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The Homology of Partitions with an Even Number of Blocks

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Abstract. Let Π_{2n}^e denote the subposet obtained by selecting even ranks in the partition lattice Π_{2n}. We show that the homology of Π_{2n}^e has dimension $\frac{(2n)!}{2^{2n-1}} E_{2n-1}$, where E_{2n-1} is the tangent number. It is thus an integral multiple of both the Genocchi number and an André or simsun number. Using the general theory of rank-selected homology representations developed in [22], we show that, for the special case of Π_{2n}^e, the character of the symmetric group S_{2n} on the homology is supported on the set of involutions. Our proof techniques lead to the discovery of a family of integers $b_i(n)$, $2 \le i \le n$, defined recursively. We conjecture that, for the full automorphism group S_{2n}, the homology is a sum of permutation modules induced from Young subgroups of the form $S_2^i \times S_1^{2n-2i}$, with *nonnegative* integer multiplicity $b_i(n)$. The nonnegativity of the integers $b_i(n)$ would imply the existence of new refinements, into sums of powers of 2, of the tangent number and the André or simsun number $a_n(2n)$.

Similarly, the restriction of this homology module to S_{2n-1} *yields a family of integers* $d_i(n)$, $1 \le i \le n-1$, such that the numbers $2^{-i}d_i(n)$ refine the Genocchi number G_{2n}. We conjecture that $2^{-i}d_i(n)$ is a positive integer for all i.

Finally, we present a recursive algorithm to generate a family of polynomials which encode the homology representations of the subposets obtained by selecting the top k ranks of Π_{2n}^e, $1 \le k \le n-1$. We conjecture that these are all permutation modules for S_{2n}.

Keywords: homology representation, permutation module, André permutations, simsun permutation, tangent and Genocchi number

0. Introduction

Let Π_n^e denote the subposet of the partition lattice Π_n consisting of those set partitions of $\{1, \ldots, n\}$ which have an even number of blocks. This subposet may also be obtained by selecting alternate ranks in the partition lattice. It is well known that rank-selected subposets of a Cohen-Macaulay poset are also Cohen-Macaulay (see [3, Theorem 6.4], [13, Corollary 6.6] and [16, Theorem 4.3]). Since Π_n is Cohen-Macaulay, the subposet Π_n^e is also. In general, Π_n^e is not a lattice.

In this paper we study the representation of the symmetric group S_n on the homology of the subposet Π_n^e. A systematic study of the homology representations of rank-selected and other Cohen-Macaulay subposets of Π_n was initiated in [22], where a number of tools were introduced. We shall use these techniques to show that, when n is even, the homology representation of Π_n^e has remarkable properties. The formulas in [22, Theorem 2.8] can be programmed using Stembridge's symmetric functions package for Maple, and first

*This research was begun while the author was on leave during the academic year 1991-92.

allowed us to compute the homology representations. This data overwhelmingly supports the surprising conjecture that the (reduced) top homology is a permutation module for the full automorphism group S_n, when n is even.

Our approach to this problem begins with a recurrence, derived in [22], for the Frobenius characteristics of these representations. The exposition of the representation-theoretic results in this paper is cast entirely in the language of symmetric functions, as in [12]. Consequently no attempt is made to define homological concepts.

For readers who have some knowledge of symmetric functions, the contents of this paper may be summarised briefly as follows. We begin with a family of symmetric functions R_{2n}, each homogeneous of even degree $2n$, defined only by means of a plethystic recurrence (Theorem 2.1). By means of a purely computational and rather unilluminating argument, we are able to show (Theorem 2.5) that the symmetric function R_{2n} is in fact a polynomial with integer coefficients in the homogeneous symmetric functions h_1 and h_2. Extensive data indicates that these integers are nonnegative, and representation-theoretic considerations suggest that these integers somehow encode mysterious refinements, into powers of 2, of several well known numbers, such as the tangent number E_{2n-1}.

The main result of Section 1 is the computation of the Möbius number of Π^e_{2n}. We also discuss some interesting enumerative aspects of this number.

In Section 2 of the paper we examine the plethystic recurrence for the Frobenius characteristic of the homology representation of Π^e_{2n} more closely. The plethystic generating function yields valuable information on the representation; we prove that the character values vanish outside the set of involutions. We show that the representation is completely determined by a two-variable polynomial with integer coefficients, by the simple condition that all terms of even degree within a given range, vanish identically. This eliminates the need for plethystic calculations, and allows us to compute explicitly the data supporting our conjecture on the nature of the homology module.

In fact we present a recursive algorithm to compute a more general polynomial $q_{2n}(x, y)$ with integer coefficients, which determines, for m such that $2n \leq m \leq 2n + 1$, (the symmetric functions encoding) the homology representation of Π^e_m as well as that of the subposet of Π^e_m consisting of the top k ranks. The results of this computation support the stronger conjecture that the homology of these subposets of Π^e_{2n}, is also a permutation module for S_{2n}, for all $1 \leq k \leq n - 1$. The algorithm has allowed us to verify this for $2 \leq 2n \leq 40$. An equivalent formulation of our conjecture is that the polynomial $q_{2n}(x, y)$ has nonnegative coefficients.

The truth of our conjecture also implies that the homology modules of these posets are permutation modules for the subgroup S_{2n-1}. This is a less surprising phenomenon in the partition lattice; see [17, Corollary 7.6] and, more generally, [22, Theorem 2.1].

Finally in Section 3 we outline the enumerative implications of our representation-theoretic conjecture. The techniques we use to analyse the homology representation lead to the discovery of a family of integers $b_i(n), 2 \leq i \leq n$, for which we give an explicit but unenlightening recurrence. The integer $b_i(n)$ may also be defined as the (virtual) multiplicity of the permutation module induced from the Young subgroup $S_2^i \times S_1^{2n-2i}$ in the homology module $\tilde{H}(\Pi^e_{2n})$. Our representation-theoretic conjecture is equivalent to the nonnegativity of the integers $b_i(n)$. If true, this conjecture would yield apparently new refinements of the tangent number, the Genocchi number, and certain André or simsun permutations.

A formula for computing the homology representations of rank-selected subposets of Π_n was given in [22, Theorem 2.13]. It is curious that of all the proper rank-selected subposets of Π_m, the class studied in this paper seems to be the only one whose homology is interesting as an S_m-module. In particular there seems to be a special significance to the selection of even ranks. A similar significance seems to be attached to the selection of alternate ranks in an Eulerian poset P with nonnegative cd-index; see [20, Corollary 1.7].

One might ask if there are other approaches to showing that a homology module is a permutation module. In the case of posets which are homotopy equivalent to a wedge of spheres, one can argue that if the homology spheres themselves are permuted by the group action, then there must be an element of the poset which is fixed by the group action, namely, an element belonging to the facet which (eventually, by contraction) serves as a common wedge point for the bouquet of spheres in homology. Clearly no partition in Π_{2n}^e is fixed by S_{2n}, and hence one cannot hope to find a basis of fundamental cycles in homology (or dually, a basis of cochains in cohomology) which is permuted by S_{2n}.

Two other instances when a (co)homology representation turns out to be a permutation module for the full automorphism group, have recently appeared in the algebraic combinatorics literature. One is the classical case of the cohomology of the flag variety, (see [12, p. 136, Example 9]), more recently studied from a combinatorial viewpoint by Garsia and Procesi. The other example arises most recently in the work of Stembridge [21], who shows that the cohomology of the toric variety associated to the Coxeter complex of a crystallographic root system is always a permutation module for the corresponding Weyl group. In both cases the cohomology is a graded version of a permutation representation; the graded components themselves are not always permutation modules. The proofs in both the above cases use formal rather than constructive methods; indeed in neither instance does it seem to be known how to exhibit a basis that is actually permuted by the group.

The isotropy groups of the induced modules appearing in this paper will always be Young subgroups. Hence notation such as $S_a \times S_b$ or $S_n^m = \underbrace{S_n \times \cdots \times S_n}_{m \text{ copies}}$ will always refer to the appropriate Young subgroups in S_{a+b} and S_{mn} respectively.

1. Betti numbers

Denote by β_n the absolute value of the Möbius number of the poset Π_n^e. By the Cohen-Macaulay property of the poset, this is also the rank of the unique nonzero homology group over the integers, that is, the Betti number of the simplicial complex of chains of the poset. Thus $\beta_{2n} = (-1)^n \mu(\Pi_{2n}^e)$.

