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Abstract. In this paper we use the approach introduced in [5] in order to analyze the homotopy groups of $L_{K(2)}V(0)$, the mod-3 Moore spectrum $V(0)$ localized with respect to Morava K-theory $K(2)$. These homotopy groups have already been calculated by Shimomura [12]. The results are very complicated so that an independent verification via an alternative approach is of interest. In fact, we end up with a result which is more precise and also differs in some of its details from that of [12]. An additional bonus of our approach is that it breaks up the result into smaller and more digestible chunks which are related to the $K(2)$-localization of the spectrum TMF of topological modular forms and related spectra. Even more, the Adams-Novikov differentials for $L_{K(2)}V(0)$ can be read off from those for TMF.

1. Introduction

Let $K(2)$ be the second Morava K-theory for the prime 3. For suitable spectra F, e.g. if F is a finite spectrum, the homotopy groups of the Bousfield localization $L_{K(2)}F$ can be calculated via the Adams-Novikov spectral sequence. By [3] this spectral sequence can be identified with the descent spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_2, (E_2)_tF) \Longrightarrow \pi_{t-s}(L_{K(2)}F)$$

for the action of the (extended) Morava stabilizer group $\mathbb{G}_2$ on $E_2 \wedge F$ where the action is via the Goerss-Hopkins-Miller action on the Lubin-Tate spectrum E_2 (see [5] for a summary of the necessary background material). Here we just recall that the homotopy groups of E_2 are non-canonically isomorphic to $\mathbb{W}_{\mathbb{F}_9}[[u_1]][u^{\pm 1}]$ where $\mathbb{W}_{\mathbb{F}_9}$ denotes the ring of Witt vectors of $\mathbb{F}_9$, where u_1 is of degree 0 and u is of degree -2. We also recall that $\mathbb{G}_2$ is a profinite group and its action on the profinite module $(E_2)_*F$ is continuous; group cohomology is, throughout this paper, taken in the continuous sense.

The cohomological dimension of $\mathbb{G}_2$ is well-known to be infinite and therefore a finite projective resolution of the trivial profinite $\mathbb{G}_2$-module $\mathbb{Z}_3$ cannot exist. However, in [5] a finite resolution of the trivial module $\mathbb{Z}_3$ was constructed in terms of permutation modules. More precisely, the group $\mathbb{G}_2$ is isomorphic to the product $\mathbb{G}_2^1 \times \mathbb{Z}_3$ of a central subgroup (isomorphic to) $\mathbb{Z}_3$ and a group $\mathbb{G}_2^1$ which is the kernel of a homomorphism $\mathbb{G}_2^1 \to \mathbb{Z}_3$, also called the reduced norm. One of the main technical achievements of [5] was the construction of a permutation resolution of the trivial module $\mathbb{Z}_3$ for the group $\mathbb{G}_2^1$. This resolution is self-dual in a suitable sense (cf. section 3.4) and has the form

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to Z_3 \to 0 \tag{1}$$

with $C_0 = C_3 = \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$ and $C_1 = C_2 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \chi$. Here G_{24} is a certain subgroup of $\mathbb{G}_2^1$ of order 24, isomorphic to the semidirect product $\mathbb{Z}/3 \rtimes Q_8$ of the cyclic group of order 3 with a non-trivial action of the quaternion group Q_8, and SD_{16} is another subgroup, isomorphic to the semidihedral group of order 16 (see section 2.2). Furthermore, χ is a suitable one-dimensional representation of SD_{16}, defined over $\mathbb{Z}_3$, and if S is a profinite $\mathbb{G}_2^1$-set we denote the corresponding profinite permutation module by $\mathbb{Z}_3[[S]]$.

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For any $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module M the resolution (1) gives rise to a first quadrant cohomological spectral sequence

$$E_1^{s,t} = \mathrm{Ext}^t_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_s, M) \Longrightarrow H^{s+t}(\mathbb{G}_2^1, M) \tag{2}$$

refered to in the sequel as the algebraic spectral sequence. By Shapiro's Lemma we have

$$E_1^{0,t} = E_1^{3,t} \cong H^t(G_{24}, M), \quad E_1^{1,t} = E_1^{2,t} \cong \begin{cases} \mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\chi, M) & t = 0 \\ 0 & t > 0 \ . \end{cases} \tag{3}$$

The bulk of our work is the calculation of this spectral sequence if $M = (E_2)_*(V(0))$. In this case the E_1-term is well understood and can be interpreted in terms of modular forms in characteristic 3. In fact, it is determined by the following result which we include for the convenience of the reader and in which v_1 denotes the well-known $\mathbb{G}_2$-invariant class $u_1u^{-2} \in M_4$. For the definition of the other classes figuring in this result the reader is referred to section 5.1.

Theorem 1.1. *Let $M = (E_2)_*(V(0))$.*

a) There are elements $\beta \in H^2(G_{24}, M_{12})$, $\alpha \in H^1(G_{24}, M_4)$ and $\widetilde{\alpha} \in H^1(G_{24}, M_{12})$, an invertible G_{24}-invariant element $\Delta \in M_{24}$, and an isomorphism of graded algebras

$$H^*(G_{24}, M) \cong \mathbb{F}_3[[v_1^6\Delta^{-1}]][\Delta^{\pm 1}, v_1, \beta, \alpha, \widetilde{\alpha}]/(\alpha^2, \widetilde{\alpha}^2, v_1\alpha, v_1\widetilde{\alpha}, \alpha\widetilde{\alpha} + v_1\beta) \ .$$

b) The ring of SD_{16}-invariants of M is given by the subalgebra $M^{SD_{16}} = \mathbb{F}_3[[u_1^4]][v_1, u^{\pm 8}]$ and $\mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\chi, M)$ is a free $M^{SD_{16}}$-module of rank 1 with generator ω^2u^4, i.e.

$$\mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\chi, M) \cong \omega^2u^4\mathbb{F}_3[[u_1^4]][v_1, u^{\pm 8}] \ . \quad \square$$

Remark We note that $v_1^6\Delta^{-1}$ is a G_{24}-invariant class in the maximal ideal of M_0 and hence a formal power series in $v_1^6\Delta^{-1}$ converges in M and is also invariant. Similarly with u_1^4. Of course, the name for Δ is chosen to emphasize the close relation with the theory of modular forms. For example we note that $M^{G_{24}}$ is isomorphic to the completion of $\mathcal{M}_3 := \mathbb{F}_3[\Delta^{\pm 1}, v_1]$ with respect to the ideal generated by $v_1^6\Delta^{-1}$, and $\mathcal{M}_3$ is isomorphic to the ring of modular forms in characteristic 3 (cf. [2] and [1]). Similarly, $M^{SD_{16}}$ is isomorphic to the completion of $\mathbb{F}_3[v_1, u^{\pm 8}]$ with respect to the ideal generated by $u_1^4 = v_1^4u^8$. The larger algebra $\mathbb{F}_3[v_1, u^{\pm 4}]$ is isomorphic to the ring $\mathcal{M}_3(2)$ of modular forms of level 2 (in characteristic 3) (cf. [1]). The relation with modular forms could be made tight if in [5] we had worked with a version of E_2 which uses a deformation of the formal group of a supersingular curve rather than that of the Honda formal group.

As $(E_2)_*(V(0))$ is a graded module, the spectral sequence is trigraded. The differentials in this spectral sequence are v_1-linear and continuous. Therefore d_1 is completely described by continuity and the following formulae in which we identify the E_1-term via Theorem 1.1.

Theorem 1.2. *There are elements*

$$\Delta_k \in E_1^{0,0,24k}, \quad b_{2k+1} \in E_1^{1,0,16k+8}, \quad \overline{b}_{2k+1} \in E_1^{2,0,16k+8}, \quad \overline{\Delta}_k \in E_1^{3,0,24k}$$

for each $k \in \mathbb{Z}$ satisfying

$$\Delta_k \equiv \Delta^k, \quad b_{2k+1} \equiv \omega^2u^{-4(2k+1)}, \quad \overline{b}_{2k+1} \equiv \omega^2u^{-4(2k+1)}, \quad \overline{\Delta}_k \equiv \Delta^k$$

(where the congruences are modulo the ideal $(v_1^6\Delta^{-1})$ resp. $(v_1^4u^8)$ and in the case of Δ_0 we even have equality $\Delta_0 = \Delta^0 = 1$) such that

$$d_1(\Delta_k) \quad = \quad \begin{cases} (-1)^{m+1}b_{2.(3m+1)+1} & k = 2m+1 \\ (-1)^{m+1}mv_1^{4.3^n-2}b_{2.3^n(3m-1)+1} & k = 2m.3^n, m \not\equiv 0 \ (3) \\ 0 & k = 0 \end{cases}$$

$$d_1(b_{2k+1}) \quad = \quad \begin{cases} (-1)^n v_1^{6.3^n+2}\overline{b}_{3^{n+1}(6m+1)} & k = 3^{n+1}(3m+1) \\ (-1)^n v_1^{10.3^n+2}\overline{b}_{3^n(18m+11)} & k = 3^n(9m+8) \\ 0 & \textit{else} \end{cases}$$

$$d_1(\overline{b}_{2k+1}) \quad = \quad \begin{cases} (-1)^{m+1}v_1^2\overline{\Delta}_{2m} & 2k+1=6m+1 \\ (-1)^{m+n}v_1^{4.3^n}\overline{\Delta}_{3^n(6m+5)} & 2k+1=3^n(18m+17) \\ (-1)^{m+n+1}v_1^{4.3^n}\overline{\Delta}_{3^n(6m+1)} & 2k+1=3^n(18m+5) \\ 0 & \textit{else} \ . \end{cases}$$

It turns out that the d_2-differential of this spectral sequence is determined by the following principles: it is non-trivial if and only if v_1-linearity and sparseness of the resulting E_2-term permit it, and in this case it is determined up to sign by these two properties. The remaining d_3-differential turns out to be trivial. More precisely we have the following result.

