}(\underline{A}) \xrightarrow{\Omega(\underline{\mu}_1)} \Omega^2\Sigma_{\mathbf{P}}^2(\underline{A}) \xrightarrow{\Omega^2(\underline{\mu}_2)} \Omega^3\Sigma_{\mathbf{P}}^3(\underline{A}) \to \cdots$$

in $\underline{\mathrm{Mod}}(\Lambda)$, and we have projective presentations of increasing length:

$$(A_1): \quad 0 \to \Omega\Sigma_{\mathbf{P}}(A) \to P_0^A \to \Sigma_{\mathbf{P}}(A) \to 0,$$

$$(A_2): \quad 0 \to \Omega^2\Sigma_{\mathbf{P}}^2(A) \to Q^{1,1} \to P_1^A \to \Sigma_{\mathbf{P}}^2(A) \to 0,$$

$$(A_3): \quad 0 \to \Omega^3\Sigma_{\mathbf{P}}^3(A) \to Q^{2,2} \to Q^{2,1} \to P_2^A \to \Sigma_{\mathbf{P}}^3(A) \to 0, \cdots$$

such that the tower $A \to \Omega\Sigma_{\mathbf{P}}(A) \to \Omega^2\Sigma_{\mathbf{P}}^2(A) \to \Omega^3\Sigma_{\mathbf{P}}^3(A) \to \cdots$ induces morphisms between the presentations. Hence we obtain a direct system of projective presentations $A \to (A_1) \to (A_2) \to \cdots$. Taking direct limits in this direct system, we obtain an exact sequence

$$0 \to \varinjlim \Omega^n\Sigma_{\mathbf{P}}^n \to \varinjlim Q^{n,n} \to \varinjlim Q^{n+1,n} \to \varinjlim Q^{n+2,n} \to \cdots$$

Since Λ is left coherent and right perfect, the above exact sequence is a coresolution of $\varinjlim \Omega^n \Sigma^n_{\mathbf{P}}(A)$ by projectives. Hence $\varinjlim \Omega^n \Sigma^n_{\mathbf{P}}(A) \in \Omega^\infty(\mathrm{Mod}(\Lambda))$, and there exists a canonical map $d^2_\infty(A) := \varinjlim d^2_n(A) : A \to \varinjlim \Omega^n \Sigma^n_{\mathbf{P}}(A)$, where $d^2_n(A) = \mu_0 \circ \Omega(\mu_1) \circ \cdots \circ \Omega^{n-1}(\mu_{n-1}) : A \to \Omega^n \Sigma^n_{\mathbf{P}}(A)$. Observe that the morphisms $d^2_n(A) : A \to \Omega^n \Sigma^n_{\mathbf{P}}(A)$ by construction are left $\Omega^n(\mathrm{Mod}(\Lambda))-$approximations of A. We claim that the map $d^2_\infty(A)$ is a left $\Omega^\infty(\mathrm{Mod}(\Lambda))-$approximation of A. Indeed let $\alpha : A \to B$ with $B \in \Omega^\infty(\mathrm{Mod}(\Lambda))$. Since $B \in \Omega^n(\mathrm{Mod}(\Lambda))$, $\forall n \geq 1$, and since $d^2_n(A)$ are left $\Omega^n(\mathrm{Mod}(\Lambda))-$approximations of A, there are morphisms $g_n : \Omega^n \Sigma^n_{\mathbf{P}}(A) \to B$ with $d^2_n(A) \circ g_n = \alpha$. Taking direct limits, we have that $d^2_\infty(A) \circ \varinjlim g_n = \alpha$. Hence $\Omega^\infty(\mathrm{Mod}(\Lambda))$ is covariantly finite. Let $\Omega^\infty(\underline{A})$ be the image of $\varinjlim \Omega^n \Sigma^n_{\mathbf{P}}(A)$ in $\underline{\mathrm{Mod}}(\Lambda)$. Then obviously the morphism $\underline{d}^2_\infty(\underline{A}) : \underline{A} \to \Omega^\infty(\underline{A})$ is the reflection of $\underline{A}$ in $\Omega^\infty(\underline{\mathrm{Mod}}(\Lambda))$ and the morphism $d^2_n(\underline{A}) : \underline{A} \to \Omega^n \Sigma^n_{\mathbf{P}}(\underline{A})$ is the reflection of $\underline{A}$ in $\Omega^n(\underline{\mathrm{Mod}}(\Lambda))$. Part (2) is left to the reader. $\square$

If Λ is right Noetherian, then there is a similar result for the categories of cosyzygy modules. We leave its formulation to the reader. Now let Λ be left coherent and right perfect. Consider the reflection $\Omega^\infty : \underline{\mathrm{Mod}}(\Lambda) \to \Omega^\infty(\underline{\mathrm{Mod}}(\Lambda))$ of $\underline{\mathrm{Mod}}(\Lambda)$ in $\Omega^\infty(\underline{\mathrm{Mod}}(\Lambda))$ constructed in the Proposition 6.2, and let $\mathbf{R} : \mathcal{R}(\underline{\mathrm{Mod}}(\Lambda)) \to \underline{\mathrm{Mod}}(\Lambda)$ be the costabilization functor. By section 3, $\mathcal{R}(\underline{\mathrm{Mod}}(\Lambda)) = \mathcal{K}_{Ac}(\mathbf{P}_\Lambda)$ is the homotopy category of acyclic complexes of projectives.

Corollary 6.3. *Let Λ be left coherent and right perfect. If* $\mathrm{Mod}(\Lambda)$ *is* $\mathbf{P}_\Lambda-$*Co-Gorenstein, then the costabilization functor* $\mathbf{R} : \mathcal{R}(\underline{\mathrm{Mod}}(\Lambda)) \to \underline{\mathrm{Mod}}(\Lambda)$ *admits a left adjoint* $\tilde{\Omega}^\infty : \underline{\mathrm{Mod}}(\Lambda) \to \mathcal{R}(\underline{\mathrm{Mod}}(\Lambda))$.

Proof. If $\mathrm{Mod}(\Lambda)$ is $\mathbf{P}_\Lambda-$Co-Gorenstein, then from section 4 we have an identification $\mathcal{R}(\underline{\mathrm{Mod}}(\Lambda)) = \Omega^\infty(\underline{\mathrm{Mod}}(\Lambda))$ and the assertion follows by Proposition 6.2. $\square$

Remark 6.4. (1) Our standard assumptions in this paper refer to pairs $(\mathcal{C}, \mathcal{X})$, where $\mathcal{C}$ is an additive category and any $\mathcal{X}-$epic has a kernel, and then we usually require that any left $\mathcal{X}-$approximation of an $\mathcal{X}-$Gorenstein object is an admissible monic. The most natural example is the pair $(\mathrm{Mod}(\Lambda), \mathbf{P}_\Lambda)$. Dually the most natural example of a pair satisfying the dual assumptions is the pair $(\mathrm{Mod}(\Lambda), \mathbf{I}_\Lambda)$.

(2) One can define $\mathcal{X}-$Gorenstein objects in $\mathrm{Mod}(\Lambda)$, choosing $\mathcal{X}$ to be the full subcategory of flat modules or the FP-injective modules or any other interesting subcategory of modules and to apply the theory. We leave the details to the reader.