We shall use the classical compositional formula, as stated in [19], to compute β_{2n}.

Theorem 1.1 ([19, Theorem 5.1.4]). *Let $\mathbf{K}$ be a field of characteristic zero, and let f and g be $\mathbf{K}$-valued functions defined on the set of nonnegative integers, with $f(0) = 0$, and* *with respective exponential generating functions E_f, E_g. Define a $\mathbf{K}$-valued function h on the nonnegative integers by*

$$h(n) = \begin{cases} \sum_{\pi=\{B_1/B_2/\cdots/B_k\}} f(|B_1|)f(|B_2|)\cdots f(|B_k|)g(k), & if\ n > 0; \\ 1, & if\ n = 0. \end{cases}$$

where the sum ranges over all set partitions $\pi = \{B_1/B_2/\cdots/B_k\}$ of a set of size n into nonempty blocks.

Then the exponential generating function for h is given by

$$E_h(x) = E_g\big(E_f(x)\big).$$

Proposition 1.2 *For $k \geq 0$ define $g(k)$ to be 0 if k is odd, and $(-1)^n\beta_{2n}$, if $k = 2n$ is even; in particular $g(0) = 1$. Set $B(x) = E_g(x) - 1 = \sum_{n\geq 1}(-1)^n \frac{\beta_{2n}}{(2n)!}x^{2n}$. Then $B(x)$ satisfies the functional equation*

$$B(x) + B\Big(\frac{x}{1+x}\Big) = \frac{-x^2}{1+x}.$$

Proof: We apply the compositional formula to the functions g and f such that $f(n) = 1$ for all $n \geq 1$, $f(0) = 0$. Note that $E_f(x) = e^x - 1$. Letting $S(m,k)$ denote as usual the Stirling number of the second kind (which counts the number of set partitions in Π_m with k blocks) we find that $h(2n) = \sum_{k=1}^{n} S(2n, 2k)(-1)^k\beta_{2k}$, since the sum in Theorem 1.1 must now range over all partitions with an even number $2k$ of blocks. Observe that the top element, denoted by $\hat{1}$, has to be artificially adjoined to Π_{2n}^e. Thus $h(2n) = \sum_{\substack{x\in\Pi_{2n}^e \\ x\neq\hat{1}}} \mu(x,\hat{1})$, which in turn is $-\mu_{\Pi_{2n}^e}(\hat{1},\hat{1}) = -1$, by definition of the Möbius function.

Similarly $h(2n+1) = \sum_{k=1}^{n} S(2n+1, 2k)(-1)^k\beta_{2k} = \sum_{\substack{x\in\Pi_{2n+1}^e \\ x\neq\hat{0},\hat{1}}} \mu(x,\hat{1}) = -1$ $-\mu(\Pi_{2n+1}^e)$, since now the top and the bottom element $(\hat{0})$ have to be artificially adjoined to the poset Π_{2n+1}^e.

Now from Theorem 1.1, using the expression for $E_g(x)$ in the statement of the proposition, we obtain

$$1 - (e^x - 1) + \sum_{n\geq 1}(-1)^n\beta_{2n-1}\frac{x^{2n-1}}{(2n-1)!} = 1 + \sum_{n\geq 1}(-1)^n\beta_{2n}\frac{(e^x-1)^{2n}}{(2n)!}. \quad \text{(A)}$$

Eliminating the odd powers of x in this identity and substituting $\log(1+x)$ for x, we obtain the required functional equation. □

We can now deduce explicitly the value of β_{2n}. Let E_{2n-1} be the nth tangent number, $n \geq 1$ (that is, $\tan(x) = \sum_{n\geq 1} E_{2n-1}\frac{x^{2n-1}}{(2n-1)!}$).

The author is grateful to Richard Stanley and Ira Gessel for showing her the following elegant technique, which eliminated the cumbersome calculations of her original proof. Define a map θ on the ring of formal power series with coefficients in a field $\mathbf{K}$ of characteristic zero as follows:

$$\theta\Big(\sum_{n\geq 0} f(n)x^n\Big) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}.$$

Lemma 1.3 *Let $F(x) = \sum_{n\geq 0} f(n)x^n$. Under the map θ, the image of the power series $\frac{1}{1-\alpha x}F(\frac{x}{1-\alpha x})$ is $e^{-\alpha x}\theta(F(x))$.*

For a proof, see [18, Exercises 37(a) and 36(a)].

Theorem 1.4 *The top homology of the poset Π_{2n}^{e} is a space of dimension $\beta_{2n} = \frac{(2n)!}{2^{2n-1}} E_{2n-1}$. Equivalently, this is the value of $(-1)^n$ times the Möbius number.*

Proof: Write $b_{2n-1} = (-1)^n \frac{\beta_{2n}}{(2n)!}$.

Note that the exponential generating function for the numbers $(-1)^n E_{2n-1}$ is $i\tan(ix)$, where $i^2 = -1$. Hence it suffices to show that the generating function

$$\sum_{n\geq 1} \frac{b_{2n-1}}{(2n-1)!} x^{2n-1}$$

for the numbers b_{2n-1} is

$$i\tan\left(\frac{ix}{2}\right) = -\tanh\left(\frac{x}{2}\right).$$

Apply Lemma 1.3 to the functional equation of Proposition 1.2, taking $F(x) = \frac{1}{x}B(x)$. The result now follows by applying the map θ, noting that $\theta(\frac{1}{1+x}) = e^{-x}$. □

Remark 1.4.1 Here we collect some identities involving the Betti numbers $\beta_{2n} = (-1)^n\mu(\Pi_{2n}^{e})$.

(i) If we use the defining equation for the Möbius function on the poset $\Pi_{2n}^{e}\cup\{1\}$, we obtain the identity $0 = \sum_{k=0}^{n}(-1)^k S(2n,2k)\beta_{2k}$, or equivalently, from the preceding result,

$$1 = \sum_{k=1}^{n}(-1)^{k-1}(2k)!2^{2n-2k}S(2n,2k)\, E_{2k-1}.$$

(ii) Recall from [18, Example 3.13.5] that an atom-ordering of a geometric lattice gives an R-labelling (in fact an admissible labelling) of the lattice; Theorem 3.13.2 of [18] then says that the Betti number of a rank-selected subposet of the lattice is given by the number of chains with descents in the positions corresponding to the ranks selected. (It is shown in [4] that an admissible labelling of a lattice is in fact an EL-labelling, and consequently defines a shelling order on the maximal chains of the lattice.) The atom-ordering corresponding to Gessel's EL-labelling for the partition lattice as described in [4] is $(12) < \cdots < (1i) < (2i) < \cdots < (i-1, i) < \cdots < (1, i+1) < \cdots < (n-1, n)$. The chains in Π_{2n} with descents in the positions corresponding to even ranks may be termed up-down chains; counting up-down chains with respect to this labelling yields the formula ([7, Theorem 1.7])

$$\beta_{2n} = \sum_{\substack{\sigma \in S_{2n-1} \\ \sigma \text{ is an up-down} \\ \text{permutation}}} \prod_{k=1}^{2n-1} (|\{i : i < k,\ \sigma(i) < \sigma(k)\}| + 1),$$

and hence, by Theorem 1.4,

$$\frac{(2n)!}{2^n} = \frac{2^{n-1}}{E_{2n-1}} \sum_{\substack{\sigma \in S_{2n-1} \\ \sigma \text{ is an up-down} \\ \text{permutation}}} \prod_{k=1}^{2n-1} (|\{i : i < k,\ \sigma(i) < \sigma(k)\}| + 1).$$

(iii) The well known recurrence

$$E_{2n-1} = \sum_{i=1}^{n-1} \binom{2n-2}{2i-1} E_{2i-1}\, E_{2n-2i-1}$$

for the tangent number E_{2n-1}, translates into the following recurrence for the Betti number β_{2n}:

$$\beta_{2n} = \frac{1}{2} \sum_{i=1}^{n-1} \binom{2n}{2i} \binom{2n-2}{2i-1} \beta_{2i}\, \beta_{2n-2i}.$$

It is unclear to us how this latter formula can be explained by means of the combinatorial interpretation of up-down chains in Π_{2n} described in (ii).