Proposition 1.3.

a) The differential $d_2 : E_2^{0,1,*} \to E_2^{2,0,*}$ is determined by

$$d_2(\Delta_k\alpha) = \begin{cases} (-1)^{m+n+1}v_1^{6.3^n+1}\overline{b}_{3^{n+1}(6m+1)} & k=2.3^n(3m+1) \\ (-1)^{m+n}v_1^{10.3^{n+1}+1}\overline{b}_{3^{n+1}(18m+11)} & k=2.3^n(9m+8) \\ 0 & \textit{else} \end{cases}$$

$$d_2(\Delta_k\widetilde{\alpha}) = \begin{cases} (-1)^m v_1^{11}\overline{b}_{18m+11} & k=6m+5 \\ 0 & \textit{else} \ . \end{cases}$$

b) The d_3-differential is trivial.

Remark 1 on notation Of course, the elements $\Delta_k\alpha$ and $\Delta_k\widetilde{\alpha}$ are only names for elements in the E_2-term which are represented in the E_1-term as products, but which are no longer products in the E_2-term. Similar abuse of notation will be used in Theorem 1.4, Proposition 1.5, Theorem 1.6 and in section 6 and 8.

Next we use that the element β of Theorem 1.1 lifts to an element with the same name in $H^2(\mathbb{G}_2^1, M_{12})$ resp. in $H^2(\mathbb{G}_2, M_{12})$. In fact this latter element detects the image of $\beta_1 \in \pi_{10}(S^0)$ in $\pi_{10}(L_{K(2)}V(0))$. The previous results yield the following E_∞-term as a module over $\mathbb{F}_3[\beta, v_1]$.

Theorem 1.4. *As an $\mathbb{F}_3[\beta, v_1]$-module the E_∞-term of the algebraic spectral sequence (2) for $M = (E_2)_*/(3)$ is isomorphic to a direct sum of cyclic modules generated by the following elements and with the following annihilator ideals:*

a) For $E_\infty^{0,*,*}$ we have the following generators with respective annihilator ideals

$1=\Delta_0$		(βv_1^2)
$\Delta_m\beta$	$m \neq 0$	(v_1^2)
α		(v_1)
$\Delta_{2m+1}\alpha$		(v_1)
$\Delta_{2.3^n(3m-1)}\alpha$	$m \not\equiv 0 \mod (3)$	(v_1)
$\Delta_{2m}\widetilde{\alpha}$		(v_1)
$\Delta_{2m+1}\widetilde{\alpha}$	$m \not\equiv 2 \mod (3)$	(v_1)
$\Delta_{2.3^n(3m+1)}\alpha\beta$		(v_1)
$\Delta_{2.3^n(3m-1)}\alpha\beta$	$m \equiv 0 \mod (3)$	(v_1)
$\Delta_{2m+1}\widetilde{\alpha}\beta$	$m \equiv 2 \mod (3)$	(v_1) .

b) For $E_\infty^{1,*,*}$ we have the following generators with respective annihilator ideals

b_1		(β)
$b_{2.3^n(3m-1)+1}$	$m \not\equiv 0 \mod (3)$	$(v_1^{4.3^n-2}, \beta)$.

c) For $E_\infty^{2,*,*}$ we have the following generators with respective annihilator ideals

$\overline{b}_{3^{n+1}(6m+1)}$		$(v_1^{6.3^n+1}, \beta)$
$\overline{b}_{3^n(6m+5)}$	$m \equiv 1 \mod (3)$	$(v_1^{10.3^n+1}, \beta)$.

d) For $E_\infty^{3,*,*}$ we have the following generators with respective annihilator ideals

$$\begin{array}{ll} \overline{\Delta}_{2m} & (v_1^2) \\ \overline{\Delta}_{3^n(6m\pm1)} & (v_1^{4.3^n}, \beta v_1^2) \\ \overline{\Delta}_m\alpha & (v_1) \\ \overline{\Delta}_m\widetilde{\alpha} & (v_1)\ . \quad \square \end{array}$$

To get at $H^*(\mathbb{G}_2^1,(E_2)_*/(3))$ we still need to know the extensions between the filtration quotients. They are given by the following result.

Proposition 1.5. *The $\mathbb{F}_3[\beta,v_1]$-module generators of the E_∞-term of Theorem 1.4 can be lifted to elements (with the same name) in $H^*(\mathbb{G}_2^1;(E_2)_*/(3))$ such that the relations defining the annihilator ideals of Theorem 1.4 continue to hold with the following exceptions*

$$\begin{array}{lcl} v_1\alpha & = & b_1 \\ v_1\Delta_{2.3^n(9m+2)}\alpha & = & (-1)^{m+1}b_{2.3^{n+1}(9m+2)+1} \\ v_1\Delta_{2.3^n(9m+5)}\alpha & = & (-1)^{m+1}b_{2.3^{n+1}(9m+5)+1} \\ & & \\ v_1\Delta_{6m+1}\widetilde{\alpha} & = & (-1)^{m}b_{2(9m+2)+1} \\ v_1\Delta_{6m+3}\widetilde{\alpha} & = & (-1)^{m+1}b_{2(9m+5)+1} \\ & & \\ \beta\overline{b}_{3^{n+1}(6m+1)} & = & \pm\overline{\Delta}_{3^n(6m+1)}\widetilde{\alpha} \\ \beta\overline{b}_{3^{n+1}(18m+11)} & = & \pm\overline{\Delta}_{3^n(18m+11)}\widetilde{\alpha} \\ \beta\overline{b}_{18m+11} & = & \pm\overline{\Delta}_{6m+4}\alpha\ . \end{array}$$

Apart from the last group of β-extensions (which are simple consequences of the calculation of $H^*(\mathbb{G}_2,(E_2)_*/(3,u_1))$, cf. [4]) one can summarize the result by saying that nontrivial v_1-extensions exist only between $E_\infty^{0,1,*}$ and $E_\infty^{1,0,*+4}$ and there is such an extension whenever sparseness permits it, and then the corresponding relation is unique up to sign. Unfortunately this is not clear a priori, but needs proof and the proof gives the exact value of the sign. In contrast determining the sign for the β-relations would require an extra effort.

The main results can now be stated as follows.

Theorem 1.6. *As an $\mathbb{F}_3[\beta,v_1]$-module $H^*(\mathbb{G}_2^1,(E_2)_*/(3))$ is isomorphic to the direct sum of the cyclic modules generated by the following elements and with the following annihilator ideals*

$$\begin{array}{lll} 1=\Delta_0 & & (\beta v_1^2) \\ \Delta_m\beta & m\neq 0 & (v_1^2) \\ \alpha & & (\beta v_1) \\ \Delta_{2m+1}\alpha & & (v_1) \\ \Delta_{2.3^n(3m-1)}\alpha & m\not\equiv 0 \mod (3) & (v_1^{4.3^{n+1}-1},\beta v_1) \\ \Delta_{2m}\widetilde{\alpha} & & (v_1) \\ \Delta_{2m+1}\widetilde{\alpha} & m\not\equiv 2 \mod (3) & (v_1^3,\beta v_1) \\ \Delta_{2.3^n(3m+1)}\alpha\beta & & (v_1) \\ \Delta_{2.3^n(3m-1)}\alpha\beta & m\equiv 0 \mod (3) & (v_1) \\ \Delta_{2m+1}\widetilde{\alpha}\beta & m\equiv 2 \mod (3) & (v_1) \\ & & \\ \overline{b}_{3^{n+1}(6m+1)} & & (v_1^{6.3^n+1},\beta v_1) \\ \overline{b}_{3^n(6m+5)} & m\equiv 1 \mod (3) & (v_1^{10.3^n+1},\beta v_1) \\ & & \\ \overline{\Delta}_{2m} & & (v_1^2) \\ \overline{\Delta}_{3^n(6m\pm1)} & & (v_1^{4.3^n},\beta v_1^2) \\ & & \\ \overline{\Delta}_{2m+1}\alpha & & (v_1) \\ \overline{\Delta}_{2m}\alpha & m\not\equiv 2 \mod (3) & (v_1) \\ \overline{\Delta}_{2m}\widetilde{\alpha} & & (v_1) \\ \overline{\Delta}_{3^n(6m+5)}\widetilde{\alpha} & m\not\equiv 1 \mod (3) & (v_1)\ . \quad \square \end{array}$$

We emphasize that even though this result is involved the mechanism which produces is quite transparent. The passage to the cohomology of $\mathbb{G}_2$ results now from the decomposition $\mathbb{G}_2 \cong \mathbb{Z}_3 \times \mathbb{G}_2^1$ and the fact that the central factor $\mathbb{Z}_3$ acts trivially on $(E_2)_*/(3)$.