Suppose now that Λ is left coherent and right perfect, so that $\mathbf{P}_\Lambda$ is functorially finite. Then as in section 2, the functors $\mathcal{E}xt^n_{\mathbf{P}_\Lambda}(A, B)$ and $\overline{\mathcal{E}xt}^n_{\mathbf{P}_\Lambda}(A, B)$ are defined. Obviously $\mathcal{E}xt^n_{\mathbf{P}_\Lambda}(-,-)$ are the usual extension functors. The subcategory $\mathbf{P}^\perp_\Lambda = \{A \in \mathrm{Mod}(\Lambda) : \overline{\mathcal{E}xt}^n_{\mathbf{P}_\Lambda}(\mathbf{P}_\Lambda, A) = 0, \forall n \geq 1 \text{ and } \overline{\mathcal{E}xt}^0_{\mathbf{P}_\Lambda}(\mathbf{P}_\Lambda, A) = \mathrm{Hom}_\Lambda(\mathbf{P}_\Lambda, A)\}$ consists of of all modules A such that there exists an exact sequence $0 \to A \to P^0 \to P^1 \to \cdots$, where P^i are projective modules, such that the sequence $\cdots \to \mathrm{Hom}_\Lambda(P^1, \mathbf{P}_\Lambda) \to \mathrm{Hom}_\Lambda(P^0, \mathbf{P}_\Lambda) \to \mathrm{Hom}_\Lambda(A, \mathbf{P}_\Lambda) \to 0$ is exact. Observe that the subcategory $\mathbf{P}^\perp_\Lambda$ is defined always, although the derived functors $\overline{\mathcal{E}xt}^n_{\mathbf{P}_\Lambda}$ may not exist globally. Then for an arbitrary ring Λ, the subcategory $\mathbf{P}^\perp_\Lambda$ denotes the full subcategory of all modules with the property described above. For an arbitrary ring Λ the definitions of section 2, take the following form.

Definition 6.5. (1) A module A is called **(projectively) stable** if $A \in {}^{\perp}\mathbf{P}_{\Lambda}$.

(2) A module A is called n**-torsion-free** if

$$\overline{\mathcal{E}xt}^{i}(\mathbf{P}_{\Lambda}, A) = 0, 1 \leq i \leq n \text{ and } \overline{\mathcal{E}xt}^{0}_{\mathbf{P}_{\Lambda}}(\mathbf{P}_{\Lambda}, A) = \mathrm{Hom}_{\Lambda}(\mathbf{P}_{\Lambda}, A).$$

(3) A is called **torsion-free** if $A \in \mathbf{P}_{\Lambda}^{\perp}$, or equivalently A is n-torsion-free, $\forall n \geq 1$.

We leave to the reader the formulation of the above definitions using the subcategory $\mathcal{P}_{\Lambda}$ and the subcategories $\mathbf{I}_{\Lambda}, \mathcal{I}_{\Lambda}$. The above terminology is natural since if Λ is a two-sided Noetherian (or more generally two-sided coherent) ring , then the modules in $\mathcal{P}_{\Lambda}^{\perp}$ are exactly the n-torsion free modules $\forall n \geq 1$, in the sense of Auslander-Bridger [3].

Definition 6.6. [20], [3] A module A is called **Gorenstein-projective** if A is stable and torsionfree, i.e. $A \in {}^{\perp}\mathbf{P}_{\Lambda} \cap \mathbf{P}_{\Lambda}^{\perp}$. Equivalently there exists an exact sequence $\cdots \to P^{-1} \to P^{0} \to P^{1} \to \cdots$ of projective modules with $Im(P^{-1} \to P^{0}) = A$ and the sequence remains exact applying $\mathrm{Hom}_{\Lambda}(-, \mathbf{P}_{\Lambda})$. The full subcategory of all Gorenstein-projective modules is denoted by $\mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda))$, and the induced stable category modulo projectives is denoted by $\mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Lambda))$. The full subcategory $\mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda))$ of **Gorenstein-injective** modules is defined dually and the induced stable category modulo injectives is denoted by $\mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Lambda))$.

By the results of sections 2, 4, $\mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda))$ is the largest resolving subcategory of $\mathrm{Mod}(\Lambda)$, such that the stable category $\mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Lambda))$, is full triangulated subcategory of $\underline{\mathrm{Mod}}(\Lambda)$ and dually $\mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda))$ is the largest coresolving subcategory of $\mathrm{Mod}(\Lambda)$, such that the stable category $\mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Lambda))$, is a full triangulated subcategory of $\overline{\mathrm{Mod}}(\Lambda)$. If Λ is a two-sided Noetherian (or coherent) ring, the finitely presented Gorenstein-projective modules $\mathcal{G}_{\mathcal{P}}(\mathrm{mod}(\Lambda))$, are exactly the modules in $\mathrm{mod}(\Lambda)$ with zero G-dimension in the sense of [3]. We note that the categories of Gorenstein-projective and Gorenstein-injective modules were introduced also in [20].

In case Λ is left and right coherent, then it is not difficult to see that the left adjoint $\Sigma_{\mathcal{P}}$ of the loop functor $\Omega : \underline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda)$ is given by $\Sigma_{\mathcal{P}} = \mathrm{Tr}\Omega\mathrm{Tr}$, where Tr is the Auslander-Bridger duality [3], and there exists an exact sequence $0 \to \mathcal{E}xt^{1}(\mathrm{Tr}A, \Lambda) \to A \to P^{A} \to \Sigma_{\mathcal{P}}(A) \to 0$, where $A \to P^{A}$ is the left $\mathcal{P}_{\Lambda}$-approximation. This follows from the easily established fact that a left $\mathcal{P}_{\Lambda}$-approximation of A can be obtained as follows. We denote by $\mathrm{d} = (-)^{*}$ both the Λ-dual functors, $\mathrm{d} = (-)^{*} = \mathrm{Hom}_{\Lambda}(-, \Lambda)$. Let $f : Q \to A^{*}$ be an epimorphism with $Q \in {}_{\Lambda}\mathcal{P}$. Then the composition $A \to A^{**} \xrightarrow{f^{*}} Q^{*}$ is a left $\mathcal{P}_{\Lambda}$-approximation of A, where $A \to A^{**}$ is the canonical morphism. Moreover, setting $\mathrm{D}_{k}^{2} := \Omega^{k}\Sigma_{\mathcal{P}}^{k}$, the unit $d_{\underline{A}}^{2} : \underline{A} \to \mathrm{D}_{k}^{2}(\underline{A})$ of the adjoint pair $(\Sigma_{\mathcal{P}}^{n}, \Omega^{n})$ is the natural morphism introduced in [3], in case Λ is Noetherian. Similarly setting $\mathrm{J}_{k}^{2} = \Sigma_{\mathcal{P}}^{k}\Omega^{k}$, the counit of the adjoint pair $(\Sigma_{\mathcal{P}}^{k}, \Omega^{k})$ is the natural morphism introduced in [3]. Note that the functors J_{k}^{2}, D_{k}^{2} play a fundamental role in [3]. One can develop the theory of [3] for arbitrary modules in a right Noetherian ring Λ, using that in the stable category $\overline{\mathrm{Mod}}(\Lambda)$ modulo injectives, the suspension functor $\Sigma : \overline{\mathrm{Mod}}(\Lambda) \to \overline{\mathrm{Mod}}(\Lambda)$ has a right adjoint $\Omega_{\mathbf{I}}$ and similarly in a left coherent and right perfect ring using that in the stable category $\underline{\mathrm{Mod}}(\Lambda)$ modulo projectives, the loop functor $\Omega : \underline{\mathrm{Mod}}(\Lambda) \to \underline{\mathrm{Mod}}(\Lambda)$ has a left adjoint $\Sigma_{\mathbf{P}}$. Hence in each case one can define the corresponding functors $\tilde{\mathrm{D}}_k^2 = \Omega_{\mathbf{I}}^k \Sigma^k$, $\tilde{\mathrm{J}}_k^2 = \Sigma^k \Omega_{\mathbf{I}}^k$, and $\hat{\mathrm{D}}_k^2 = \Omega^k \Sigma_{\mathbf{P}}^k$, $\hat{\mathrm{J}}_k^2 = \Sigma_{\mathbf{P}}^k \Omega^k$ and also the corresponding (adjunction) morphisms.

6.2. **Complete Extension Functors, Complete Resolutions and Gorenstein Rings.** A complete projective (injective) resolution of a module is a $\mathbf{P}_\Lambda-$complete ($\mathbf{I}_\Lambda-$complete) resolution in the sense of section 5. In this case the notions of complete projective or injective resolutions coincide with the notions introduced in [18]. First we note the following Corollary of section 5. The last part follows from the description of the stabilization of the stable module category in section 3 and the Morita theorem of Rickard [41].