(iv) It is worth noting that, by a result of Stanley [15], the tangent number E_{2n-1} is itself the Betti number of a subposet of the partition lattice, namely, the join sublattice Π_n^2 of Π_{2n} generated by the partitions with n blocks all of size 2. The homology representation of S_{2n} for this lattice was determined in [6].

(v) Finally, the Euler number E_{m-1} is also the Betti number of the subposet of the Boolean lattice B_m consisting of alternate ranks. The homology representation of S_m in this case was computed by Solomon ([14]) to be one of the Foulkes representations: it is the representation of S_m indexed by a skew-hook or border strip (see [12] for definitions) formed by horizontal strips all of size 2, with the exception of the top-most strip, which may be of size 1.

Remark 1.4.2 From the proof of Proposition 1.2, the Betti numbers β_{2n-1} of the poset Π_{2n-1}^e are determined by equation (A) in the proof. We are unable to obtain an explicit closed formula for β_{2n-1}. The first few values, for $n = 1, 2, 3$ and 4, are 1, 2, 46 and 5522.

Next we present some numerical evidence which suggests the following: The numbers β_{2n} are related to the number of maximal chains in the full partition lattice Π_{2n} which are orbit representatives of chains fixed by the action of S_2^n, or equivalently, representatives of orbits of size $\frac{(2n)!}{2^n}$. This in turn implies a connection with certain types of André permutations [8].

In [22], the action of S_n on the maximal chains of Π_n was studied, and the following theorem was proved.

Define positive integers $a_i(n)$ by means of the recurrence

$$a_i(n+1) = ia_i(n) + (n - 2i + 2)a_{i-1}(n),$$

with initial conditions

$$a_0(1) = 1 = a_1(2),\ a_0(n) = 0,\ n > 1, \text{ and } a_i(n) = 0 \text{ if } 2i > n.$$

We have

Theorem 1.5 ([22, Theorem 3.2]). *The action of S_n on the chains of Π_n decomposes into orbits as follows:*

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(n) W_i,$$

where W_i denotes the transitive permutation representation of S_n acting on the cosets of $S_2^i \times S_1^{n-2i}$.

It follows from [17, Theorem 7.7] that the positive integers $a_i(n)$ refine the Euler number E_{n-1}.

R. Simion and this author observed from the above recurrence that the numbers $a_i(n)$ admit the following combinatorial interpretation: $a_i(n)$ is the number of permutations in S_{n-2} with exactly $i-1$ descents, none consecutive, and with the hereditary property that when the letters $n-2, n-3, \ldots, 3, 2, 1$, are erased in succession, the property of not having any consecutive descents is preserved after each erasure. We call these simsun permutations. The equinumerous set of André permutations appears in work of Foata and Schützenberger (see [23] and also the references in [22]). Both sets reappear in recent work of Purtill and Stanley on the cd-index (see [20]). With the Foata-Schützenberger interpretation, $a_i(n)$ is the number of André permutations in S_{n-1} with exactly i peaks, where j is a peak if either $j = 1$ and $\sigma(1) > \sigma(2)$, or if $2 \leq j \leq n-2$ and $\sigma(j) > \max(\sigma(j-1), \sigma(j+1))$.

Note in particular that $a_n(2n)$ is the number of alternating (up-down or down-up) permutations in S_{2n-2} with the hereditary property of no consecutive descents described above. It now transpires that the Betti number β_{2n} is an exact multiple of the number $a_n(2n)$:

Proposition 1.6 *We have*

$$E_{2n-1} = 2^{n-1} a_n(2n) \quad \text{and} \quad \beta_{2n} = \frac{(2n)!}{2^n} a_n(2n).$$

Proof: From Theorem 1.5, we see that $a_n(2n)$ is the number of orbits of maximal saturated chains in Π_{2n} which are fixed point-wise under the action of a subgroup conjugate to S_2^n. In order that a partition be fixed by all involutions generated by $(1,2), \ldots (2i-1, 2i), \ldots, (2n-1, 2n)$, it must be the case that either $2i-1$ and $2i$ are in the same block, or they are both singletons. Hence $a_n(2n)$ is the number of orbits of chains in the subposet of partitions with the property that the only blocks of odd size are the singletons, and the latter occur in pairs. We use this fact and a counting argument completely analogous to [17, Theorem 7.7] to obtain the recurrence

$$a_n(2n) = \frac{1}{2} \sum_{i=1}^{n-1} \binom{2n-2}{2i-1} a_i(2i)\, a_{n-i}(2n-2i).$$

To see this, first choose the element of maximal rank in the representative chain; such an element is of the form $\pi = B_1/B_2$, where B_1 and B_2 are blocks of sizes $2i$ and $2n-2i$ respectively, $1 \leq i \leq n-1$. Since we are counting orbit representatives it is enough to choose one pair of blocks for each pair of sizes. Now choose representative chains from $\Pi_{|B_1|}$ in $a_i(2i)$ ways and from $\Pi_{|B_2|}$ in $a_{n-i}(2n-2i)$ ways; these can be intertwined in $\binom{2n-2}{2i-1}$ ways to produce a maximal chain in Π_{2n}.

Now from (iii) of Remark 1.4.1, it is clear that this coincides with the recurrence satisfied by the numbers $\frac{E_{2n-1}}{2^{n-1}}$. □

Remark 1.6.1 Let c_n denote the number of maximal saturated chains in Π_n. Counting from the bottom, we have $c_n = \frac{n!(n-1)!}{2^{n-1}}$. We then have (see [22, Remark 3.2.1])

$$(n-1)! = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(n) 2^{n-1-i};$$

this decomposition is explained combinatorially in [9] and [10], by means of an action of the subgroup $S_2^{\lfloor \frac{n-1}{2} \rfloor}$ on the set of $(n-1)!$ permutations. See also [23].

It would be interesting to investigate combinatorially the various enumerative formulas of this section.

2. The homology representation

It is instructive to look at the first nontrivial example, Π_4^e. Here the elements are all at rank 1; there are four partitions in the orbit of $/123/4/$, and three in the orbit of $/12/34/$.

The reduced homology is concentrated in dimension zero, and as an S_4-module it is a sum of two transitive permutation representations, with one copy of the trivial representation removed. A little thought reveals the following surprising fact: the homology module is simply the (single) transitive permutation module induced from the Young subgroup $S_2 \times S_2$ (although, as observed in the introduction, this is not obvious from any basis of fundamental cycles). This is our first hint that the homology modules of Π_{2n}^e have interesting properties.

We refer to [12] for all background on symmetric functions. As in [12], we denote the plethysm of symmetric functions f, g by $f[g]$. If f and g are respectively Frobenius characteristics of the S_m-module V and the S_n-module W, then the plethysm $f[g]$ is the characteristic of the module $V[W]$ induced up to S_{mn} from the wreath product subgroup $S_m[S_n]$, where S_m acts on m copies of S_n. (Note that $S_m[S_n]$ is the normaliser in S_{mn} of the Young subgroup $\underbrace{S_n \times \cdots S_n}_{m \text{ copies}}$.)

The remainder of this section is devoted to studying the symmetric functions associated to the homology representations, via the Frobenius characteristic map. We begin with a family of symmetric functions R_n. For the purposes of this paper, the plethystic recurrence of Theorem 2.1 below may be taken to be the defining equation for the functions R_n. (We note in passing that because R_n is the Frobenius characteristic of a representation, we know in addition that it is a nonnegative integer combination of Schur functions.)

Theorem 2.1 (See [22, Theorem 2.8]). *Let R_n denote the Frobenius characteristic of the representation of S_n on the top homology of Π_n^e. Then*

(i) $$\sum_{n\geq 0}(-1)^n R_{2n+1} = \left(h_1 - R_2 + R_4 - \cdots\right)\left[h_1 + h_2 + \cdots\right].$$

Equivalently, separating terms of even and odd degree, we find that R_{2n} (respectively $(-1)^{n-1}R_{2n-1}$) is the degree $(2n)$ (respectively degree $(2n-1)$) term in the alternating sum of plethysms

$$(R_{2(n-1)} - R_{2(n-2)} + \cdots + (-1)^{n-2}R_2 + (-1)^{n-1}h_1)\left[\sum_{j\geq 1} h_j\right],$$

while terms in this sum of even degree smaller than $(2n)$ vanish identically.