Theorem 1.7. *There is a class $\zeta \in H^1(\mathbb{G}_2, (E_2)_0/(3))$ and an isomorphism of graded algebras*

$$H^*(\mathbb{G}_2^1; (E_2)_*/(3)) \otimes_{\mathbb{Z}_3} \Lambda_{\mathbb{Z}_3}(\zeta) \cong H^*(\mathbb{G}_2, (E_2)_*/(3)) \ . \quad \square$$

Remark We warn the reader that there is something subtle about this Künneth type isomorphism. In fact, the class α of Theorem 1.1 is defined via the Greek letter formalism in $H^1(G_{24}, -)$ as the Bockstein of the class v_1 with respect to the obvious short exact sequence

$$0 \to (E_2)_*/(3) \xrightarrow{3} (E_2)_*/(9) \longrightarrow (E_2)_*/(3) \to 0 \ .$$

The same formalism allows to define classes $\alpha(F) \in H^1(F, (E_2)_4/(3))$ for any closed subgroup F of $\mathbb{G}_2$ and these classes are well compatible with respect to restrictions among different subgroups. However, with respect to the isomorphism of Theorem 1.7 the class $\alpha(\mathbb{G}_2)$ corresponds to $\alpha(\mathbb{G}_2^1) - v_1\zeta$ (cf. Corollary 7.2). We will insist on the notation $\alpha(\mathbb{G}_2)$ and $\alpha(\mathbb{G}_2^1)$ in order to avoid possible confusion when we deal with $H^*(\mathbb{G}_2, -)$. Fortunately similar notation is unnecessary for the classes $\widetilde{\alpha}$ and β (cf. Corollary 7.2). The difference between $\alpha(\mathbb{G}_2)$ and $\alpha(\mathbb{G}_2^1)$ turns out to be important for studying the differentials in the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(0))$.

In fact, these differentials can be derived from those of the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(1))$ which have been determined in [4].

Remark 2 on notation In the following theorem we give the E_∞-term of the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(0))$ as a subquotient of its E_2-term which itself has been described in Theorem 1.6 and Theorem 1.7 as a module over $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$ with generators represented in the E_1-term of the algebraic spectral sequence (2) for $M = (E_2)_*V(0)$. As before, generators of E_∞ which are represented by products in this E_1-term are not necessarily products in E_∞. In order to distinguish between module multiplication and the name of a generator we write β and v_1 as right hand factors in such a product if they are only part of the name of a generator, e.g. in the case of $\Delta_{6m+1}\beta v_1$. We have also renamed (for reasons which will be explained below) generators involving $\overline{\Delta}_k$ by $\Sigma^{48}\overline{\Delta}_{k-2}$.

Theorem 1.8. *As a module over $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$ the E_∞-term of the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(0))$ is the quotient of the direct sum of cyclic $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$-modules with the following generators and annihilator ideals*

$1 = \Delta_0$		$(\beta v_1^2, \beta^3 v_1, \beta^6)$
$\Delta_{3m}\beta,$	$m \neq 0$	$(v_1^2, \beta^2 v_1, \beta^5)$
$\Delta_{6m+1}\beta v_1$		(v_1, β^2)
$\Delta_{6m+4}\beta v_1$		(v_1, β^3)
$\alpha(\mathbb{G}_2^1)$		$(\beta v_1, \beta^3)$
$\Delta_{2m+1}\alpha(\mathbb{G}_2^1)$	$m \not\equiv 2 \mod (3)$	(v_1, β^3)
$\Delta_{2.3^n(3m-1)}\alpha(\mathbb{G}_2^1)$	$m \not\equiv 0 \mod (3),\ n \geq 1$	$(v_1^{4.3^{n+1}-1}, \beta v_1, \beta^3)$
$\Delta_{2(3m-1)}\alpha(\mathbb{G}_2^1)$	$m \not\equiv 0 \mod (3)$	$(v_1^{11}, \beta v_1, \beta^4)$
$\Delta_{6m}\widetilde{\alpha}$		(v_1, β^5)
$b_{2(9m+2)+1}$		(v_1^2, β)
$\Delta_{6m+3}\widetilde{\alpha}$		$(v_1^3, \beta v_1, \beta^5)$
$\Delta_{2.3^n(3m+1)}\alpha(\mathbb{G}_2^1)\beta$	$n \geq 1$	(v_1, β^2)
$\Delta_{2.3^n(3m-1)}\alpha(\mathbb{G}_2^1)\beta$	$m \equiv 0 \mod (3),\ n \geq 1$	(v_1, β^2)
$\Delta_{2(3m-1)}\alpha(\mathbb{G}_2^1)\beta$	$m \equiv 0 \mod (3)$	(v_1, β^3)

$\overline{b}_{3^{n+1}(6m+1)}v_1$	$n \geq 0$	$(v_1^{6.3^n}, \beta)$
$\overline{b}_{3^n(6m+5)}v_1$	$m \equiv 1 \mod (3),\ n \neq 1$	$(v_1^{10.3^n}, \beta)$
$\overline{b}_{3(6m+5)}$	$m \equiv 1 \mod (3)$	$(v_1^{31}, \beta v_1, \beta^5)$
$\Sigma^{48}\overline{\Delta}_{3^n(6m+1)-3}$	$n \geq 1$	$(v_1^2, \beta^2 v_1, \beta^5)$
$\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}$	$m \not\equiv 1 \mod (3),\ n \geq 1$	$(v_1^2, \beta^3 v_1, \beta^5)$
$\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}$	$m \equiv 1 \mod (3),\ n \geq 1$	$(v_1^2, \beta^2 v_1, \beta^5)$
$\Sigma^{48}\overline{\Delta}_{(6m+1)-3}v_1$		(v_1, β^2)
$\Sigma^{48}\overline{\Delta}_{3^n(6m\pm1)-2}v_1$	$n \geq 1$	$(v_1^{4.3^n-1}, \beta v_1, \beta^3)$
$\Sigma^{48}\overline{\Delta}_{(6m+1)-2}v_1^2$		(v_1^2, β)
$\Sigma^{48}\overline{\Delta}_{(6m+5)-2}$		$(v_1^4, \beta v_1^2, \beta^3 v_1, \beta^5)$
$\Sigma^{48}\overline{\Delta}_{2m-1}\alpha(\mathbb{G}_2^1)$	$m \not\equiv 0 \mod (3)$	(v_1, β^3)
$\Sigma^{48}\overline{\Delta}_{3^n(6m+1)-3}\alpha(\mathbb{G}_2^1)$	$n \geq 0$	(v_1, β^2)
$\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}\alpha(\mathbb{G}_2^1)$	$m \not\equiv 1 \mod (3),\ n \geq 1$	(v_1, β^3)
$\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}\alpha(\mathbb{G}_2^1)$	$m \equiv 1 \mod (3),\ n \geq 1$	(v_1, β^2)
$\Sigma^{48}\overline{\Delta}_{6m}\widetilde{\alpha}$		(v_1, β^4)
$\Sigma^{48}\overline{\Delta}_{6m+3}\widetilde{\alpha}$	$m \not\equiv 1 \mod (3)$	(v_1, β^4)

modulo the following relations (in which module generators are put into brackets in order to distinguish between module multiplications and generators.)

$$
\begin{array}{llll}
\beta^3[\Delta_k\alpha(\mathbb{G}_2^1)] = \beta^2\zeta[\Delta_k\beta v_1] & k = 2(3m-1) & m \not\equiv 0 \mod (3) \\
\beta^2[\Delta_k\alpha(\mathbb{G}_2^1)\beta] = \beta^2\zeta[\Delta_k\beta v_1] & k = 2(3m-1) & m \equiv 0 \mod (3) \\
\beta^4[\Delta_k\beta] = \beta^4\zeta[\Delta_k\widetilde{\alpha}] & k = 6m+3 & \\
\beta^2[\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2^1)] = \beta^2\zeta[\Sigma^{48}\overline{\Delta}_k v_1] & k = 6m+1 & \\
\beta^2[\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2^1)] = \beta^2 v_1\zeta[\Sigma^{48}\overline{\Delta}_k] & k = 6m+3 & \\
\beta^2[\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2^1)] = \beta^2 v_1\zeta[\Sigma^{48}\overline{\Delta}_k] & k = 3^n(6m+5)-3 & m \not\equiv 1 \mod (3),\ n \geq 1 \\
\beta^4[\Sigma^{48}\overline{\Delta}_k] = \beta^3\zeta[\Sigma^{48}\overline{\Delta}_k\widetilde{\alpha}] & k = 6m & .
\end{array}
$$

We remark that some but not all of the relations figuring in this result could have been avoided by choosing different generators, e.g. if we had chosen, for $k = 2(3m-1)$ and $m \equiv 0$ mod (3), $\Delta_k\alpha(\mathbb{G}_2)\beta$ as a generator instead of $\Delta_k\alpha(\mathbb{G}_2^1)\beta$.

Furthermore we remark that this description of the E_∞-term as an $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$-module does not lift to $\pi_*(L_{K(2)}V(0))$. In fact, it is not hard to see that there are exotic relations like $v_1\Delta\alpha = \beta^2\widetilde{\alpha}$ which hold in $\pi_*(E_2^{hG_{24}} \wedge V(0))$, in particular $v_1[\Delta\alpha(\mathbb{G}_2^1)] \neq 0$ in $\pi_*(L_{K(2)}V(0))$.