Corollary 6.7. (1) *A right $\Lambda-$module A has a complete projective resolution iff A has finite Gorenstein-projective resolution dimension.*

(2) *A right $\Lambda-$module A has a complete injective resolution iff A has finite Gorenstein-injective resolution dimension.*

(3) $\forall B \in \mathrm{Mod}(\Lambda)$, *the complete projective extension functors are computed as:*

$$\widehat{\mathrm{H}}_{\mathbf{P}}^*(-,B) \cong \widehat{\mathrm{Ext}}_{\mathbf{P}}^*(-,B) : \ \mathcal{G}_{\mathbf{P}}(\widehat{\mathrm{Mod}(\Lambda)})^{op} \longrightarrow \mathcal{A}b.$$

(4) $\forall A \in \mathrm{Mod}(\Lambda)$, *the complete injective extension functors are computed as:*

$$\widehat{\mathrm{H}}_*^{\mathbf{I}}(A,-) \cong \widehat{\mathrm{Ext}}_*^{\mathbf{I}}(A,-) : \ \mathcal{G}_{\mathbf{I}}(\widetilde{\mathrm{Mod}(\Lambda)}) \longrightarrow \mathcal{A}b.$$

(5) *If Λ and Γ are derived equivalent rings (and Λ,Γ are right coherent), then the projective stabilizations of their modules categories are triangle equivalent:*

$$\mathcal{S}(\underline{\mathrm{Mod}}(\Lambda)) \approx \mathcal{S}(\underline{\mathrm{Mod}}(\Gamma)) \quad (\text{and} \quad \mathcal{S}(\underline{\mathrm{mod}}(\Lambda)) \approx \mathcal{S}(\underline{\mathrm{mod}}(\Gamma))).$$

Definition 6.8. An arbitrary ring Λ is called a **right Gorenstein ring** if any projective right module has finite injective dimension and any injective right module has finite projective dimension.

The following result presents various characterizations and properties of right Gorenstein rings. Its proof is a direct consequence of the results of the previous sections and of Propositions 6.1, 6.2.

Theorem 6.9. *Let Λ be an arbitrary ring. Then the following are equivalent.*

(1) *Λ is a right Gorenstein ring.*
(2) $\mathrm{Mod}(\Lambda)$ *is $\mathbf{P}_\Lambda-$Gorenstein category.*
(3) $\mathrm{Mod}(\Lambda)$ *is $\mathbf{I}_\Lambda-$Gorenstein category.*
(4) $({}^{\perp}\mathbf{P}_\Lambda, \mathbf{P}_\Lambda^\infty, \mathbf{P}_\Lambda)$ *is an Auslander-Buchweitz context.*
(5) $(\mathbf{I}_\Lambda^\perp, \mathbf{I}_\Lambda^\infty, \mathbf{I}_\Lambda)$ *is a dual Auslander-Buchweitz context.*
(6) $\mathbf{P}_\Lambda^\infty = \mathbf{I}_\Lambda^\infty$
(7) $d := sup\{\mathrm{p.d.}I : I \in \mathbf{I}_\Lambda\} = sup\{\mathrm{i.d.}P : P \in \mathbf{P}_\Lambda\} < \infty$.
(8) *The functor ${}^{\perp}\mathbf{P}_\Lambda/\mathbf{P}_\Lambda \to \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{P}_\Lambda)$ is a triangle equivalence.*
(9) *The functor $\mathbf{I}_\Lambda^\perp/\mathbf{I}_\Lambda \to \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{I}_\Lambda)$ is a triangle equivalence.*
(10) *$\mathcal{K}^b(\mathbf{I}_\Lambda) = \mathcal{K}^b(\mathbf{P}_\Lambda)$ as full subcategories of $\mathcal{D}^b(\mathrm{Mod}(\Lambda))$.*

(11) *Any right Λ–module has a complete projective resolution.*
(12) *Any right Λ–module has a complete injective resolution.*
(13) *Any right Λ–module has finite Gorenstein-projective resolution dimension.*
(14) *Any right Λ–module has finite Gorenstein-injective resolution dimension.*
(15) *The natural functor $\mathcal{D}^b(\mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda))) \to \mathcal{D}^b(\mathrm{Mod}(\Lambda))$ is a triangle equivalence.*
(16) *The natural functor $\mathcal{D}^b(\mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda))) \to \mathcal{D}^b(\mathrm{Mod}(\Lambda))$ is a triangle equivalence.*

If one of the above equivalent statements is true, then we have the following:

(α) $\mathrm{Mod}(\Lambda)$ *is* $\mathbf{P}_\Lambda$*–Co-Gorenstein and* $\mathbf{I}_\Lambda$*–Co-Gorenstein. Moreover:*

$$\mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda))\text{–gl.res.dim}(\mathrm{Mod}(\Lambda)) = \mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda))\text{–gl.cores.dim}(\mathrm{Mod}(\Lambda)) = d.$$

(β) *The categories $\mathbf{P}_\Lambda^\infty = \mathbf{I}_\Lambda^\infty$ are functorially finite, the category $\mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda))$ is contravariantly finite and the category $\mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda))$ is covariantly finite.*

(γ) *If Λ is left coherent and right perfect, then $\mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda))$ is functorially finite. If Λ is right Noetherian, then $\mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda))$ is functorially finite.*

(δ) *We have: $\Omega^\infty(\mathrm{Mod}(\Lambda)) = \Omega^d(\mathrm{Mod}(\Lambda)) = \mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda))$ and $\Sigma^\infty(\mathrm{Mod}(\Lambda)) = \Sigma^d(\mathrm{Mod}(\Lambda)) = \mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda))$. Hence $\mathbf{P}_\Lambda^\infty = \mathbf{P}_\Lambda^{\leq d}$ and $\mathbf{I}_\Lambda^\infty = \mathbf{I}_\Lambda^{\leq d}$, where $\mathbf{P}_\Lambda^{\leq d}$, resp. $\mathbf{I}_\Lambda^{\leq d}$, is the full subcategory of all modules having projective, resp. injective, dimension bounded by d. The right finitistic projective dimension* $\mathrm{FPD}(\Lambda)$ *and the right finitistic injective dimension* $\mathrm{FID}(\Lambda)$ *of Λ are finite:* $\mathrm{FPD}(\Lambda) = \mathrm{FID}(\Lambda) = d < \infty$.

(ϵ) *There are triangle equivalences:*

$$\mathcal{K}_{Ac}(\mathbf{P}_\Lambda) \approx \mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Lambda)) \approx {}^\perp\mathbf{P}_\Lambda/\mathbf{P}_\Lambda \approx {}^\perp(\mathbf{P}_\Lambda^\infty)/\mathbf{P}_\Lambda \approx \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{P}_\Lambda) \approx$$
$$\approx \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{I}_\Lambda) \approx (\mathbf{I}_\Lambda^\infty)^\perp/\mathbf{I}_\Lambda \approx \mathbf{I}_\Lambda^\perp/\mathbf{I}_\Lambda \approx \mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Lambda)) \approx \mathcal{K}_{Ac}(\mathbf{I}_\Lambda).$$

The costabilization functors are the inclusions

$$\mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Lambda)) \hookrightarrow \underline{\mathrm{Mod}}(\Lambda), \quad \mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Lambda)) \hookrightarrow \overline{\mathrm{Mod}}(\Lambda)$$

and the stabilization functors are given by

$$\Omega^{-d}\Omega^d : \underline{\mathrm{Mod}}(\Lambda) \to \mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Lambda)), \quad \Sigma^{-d}\Sigma^d : \overline{\mathrm{Mod}}(\Lambda) \to \mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Lambda)).$$