(ii) *Let $\Pi_m^e(\bar{k})$ denote the rank-selected subposet of Π_m^e consisting of the top k nontrivial ranks $\{n - 1, \ldots n - k\}$, $0 \leq k \leq n$, for $2n \leq m \leq 2n + 1$. (In particular, if $k = n - 1$, this is the whole poset Π_m^e). Then the Frobenius characteristic of the top homology of this subposet as an S_m-module is given by the degree m term in the alternating sum*

$$(R_{2k} - R_{2(k-1)} + \cdots + (-1)^{k-1}R_2 + (-1)^k h_1)\left[\sum_{j\geq 1} h_j\right].$$

In this section we shall use the defining equation above to investigate the representations R_{2n}. The following well known expansion of the plethysm $h_2[h_n]$ into Schur functions is easily established by routine computation. It also provides an explanation for the phenomenon noticed at the beginning of this section, for the representation R_4.

Proposition 2.2 $h_2[h_n] = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} s_{(2n-2k,2k)}$.

Now write H_e for the sum $\sum_{n\geq 1} h_{2n}$, $H_o = \sum_{n\geq 0} h_{2n+1}$, and $H = H_e + H_o$. The following identity is the essential trick to simplifying the plethystic recurrence, and is the crucial ingredient in the proof of Theorem 2.5 below.

Proposition 2.3 $h_2[H] = H_e(1+H) = H_e(1+H_e) + H_oH_e$.

Proof: From the rules for computing plethysm of symmetric functions, we have $h_2[H] = \sum_{n\geq 1} h_2[h_n] + \sum_{n\geq 1}\sum_{i:1\leq i<n-i} h_ih_{n-i}$. Using Proposition 2.2, the result follows. $\square$

The next observation will be used frequently:

Lemma 2.4 *The lowest degree term in $H_e^iH_o^{2n-2i}$ is of degree $2n$, and equals $h_2^ih_1^{2n-2i}$.*

We are now ready to establish the following surprising result:

Theorem 2.5 *The Frobenius characteristic R_{2n} is a polynomial in the homogeneous symmetric functions h_2 and h_1. Equivalently, the character values of S_{2n} acting on the top homology of Π_{2n}^e are supported on involutions.*

Proof: We shall proceed by induction on n. Note first that, using the defining equation, we have $R_2 = h_2$, since $h_1[H] = H_e + H_o$. Now R_4 is the degree 4 term in $(R_2 - h_1)[H]$, that is, (using Propostion 2.3), the degree 4 term in $H_e^2 + H_o(H_e - 1)$. Hence $R_4 = h_2^2$. Our induction hypothesis will be that the plethysm

$$(R_{2(n-1)} - R_{2(n-2)} + \cdots + (-1)^{n-1}R_2 + (-1)^n h_1)[H]$$

is a polynomial in H_e and H_o with integer coefficients. More specifically, isolating the terms of even and odd degree with respect to the grading $\deg(H_e) = 2$, $\deg(H_o) = 1$, we assume that there exist polynomials $Q_{2(n-1)}(x,y)$ and $Q'_{2(n-1)}(x,y)$ in the variables x and y, with integer coefficients, such that

$$\begin{aligned}(R_{2(n-1)} &- R_{2(n-2)} + \cdots + (-1)^{n-1}R_2 + (-1)^n h_1)[H] \\ &= Q_{2(n-1)}(H_e, H_o^2) + H_oQ'_{2(n-1)}(H_e, H_o^2);\end{aligned} \tag{A}$$

we further assume that $Q_{2(n-1)}(x,y)$ is divisible by x^2 for $n \geq 3$. (Note in particular that with respect to the above grading, $Q_{2(n-1)}(H_e, H_o^2)$ and $Q'_{2(n-1)}(H_e, H_o^2)$ are both of even degree.)

From the first paragraph of the proof, we have $Q_2(H_e, H_o^2) = H_e$, $Q'_2(H_e, H_o^2) = 1$, and $Q_4(H_e, H_o^2) = H_e^2$, $Q'_4(H_e, H_o^2) = H_e - 1$; hence these hypotheses are satisfied for $n = 2, 3$.

Note that by (A), only the terms in $Q_{2(n-1)}(H_e, H_o^2)$ contribute to R_{2n}. The defining equation of Theorem 2.1 (i) implies that R_{2n} is the degree $2n$ term in $Q_{2(n-1)}(H_e, H_o^2)$; by Lemma 2.4, since all terms of smaller even degree in the left-hand side of (A) must vanish identically, this is the lowest degree term in $Q_{2(n-1)}(H_e, H_o^2)$, as a polynomial in H_e and H_o^2.

Now let $n \geq 3$. By induction hypothesis the lowest degree term in $Q_{2n-2}(H_e, H_o^2)$ is of the form

$$\sum_{i=2}^{n} b_i(n) H_e^i (H_o^2)^{n-i},$$

for some integers $b_i(n)$, $2 \leq i \leq n$. By Lemma 2.4, R_{2n} equals $\sum_{i=2}^{n} b_i(n) h_2^i h_1^{2n-2i}$, and hence, using Proposition 2.3, $R_{2n}[H]$ equals

$$\sum_{i=2}^{n} b_i(n)\ (h_2[H])^i H^{2n-2i} = \sum_{i=2}^{n} b_i(n)\ H_e^i (1 + H_e + H_o)^i (H_e + H_o)^{2n-2i}. \tag{B}$$

In particular, $R_{2n}[H]$ is divisible by H_e^2.

It follows that there are polynomials $Q_{2n}(x, y)$ and $Q'_{2n}(x, y)$ with integer coefficients, such that

$$\begin{aligned}(R_{2n} - R_{2(n-1)} + \cdots + (-1)^{n-2} R_2 + (-1)^{n-1} h_1)[H] \\ = Q_{2n}(H_e, H_o^2) + H_o Q'_{2n}(H_e, H_o^2),\end{aligned} \tag{C}$$

where, by inspecting the right-hand side of (B), since

$$Q_{2n}(H_e, H_o^2) = \text{ even degree terms in } R_{2n}(H_e, H_o^2) - Q_{2n-2}(H_e, H_o^2),$$

it is clear that $Q_{2n}(H_e, H_o^2)$ is divisible by H_e^2.

Note also that, with respect to the grading $\deg(H_e) = 2$, $\deg(H_o) = 1$, the lowest degree term in (B) coincides with the lowest degree term in $Q_{2n-2}(H_e, H_o^2)$, and has degree $2n$. Hence the lowest degree term in $Q_{2n}(H_e, H_o^2)$ has degree $(2n + 2)$, and is of the form

$$\sum_{i=2}^{n+1} b_i(n+1) H_e^i (H_o^2)^{n+1-i}, \tag{D}$$

for some integers $b_i(n + 1)$, $2 \leq i \leq n + 1$.

It now follows, by another application of Lemma 2.4, that $R_{2(n+1)}$, which by the defining equation is the term of degree $2(n + 1)$ in the symmetric function $Q_{2n}(H_e, H_o^2)$, is of the form

$$\sum_{i=2}^{n+1} b_i(n+1) h_2^i h_1^{2(n+1)-2i},$$

where the $b_i(n + 1)$ are integers, uniquely determined from (A), (B) and (C).

The second assertion in the statement of the theorem follows from the fact that the character of the permutation representation on cosets of the Young subgroup $S_2^i \times S_1^{2n-2i}$ is supported on the set of involutions. □

Now let $n \geq 2$, and let $b_i(n)$ be the unique integers such that

$$R_{2n} = \sum_{i=2}^{n} b_i(n) h_2^i h_1^{2n-2i}.$$

Remark 2.5.1 Converting the homogeneous symmetric functions to power-sums, we see that the character of the homology representation on involutions of type $(2^i, 1^{2n-2i})$ is given by

$$\sum_{j=2}^{n} b_j(n) \frac{1}{2^j} \binom{j}{i} 2^i i! (2n-2i)!.$$

In particular, the character value on fixed-point-free involutions is $b_n(n)n!$.