As stated above this result is obtained without much trouble from the calculation of the Adams-Novikov E_2-term given in Theorem 1.6 and Theorem 1.7 by using knowledge of the Adams-Novikov differentials for $L_{K(2)}V(1)$. However, even if the rigorous proof proceeds this way, we feel that the final result can be better appreciated from the following point of view. In [5] the algebraic resolution (1) for $\mathbb{G}_2^1$ resp. the companion resolution for $\mathbb{G}_2$ (obtained by tensoring with a minimal resolution for $\mathbb{Z}_3$) was "realized" by resolutions of the homotopy fixed point spectrum $E_2^{h\mathbb{G}_2^1}$ resp. of $L_{K(2)}S^0$ via homotopy fixed point spectra with respect to the corresponding finite subgroups of $\mathbb{G}_2$. In particular there is a resolution

$$(4) \qquad * \to L_{K(2)}S^0 \to X_0 \to X_1 \to X_2 \to X_3 \to X_4 \to *$$

with $X_0 = E_2^{hG_{24}}$, $X_1 = E_2^{hG_{24}} \vee \Sigma^8 E_2^{hSD_{16}}$, $X_2 = \Sigma^8 E_2^{hSD_{16}} \vee \Sigma^{40} E_2^{hSD_{16}}$, $X_3 = \Sigma^{48} E_2^{hG_{24}} \vee \Sigma^{40} E_2^{hSD_{16}}$ and $X_4 = \Sigma^{48} E_2^{hG_{24}}$. We note that the 48-fold suspension appearing in the definition of X_3 and X_4 is the reason for the (abusive) change of notation from Δ_k in Theorem 1.6 to $\Sigma^{48}\Delta_{k-2}$ in Theorem 1.8. Furthermore, the spectrum $E_2^{hG_{24}}$ can be identified with the $K(2)$-localization of the spectrum TMF of topological modular forms and $E_2^{hSD_{16}}$ with "half" of the $K(2)$-localization of the spectrum $TMF_0(2)$ of topological modular forms of level 2 (cf. [1]). These spectra are of considerable independent interest and their Adams-Novikov spectral sequences and their homotopy is well understood (cf. the appendix or [1], [5]). The Adams-Novikov differentials for $L_{K(2)}V(0)$ can be completely understood by those for $E_2^{hG_{24}} \wedge V(0)$ (cf. the remarks following Lemma 8.1 and Lemma 8.4 for more precise statements). The complicated final result described in Theorem 1.8 can thus be deduced, just as in the case of Theorem 1.6, from more basic structures by an essentially simple though elaborate mechanism.

We believe that our results have the following advantages over those by Shimomura [12]. In our approach the final result relates well to modular forms and the homotopy of the spectrum TMF of topological modular forms; in particular the approach helps to understand how the complicated structure of $\pi_*(L_{K(2)}V(0))$ is built from that of the comparatively simple homotopy of TMF. This is also reflected in our notation which is very different from that in [11] where classical chromatic notation is used. Furthermore we determine E_∞ as a module over $\mathbb{F}_3[\beta, v_1]$. In contrast Theorem 2.8. in [12] gives a direct sum decomposition as an $\mathbb{F}_3[v_1]$-module (of E_∞ and not as claimed in [12] of $\pi_*(L_{K(2)}V(1))$) and this decomposition only partially reflects the $\mathbb{F}_3[\beta, v_1]$-module structure. In fact, many non-trivial β-multiplications are not recorded in [12], for example those on the classes $\Delta_{2.3^n(9m+2)}\alpha(\mathbb{G}_2^1)$, $\Delta_{2.3^n(9m+8)}\alpha(\mathbb{G}_2^1)$, $\overline{b}_{3(18m+11)}$, $\cdots$, nor are the additional β-relations of Theorem 1.8. There are related discrepancies on the height of β-torsion; for example, in [12] all elements in the same bidegree as the elements $\Sigma^{48}\overline{\Delta}_{6m}$ appear to be already killed by β^4. Finally the classes $v_2^{9m+2}\xi/v_1^3$ of [12] which correspond to $v_1\Sigma^{48}\overline{\Delta}_{(6m+1)-2}$ in our notation and which support a non-trivial Adams-Novikov d_9-differential (cf. Lemma 8.4) seem to be permanent cycles in [12].

The paper is organized as follows. In section 2 we recall background material on the stabilizer groups, we introduce important elements of $\mathbb{G}_2$ and we recall the definition of its subgroups SD_{16} and G_{24} as well as that of an important torsionfree subgroup K of finite index in $\mathbb{G}_2^1$. In section 3 we study the maps in the permutation resolution (1). In fact, in [5] the maps $C_3 \to C_2$ and $C_2 \to C_1$ of the permutation resolution (1) were not described explicitly so that the resolution was not ready yet to be used for detailed calculations. The subgroup K plays a crucial role in finding an approximation to the map $C_2 \to C_1$. We also show that the map $C_3 \to C_2$ is in a suitable sense dual to the map $C_1 \to C_0$. In section 4 we study the action of the stabilizer group on $(E_2)_*/(3)$ and we derive formulae for the action of the elements of $\mathbb{G}_2$ introduced in section 2. In section 5 we comment on Theorem 1.1 and we verify Theorem 1.2 (cf. Proposition 5.7, Proposition 5.10 and Proposition 5.12). Most of the new results of these sections, in particular the formulae for the action of the stabilizer group, the approximation of the map $C_2 \to C_1$ and the evaluation of the induced map

$$E_1^{1,0,*} = \mathrm{Ext}^0_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_1, (E_2)_*/(3)) \to \mathrm{Ext}^0_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_2, (E_2)_*/(3)) = E_1^{2,0,*}$$

are taken from the second author's thesis [8]. The evaluation of this map is by far the hardest calculation in our approach. In section 6 we prove Proposition 1.3 and Proposition 1.5. In a short section 7 we discuss the subtleties of the Künneth isomorphism of Theorem 1.7 and section 8 contains the discussion of the differentials in the Adams-Novikov spectral sequence and proves Theorem 1.8. For the convenience of the reader we have collected the description of the related Adams-Novikov spectral sequences for $E_2^{hG_{24}} \wedge V(0)$, for $E_2^{hG_{24}} \wedge V(1)$ and for $L_{K(2)}V(1)$ in an appendix.

2. Background on the Morava Stabilizer Group

In the sequel we will recall some of the basic properties of the Morava stabilizer groups $\mathbb{S}_n$ resp. $\mathbb{G}_n$. The reader is refered to [10] for more details (see also [7] and [5] for a summary of what will be important in this paper).

2.1. **Generalities.** We recall that the Morava stabilizer group $\mathbb{S}_n$ is the group of automorphisms of the p-typical formal group law Γ_n over the field $\mathbb{F}_q$ (with $q = p^n$) whose $[p]$-series is given by $[p]_{\Gamma_n}(x) = x^{p^n}$. Because Γ_n is already defined over $\mathbb{F}_p$ the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ of the finite field extension $\mathbb{F}_p \subset \mathbb{F}_q$ acts naturally on $\mathrm{Aut}(\Gamma_n) = \mathbb{S}_n$ and $\mathbb{G}_n$ can be identified with the semidirect product $\mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$.

The group $\mathbb{S}_n$ is also equal to the group of units in the endomorphism ring of Γ_n, and this endomorphism ring can be identified with the maximal order $\mathcal{O}_n$ of the division algebra $\mathbb{D}_n$ over $\mathbb{Q}_p$ of dimension n^2 and Hasse invariant $\frac{1}{n}$. In more concrete terms, $\mathcal{O}_n$ can be described as follows: let $\mathbb{W} := \mathbb{W}_{\mathbb{F}_q}$ denote the Witt vectors over $\mathbb{F}_q$. Then $\mathcal{O}_n$ is the non-commutative ring extension of $\mathbb{W}$ generated by an element S which satisfies $S^n = p$ and $Sw = w^\sigma S$, where $w \in \mathbb{W}$ and w^σ is the image of w with respect to the lift of the Frobenius automorphism of $\mathbb{F}_q$. The element S generates a two sided maximal ideal $\mathfrak{m}$ in $\mathcal{O}_n$ with quotient $\mathcal{O}_n/\mathfrak{m}$ canonically isomorphic to $\mathbb{F}_q$. Inverting p in $\mathcal{O}_n$ yields the division algebra $\mathbb{D}_n$, and $\mathcal{O}_n$ is its maximal order.

Reduction modulo $\mathfrak{m}$ induces an epimorphism $\mathcal{O}_n^\times \longrightarrow \mathbb{F}_q^\times$. Its kernel will be denoted by S_n and is also called the strict Morava stabilizer group. The group S_n is equipped with a canonical filtration by subgroups F_iS_n, $i = \frac{k}{n}$, $k = 1, 2, \ldots$, defined by

$$F_iS_n := \{g \in S_n | g \equiv 1 \bmod (S^{in})\} \ .$$

The intersection of all these subgroups contains only the element 1 and S_n is complete with respect to this filtration, i.e. we have $S_n = \lim_i S_n/F_iS_n$. Furthermore, we have canonical isomorphisms

$$F_iS_n/F_{i+\frac{1}{n}}S_n \cong \mathbb{F}_q$$

induced by

$$x = 1 + aS^{in} \mapsto \bar{a} \ .$$

Here a is an element in $\mathcal{O}_n$, i.e. $x \in F_iS_n$ and $\bar{a}$ is the residue class of a in $\mathcal{O}_n/\mathfrak{m} \cong \mathbb{F}_q$.

The associated graded object grS_n with $gr_iS_n := F_iS_n/F_{i+\frac{1}{n}}S_n$, $i = \frac{1}{n}, \frac{2}{n}, \ldots$ becomes a graded Lie algebra with Lie bracket $[\bar{a}, \bar{b}]$ induced by the commutator $[x, y] := xyx^{-1}y^{-1}$ in S_n. Furthermore, if we define a function φ from the positive real numbers to itself by $\varphi(i) := \min\{i+1, pi\}$ then the p-th power map on S_n induces maps $P : gr_iS_n \longrightarrow gr_{\varphi(i)}S_n$ which define on grS_n the structure of a mixed Lie algebra in the sense of Lazard (cf. Chap. II.1. of [9]). If we identify the filtration quotients with $\mathbb{F}_q$ as above then the Lie bracket and the map P are explicitly given as follows (cf. Lemma 3.1.4 in [7]).