(ζ) *The complete projective extension functors are given $\forall B \in \mathrm{Mod}(\Lambda)$, by*

$$\mathrm{Ext}^n_{\mathbf{P}}(-,B) = \underline{\mathrm{Mod}}(\Lambda)[\Omega^n\Omega^d(-), \Omega^d(\underline{B})], \quad \forall n \in \mathbb{Z}.$$

The complete injective extension functors are given $\forall A \in \mathrm{Mod}(\Lambda)$, by

$$\mathrm{Ext}_n^{\mathbf{I}}(A,-) = \overline{\mathrm{Mod}}(\Lambda)[\Sigma^d(\underline{A}), \Sigma^n\Sigma^d(-)], \quad \forall n \in \mathbb{Z}.$$

The triangle equivalence $\mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Lambda)) \approx \mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Lambda))$ in (ϵ), induces isomorphisms

$$\mathrm{Ext}^*_{\mathbf{P}}(-,B)(A) \cong \mathrm{Ext}_*^{\mathbf{I}}(A,-)(B).$$

Hence the complete extension bifunctor is defined:

$$\mathrm{Ext}^*(-,-) : \mathrm{Mod}(\Lambda)^{op} \times \mathrm{Mod}(\Lambda) \to \mathcal{A}b.$$

(η) *If Λ and Γ are derived equivalent right Gorenstein rings then there are triangle equivalences:*

$$\mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Lambda)) \approx \mathcal{G}_{\mathbf{P}}(\underline{\mathrm{Mod}}(\Gamma)) \quad \text{and} \quad \mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Lambda)) \approx \mathcal{G}_{\mathbf{I}}(\overline{\mathrm{Mod}}(\Gamma)).$$

(θ) *Λ is a QF-ring* $\Leftrightarrow \mathcal{K}_{Ac}(\mathbf{P}_\Lambda) \approx \underline{\mathrm{Mod}}(\Lambda) \approx \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{P}_\Lambda) \Leftrightarrow \mathcal{K}_{Ac}(\mathbf{I}_\Lambda) \approx \overline{\mathrm{Mod}}(\Lambda) \approx \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{I}_\Lambda) \Leftrightarrow \mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda)) = \mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda)) \Leftrightarrow \Lambda$ *is a right Gorenstein ring of dimension zero* $\Leftrightarrow$ *the above equivalences are true for* Λ^{op}.

(ι) *Λ is (homologically) regular, i.e. any right $\Lambda-$module has finite projective dimension* $\Leftrightarrow \mathcal{G}_{\mathbf{P}}(\mathrm{Mod}(\Lambda)) = \mathbf{P}_\Lambda \Leftrightarrow \mathcal{G}_{\mathbf{I}}(\mathrm{Mod}(\Lambda)) = \mathbf{I}_\Lambda \Leftrightarrow$ *the stabilization of* $\underline{\mathrm{Mod}}(\Lambda)$ *is trivial* $\Leftrightarrow$ *the stabilization of* $\overline{\mathrm{Mod}}(\Lambda)$ *is trivial. Similarly for left $\Lambda-$modules.*

It follows by the above Theorem and also by the results of sections 4, 5 that the complete projective extension functors coincide with the Tate-Vogel cohomology functors, see [17], [18], [25], [26], [39]. Our results generalize the corresponding results of the above papers in much more general situations. In particular the above Theorem shows that for module categories, the Gorenstein property of Mod(Λ) with respect to the projectives and the injectives coincides.

From the above Theorem if Λ is a QF-ring or if r.gl.dim$\Lambda < \infty$ (more generally if any right module has finite projective dimension), then Λ is right Gorenstein. Observe however that in these cases the theory is trivial.

Remark 6.10. The general setting of sections 3, 4, 5 can be applied directly to the study of lattices over *(Gorenstein) orders* in the sense of Auslander [2], with analogous results. We leave the details to the reader.

Corollary 6.11. *For a Noetherian ring Λ the following are equivalent.*

(1) *Λ is a right Gorenstein ring.*
(2) *Λ is a left Gorenstein ring.*
(3) $\mathrm{i.d}_\Lambda\Lambda < \infty$ *and* $\mathrm{i.d}\Lambda_\Lambda < \infty$.

Proof. (1) $\Leftrightarrow$ (3) If Λ is a right Gorenstein ring, then $\mathrm{i.d}\Lambda_\Lambda < \infty$. By a result of Iwanaga [35], $\mathrm{i.d}_\Lambda\Lambda = sup\{\mathrm{flat.dim}E_\Lambda; E_\Lambda \in \mathbf{I}_\Lambda\}$. Since any right injective has finite projective dimension bounded by d, we have $\mathrm{i.d}_\Lambda\Lambda < \infty$. The converse follows from the results of [35]. The equivalence (2) $\Leftrightarrow$ (3) follows similarly. □

So the Gorenstein property is symmetric for Noetherian rings. In this case a Noetherian left (or right) Gorenstein ring is called simply a Gorenstein ring. By the above Corollary it follows that our definition of a Gorenstein ring agrees in the Noetherian case, with the definition introduced by Iwanaga [35] and used extensively by Enochs-Jenda, et al, in a large list of papers, see for instance [20], [22]. Hence our theory covers, presents new features, and generalizes the corresponding theory developed in [20], [22]. In particular we recover the results of [5] which were obtained using tilting theory, since an Artin algebra is called Gorenstein in the sense of Auslander-Reiten iff $\mathrm{i.d}_\Lambda\Lambda < \infty$ and $\mathrm{i.d}\Lambda_\Lambda < \infty$.

Example 6.12. Let Λ be a Noetherian ring.

(1) Trivially Λ is Gorenstein in case Λ is QF or of finite global dimension.

(2) If Λ is Gorenstein and G a finite group, then the group ring ΛG is Gorenstein. Any Quasi-Frobenius extension of a Gorenstein ring is Gorenstein [35]. It is not difficult to see that the ring of the lower triangular matrices of any size over a Gorenstein ring is Gorenstein.

(3) If Λ is an Artin algebra and G a finite group of automorphisms of Λ, then the skew group ring ΛG is Gorenstein iff Λ is Gorenstein [5]. If Λ, Γ are finite-dimensional algebras over a field k, then $\Lambda \otimes_k \Gamma$ is Gorenstein iff Λ, Γ are Gorenstein

[5]. Also it is not difficult to see [13] that if Λ is a Cohen-Macaulay Artin algebra [5] with dualizing bimodule ω, then the trivial extension $\Lambda \ltimes \omega$ is Gorenstein.

(4) Let Λ be an $\mathcal{F}-$Gorenstein Artin algebra, were $\mathcal{F}$ is an additive subfunctor of $\mathcal{E}xt^1_\Lambda(-,-)$ with enough projectives and injectives, in the sense of Auslander-Sølberg [9]. Then $\mathrm{mod}(\Lambda)$ is $\mathcal{P}(\mathcal{F})-$Gorenstein and $\mathcal{I}(\mathcal{F})-$Gorenstein category in the sense of section 4, where $\mathcal{P}(\mathcal{F})$ and $\mathcal{I}(\mathcal{F})$ are the categories of $\mathcal{F}-$projective and $\mathcal{F}-$injective modules respectively.

(5) If Λ is a local Artin algebra with $\mathcal{J}ac(\Lambda)^2 = 0$, then: Λ is Gorenstein iff Λ is representation-finite.

(6) There is a extensive literature concerning commutative (local) Noetherian Gorenstein rings. For more information we refer to [3], [11].

Definition 6.13. A ring Λ is called **right Co-Gorenstein** if any arbitrary syzygy module is $\mathbf{P}_\Lambda-$torsion-free or equivalently if $\mathrm{Mod}(\Lambda)$ is $\mathbf{P}_\Lambda-$Co-Gorenstein.

We have seen that if Λ is Gorenstein then Λ is left (and right) Co-Gorenstein. We don't know if the converse is true. We don't know also if for a (Noetherian) ring Λ, being left Co-Gorenstein is equivalent to being right Co-Gorenstein.