Let $P_{2n+2}(x, y)$ be the polynomial defined recursively by $P_2 = x,\ P_4 = x^2$, and

$$\begin{aligned} P_{2n+2}(x,y) &= (x+y) - b_1(1)x(1+x+y) \\ &\quad + \sum_{k=2}^{n} (-1)^k P_{2k}(x \leftarrow x(1+x+y), y \leftarrow x+y) \\ &= (x+y) - x(1+x+y) + \sum_{k=2}^{n} (-1)^k \\ &\quad \times \sum_{i=2}^{k} b_i(k) x^i (1+x+y)^i (x+y)^{2k-2i}. \end{aligned}$$

We have

Corollary 2.6 *With respect to the grading* $\deg x = 2$, $\deg y = 1$, *all terms of total (even) degree at most* $2n$ *in* $P_{2n+2}(x, y) + P_{2n+2}(x, -y)$ *vanish identically, and the resulting equations determine the* $b_i(n)$, *for* $n \geq 2$.

This gives the recurrences $b_2(n) = 1$ *and* $b_3(n) = 3b_3(n-1) + 2(2n-5)$, $b_3(3) = 2$, *and* $b_4(n) = (6b_4(n-1) - b_4(n-2)) + (2n-7)(3b_3(n-1) - b_3(n-2))$.

Proof: Define $b_1(1) = 1$. Then $R_2 = b_1(1)h_2$. The defining equation of Theorem 2.1 implies that all terms of even degree at most $2n$ vanish in the symmetric functions expansion of

$$R_1[H] - R_2[H] + \cdots + (-1)^n R_{2n}[H].$$

Using Proposition 2.3, we find that all terms of even degree at most $2n$ in the symmetric function expansion of

$$(H_e + H_o) - b_1(1)H_e(1 + H_e + H_o) + \sum_{k=2}^{n}(-1)^k \times \sum_{i=2}^{k} b_i(k)H_e^i(1 + H_e + H_o)^i(H_e + H_o)^{2k-2i}$$

must vanish identically. Recall that H_e (respectively H_o) contributes only even (respectively odd) degree terms. Write $P_{2n+2}(x, y)$ for the polynomial obtained by substituting x for H_e and y for H_o, i.e.,

$$P_{2n+2}(x, y) = (x + y) - x(1 + x + y) + \sum_{k=2}^{n}(-1)^k \times \sum_{i=2}^{k} b_i(k)x^i(1 + x + y)^i(x + y)^{2k-2i}.$$

The defining equation for the $b_i(n)$ now translates into the statement that, with respect to the grading $\deg x = 2$, $\deg y = 1$, all terms of total even degree at most $2n$ in $P_{2n+2}(x, y) + P_{2n+2}(x, -y)$ vanish identically.

In particular, extracting the coefficient of x^2y^{2k-4} in $P_{2n+2}(x, y)$, we find that $b_2(k) - b_2(k-1) = 0$ for all $k \geq 2$. Since $b_2(2) = 1$, it follows that $b_2(k) = 1$ for all $k \geq 2$.

Similarly, extracting the coefficient of x^3y^{2k-6}, we obtain $b_3(k) = 3b_3(k-1) + 2(2k - 5)b_2(k)$. □

The recurrences of Corollary 2.6 imply that $b_i(n) > 0$ for $i = 2, 3,$ and 4. To see that $b_4(n)$ is positive, note first that $b_3(n) \geq b_3(n-1)$ for all n. Assume inductively that $b_4(n-1) > b_4(n-2)$. (This is true for $n - 1 = 4$.) Then the recurrence shows that $b_4(n) > b_4(n-1) > 0$ for all $n \geq 5$.

Remark 2.6.1 The general recurrence for the $b_i(n)$ can be extracted from this corollary, and is recorded in Conjecture 3.1 of Section 3.

The proof of Theorem 2.5 yields an efficient way to compute the characteristic R_{2n}, without going through laborious plethystic calculations: It is enough to extract the coefficient of t^{2n} in the polynomial $P_{2n}(t^2h_2, th_1)$. We obtain, up to $n = 7$:

$$\begin{aligned}
R_2 &= h_2 \\
R_4 &= h_2^2 \\
R_6 &= 2h_2^3 + h_2^2h_1^2 \\
R_8 &= 6h_2^4 + 12h_2^3h_1^2 + h_2^2h_1^4 \\
R_{10} &= 28h_2^5 + 138h_2^4h_1^2 + 46h_2^3h_1^4 + h_2^2h_1^6 \\
R_{12} &= 208h_2^6 + 1904h_2^5h_1^2 + 1452h_2^4h_1^4 + 152h_2^3h_1^6 + h_2^2h_1^8 \\
R_{14} &= 2,336h_2^7 + 33,360h_2^6h_1^2 + 45,320h_2^5h_1^4 + 11,444h_2^4h_1^6 + 474h_2^3h_1^8 + h_2^2h_1^{10}
\end{aligned}$$

These computations lead us to make the following:

Conjecture 2.7 *The homology of Π_{2n}^e is in fact a permutation module for S_{2n}: it is a sum of induced modules whose isotropy groups are Young subgroups of the form $S_2^i \times S_1^{2n-2i}$, with nonnegative integer multiplicity $b_i(n)$, $2 \le i \le n$. The integers $b_i(n)$ are defined as in Corollary 2.6.*

It is also clear from Theorem 2.5 that R_{2n} may be viewed as a sum of (possibly virtual) representations induced from the Young subgroup S_2^n. In fact we have a second way of writing R_{2n} as such:

Corollary 2.8 *For $2 \le i \le n$, let V_i be the irreducible representation of S_2^n which acts like the trivial representation on i copies of S_2, and like the sign on the remaining $(n-i)$ copies. Define integers $E_k(n)$ by*

$$E_k(n) = \sum_{i=2}^{n} b_i(n)\binom{n-i}{k-i}, \quad 2 \le k \le n.$$

Then one has a (possibly virtual) direct sum decomposition

$$\tilde{H}(\Pi_{2n}^e) = \left(\sum_{i=2}^{n} E_i(n)\, V_i\right)\Big\uparrow_{S_2^n}^{S_{2n}}.$$

In particular, $E_2(n) = 1$, $E_3(n) = 3E_3(n-1) + (2n-3)$, and $E_n(n)$ is the multiplicity of the trivial representation in the top homology of $\tilde{H}(\Pi_{2n}^e)$.

Proof: Make the substitution $h_1^2 = h_2 + e_2$ in the characteristic R_{2n}. □

It follows that Conjecture 2.7, if true, would also imply the positivity of the integers $E_i(n)$. We already know that this is the case for $E_n(n)$, since it is the multiplicity of the trivial representation in $\tilde{H}(\Pi_{2n}^e)$.

The first few values of the $E_i(n)$, up to $n = 6$, are given below, as the coefficient of $h_2^i e_2^{n-i}$, the characteristic of the module V_i of Corollary 2.8.

$$\begin{aligned}
R_2 &= h_2\\
R_4 &= h_2^2\\
R_6 &= 3h_2^3 + h_2^2 e_2\\
R_8 &= 19h_2^4 + 14h_2^3 e_2 + h_2^2 e_2^2\\
R_{10} &= 213h_2^5 + 233h_2^4 e_2 + 49h_2^3 e_2^2 + h_2^2 e_2^3\\
R_{12} &= 3717h_2^6 + 5268h_2^5 e_2 + 1914h_2^4 e_2^2 + 156h_2^3 e_2^3 + h_2^2 e_2^4
\end{aligned}$$

Finally, Theorem 1.1 can also be used to compute the Whitney homology of the dual of Π_{2n}^e; combined with Proposition 1.9 of [22], one can then compute the homology of the rank $k+1$ subposets $\Pi_{2n}^e(\bar{k})$ obtained by selecting the top k nontrivial ranks of Π_{2n}^e, $(1 \le k \le n-1)$. We have the following particular cases:

Proposition 2.9

(i) *The characteristic of* $\tilde{H}_0(\Pi^e_{2n}(\bar{1}))$ *is equal to*

$$2 \sum_{1 \le k < n-k} h_{2k} h_{2n-2k} + h_n^2 \delta_{n,\,\mathrm{even}},$$

where $\delta_{n,\,\mathrm{even}}$ *equals* 1 *if* n *is even, and* 0 *otherwise.*