Lemma 2.1. *Let $\bar{a} \in gr_iS_n$, $\bar{b} \in gr_jS_n$. Then*

a)

$$[\bar{a}, \bar{b}] = \bar{a}\bar{b}^{p^{ni}} - \bar{b}\bar{a}^{p^{nj}} \in gr_{i+j}S_n$$

b)

$$P\bar{a} = \begin{cases} \bar{a}^{1+p^{ni}+\ldots+p^{(p-1)ni}} & i < (p-1)^{-1} \\ \bar{a} + \bar{a}^{1+p^{ni}+\ldots+p^{(p-1)ni}} & i = (p-1)^{-1} \\ \bar{a} & i > (p-1)^{-1}. \quad \square \end{cases}$$

The right action of $\mathbb{S}_n$ on $\mathcal{O}_n$ determines a group homomorphism $\mathbb{S}_n \to GL_n(\mathbb{W})$. The resulting determinant homomorphism $\mathbb{S}_n \to \mathbb{W}^\times$ extends to a homomorphism

$$\mathbb{G}_n \to \mathbb{W}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

which factors through $\mathbb{Z}_p^\times \times \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. By choosing a fixed isomorphism between the quotient of $\mathbb{Z}_p^\times$ by its maximal finite subgroup with $\mathbb{Z}_p$ we get the "reduced determinant" homomorphism

$$\mathbb{G}_n \to \mathbb{Z}_p \ .$$

We denote its kernel by $\mathbb{G}_n^1$ and the intersection of $\mathbb{G}_n^1$ with $\mathbb{S}_n$ resp. $\mathbb{S}_n^1$ by S_n resp. S_n^1. The center of $\mathbb{G}_n$ is equal to the center of $\mathbb{S}_n$ and can be identified with $\mathbb{Z}_p^\times$ (if we identify $\mathbb{S}_n$ with $\mathcal{O}_n^\times$) and the composite

$$\mathbb{Z}_p^\times \to \mathbb{G}_n \to \mathbb{Z}_p^\times$$

sends z to z^n. Thus if p does not divide n we get an isomorphism

$$\mathbb{G}_n \cong \mathbb{Z}_p \times \mathbb{G}_n^1 \ .$$

2.2. **Important subgroups in the case $n = 2$ and $p = 3$.** From now on we assume $n = 2$ and $p = 3$. Let ω be a primitive eighth root of unity in $\mathbb{W}^\times := \mathbb{W}_{\mathbb{F}_9}^\times$. Then

$$a = -\frac{1}{2}(1 + \omega S) \tag{5}$$

is an element of $\mathbb{S}_2^1$ of order 3. (This element was denoted s in [4] and [5].) We can and will in the sequel choose ω such that we have the following relation in $\mathbb{W}/(3) \cong \mathbb{F}_9$

$$\omega^2 + \omega - 1 \equiv 0 \ . \tag{6}$$

Next we let $t := \omega^2$. Then we have $ta = a^2 t$. Furthermore, if $\phi \in \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ is the Frobenius automorphism then the elements $\psi := \omega\phi$ and t generate a subgroup Q_8 which normalizes the subgroup generated by a so that ψ, t and a generate a subgroup G_{24} of $\mathbb{S}_2^1$ of order 24 which is isomorphic to a semidirect product $\mathbb{Z}/3 \rtimes Q_8$.

The elements ω and ϕ generate a subgroup SD_{16} of $\mathbb{S}_2^1$ of order 16, isomorphic to the semidihedral group of order 16.

Finally there is a torsionfree subgroup K of S_2^1 which has already played an important role in [7]. It is defined as follows: Lemma 2.1 implies that an element $1 + xS$ in S_2 of order 3 satisfies $\overline{x} \neq 0$ and $\overline{x} + \overline{x}^{1+3+9} = 0$, i.e. $\overline{x}^4 = -1$ where $\overline{x}$ is the class of x in $gr_{\frac{1}{2}} S_2^1 \cong \mathbb{F}_9$. There are no such elements x such that $\overline{x} \in \mathbb{F}_3$. Hence, if we define K to be the kernel of the homomorphism

$$S_2^1 \to gr_{\frac{1}{2}} S_2^1 \cong \mathbb{F}_9 \to \mathbb{F}_9/\mathbb{F}_3$$

then K is torsion free, and we have a split short exact sequence

$$1 \to K \to S_2^1 \to \mathbb{Z}/3 \to 1 \ . \tag{7}$$

K inherits a complete filtration from S_2 by setting $F_{\frac{k}{2}} K = F_{\frac{k}{2}} S_2 \cap K$ and it is easy to check that the associated graded is given as

$$gr_{\frac{k}{2}} K = \begin{cases} \mathrm{Ker}(Tr : \mathbb{F}_9 \to \mathbb{F}_3) & k > 0 \text{ even} \\ \mathbb{F}_3 & k = 1 \\ \mathbb{F}_9 & k > 1 \text{ odd} \end{cases} \tag{8}$$

where Tr denotes the trace from $\mathbb{F}_9$ to $\mathbb{F}_3$.

The following elements will play an important role in our later calculations.

$$b := [a, \omega], \qquad c := [a, b], \qquad d := [b, c] \ . \tag{9}$$

In the next lemma we record approximations to these elements which we will use repeatedly.

Lemma 2.2.

a) $a \equiv 1 + \omega S + S^2 \quad \mathrm{mod} \ (S^3)$
b) $b \equiv 1 - S - \omega S^2 \quad \mathrm{mod} \ (S^3)$
c) $c \equiv 1 - \omega^2 S^2 - \omega S^3 \quad \mathrm{mod} \ (S^4)$
d) $d \equiv 1 + \omega^2 S^3 \quad \mathrm{mod} \ (S^4)$

Proof. a) The approximation for a is immediate from its definition.

b) By explicit calculation in $\mathcal{O}_2$ we find

$$b = \frac{1}{4}(1 + \omega S)\omega(1 - \omega S)\omega^{-1} = \frac{1}{4}(1 + \omega S)(1 - \omega^{-1} S) = \frac{1}{4}(1 + 3\omega^2 + (\omega + \omega^3)S)$$

and then we use that our choice of ω yields $\omega^3 + \omega = -1$ and $\omega^2 - 1 = -\omega$ in $\mathbb{F}_9$.

c) Similarly we get

$$c = \frac{1}{64}(1 + \omega S)(1 + 3\omega^2 + (\omega + \omega^3)S)(1 - \omega S)(1 - 3\omega^2 - (\omega + \omega^3)S) = -\frac{1}{8}(1 + 6\omega^2 - 3\omega S) \ .$$

d) Finally the formula for d can be obtained from Lemma 2.1. $\square$

The information in the following proposition will be important for a closer inspection of the permutation resolution (1).

Proposition 2.3.

a) *$H^*(K;\mathbb{F}_3)$ is a Poincaré duality algebra of dimension 3.*
b) *$H_2(K;\mathbb{F}_3)\cong H_1(K;\mathbb{F}_3)\cong(\mathbb{F}_3)^2$.*
c) *$H_1(K;\mathbb{Z}_3)\cong\mathbb{Z}/9\oplus\mathbb{Z}/3$ where $\mathbb{Z}/9$ resp. $\mathbb{Z}/3$ is generated by b resp. by c.*
d) *$H_2(K;\mathbb{Z}_3)=0$.*
e) *$H_0(K;\mathbb{Z}_3)\cong H_3(K,\mathbb{Z}_3)\cong\mathbb{Z}_3$.*

Proof. Parts (a) and (b) have already been shown in Proposition 4.4. of [7].

For part (c) we note that $H_1(K;\mathbb{Z}_3)\cong K/\overline{[K,K]}$ where $\overline{[K,K]}$ is the closure of the commutator subgroup $[K,K]$ of K. By Lemma 2.1 the commutator map

$$gr_iK\times gr_jK\to gr_{i+j}K,\quad (x,y)\mapsto[x,y]$$

is surjective when $i=\frac{3}{2}$ and $j=\frac{k}{2}$ with k even, and also when $i=\frac{1}{2}$ and $j=\frac{l}{2}$ with $l>1$ odd. Thus $F_2K\subset[K,K]$. If $i=\frac{1}{2}$ and $j=1$ then the image of the commutator map is the kernel of the trace map. Together with part (b) of Lemma 2.1 this shows that $K/\overline{[K,K]}\cong\mathbb{Z}/9\oplus\mathbb{Z}/3$ and it is easy to check that b and c generate $\mathbb{Z}/9$ resp. $\mathbb{Z}/3$.

The remaining parts (d) and (e) now follow from a simple Bockstein calculation. □

3. The maps in the permutation resolution

3.1. **Generalities.** Let G be a profinite p-group. We say that G is finitely generated if $H_1(G,\mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p$. The kernel of the augmentation $\mathbb{Z}_p[[G]]\to\mathbb{Z}_p$ is denoted by IG, or simply by I. We say that a $\mathbb{Z}_p[[G]]$-module M is I-complete if the filtration by the submodules I^nM, $n\geq 0$, is complete. As in [5] we use a Nakayama type lemma to show that certain homomorphisms are surjective. Its proof is the same as that of Lemma 4.3 of [5].

Lemma 3.1. *Let G be a finitely generated profinite p-group and $f:M\to N$ a homomorphism of IG-complete $\mathbb{Z}_p[[G]]$-modules. Suppose that $H_0(f):H_0(G,M)\to H_0(G,N)$ is surjective. Then f is surjective.* □

In [5] we used the analogous Lemma 4.3 for $G=S_2^1$ in order to show that certain $\mathbb{Z}_3[[S_2^1]]$-linear maps are surjective. Here we use Lemma 3.1 for $G=K$ together with the action of the element a on $H_0(K,-)$ resulting from the exact sequence (7) in order to show that the same maps are surjective. The advantage of working with K will become clear when we will discuss the kernel of the map $C_1\to C_0$ (see the remark after Proposition 3.4 below). We begin the construction of the permutation resolution exactly as in [5].