6.3. **Artin Algebras.** For the remaining of this section, we assume that Λ is an Artin algebra [10]. We denote by $\mathrm{d} = \mathrm{Hom}_\Lambda(-,\Lambda)$ or by $(-)^*$ both the $\Lambda-$dual functors and by D the usual duality of Artin algebras. All the results of this and the previous sections are true for the module categories $\mathrm{Mod}(\Lambda)$ or $\mathrm{mod}(\Lambda)$ of a (Gorenstein) Artin Algebra Λ. In particular we have the following.

Corollary 6.14. (i) *The following are equivalent.*

(1) Λ *is Gorenstein.*
(2) $\mathrm{mod}(\Lambda)$ *or equivalently* $\mathrm{Mod}(\Lambda)$ *is a Gorenstein category.*
(3) $\mathcal{P}^\infty_\Lambda = \mathcal{I}^\infty_\Lambda$.
(4) $\mathrm{i.d}_\Lambda\Lambda < \infty$ *and* $\mathrm{i.d}\Lambda_\Lambda < \infty$, *in which case* $\mathrm{i.d}_\Lambda\Lambda = \mathrm{i.d}\Lambda_\Lambda$.
(5) *There exists a triangle equivalence* $\mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Lambda)) \approx \mathcal{D}^b(\mathrm{mod}(\Lambda))/\mathcal{K}^b(\mathcal{P}_\Lambda)$ *or equivalently a triangle equivalence* $\mathcal{G}_\mathbf{P}(\underline{\mathrm{Mod}}(\Lambda)) \approx \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{P}_\Lambda)$.
(6) *There exists a triangle equivalence* $\mathcal{G}_\mathcal{I}(\overline{\mathrm{mod}}(\Lambda)) \approx \mathcal{D}^b(\mathrm{mod}(\Lambda))/\mathcal{K}^b(\mathcal{I}_\Lambda)$ *or equivalently a triangle equivalence* $\mathcal{G}_\mathbf{I}(\overline{\mathrm{Mod}}(\Lambda)) \approx \mathcal{D}^b(\mathrm{Mod}(\Lambda))/\mathcal{K}^b(\mathbf{I}_\Lambda)$.
(7) *The left-hand side analogues of* (2), (3), (5) *and* (6).

If this is the case, the categories $\mathcal{P}^\infty_\Lambda, \mathcal{I}^\infty_\Lambda$, $\mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda))$, $\mathcal{G}_\mathcal{I}(\mathrm{mod}(\Lambda))$ *are functorially finite in* $\mathrm{mod}(\Lambda)$ *and any finitely presented right* $\Lambda-$*module has a minimal left and right* $\mathcal{X}-$*approximation, where* $\mathcal{X} = \mathcal{P}^\infty_\Lambda, \mathcal{I}^\infty_\Lambda$, $\mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda))$, *or* $\mathcal{G}_\mathcal{I}(\mathrm{mod}(\Lambda))$. *The categories* $\mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda))$, $\mathcal{G}_\mathcal{I}(\mathrm{mod}(\Lambda))$, $\mathcal{P}^\infty_\Lambda, \mathcal{I}^\infty_\Lambda$ *have Auslander-Reiten sequences, and the triangulated categories* $\mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Lambda))$, $\mathcal{G}_\mathcal{I}(\overline{\mathrm{mod}}(\Lambda))$ *are triangle equivalent and they have Auslander-Reiten triangles.*

(ii) (α) *If* Λ *is Gorenstein, then there are isomorphisms*

$$\mathrm{K}_0(\mathrm{mod}(\Lambda)) \cong \mathrm{K}_0(\mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda))) \quad \text{and} \quad \mathrm{K}_0(\underline{\mathrm{mod}}(\Lambda)) \cong \mathrm{K}_0(\mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Lambda))).$$

(β) *If* Λ, Γ *are derived equivalent Gorenstein algebras, then*

$$\mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Lambda)) \approx \mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Gamma)), \quad |\mathrm{Det}(c_\Lambda)| = |\mathrm{Det}(c_\Gamma)|, \quad \mathrm{Ker}(c_\Lambda) = \mathrm{Ker}(c_\Gamma).$$

Moreover the triangle equivalence $\mathcal{G}_{\mathcal{P}}(\underline{\mathrm{mod}}(\Lambda)) \approx \mathcal{G}_{\mathcal{P}}(\underline{\mathrm{mod}}(\Gamma))$ *lifts to a triangle equivalence* $\underline{\mathrm{mod}}(\Lambda) \approx \underline{\mathrm{mod}}(\Gamma)$ *iff* Λ, Γ *are selfinjective.*

(γ) *If two Gorenstein Artin algebras have equivalent categories of Gorenstein -projective or Gorenstein-injective modules, then they are derived equivalent.*

(δ) Λ *is selfinjective* $\Leftrightarrow \mathcal{K}_{Ac}(\mathcal{P}_\Lambda) \approx \underline{\mathrm{mod}}(\Lambda) \approx \mathcal{D}^b(\mathrm{mod}(\Lambda))/\mathcal{K}^b(\mathcal{P}_\Lambda) \Leftrightarrow \mathcal{K}_{Ac}(\mathcal{I}_\Lambda) \approx \overline{\mathrm{mod}}(\Lambda) \approx \mathcal{D}^b(\mathrm{mod}(\Lambda))/\mathcal{K}^b(\mathcal{I}_\Lambda) \Leftrightarrow$ *the above equivalences are true for* Λ^{op}.

Remark 6.15. The equivalence (**i**)(1) $\Leftrightarrow$ (2) has been proved first by Hoshino [34] and by Auslander-Reiten [5] and in case Λ is a commutative local Noetherian ring it has been proved by Auslander-Bridger [3]. The last part of (**i**) has been proved first by Auslander-Reiten [5], using tilting theory. The direction (1) $\Rightarrow$ (4) in (**i**) of the above Corollary was proved first by Rickard [42] in the selfinjective case and then by Happel [32] in the Gorenstein case. The generalized form of the Happel-Rickard Theorem can be stated as follows (this is also a consequence of a general result due to Keller-Vossieck, see [36]):

- For any right coherent ring Λ the stabilization of $\underline{\mathrm{mod}}(\Lambda)$ is triangle equivalent to $\mathcal{D}^b(\mathrm{mod}(\Lambda))/\mathcal{K}^b(\mathcal{P}_\Lambda)$. Further if two right coherent rings Λ and Γ are derived equivalent, then their stable module categories $\underline{\mathrm{mod}}(\Lambda)$, $\underline{\mathrm{mod}}(\Gamma)$ have triangle equivalent stabilizations.

An Artin algebra Λ is called *right* $\mathcal{P}_\Lambda$*-Co-Gorenstein* if any finitely presented arbitrary syzygy right module is torsion-free or equivalently if mod(Λ) is $\mathcal{P}_\Lambda$−Co-Gorenstein category. Similarly we can define, using the costabilization of $\overline{\mathrm{mod}}(\Lambda)$, when Λ is *right* $\mathcal{I}_\Lambda$ *− Co-Gorenstein.* Since the duality D induces an exact duality D : $\mathcal{R}(\underline{\mathrm{mod}}(\Lambda)) \to \mathcal{R}(\overline{\mathrm{mod}}(\Lambda^{op}))$ and a duality D : $\Omega^\infty(\mathrm{mod}(\Lambda)) \to \Sigma^\infty(\mathrm{mod}(\Lambda^{op}))$, we have that Λ is right $\mathcal{P}_\Lambda$−Co-Gorenstein iff Λ is left ${}_\Lambda\mathcal{I}$−Co-Gorenstein. From now on we call a $\mathcal{P}_\Lambda$−Co-Gorenstein algebra, simply right Co-Gorenstein.

Corollary 6.16. (1) *Suppose that* f.p.d$\Lambda < \infty$. *Then for the costabilization functor* $\mathbf{R} : \mathcal{R}(\underline{\mathrm{mod}}(\Lambda)) \to \underline{\mathrm{mod}}(\Lambda)$ *we have:* Ker$\mathbf{R} = 0$.