(ii) *If* μ *is an integer partition of* n, *denote by* $\ell(\mu)$ *the number of parts of* μ. *Also write* $m_i(\mu)$ *for the multiplicity of the part* i *in* μ. *The characteristic of* $\tilde{H}_1(\Pi^e_{2n}(\bar{2}))$ *is equal to*

$$\sum_{\mu \vdash 2n,\, \ell(\mu)=4} \binom{\ell(\mu)}{m_2(\mu), m_4(\mu), \ldots} h_\mu + 2 \sum_{\mu \vdash 2n,\, \ell(\mu)=3} \binom{\ell(\mu)}{m_2(\mu), m_4(\mu), \ldots} h_\mu$$
$$+ \sum_{k=2}^{n-1} \sum_{\substack{\mu \vdash 2k,\, \ell(\mu)=2 \\ \nu \vdash 2n-2k,\, \ell(\nu)=2}} \binom{\ell(\mu)}{m_2(\mu), m_4(\mu), \ldots} \binom{\ell(\nu)}{m_1(\mu), m_3(\mu), \ldots} h_\mu h_\nu,$$

where the first and second sums run over partitions μ *with all parts even, and the third sum runs over pairs of partitions* (μ, ν) *such that all the parts of* μ *are even, and all the parts of* ν *are odd.*

Proof:

(i) From Theorem 2.1 (2), the characteristic is the degree $2n$ term in $R_2[H] - H = H_e(1 + H_e + H_o) - H$, hence equals the degree $2n$ term in H_e^2.

(ii) This time the characteristic is the degree $2n$ term in $R_4[H] - R_2[H] + H = H_e^2(1 + H_e + H_o)^2 - H_e(1 + H_e + H_o) + H$, hence equals the degree $2n$ term in $H_e^4 + (2H_e^3 + H_e^2 H_o^2)$. The results follow. □

Notice that both homology modules of Proposition 2.9 are permutation modules for S_{2n}. This observation is reinforced by data from further computations. We can now strengthen Conjecture 2.7 by the following representation-theoretic conjecture:

Conjecture 2.10 *The homology* $\tilde{H}_{k-1}(\Pi^e_{2n}(\bar{k}))$, *for* $1 \le k \le n-1$, *is a permutation module for the action of* S_{2n}, *and can be written as a sum of induced modules whose point stabilisers are all Young subgroups.*

In general we are unable to discern any particular distinguishing qualities for the isotropy subgroups, beyond the fact that Young subgroups indexed by the conjugacy classes of involutions appear only for $k = n - 1$.

Remark 2.10.1 For the case $k = 1$, we can describe a basis for the (co)homology which is permuted by the group action:

For each i such that $1 \leq i \leq n$, let τ_i be the partition whose blocks are $\{1, 2, \ldots, i\}$ and $\{i+1, \ldots, 2n\}$. For each k such that $1 \leq 2k \leq 2n - 2k < 2n$, consider the sums of fundamental cycles

$$x_k = \sum_{\substack{\sigma \\ \in T_{2k} \setminus T_{2k-1}}} \sigma.(\tau_{2n-2k} - \tau_{2n-2k+1}),$$
$$y_k = \sum_{\substack{\sigma \\ \in S_{2n-2k} \setminus S_{2n-2k-1}}} \sigma.(\tau_{2n-2k} - \tau_{2n-2k-1}),$$

where $\sigma(\tau)$ is the image of the partition τ under the permutation σ, and T_i is the subgroup of S_{2n} which fixes the first $2n - i$ letters point-wise. Here we identify S_i with the subgroup of S_{2n} which fixes the $2n - i$ largest letters point-wise.

The stabilisers of x_k and y_k are respectively $X_k = S_{\{1,\ldots,2k\}} \times S_{\{2k+1,\ldots,2n\}}$ and $Y_k = S_{\{1,\ldots,2n-2k\}} \times S_{\{2n-2k+1,\ldots,2n\}}$, if $2k < 2n - 2k$. If $2k = 2n - 2k$, then the stabiliser is $Z_k = S^2_{\frac{n}{2}}$. Also the elements $\{\sigma(x_k), \rho(y_k)\}$, where σ and ρ range respectively over the coset representatives of X_k and Y_k, are all independent. When n is even and $2k = n$, similarly the collection $\{\rho(x_k)\}$ for ρ ranging over the coset representatives of X_k, is independent of all the others. A more detailed analysis shows that in this case also, the set $\{\rho(x_k)\}$ is itself independent.

Hence for $2k < 2n - 2k$, x_k (respectively y_k) generates the transitive permutation module with stabiliser subgroup X_k (respectively Y_k). A similar statement holds for the case $2k = 2n - 2k$.

Remark 2.10.2 Note that the Betti number of the poset $\Pi^e_{2n}(\bar{k})$ can be computed using the defining recurrence for the Möbius function (see also [22, Proposition 2.16]). We then have

$$(-1)^{k-1}\mu(\Pi^e_{2n}(\bar{k})) = \sum_{i=0}^{k-1}(-1)^i \beta_{2k-2i} S(2n, 2i) + (-1)^k,$$

where β_{2k} is the Betti number of the poset Π^e_{2k}.

The proof of Theorem 2.5 shows that the representation $\tilde{H}(\Pi^e_{2n}(\bar{k}))$ is in fact determined by the degree $2n$ term in the polynomial $Q_{2k}(H_e, H_o^2)$, which was shown to have integral coefficients. Hence Conjecture 2.10 is equivalent to the conjecture that this polynomial has *nonnegative* integral coefficients. From the proof of Theorem 2.5, we extract the following recursive algorithm for computing the homology representations:

Algorithm 2.11 *Define polynomials* $F_{2n}(x, y)$, $n \geq 1$, *recursively by setting*

(i) $F_2(x, y) = t^2 x + ty$;

(ii) $q_{2n}(x, y) = \frac{1}{2}(F_{2n}(x, y) + F_{2n}(x, -y))$;

(iii) $r_{2n}(x, y) =$ coefficient of t^{2n} in $q_{2n}(x, y) =$ coefficient of t^{2n} in $F_{2n}(x, y)$;

(iv) $F_{2n+2}(x,y) = t^{2n} r_{2n}(x(t^2x+ty+1), tx+y) - F_{2n}(x,y)$. *Then*

(a) *The characteristic* R_{2n} *of the top homology of* Π_{2n}^e *is given by* $r_{2n}(h_2, h_1)$;

(b) *The characteristic of the top homology of* $\Pi_{2n}^e(\bar{k})$ *(respectively* $\Pi_{2n-1}^e(\bar{k})$*), for* $1 \le k \le n-1$*, is given by substituting* $x = H_e, y = H_o$ *in* $q_{2k+2}(x,y)$ *(respectively* $F_{2k+2}(x,y) - q_{2k+2}(x,y)$*) and then extracting those terms which are homogeneous polynomials of degree* $2n$ *(respectively* $2n-1$*) in the* $\{h_i\}_{i \ge 1}$.

We may now summarise the content of the preceding conjectures by the statement that $q_{2n}(x,y)$ is a polynomial in t, x and y, all of whose coëfficients are nonnegative integers.

Conjecture 2.7 implies that the Whitney homology (see [2], [5] and [22]) of the dual poset is also a permutation module. However, computations show that the Whitney homology of Π_{2n}^e is not, in general, a permutation module.

The first few polynomials $q_{2n}(x,y)$ are recorded below. (The coefficient of t^{2n} is r_{2n}.)

$$
\begin{aligned}
q_2 &= t^2(x)\\
q_4 &= t^4(x^2)\\
q_6 &= t^6(2x^3 + x^2y^2) + t^8x^4\\
q_8 &= t^8(6x^4 + 12x^3y^2 + x^2y^4) + t^{10}(8x^5 + 12x^4y^2) + t^{12}(3x^6)\\
q_{10} &= t^{10}(28x^5 + 138x^4y^2 + 46x^3y^4 + x^2y^6)\\
&\quad + t^{12}(70x^6 + 308x^5y^2 + 81x^4y^4)\\
&\quad + t^{14}(62x^7 + 171x^6y^2) + t^{16}(19x^8)\\
q_{12} &= t^{12}(1904x^5y^2 + 1452x^4y^4 + 152x^3y^6 + 208x^6 + x^2y^8)\\
&\quad + t^{14}(816x^7 + 7032x^6y^2 + 5040x^5y^4 + 488x^4y^6)\\
&\quad + t^{16}(1228x^8 + 8472x^7y^2 + 3890x^6y^4)\\
&\quad + t^{18}(832x^9 + 3344x^8y^2) + t^{20}(213x^{10}).
\end{aligned}
$$

For odd m, the representations R_m do not seem to have especially nice properties. We have, using Algorithm 2.11,

$$
\begin{aligned}
R_1 &= h_1,\ R_3 = h_2h_1 - h_3,\\
R_5 &= 2h_2^2h_1 - h_2h_3 - h_4h_1 + h_5,\\
R_7 &= 6h_2^3h_1 + 2h_2^2h_1^3 - 2h_2^2h_3 - 4h_2h_4h_1 + h_6h_1 + h_4h_3 + h_2h_5 - h_7.
\end{aligned}
$$

Using the techniques in [22, Section 4], we obtain from the defining equation of Theoerm 2.1 a relation between the multiplicities of (Schur functions indexed by) hooks in the R_{2n-1} and the multiplicity of the trivial representation in R_{2n}.