3.2. **The homomorphism ∂_1.** Let $C_0=\mathbb{Z}_3[[\mathbb{G}_2^1]]\otimes_{\mathbb{Z}_3[G_{24}]}\mathbb{Z}_3$ and $e_0=e\otimes 1\in C_0$ if e is the unit in $\mathbb{G}_2^1$. Let $\partial_0:C_0\to\mathbb{Z}_3$ be the standard augmentation and let N_0 be the kernel of ∂_0 so that we have a short exact sequence

$$0\to N_0\to C_0\to\mathbb{Z}_3\to 0\ . \tag{10}$$

Proposition 3.2.

a) *As $\mathbb{Z}_3[[K]]$-module N_0 is generated by the elements $f_1:=(e-\omega)e_0$, $f_2:=(e-b)e_0$ and $f_3:=(e-c)e_0$. If we denote the class of f_i in $H_0(K,N_0)$ by $\overline{f_i}$ then we have an isomorphism*
$$H_0(K;N_0)\cong\mathbb{Z}_3\{\overline{f_1}\}\oplus\mathbb{Z}/9\{\overline{f_2}\}\oplus\mathbb{Z}/3\{\overline{f_3}\}\ .$$
b) *The action of a on $H_0(K,N_0)$ is given by :*
$$a_*\overline{f_1}=\overline{f_1}+\overline{f_2},\qquad a_*\overline{f_2}=\overline{f_2}+\overline{f_3},\qquad a_*\overline{f_3}=\overline{f_3}-3\overline{f_2}\ .$$

c) $H_1(K;N_0)=0$.

Proof. a) We consider the long exact sequence in $H_*(K,-)$ associated to the short exact sequence (10). As C_0 is free K-module we have $H_1(K;C_0)=0$. Furthermore, $H_0(K;C_0)\cong\mathbb{Z}_3^2$, generated by the classes of e_0 and ωe_0 so that the end of this long exact sequence has the following form

$$0\to H_1(K;\mathbb{Z}_3)\cong\mathbb{Z}/9\oplus\mathbb{Z}/3\to H_0(K;N_0)\to H_0(K;C_0)\cong\mathbb{Z}_3^2\to H_0(K;\mathbb{Z}_3)\cong\mathbb{Z}_3\to 0\ .$$

Now (a) follows from Proposition 2.3 and the identification of $H_1(K,\mathbb{Z}_3)$ with $IK/(IK)^2$ which sends $x\in K$ to the class of $x-e$ in $IK/(IK)^2$.

b) By definition we have $ae_0=e_0$ and thus

$$a\omega e_0=a\omega a^{-1}\omega^{-1}\omega ae_0=[a,\omega]\omega e_0=b\omega e_0\ .$$

Consequently

$$a(e-\omega)e_0=(e-a\omega)e_0=(e-b\omega)e_0=b(e-\omega)e_0+(e-b)e_0 \tag{11}$$

and we obtain the first formula by passing to K-coinvariants.

Similarly, $ab=cba$ and $ae_0=e_0$ imply

$$a(e-b)e_0=(e-ab)e_0=(e-cba)e_0=(e-cb)e_0=c(e-b)e_0+(e-c)e_0 \tag{12}$$

and by passing to K-coinvariants we get the second formula.

The third formula can now be deduced from the fact that $a_*^3\overline{f}_1=\overline{f}_1$.

c) This follows from the long exact sequence in $H_0(K,-)$ associated to the exact sequence (10) by using that $H_i(K,C_0)=0$ for $i=1,2$ and $H_2(K,\mathbb{Z}_3)=0$. □

Let $C_1=\mathbb{Z}_3[[\mathbb{G}_2^1]]\otimes_{\mathbb{Z}_3[SD_{16}]}\chi$ where χ is the non trivial character of SD_{16} defined over $\mathbb{Z}_3$ on which ω and ϕ both act by multiplication by -1. Let e_1 be the generator of C_1 given by $e\otimes 1$ where e is as before the unit in $\mathbb{G}_2^1$.

Corollary 3.3. *There is a $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-linear epimorphism $\partial_1: C_1\to N_0$ given by $e_1\mapsto(e-\omega)e_0$.*

Proof. The elements ω^2, $\phi\omega$ and $\omega^{-1}\phi$ all belong to G_{24} and hence they act trivially on e_0. Therefore we have

$$\omega(e-\omega)e_0=(\omega-\omega^2)e_0=-(e-\omega)e_0$$

and

$$\phi(e-\omega)e_0=(\phi-\phi\omega)e_0=(\omega(\omega^{-1}\phi)-e)e_0=-(e-\omega)e_0\ .$$

This implies that there is a well defined homomorphism $C_1\to N_0$ which sends e_1 to $(e-\omega)e_0$. To see that this homomorphism is surjective we note that C_1 is free as $\mathbb{Z}_3[[K]]$-module of rank 3 with generators e_1, ae_1 and a^2e_1. Then we use Lemma 3.1 and Proposition 3.2. □

3.3. **The homomorphism ∂_2.** Now we turn towards the construction of the second homomorphism in our permutation resolution. This is substantially more intricate; in [5] its existence was established but no explicit formula was given.

Let N_1 be the kernel of ∂_1 so that we have a short exact sequence

$$0\to N_1\to C_1\to N_0\to 0\ . \tag{13}$$

Proposition 3.4.

a) $H_0(K;N_1)\cong\mathbb{Z}_3^2$. *The inclusion of N_1 into C_1 induces an injection $H_0(K,N_1)\to H_0(K,C_1)$ and identifies $H_0(K,N_1)$ with the submodule generated by the classes $\overline{g}_i$, $i=1,2$, of $g_1=3(a-e)^2e_1$ and $g_2=9(a-e)e_1$.*

b) *The action of a on $H_0(K,N_1)$ is determined by*

$$a_*\overline{g}_1=-2\overline{g}_1-\overline{g}_2,\qquad a_*\overline{g}_2=3\overline{g}_1+\overline{g}_2\ .$$

c) *$H_0(S_2^1, N_1) \cong \mathbb{Z}/3$ and if n_1 is any element of N_1 which agrees in $H_0(K, C_1)$ with $\overline{g}_1$ then its class in $H_0(S_2^1, N_1)$ is non-trivial.*
d) *The elements ω and ϕ both act on $H_0(S_2^1, N_1)$ by multiplication by -1.*

Proof. a) We observe that C_1 is a free K-module of rank 3 generated by e_1, ae_1 and ae_2. Then (a) follows from Proposition 3.2 by using the long exact sequence in $H_0(K, -)$ associated to the short exact sequence (13) .

b) The formulae for the action of a already hold in $H_0(K, C_1)$.

c) This follows from part (b) by passing to coinvariants with respect to the action of a.

d) This has already been observed in Lemma 4.6. of [5]. □

Remark We remark that working with S_2^1-coinvariants only (as in [5]) does not give us a good hold on a generator of N_1. The reason is that the map $H_0(S_2^1, N_1) \cong \mathbb{Z}/3 \to H_0(S_2^1, C_1) \cong \mathbb{Z}_3$ is necessarily trivial and therefore such a generator cannot be easily associated with an element in C_1. Working with K-coinvariants gives us a starting point, namely the element $g_1 \in C_1$, from which we can try to construct an element n_1 of N_1 whose class in $H_0(K, C_1)$ agrees with that of $\overline{g}_1$ and thus projects to a generator of $H_0(S_2^1, N_1)$. A first step in the direction of finding such a generator n_1 is taken in Lemma 3.6 below.

Corollary 3.5. *Let $C_2 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \chi$, let $n_1 \in N_1$ be any element which projects non-trivially to the coinvariants $H_0(S_2^1; N_1)$ and let*

$$n_1' := \frac{1}{16} \sum_{g \in SD_{16}} \chi(g^{-1}) g(n_1) \ .$$

Then there is a $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-linear epimorphism $\partial_2 : C_2 \to N_2$ given by $e_2 \mapsto n_1'$.

Proof. By construction the group SD_{16} acts on n_1' via the mod-3 reduction of the character χ and thus there is a homomorphism ∂_2 as claimed. Surjectivity of ∂_2 follows from Lemma 3.1. □

Lemma 3.6.

a) *Let $l_1 = (a - b)e_1$, $l_2 = (a - c)l_1$ and $l_3 = 3cl_2 + (e - c)^2 l_2$. Then*
$$\partial_1(l_1) = (e - b)e_0, \quad \partial_1(l_2) = (e - c)e_0 \ , \quad \partial_1(l_3) = (e - c^3)e_0 \ .$$
b) *There exist elements $x, y \in IK$ such that $e - c^3 = x(e - b) + y(e - c)$.*
c) *If $x, y \in IK$ satisfy $e - c^3 = x(e - b) + y(e - c)$ then*
$$n_1 := l_3 - xl_1 - yl_2$$
belongs to N_1 and projects non-trivially to $H_0(S_2^1, N_1)$.