(2) *If the costabilization functor* $\mathbf{R} : \mathcal{R}(\underline{\mathrm{mod}}(\Lambda)) \to \underline{\mathrm{mod}}(\Lambda)$ *satisfies* Ker$\mathbf{R} = 0$ (*in particular if* Λ *is right Co-Gorenstein*), *then* Λ *satisfies the so-called Nunke condition for finitely presented left modules: if* A *is a finitely generated left* Λ*−module satisfying* $\mathcal{E}xt^n_\Lambda(A, \Lambda) = 0, \forall n \geq 0$, *then* $A = 0$. *In particular if* Λ *is right Co-Gorenstein, then* Λ *satisfies the Generalized Nakayama Conjecture* [10].

Proof. (1) If Λ satisfies the finitistic dimension conjecture with f.p.d$\Lambda = d < \infty$ and $P^\bullet$ is a non contractible complex in Ker$\mathbf{R}$, then we have a non contractible complex of projectives $0 \to P^0 \xrightarrow{f_0} P^1 \xrightarrow{f_1} \cdots$ in mod(Λ). Hence $\exists t \geq 0$ such that $Im(f_t)$ is not projective. Since p.d.$Im(f_{t+d}) < \infty$, we have p.d.$Im(f_{t+d}) \leq d$. Hence $Im(f_t)$ is projective and this is not the case. So Ker$\mathbf{R} = 0$.

(2) Let $P^\bullet \to A$ be a projective resolution of A. Then we have an acyclic complex of projectives $P^{\bullet *}$ in mod(Λ). Viewing $P^{\bullet *}$ as an object of the costabilization $\mathcal{R}(\underline{\mathrm{mod}}(\Lambda))$, we have that $\mathbf{R}(P^{\bullet *}) = 0$. Since Ker$\mathbf{R} = 0$, $P^{\bullet *} = 0$ in $\mathcal{R}(\underline{\mathrm{mod}}(\Lambda))$, i.e. $P^{\bullet *}$ is contractible. But then $P^\bullet = P^{\bullet **}$ is contractible, and then $A = 0$. ☐

Proposition 6.17. (1) Λ *is left Co-Gorenstein iff* $\Omega^\infty(\mathrm{mod}(\Lambda^{op})) \subseteq {}^{\perp}_{\Lambda}\mathcal{P}$. *In this* *case we have:* $\Omega^\infty(\text{mod}(\Lambda^{op})) = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda^{op})) \subseteq {}_\Lambda^\perp\mathcal{P}$ *and* ${}^\perp\mathcal{P}_\Lambda = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda)) \subseteq \Omega^\infty(\text{mod}(\Lambda))$.

(2) Λ *is right Co-Gorenstein iff* $\Omega^\infty(\text{mod}(\Lambda)) \subseteq {}^\perp\mathcal{P}_\Lambda$. *In this case we have:* $\Omega^\infty(\text{mod}(\Lambda)) = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda)) \subseteq {}^\perp\mathcal{P}_\Lambda$ *and* ${}_\Lambda^\perp\mathcal{P} = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda^{op})) \subseteq \Omega^\infty(\text{mod}(\Lambda^{op}))$.

Proof. (1) By our previous results it suffices to show that if Λ is left Co-Gorenstein then ${}^\perp\mathcal{P}_\Lambda \subseteq \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda))$. Let A in ${}^\perp\mathcal{P}_\Lambda$. Then $\text{Tr}(A) \in \Omega^\infty(\text{mod}(\Lambda^{op}))$. Since Λ is left Co-Gorenstein, $\Omega^\infty(\text{mod}(\Lambda^{op})) \subseteq {}_\Lambda^\perp\mathcal{P}$. Hence $\text{Tr}(A) \in {}_\Lambda^\perp\mathcal{P}$ and A is Gorenstein-projective [3]. Part (2) is dual. □

The following is a direct consequence of the above Proposition.

Corollary 6.18. (1) Λ *is right Co-Gorenstein iff the* $\Lambda-$*dual functor* d *induces a duality* $\text{d} : {}_\Lambda^\perp\mathcal{P} \to \Omega^\infty(\text{mod}(\Lambda))$ *or equivalently an exact duality* $\text{d} : \mathcal{G}_\mathcal{P}(\underline{\text{mod}}(\Lambda^{op})) \to \mathcal{R}(\underline{\text{mod}}(\Lambda))$.

(2) Λ *is left Co-Gorenstein iff the* $\Lambda-$*dual functor* d *induces a duality* $\text{d} : {}^\perp\mathcal{P}_\Lambda \to \Omega^\infty(\text{mod}(\Lambda^{op}))$ *or equivalently an exact duality* $\text{d} : \mathcal{G}_\mathcal{P}(\underline{\text{mod}}(\Lambda)) \to \mathcal{R}(\underline{\text{mod}}(\Lambda^{op}))$.

(3) Λ *is left and right Co-Gorenstein iff the* $\Lambda-$*dual functor* d *induces an exact duality* $\text{d} : \mathcal{R}(\underline{\text{mod}}(\Lambda)) \to \mathcal{R}(\underline{\text{mod}}(\Lambda^{op}))$. *In this case we have:*

$$\Omega^\infty(\text{mod}(\Lambda)) = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda)) = {}^\perp\mathcal{P}_\Lambda \text{ and } {}_\Lambda^\perp\mathcal{P} = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda^{op})) = \Omega^\infty(\text{mod}(\Lambda^{op})).$$

Lemma 6.19. (1) *The following are equivalent:*

(α) $\text{i.d}\Lambda_\Lambda < \infty$.
(β) $\mathcal{P}_\Lambda^\infty \subseteq \mathcal{I}_\Lambda^\infty$.
(γ) $\exists d \geq 0 : \Omega^d(\text{mod}(\Lambda)) \subseteq {}^\perp\mathcal{P}_\Lambda$.

If this is the case, then: ${}^\perp\mathcal{P}_\Lambda^\infty = {}^\perp\mathcal{P}_\Lambda$ *and* ${}_\Lambda^\perp\mathcal{P} = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda^{op}))$.

(2) *If* $\text{i.d}\Lambda_\Lambda < \infty$, *then* Λ *is right Co-Gorenstein.*

(3) *If the category* $\Omega^d(\text{mod}(\Lambda))$ *is closed under extensions,* $\forall d \geq 1$, *then* Λ *is right Co-Gorenstein.*

Proof. Part (1) is easy and the proof is left to the reader. Suppose $\text{i.d}\Lambda_\Lambda = d < \infty$. Then by (1) we have $\Omega^d(\text{mod}(\Lambda)) \subseteq {}^\perp\mathcal{P}_\Lambda$. But then $\Omega^\infty(\text{mod}(\Lambda)) \subseteq {}^\perp\mathcal{P}_\Lambda$. Then Λ is right Co-Gorenstein by Proposition 6.17. Part (3) follows from [7]. □

Proposition 6.20. *The following are equivalent.*

(α) Λ *is Gorenstein.*
(β) Λ *is left Co-Gorenstein and* $\text{i.d}\Lambda_\Lambda < \infty$.
(γ) Λ *is right Co-Gorenstein and* $\text{i.d}_\Lambda\Lambda < \infty$.

Proof. We prove only that (α) is equivalent to (β), since the proof of the other parts is similar. By our previous results condition (α) implies (β). Suppose that (β) is true. Since $\text{i.d}\Lambda_\Lambda = d < \infty$, we have by Lemma 6.19 that $\Omega^d(\text{mod}(\Lambda)) \subseteq {}^\perp\mathcal{P}_\Lambda$. Hence by Proposition 6.17(1), we have $\Omega^d(\text{mod}(\Lambda)) \subseteq {}^\perp\mathcal{P}_\Lambda = \mathcal{G}_\mathcal{P}(\text{mod}(\Lambda)) \subseteq \Omega^\infty(\text{mod}(\Lambda)) \subseteq \Omega^d(\text{mod}(\Lambda))$. This implies that $\mathcal{G}_\mathcal{P}(\text{mod}(\Lambda)) = \Omega^d(\text{mod}(\Lambda))$. Then by Corollaries 4.11, 6.14 we have that Λ is Gorenstein. □

The next result includes simple proofs of some results due to Auslander-Reiten [7], see also [29]. For the notion of a $k-$Gorenstein algebra we refer to [7].