Proposition 2.12

(i) $\tilde{H}(\Pi_m^e)$ *is not a permutation module for* S_m *when* $m \ge 3$ *is odd.*

(ii) *Let $E_n(n)$ denote the multiplicity of the trivial representation in $\tilde{H}(\Pi_{2n}^e)$ (cf. Corollary 2.8), and let $c_i(j)$ be the multiplicity in $\tilde{H}(\Pi_{2i+1}^e)$ of the irreducible indexed by the hook $(2i+1-j, 1^j)$, for i such that $n \leq 2i+1 \leq 2n$. Then*

$$E_n(n) = \sum_{i \geq \frac{n-1}{2}}^{n-1} (-1)^{i+1+n}(c_i(2n-2i-1) + c_i(2n-2i-2)).$$

Proof:

(i) From Algorithm 2.11 we see by induction that the polynomial $F_{2n}(x,y) - q_{2n}(x,y)$ contains the term $(-1)^{n-1}(t^3xy - ty)$. This contributes $(-1)^n(\sum_{i=1}^{n-1} h_{2i}h_{2n-1-2i} - h_{2n-1})$ to the expansion of R_{2n-1} in the homogeneous symmetric functions. It follows that if n is odd, the character value of R_{2n-1} is (-1) on a $(2n-3)$-cycle, while if n is even, the character value is (-1) on a $(2n-1)$-cycle. Consequently the homology of Π_{2n-1}^e is never a permutation module for $n \geq 2$.

(ii) Use [22, Lemma 4.1] and the fact that the plethystic inverse of $\sum_{i\geq 1} h_i$ is $\sum_{i\geq 1}(-1)^{i-1}\pi_i$ where π_n is the characteristic of the representation of S_n on the top homology of the full partition lattice Π_n. (See for example [22, Example 1.6 (i)].) □

Note that the truth of our conjectures would imply that the homology of the subposet $\Pi_{2n}(\bar{k})$ is a permutation module for S_{2n-1}, for all $k = 1, \ldots, n-1$. In particular the characteristic of the restriction of the homology of Π_{2n}^e to S_{2n-1} may be easily computed from the polynomial expansion of $R_{2n}(h_2, h_1)$ given by Theorem 2.5. We have

Corollary 2.13 *As an S_{2n-1}-module, the homology of Π_{2n}^e decomposes into a (possibly virtual) direct sum of induced modules as follows:*

$$\tilde{H}(\Pi_{2n}^e) = \sum_{i=1}^{n-1} d_i(n)\, 1\uparrow_{S_2^i \times S_1^{2n-1-2i}}^{S_{2n-1}}.$$

The integers $d_i(n)$ are determined by the equations

$$d_i(n) = 2(n-i)b_i(n) + (i+1)b_{i+1}(n),\ n \geq 2.$$

Conjecture 2.7 implies that the integers $d_i(n)$ are nonnegative for all $i = 1, \ldots, n-1$. (From Corollary 2.6, $d_1(n) = 2$ and $d_2(n) = 3d_2(n-1) + 8(n-2) \geq 1$ for all $n \geq 2$.)

We conclude this section with some remarks on other methods for showing that a homology module is a permutation module. As argued in the introduction, and as shown by the basis for $\Pi_{2n}^e(\bar{1})$ constructed in Remark 2.10.1, it is not true that the homology spheres themselves are permuted by S_{2n}. On the other hand, since $\Pi_{2n}^e(\bar{1})$ does contain a unique element fixed by S_{2n-1}, namely, the two-block partition $\tau_{2n-1} = /1, 2, \ldots, (2n-1)/2n/$ in which $2n$ is a singleton, there is some hope that the homology spheres of the subposets

$\Pi^e_{2n}(\bar{k})$, $1 \le k \le n-1$, (which in fact contain a maximal chain fixed by S_{2n-1}) are permuted by the action of the subgroup S_{2n-1}.

While it seems unusual for homology modules to be permutation modules for the action of the full automorphism group, there are several examples of this phenomenon when the automorphism group is restricted to a subgroup. The examples we have in mind arise as subposets of the partition lattice Π_n, considered as modules for the subgroup S_{n-1}. It was shown in [22] that the action of S_{n-1} on the homology of the subposets $\Pi_n(\bar{k})$ obtained by taking the top k ranks of Π_n, excluding the top element of rank $n-1$, is a permutation module for S_{n-1} (see [22, Theorem 2.1 (2)] for an explicit determination of the orbits). Here $1 \le k \le n-1$. (For $k = n-2$ one obtains the regular representation of S_{n-1}, a result first obtained in [17].) The proof given in [22] for arbitrary k is by formal manipulation of the Frobenius characteristics. Wachs ([24]) has constructed bases of fundamental cycles for these subposets which are indeed permuted by S_{n-1}.

3. Enumerative implications

The results of the preceding section have consequences which may be described in a purely enumerative setting.

By combining Theorems 1.1 and 2.5, we see that our conjecture would have the following intriguing consequence: denote as before by E_{2n-1} the number of alternating (up-down) permutations on $(2n-1)$ letters.

Conjecture 3.1

(i) *The integers* $\{b_i(n),\ 2 \le i \le n\}$, $b_2(n) = 1, n \ge 2$, $(b_i(n) = 0$ *unless* $2 \le i \le n)$, *defined by the recurrence*

$$b_i(n) = \sum_{k\ge 0} \binom{2n-2i+k}{k} \sum_{r\ge 1} (-1)^{r-1} \binom{i-k}{i-2r} b_{i-k}(n-r),$$

are nonnegative.

(ii) *The integers* $\{E_i(n),\ 2 \le i \le n\}$ *defined by the transformation* $E_k(n) = \sum_{i=2}^{n} b_i(n) \binom{n-i}{k-i}$, $2 \le k \le n$, *are also nonnegative.*

Since furthermore we have

$$E_{2n-1} = \sum_{i=2}^{n} b_i(n)\, 2^{2n-1-i} = 2^{n-1} \sum_{i=2}^{n} E_i(n),$$

this would yield apparently new refinements of the numbers E_{2n-1}.

It is difficult to see how to make an inductive argument in favour of the positivity of the $b_i(n)$, from the above recurrence.

Notice that the decomposition of Conjecture 2.7 is strongly reminiscent of the orbit decomposition (Theorem 1.5) for the action of S_{2n} on the chains of Π_{2n}; the characteristic of this action is

$$\sum_{i=1}^{n} a_i(2n) h_2^i h_1^{2n-2i},$$

where the $a_i(2n)$ count the number of simsun permutations in S_{2n-2} with exactly $i-1$ descents (see Section 1).