Proof. We start with the following two observations:

- Proposition 3.2 implies that $\partial_1((a - e)^2 e_1) \equiv (e - c)e_0 \mod (IK)N_0$.
- In $\mathbb{Z}_3[[K]]$ we have the relation $3c(e - c) + (e - c)^3 = e - c^3$.

a) By equations (11) and (12) of the proof of Proposition 3.2 we see that
$$\partial_1(l_1) = (e - b)e_0 \ \text{ and } \ \partial_1(l_2) = (e - c)e_0 \ .$$
The result for $\partial_1(l_3)$ is now obvious.

b) By Proposition 2.3.c we know that IK is generated by $e - b$ and $e - c$. Furthermore c^3 belongs to F_2K, hence it is trivial in $H_1(K, \mathbb{Z}_3)$ by Proposition 2.3.c . Therefore $e - c^3$ belongs to IK^2 and we get the existence of $x, y \in IK$ as required in (b).

c) By (a) and (b) n_1 belongs to N_1. Furthermore it is clear that n_1 and $3(a - e)^2 e_1$ agree in $H_0(K, C_1)$, hence n_1 projects non-trivially to $H_0(S_2^1, N_1)$. □

The question becomes now how we can determine x and y. In fact, we do not have explicit formulae for x and y. However, in section 5.3 we will give approximations for them which are sufficient for our homological calculations.

3.4. **The homomorphism ∂_3.** In [5] it was shown (by using that K is a Poincaré duality group) that the kernel of ∂_2 can be identified with $\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$. However, the identification and thus the construction of ∂_3 was not explicit. The following result shows that ∂_3 can be replaced by the dual of ∂_1, at least up to isomorphism.

If G is a profinite group and M a continuous left $\mathbb{Z}_p[[G]]$-module then we define its dual M^* by $\mathrm{Hom}_{\mathbb{Z}_p[[G]]}(M,\mathbb{Z}_p[[G]])$. This becomes a left $\mathbb{Z}_p[[G]]$-module via $(g.\varphi)(m)=\varphi(m)g^{-1}$ if $g\in G$, $\varphi\in \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(M,\mathbb{Z}_p[[G]])$ and $m\in M$. We observe that for a finite subgroup H there is a canonical $\mathbb{Z}_p[[G]]$-linear isomorphism

$$\mathbb{Z}_p[[G/H]]\to \mathbb{Z}_p[[G/H]]^*\cong \mathbb{Z}_p[[G]]^H,\quad g\mapsto \Big(g^*:\widetilde{g}\mapsto \widetilde{g}(\sum_{h\in H}h)g^{-1}\Big)\leftrightarrow (\sum_{h\in H}h)g^{-1}\ . \tag{14}$$

Proposition 3.7.

a) *There is an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules*
$$0\longrightarrow C_0^*\xrightarrow{\partial_1^*}C_1^*\xrightarrow{\partial_2^*}C_2^*\xrightarrow{\partial_3^*}C_3^*\xrightarrow{\overline{\epsilon}}\mathbb{Z}_3\longrightarrow 0$$
in which ∂_i^* is the dual of ∂_i for $i=1,2,3$.

b) *There is an isomorphism of complexes of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules*
$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & C_0^* & \xrightarrow{\partial_1^*} & C_1^* & \xrightarrow{\partial_2^*} & C_2^* & \xrightarrow{\partial_3^*} & C_3^* & \xrightarrow{\overline{\epsilon}} & \mathbb{Z}_3 & \longrightarrow & 0\\ & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow = & & \\ 0 & \longrightarrow & C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z}_3 & \longrightarrow & 0 \end{array}\ .$$
such that f_i induces the identity on $\mathrm{Tor}_0^{\mathbb{Z}_3[[S_2^1]]}(\mathbb{F}_3,C_i)$ for $i=2,3$ if we identify C_i^* with C_{3-i} via the isomorphism of (14).

c) *The homomorphism $\partial_1^*:C_0^*\to C_1^*$ is given by $e_0^*\mapsto(e+a+a^2)e_1^*$.*

Proof. a) Each C_i is free as a $\mathbb{Z}_3[[K]]$-module and therefore the complex
$$0\longrightarrow C_3\xrightarrow{\partial_3}C_2\xrightarrow{\partial_2}C_1\xrightarrow{\partial_1}C_0\xrightarrow{\epsilon}\mathbb{Z}_3\longrightarrow 0$$
is a free $\mathbb{Z}_3[[K]]$-resolution of $\mathbb{Z}_3$. Because K is of finite index in $\mathbb{G}_2^1$ the coinduced module of $\mathbb{Z}_3[[K]]$ is isomorphic to $\mathbb{Z}_3[[\mathbb{G}_2^1]]$. Therefore there are natural isomorphisms
$$C_i^*=\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_i,\mathbb{Z}_3[[\mathbb{G}_2^1]])\cong \mathrm{Hom}_{\mathbb{Z}_3[[K]]}(C_i,\mathbb{Z}_3[[K]])$$
and the n-th cohomology of the complex $\mathrm{Hom}_{\mathbb{Z}_3[[K]]}(C_i;\mathbb{Z}_3[[K]])$ is $H^n(K;\mathbb{Z}_3[[K]])$. Because K is a Poincaré duality group this is zero except when $n=3$ and then it is isomorphic to $\mathbb{Z}_3$. Finally, one sees as in Proposition 5 of [13] that the $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module structure on $H^n(\mathbb{G}_2^1;\mathbb{Z}_3[[\mathbb{G}_2^1]])\cong H^n(K;\mathbb{Z}_3[[K]])$ is trivial.

b) The augmentation $\mathbb{Z}_3[[\mathbb{G}_2^1]]\to\mathbb{Z}_3$ induces an isomorphism
$$\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(\mathbb{Z}_3,\mathbb{Z}_3)\cong \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(\mathbb{Z}_3[[\mathbb{G}_2^1]],\mathbb{Z}_3)\ .$$
Thus the right hand square is commutative up to a unit in $\mathbb{Z}_3$ if we choose for f_0 the isomorphism given in (14), and we can modify f_0 by a unit so that it commutes on the nose. Then f_0 induces an isomorphism $\mathrm{Ker}\epsilon\cong\mathrm{Ker}\overline{\epsilon}$ and
$$C_2\xrightarrow{\partial_2}C_1\xrightarrow{\partial_1}\mathrm{Ker}\epsilon\quad\text{resp.}\quad C_1^*\xrightarrow{\partial_2^*}C_2^*\xrightarrow{\partial_3^*}\mathrm{Ker}\overline{\epsilon}$$
is the beginning of a resolution of $\mathrm{Ker}\epsilon$ resp. $\mathrm{Ker}\overline{\epsilon}$ by projective $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules and the isomorphism induced by f_0 lifts to a chain map $f_\bullet$ between the projective resolutions. By Lemma 4.5 of [5] we have $\mathrm{Tor}_i^{\mathbb{Z}_3[[S_2^1]]}(\mathbb{F}_3,\mathrm{Ker}\epsilon)\cong\mathbb{F}_3$ if $i=0,1$ and this implies that the maps $f_1:C_1\to C_2^*$ and $f_2:C_2\to C_1^*$ induce isomorphisms on $\mathrm{Tor}_0^{\mathbb{Z}_3[[S_2^1]]}(\mathbb{F}_3,-)$ and hence they are

themselves isomorphisms by Lemma 4.3 of [5]. Finally, f_3 is trivially an isomorphism because f_2 and f_1 are isomorphisms.

The isomorphism of chain complexes (considered as automomorphism via (14)) induces an automorphism of spectral sequences

$$\mathrm{Tor}_j^{\mathbb{Z}_3[[S_2^1]]}(\mathbb{F}_3, C_i) \Rightarrow \mathrm{Tor}_{i+j}^{\mathbb{Z}_3[[S_2^1]]}(\mathbb{F}_3, \mathbb{Z}_3)$$

which converges towards the identity, and this easily implies the remaining part of (b).

c) By (14) we have for each $\widetilde{g} \in \mathbb{G}_2^1$

$$(e+a+a^2)^*(e_1^*)(\widetilde{g}) = \widetilde{g}\big(\sum_{h\in SD_{16}} \chi(h^{-1})h\big)(e+a^{-1}+a^{-2}) = \widetilde{g}\big(\sum_{h\in SD_{16}} \chi(h^{-1})h\big)(e+a+a^2)$$

and

$$\partial_1^*(e_0^*)(\widetilde{g}) = e_0^*(\widetilde{g}(e-\omega)) = \widetilde{g}(e-\omega)\big(\sum_{h\in G_{24}} h\big) = \widetilde{g}(e-\omega)\big(\sum_{h\in Q_8} h\big)(e+a+a^2)\ .$$

Then we conclude via the identity $(e-\omega)\big(\sum_{h\in Q_8} h\big) = \sum_{h\in SD_{16}} \chi(h^{-1})h$. □

4. On the action of the stabilizer group

In this section we will produce formulae for the action of the elements a, b, c and d of the stabilizer group $\mathbb{G}_2$ on $\mathbb{F}_9[[u_1]][u^{\pm 1}]$, at least modulo suitable powers of the invariant ideal generated by u_1. It turns out that it is sufficient to have a formula for the action of a and b on u modulo (u_1^6), and for the action of c and d on u modulo (u_1^{10}).