Corollary 6.21. (1) *Suppose that Λ is $k-$Gorenstein $\forall k$. Then Λ is left and right Co-Gorenstein and Λ satisfies the Generalized Nakayama Conjecture.*

(2) *The following are equivalent:*

(α) *Λ is Gorenstein.*
(β) *Λ is $k-$Gorenstein $\forall k$ and* $\mathrm{i.d}\Lambda_\Lambda < \infty$ (*or* $\mathrm{i.d}_\Lambda\Lambda < \infty$).
(γ) *Λ is $k-$Gorenstein $\forall k$ and $\mathcal{P}^\infty_\Lambda$ (or $\mathcal{P}^\infty_{\Lambda^{op}}$) is contravariantly finite.*
(δ) *Λ is $k-$Gorenstein $\forall k$ and* $\mathrm{f.p.d}\Lambda < \infty$ (*or* $\mathrm{f.i.d}\Lambda < \infty$.
(ϵ) *Λ is right Co-Gorenstein and* $\mathrm{f.p.d}\Lambda < \infty$ (*or* $\mathrm{f.i.d}\Lambda < \infty$).
(ζ) *Λ is right Co-Gorenstein and* $\Omega^\infty(\widehat{\mathrm{mod}(\Lambda)}) = \mathrm{mod}(\Lambda)$.

Proof. (1) Suppose that Λ is $k-$Gorenstein for all k. Then by [3] we have that $\forall d \geq 0$, any $d-$syzygy module is $d-$torsionfree. Hence the category $\Omega^\infty(\mathrm{mod}(\Lambda))$ coincides with the category of torsionfree modules. Then by definition Λ is right Co-Gorenstein. Since the notion of a $k-$Gorenstein algebra is left-right symmetric, we have also that Λ is left Co-Gorenstein.

(2) (α) $\Leftrightarrow$ (β) Follows directly from (1) and the above Proposition. (α) $\Rightarrow$ (γ) follows from Corollary 6.14. (γ) $\Rightarrow$ (δ) holds for any Artin algebra (see [6]) and (δ) $\Rightarrow$ (ϵ) follows from (1). (ϵ) $\Rightarrow$ (α) Let $\mathrm{fin.p.dim}\Lambda = d < \infty$. Then $\mathcal{P}^{\leq d}(\mathrm{mod}(\Lambda)) = \mathcal{P}^\infty(\mathrm{mod}(\Lambda))$, where $\mathcal{P}^{\leq d}(\mathrm{mod}(\Lambda))$ is the full subcategory of all modules with projective dimension bounded by d. By [7], we have that $\Omega^d(\mathrm{mod}(\Lambda)) = \Omega^{d+t}(\mathrm{mod}(\Lambda))$, $\forall t \geq 0$. Hence $\Omega^\infty(\mathrm{mod}(\Lambda)) = \Omega^d(\mathrm{mod}(\Lambda))$. Since Λ is right Co-Gorenstein, by Theorem 4.10 we have $\Omega^\infty(\mathrm{mod}(\Lambda)) = \mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda))$. Hence $\mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda)) = \Omega^d(\mathrm{mod}(\Lambda))$, and by Corollary 6.14, Λ is Gorenstein. The equivalence (α) $\Leftrightarrow$ (ζ) follows from our previous results. The parenthetical cases are treated similarly. □

By the above results it is reasonable to conjecture:

- *Λ is right Co-Gorenstein $\Leftrightarrow$ Λ is left Co-Gorenstein.*

If the conjecture is true, by Proposition 6.18 and Lemma 6.17(2), the Auslander-Reiten Conjecture, that any Artin algebra is Gorenstein if $\mathrm{i.d}\Lambda_\Lambda < \infty$, is true. In any case we have the inclusions $\mathbf{Gor} \subseteq \forall k-\mathbf{Gor} \subseteq \mathbf{Co}-\mathbf{Gor}$ between Gorenstein algebras, $k-$Gorenstein algebras $\forall k$ and (left and right) Co-Gorenstein algebras. The above inclusions also show that if any (left and right) Co-Gorenstein algebra is Gorenstein, then another Conjecture due to Auslander-Reiten is true, namely that any $k-$Gorenstein algebra $\forall k$ is Gorenstein.

The next Corollary follows directly from the above results and its proof is left to the reader. We note only that if $\mathrm{dom.dim}\Lambda = \infty$ then $\Omega^\infty(\mathrm{mod}(\Lambda)) = \mathcal{D}om(\Lambda)$, is the full subcategory of modules of infinite dominant dimension [37].

Corollary 6.22. *The following are equivalent.*

(1) *Λ is selfinjective.*
(2) *Λ is Gorenstein and* $\mathrm{dom.dim}\Lambda = \infty$.
(3) *Λ is (left or right) Co-Gorenstein and* $\mathrm{dom.dim}\Lambda = \infty$.

Clearly the $\Lambda-$dual functors induce a duality $\mathrm{d} : \mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda)) \to \mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda^{op}))$ and an exact duality $\underline{\mathrm{d}} : \mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Lambda)) \to \mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Lambda^{op}))$, and the Nakayama functor N^+ induces an equivalence $\mathrm{N}^+ : \mathcal{G}_\mathcal{P}(\mathrm{mod}(\Lambda)) \to \mathcal{G}_\mathcal{I}(\mathrm{mod}(\Lambda))$ and a triangle equivalence $\underline{\mathrm{N}}^+ : \mathcal{G}_\mathcal{P}(\underline{\mathrm{mod}}(\Lambda)) \to \mathcal{G}_\mathcal{I}(\overline{\mathrm{mod}}(\Lambda))$. The next result contains another characterization of Gorenstein algebras, which is based on the notion of duality of derived categories. It follows that the Gorenstein property is invariant under derived equivalence.

Theorem 6.23. (1) *For an Artin algebra* Λ, *the following are equivalent:*

(α) Λ *is Gorenstein.*

(β) *The functor* $\mathrm{d} = \mathrm{Hom}_\Lambda(-,\Lambda)$ *induces an exact duality:*

$$\underline{\underline{\mathrm{R}}}^b\mathrm{Hom}(-,\Lambda) = \underline{\underline{\mathrm{R}}}^b\mathrm{d} : \mathcal{D}^b(\mathrm{mod}(\Lambda)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(\Lambda^{op})).$$

(γ) *The functor* $\mathrm{N}^+ = - \otimes_\Lambda \mathrm{D}(\Lambda)$ *induces a triangle equivalence:*

$$- \otimes^{\mathrm{L}}_\Lambda \mathrm{D}(\Lambda) = \underline{\underline{\mathrm{L}}}^b\mathrm{N}^+ : \mathcal{D}^b(\mathrm{mod}(\Lambda)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(\Lambda)).$$