From Proposition 1.6, we obtain an equivalent implication of Conjecture 2.7, which is perhaps more natural from the enumerative point of view:

Conjecture 3.2 *The number of simsun permutations in S_{2n-2} with exactly $(n-1)$ descents (and hence the number of André permutations in S_{2n-1} with n peaks) admits the further refinements (into nonnegative integers)* :

$$a_n(2n) = \sum_{i=2}^{n} b_i(n) 2^{n-i} = \sum_{i=2}^{n} E_i(n).$$

In fact the dimension equality of Proposition 1.6 strongly suggests that the homology representation of Π_{2n}^e is isomorphic to an induced module V, where V is a permutation module for S_2^n of dimension $a_n(2n)$, whose orbit decomposition is specified by the numbers $b_i(n)$ (cf. Remark 1.6.1).[1]

By examining the dimensions of each induced module in the decomposition of Corollary 2.13 for the restriction of R_{2n} to S_{2n-1}, we are led to conjecture the existence of the following apparently new refinement for the Genocchi number G_{2n}. One way to define the Genocchi numbers is by means of the exponential generating function [19, Chapter 5, Exercise 5 (d)-(f)]

$$\sum_{n\geq 1} (-1)^n \frac{G_{2n}}{(2n)!} x^{2n} = -x \tanh\left(\frac{x}{2}\right).$$

The Betti number of Π_{2n}^e is therefore related to the Genocchi number by the equation $\beta_{2n} = (2n-1)!\, G_{2n}$.

Conjecture 3.3 *The integers $d_i(n)$ of Corollary 2.13 are positive, and are divisible by 2^i, for all $i = 1, \ldots, n-1$. Consequently, we have the following refinement of the nth Genocchi number into a sum of $(n-1)$ positive integers for $n \geq 2$:*

$$G_{2n} = \sum_{i=1}^{n-1} \frac{d_i(n)}{2^i}.$$

For $n \geq 2$, define polynomials $R'_{2n}(x) = \sum_{i=1}^{n-1} \frac{d_i(n)}{2^i} x^i$. Then $R'_{2n}(1) = G_{2n}$. The first few coefficients of the polynomials R'_{2n} are as follows:

$$\begin{aligned}
R'_4 &= x \\
R'_6 &= 2x^2 + x \\
R'_8 &= 6x^3 + 10x^2 + x \\
R'_{10} &= 26x^4 + 92x^3 + 36x^2 + x \\
R'_{12} &= 158x^5 + 958x^4 + 840x^3 + 116x^2 + x.
\end{aligned}$$

Numerical verification reveals that this refinement does not coincide with any of the refinements for the Genocchi numbers discussed in the seemingly exhaustive survey article of Viennot [23].

All of these phenomena are in need of a combinatorial explanation.

We have been able to find a refinement of the Genocchi numbers into sums of powers of 2, by defining an equivalence relation on a certain combinatorial model for objects counted by the Genocchi numbers. The combinatorial interpretation that we use is due to Viennot. A more recent reference is [19, Chapter 5, Exercise 5(f)(ii)].

Proposition 3.4 [23, Proposition 5.5]. *The Genocchi number G_{2n} is the number of permutations σ in S_{2n-1} with the property that $\sigma(i) < \sigma(i+1)$ iff $\sigma(i)$ is even, for $1 \leq i \leq 2n-2$.*

Now consider the representation, due to Foata and Schützenberger, of a permutation as an increasing binary tree (see [8], [23] and [18]). Then $\sigma(i) < \sigma(i+1)$ iff $\sigma(i+1)$ is a right child of $\sigma(i)$. Thus G_{2n} is also the number of increasing binary trees on $(2n-1)$ nodes, with the property that a node has a right child if and only if it is even. We shall refer to such a tree as a Genocchi tree.

We define the following operation on a Genocchi tree: choose a node of degree 2, and exhange the left and right subtrees of this node. Note that a node of degree 2 necessarily has one odd child and one even child, and is itself even. Call two Genocchi trees equivalent if one can be obtained from another by a series of such reflections. Clearly this defines an equivalence relation on the set of G_{2n} Genocchi trees, whose classes are of size 2^i, where i is the number of nodes of degree 2 in any one representative of the class.

It is easy to see that there is a unique class of size 1, corresponding to the tree in which all nodes except $2n-1$ have degree 1. Also the maximum number of nodes of degree 2 is $(n-2)$, and is attained when all the even nodes, with the exception of $(2n-2)$, have degree 2. In this case, working backwards from the node $2n-4$, it is clear that the children of $2n-2i-4$ must be $2n-2i-3$ and $2n-2i-2$, for $i = 0, \ldots, n-3$, and hence the representative tree for a class of size 2^{n-2} is uniquely determined.

Proposition 3.5 *Let $g_i(n)$ denote the number of equivalence classes of size 2^i under the above equivalence relation. The enumerative refinement of the Genocchi number G_{2n} arising from this relation is thus*

$$G_{2n} = \sum_{i=0}^{n-2} g_i(n) 2^i,$$

where $g_0(n) = g_{n-2}(n) = 1$ for all $n \geq 2$.

For example, $G_4 = 1$, $G_6 = 1 + (1)2$, $G_8 = 1 + (6)2 + (1)2^2$ and $G_{10} = 1 + (21)2 + (26)2^2 + (1)2^3$.

4. Partitions with number of blocks divisible by d

Denote by $P_{m,d}$ the rank-selected subposet of Π_m consisting of all partitions such that the number of blocks is divisible by d. (In the preceding sections we wrote Π_{2n}^e for the poset $P_{2n,2}$.) It is natural to wonder whether these posets yield interesting homology representations for $d > 2$. Again the representations may be computed from the plethystic generating functions of [22, Theorem 2.8].

It is easy to see that these representations are not, in general, permutation modules, for instance by examining the simplest case of partitions of $2d$ consisting of d blocks. The only nontrivial partition left invariant under the action of the $(2d-1)$-cycle which fixes $2d$, is the two-block partition with $2d$ forming a block by itself. Hence for $d > 2$, the trace of a $(2d-1)$-cycle on the 0-dimensional chains is 0, and consequently on the reduced homology it is (-1).

Let $\beta_{m,d}$ denote the absolute value of the Möbius number of $P_{m,d}$. (In particular $\beta_{2n} = \beta_{2n,2}$.) As in the proof of Proposition 1.2, we obtain the following generating function for the Betti numbers $\beta_{m,d}$:

Proposition 4.1

$$(1 - e^x) + \sum_{i=1}^{d-1} \sum_{n \geq 0} (-1)^{n+1} \beta_{nd+i,d} \frac{x^{nd+i}}{(nd+i)!} = \sum_{n \geq 1} (-1)^n \beta_{nd,d} \frac{(e^x - 1)^{nd}}{(nd)!}.$$

Denote by Π_n^d the d-divisible lattice first studied in [6], that is, the sublattice of Π_{nd} consisting of partitions whose block sizes are divisible by d. Remark 1.4.1 (iv) points out the relation between the Betti numbers of $\Pi_{2n}^e = P_{2n,2}$ and Π_n^2. We do not know if there is any relation between the Betti number $\beta_{nd,d}$ of the poset $P_{nd,d}$ and the Betti number of the poset Π_n^d.

There is some enumerative motivation for studying these posets. Consider the case of $P_{nd,d}$. Define an $(n+1)$ by $(n+1)$ matrix M_n by letting the element in row i and column j, $0 \leq i, j \leq n$, be the Stirling number of the second kind $S(di, dj)$ if $i, j \geq 1$; define the $(i, 0)$ entry to be 1, for all $i \geq 0$, and the $(0, j)$ entry to be zero for all $j \geq 1$. The matrix M_n is lower triangular with 1's on the diagonal, hence its inverse is a lower triangular integer matrix with 1's on the diagonal. Let $b_{i,j}$ denote the (i, j) entry of the inverse. From the recursive definition of the Möbius function it follows that $b_{i,1} = (-1)^{i-1}\beta_{(i-1)d,d}$. (Remark 1.4.1 (i) is a special case of this observation.) More generally, we have that $b_{i,j}$ is the Whitney number $w_{i,i-j}$ of the first kind of the poset $P_{nd,d}$. See [1, Proposition 4.21]. (The Whitney numbers of the second kind are the Stirling numbers $S(di, dj)$.) Here $w_{i,k} = \sum_{x \in P_{id,d}, \mathrm{rank}(x)=i-k} \mu(\hat{0}, x) = \sum_{x \in P_{id,d}, \mathrm{rank}(x)=i-k} (-1)^{i-k} \beta(\tilde{H}(\hat{0}, x)_{P_{id,d}})$, where β denotes the Betti number.

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Notes

1. A refinement of $a_n(2n)$ into sums of powers of 2, different from the one predicted by Conjecture 3.2, has recently been discovered by Wachs ([25]), who defines an action of S_2^{n-1} on a subset of the symmetric group S_{2n-1}, which is equinumerous with the set of simsun permutations counted by $a_n(2n)$.

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