In this case we have the following commuting diagram of localization sequences, where all vertical arrows are exact dualities:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}^b(\mathcal{P}_\Lambda) \arrow[r] \arrow[d, "\mathcal{K}^b(\mathrm{d})"'] & \mathcal{D}^b(\mathrm{mod}(\Lambda)) \arrow[r] \arrow[d, "\underline{\underline{\mathrm{R}}}^b\mathrm{d}"'] & \mathcal{G}_{\mathcal{P}}(\underline{\mathrm{mod}}(\Lambda)) \arrow[r] \arrow[d, "\underline{\mathrm{d}}"'] & 0 \\
0 \arrow[r] & \mathcal{K}^b({}_\Lambda\mathcal{P}) \arrow[r] & \mathcal{D}^b(\mathrm{mod}(\Lambda^{op})) \arrow[r] & \mathcal{G}_{\mathcal{P}}(\underline{\mathrm{mod}}(\Lambda^{op})) \arrow[r] & 0
\end{tikzcd}$$

and a commuting diagram of localization sequences, extending the "exact sequences"

$$0 \to \mathcal{P}_\Lambda \hookrightarrow \mathrm{mod}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda) \to 0, \quad 0 \to \mathcal{I}_\Lambda \hookrightarrow \mathrm{mod}(\Lambda) \to \overline{\mathrm{mod}}(\Lambda) \to 0$$

where the vertical arrows are triangle equivalences:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}^b(\mathcal{P}_\Lambda) \arrow[r] \arrow[d, "\mathcal{K}^b(\mathrm{N}^+)"'] & \mathcal{D}^b(\mathrm{mod}(\Lambda)) \arrow[r] \arrow[d, "\underline{\underline{\mathrm{L}}}^b\mathrm{N}^+"'] & \mathcal{G}_{\mathcal{P}}(\underline{\mathrm{mod}}(\Lambda)) \arrow[r] \arrow[d, "\underline{\mathrm{N}}^+"'] & 0 \\
0 \arrow[r] & \mathcal{K}^b(\mathcal{I}_\Lambda) \arrow[r] & \mathcal{D}^b(\mathrm{mod}(\Lambda)) \arrow[r] & \mathcal{G}_{\mathcal{I}}(\overline{\mathrm{mod}}(\Lambda)) \arrow[r] & 0
\end{tikzcd}$$

(2) *If* Λ *and* Γ *are derived equivalent, then:* Λ *is Gorenstein* $\Leftrightarrow$ Γ *is Gorenstein.*

Proof. (1) (α) $\Rightarrow$ (β) If Λ is Gorenstein, then since the functor d induces a duality between the left and right Gorenstein projective modules, condition (β) follows from parts (15), (16) of Theorem 6.9 and Corollary 6.14. (β) $\Rightarrow$ (γ) Trivial. (γ) $\Rightarrow$ (α) It is well-known that under condition (γ), the module $\mathrm{D}(\Lambda)$ is a (generalized) tilting module, and this is equivalent to $\mathrm{i.d}_\Lambda\Lambda < \infty$ and $\mathrm{i.d}\Lambda_\Lambda < \infty$, i.e. Λ is a Gorenstein algebra. Clearly the above diagrams commute by construction.

(2) If $F : \mathcal{D}^b(\mathrm{mod}(\Lambda)) \to \mathcal{D}^b(\mathrm{mod}(\Gamma))$ is a triangle equivalence, then by [43] we have that F commutes with the total derived functors $\underline{\underline{\mathrm{L}}}^b\mathrm{N}^+_\Lambda$ and $\underline{\underline{\mathrm{L}}}^b\mathrm{N}^+_\Gamma$. Hence the assertion follows from part (1). □

We note that if Λ is Gorenstein, then a Gorenstein-projective module is called a Cohen-Macaulay module in [5]. Since D induces a duality ${}^\perp_\Lambda\mathcal{P} \approx \mathcal{I}^\perp_\Lambda$, if Λ is Gorenstein, the Gorenstein-injective modules $\mathcal{G}_{\mathcal{I}}(\mathrm{mod}(\Lambda)) = \mathcal{I}^\perp_\Lambda$ coincide with the Co-Cohen-Macaulay modules in the sense of Auslander-Reiten [5], i.e. with the full subcategory $\mathrm{D}({}^\perp_\Lambda\mathcal{P})$. Note that many of the above results for Artin algebras can be extended easily to (Gorenstein) rings with a Matlis duality in the sense of [21].

We close this section discussing briefly the complete projective or injective extension functors for an Artin algebra Λ and the relative homology induced in $\mathrm{mod}(\Lambda)$ using the contravariant finiteness of $\mathcal{I}_\Lambda$ and the covariant finiteness of $\mathcal{P}_\Lambda$. Of course the relative homology using the covariant finiteness of $\mathcal{I}_\Lambda$ and the contravariant finiteness of $\mathcal{P}_\Lambda$ is the usual (absolute) homology in $\text{mod}(\Lambda)$. Define $\mathcal{P}_\Lambda-\text{gl.codim}\Lambda = sup\{\mathcal{P}_\Lambda-\text{codim}A; A \in \text{mod}(\Lambda)\}$, $\mathcal{I}_\Lambda-\text{gl.dim}\Lambda = sup\{\mathcal{I}_\Lambda-\text{dim}A; A \in \text{mod}(\Lambda)\}$ as in section 2, using the covariant finiteness of $\mathcal{P}_\Lambda$ and the contravariantly finiteness of $\mathcal{I}_\Lambda$. Since $\mathcal{P}_\Lambda, \mathcal{I}_\Lambda$ are functorially finite the complete functors $\widehat{\mathrm{H}}^*_{\mathcal{P}}(-,B)$, $\widehat{\mathrm{H}}^{\mathcal{I}}_*(A,-)$ are defined and it is easy to see that:

$$\widehat{\mathrm{H}}^{n+1}_{\mathcal{P}}(A,B) = \mathrm{D}\widehat{\mathrm{H}}^{-n}_{\mathcal{P}}(\mathrm{TrD}B,A) \text{ and } \widehat{\mathrm{H}}^{-n}_{\mathcal{P}}(A,B) = \mathrm{D}\widehat{\mathrm{H}}^{n+1}_{\mathcal{P}}(B,\mathrm{DTr}A), \ \forall n \geq 0.$$

Similarly for the complete functors $\widehat{\mathrm{H}}^{\mathcal{I}}_*(A,-)$. Having describing the left projective approximation of a right Λ-module, it is not difficult to see that a right injective approximation of A is the composition $\mathrm{N}^+(P) \to \mathrm{N}^+\mathrm{N}^-(A) \to A$, where $P \to \mathrm{N}^-(A)$ is an epimorphism with P projective and $\mathrm{N}^+\mathrm{N}^-(A) \to A$ is the counit of the adjoint pair $(\mathrm{N}^+,\mathrm{N}^-)$. In particular the right adjoint of the usual suspension functor Σ^n in $\overline{\text{mod}}(\Lambda)$ is given by $\mathrm{DTr}\Omega^n\mathrm{TrD}$, $\forall n \geq 0$, and there exists an exact sequence $0 \to \mathrm{DTr}\Omega\mathrm{TrD}(A) \to \mathrm{N}^+(P) \to A \to \mathrm{D}\mathcal{E}xt^1_\Lambda(\mathrm{TrD}(A),\Lambda) \to 0$. The proof of our final result is left to the reader, noting that most of the assertions can be generalized to right Noetherian or left coherent and right perfect rings.

Corollary 6.24. (1) $\mathcal{P}_\Lambda-\text{gl.codim}\Lambda = 0$ *iff* $\text{gl.dim}\Lambda \leq 2$ *iff* $\mathcal{I}_\Lambda-\text{gl.dim}\Lambda = 0$ *iff* $\mathcal{P}_\Lambda$ *is a reflective subcategory of* $\text{mod}(\Lambda)$ *iff* $\mathcal{I}_\Lambda$ *is a coreflective subcategory of* $\text{mod}(\Lambda)$. *In this case the reflection of A in $\mathcal{P}_\Lambda$ is given by the natural morphism $A \to A^{**}$ and the coreflection of A in $\mathcal{I}_\Lambda$ is given by the natural morphism* $\mathrm{N}^+\mathrm{N}^-(A) \to A$.
(2) *If* $\text{gl.dim}\Lambda \geq 2$, *then:* $\mathcal{P}_\Lambda-\text{gl.codim}\Lambda = \text{gl.dim}\Lambda - 2 = \mathcal{I}_\Lambda-\text{gl.dim}\Lambda$.

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