4.1. **Generalities.** We recall (cf. [10]) that $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ where the Araki generators v_i satisfy the following equation (in $BP_* \otimes \mathbb{Q}$)

$$p\lambda_k = \sum_{0\le i\le k} \lambda_i v_{k-i}^{p^i}\ . \tag{15}$$

Here the $\lambda_i \in BP_* \otimes \mathbb{Q}$ are the coefficients of the logarithm of the universal p-typical formal group law F on BP_*,

$$\log_F(x) = \sum_{i\ge 0} \lambda_i x^{p^i}$$

(with $\lambda_0 = 1$), and thus the $[p]$-series of F is given by

$$[p]_F(x) = \sum_{i\ge 0}^F v_i x^{p^i}\ .$$

The homomorphism

$$BP_* \to (E_n)_* = \mathbb{W}_{\mathbb{F}_{p^n}}[[u_1, \ldots, u_{n-1}]][u^{\pm 1}], \quad v_i \mapsto \begin{cases} u_i u^{1-p^i} & i<n \\ u^{1-p^n} & i=n \\ 0 & i>n \end{cases}$$

defines a p-typical formal group law F_n over $(E_n)_*$. Then the formal group law G_n over $(E_n)_0$ defined by $G_n(x,y) = u^{-1}F_n(ux,uy)$ is a universal deformation of Γ_n and is p-typical with p-series

$$[p]_{G_n}(x) = px +_{G_n} u_1 x^p +_{G_n} \cdots +_{G_n} u_{n-1} x^{p^{n-1}} +_{G_n} x^{p^n}\ . \tag{16}$$

Next we recall how one can get at the action of an element $g \in \mathbb{S}_n$ on $(E_n)_0$. For a given g we choose a lift $\widetilde{g} \in (E_n)_0[[x]]$ of g and let $\widetilde{G}$ be the formal group law defined by

$$\widetilde{G}(x,y) = \widetilde{g}^{-1}\big(G_n(\widetilde{g}(x), \widetilde{g}(y))\big)\ .$$

Then there is a unique ring homomorphism $g_* : (E_n)_0 \to (E_n)_0$ and a unique $*$-isomorphism from g_*G_n to $\widetilde{G}$ such that the composition

$$h_g : g_*G_n \to \widetilde{G} \xrightarrow{g} G_n$$

is an isomorphism of p-typical formal group laws and can therefore be written (cf. Appendix 2 of [10]) as

$$h_g(x) = \sum_{i\geq 0}{}^{G_n} t_i(g)x^{p^i}$$

for unique continuous functions

$$t_i : \mathbb{S}_n \to (E_n)_0 \ . \tag{17}$$

We note that

$$t_i(g) \equiv g_i \bmod (3, u_1) \tag{18}$$

if $g = \sum_i g_i S^i \in \mathbb{S}_n$ with $g_i^{p^2} = g_i$. Then we have

$$g_*(u) = t_0(g)u \tag{19}$$

and the equation

$$h_g([p]_{g_*G_n}(x)) = [p]_{G_n}(h_g(x)) \tag{20}$$

can be used to recursively find better and better approximations for $t_0(g)$ as well as for the action of g on the deformation parameters $u_1, \ldots, u_{n-1}$.

4.2. **A formula modulo** (3) **for the formal group law** G_2**.** From now on we restrict attention to the case $n = 2$ and $p = 3$.

Lemma 4.1. *The logarithm and exponential of the formal group law G_2 satisfy*

$$\begin{aligned}
\log_{G_2}(x) &= x - \tfrac{u_1}{24}x^3 + \tfrac{1}{1-3^8}\big(\tfrac{1}{3} - \tfrac{u_1^4}{72}\big)x^9 && \bmod \ (x^{27}) \\
\exp_{G_2}(x) &= x + \tfrac{u_1}{24}x^3 + \tfrac{3u_1^2}{24^2}x^5 + \tfrac{12u_1^3}{24^3}x^7 - \Big(\tfrac{1}{1-3^8}\big(\tfrac{1}{3} - \tfrac{u_1^4}{72}\big) + \tfrac{55u_1^4}{24^4}\Big)x^9 && \bmod \ (x^{11}) \ .
\end{aligned}$$

Proof. From (15) we get $\lambda_1 = -\frac{v_1}{24}$ and $\lambda_2 = \frac{1}{1-3^8}\big(\frac{v_2}{3} - \frac{v_1^4}{72}\big)$. To obtain the result for $\log_{G_2}$ we use the classifying homomorphism for F_2 and that $\log_{G_2}(x) = u^{-1}\log_{F_2}(ux)$. □

Corollary 4.2. *The formal group law G_2 satisfies*

$$\begin{aligned}
x +_{G_2} y \ \equiv\ & x + y - u_1(xy^2 + x^2y) + u_1^2(xy^4 + x^4y) \\
& -u_1^3(xy^6 + x^6y) - u_1^3(x^3y^4 + x^4y^3) \\
& -(x^3y^6 + x^6y^3) + u_1^4(x^4y^5 + x^5y^4) \qquad \bmod (3, (x,y)^{11}) \ .
\end{aligned}$$

Proof. This follows directly from $x +_{G_2} y = \exp_{G_2}(\log_{G_2}(x) + \log_{G_2}(y))$. □

4.3. **Formulae for the action modulo** (3)**.** To simplify notation we will denote in the remainder of this section the mod-3 reduction of the value of the function t_i of (17) on an element g again by $t_i(g)$, or even by t_i if g is clear from the context.

Proposition 4.3. *Let $g \in \mathbb{S}_n$ and let u_1 be Araki's u_1. Then the following equations hold*

a) $g_*(u_1) = t_0^2u_1$
b) $t_0 + t_0^6t_1u_1^3 = t_0^9 + t_1^3u_1$
c) $t_1 - t_0^8t_1u_1^4 \equiv t_1^9 + t_2^3u_1 - t_0^{18}t_1^3u_1^2 - t_0^9t_1^6u_1^3 \quad \bmod \ (u_1^7)$
d) *If* $g \equiv 1 \bmod (S^2)$ *then* $t_2 \equiv t_2^9 + t_3^3u_1 \quad \bmod \ (u_1^2)$.

Remark If we want to know $g_*(u_1)$ modulo (u_1^7) then (a) shows that it is enough to know t_0 modulo (u_1^6) and this can be calculated from (b) if we know t_0 modulo (u_1) and t_1 modulo (u_1^3). Furthermore, t_1 can be calculated modulo (u_1^3) from (c) if we know t_0, t_1 and t_2 modulo (u_1). In the same manner we can even calculate $g_*(u_1)$ modulo (u_1^8).

Similarly, if we want to know $g_*(u_1)$ modulo (u_1^{11}) then (a) shows that it is enough to know t_0 modulo (u_1^{10}) and this can be calculated from (b) if we know t_0 modulo (u_1) and t_1 modulo (u_1^7). Furthermore, t_1 can be calculated modulo (u_1^7) from (c) if we know t_0 modulo (u_1^3), t_1 modulo (u_1) and t_2 modulo (u_1^2). Finally (d) can be used to calculate t_2 modulo (u_1^2) if we know t_2 and t_3 modulo (u_1).

Proof. In this proof we abbreviate G_2 simply by G. We consider equation (20)

$$h_g([3]_{g_*G})(x) = [3]_G(h_g(x))$$

over $(E_2)_0/(3)[[x]]$ and compare coefficients of x^{3^k} for $k = 1,2,3,4$. By (16) we have

$$\begin{aligned} h_g([3]_{g_*G})(x) &\equiv t_0\big(g_*(u_1)x^3 +_{g_*G} x^9\big) +_G t_1\big(g_*(u_1)x^3 +_{g_*G} x^9\big)^3 \\ &\quad +_G t_2\big(g_*(u_1)x^3 +_{g_*G} x^9\big)^9 +_G t_3\big(g_*(u_1)x^3 +_{g_*G} x^9\big)^{27} \mod (x^{82}) \\ [3]_G(h_g(x)) &\equiv u_1(t_0x +_G t_1x^3 +_G t_2x^9 + t_3x^{27})^3 \\ &\quad +_G(t_0x +_G t_1x^3 +_G t_2x^9)^9 \mod (x^{82}) . \end{aligned}$$

a) For the coefficient of x^3 we obviously get $g_*(u_1)t_0 = u_1t_0^3$ which proves (a).

b) The coefficient of x^9 in $h_g([3]_{g_*G})(x)$ is equal to $t_0 + g_*(u_1)^3t_1$ which by (a) is equal to $t_0 + u_1^3t_0^6t_1$. The coefficient of x^9 in $[3]_G(h_g(x))$ is equal to the same coefficient in

$$u_1(t_0x +_G t_1x^3)^3 +_G (t_0x)^9$$

which is clearly equal to $u_1t_1^3 + t_0^9$ and hence we get (b).

c) The coefficient of x^{27} in $h_g([3]_{g_*G})(x)$ is equal to the coefficient of x^{27} in

$$t_0\big(g_*(u_1)x^3 +_{g_*G} x^9\big) +_G t_1\big(g_*(u_1)x^3 +_{g_*G} x^9\big)^3 +_G t_2\big(g_*(u_1)x^3\big)^9$$

and the latter coefficient is equal to

$$t_1 + t_2g_*(u_1)^9 + c$$

where c is the coefficient of x^{27} in

$$t_0\big(g_*(u_1)x^3 +_{g_*G} x^9\big) +_G t_1g_*(u_1)^3x^9 .$$

Next we observe that Corollary 4.2 yields

$$\begin{aligned} g_*(u_1)x^3 +_{g_*G} x^9 &\equiv g_*(u_1)x^3 + x^9 - g_*(u_1)^3x^{15} \\ &\quad - g_*(u_1)^2x^{21} + g_*(u_1)^6x^{21} - g_*(u_1)^9x^{27} \mod (x^{28}) . \end{aligned}$$

Applying Corollary 4.2 once more and using (a) and calculating modulo (u_1^7) we obtain

$$c \equiv -u_1t_0^2t_1g_*(u_1)^3 = -u_1^4t_0^8t_1 \mod (u_1^7)$$

and hence modulo (u_1^7) the coefficient of x^{27} in $h_g([3]_{g_*G})(x)$ is equal to

$$t_1 - u_1^4t_0^8t_1 .$$

On the other hand the coefficient of x^{27} in $[3]_G(h_g(x))$ is equal to the same coefficient in

$$u_1(t_0x +_G t_1x^3 +_G t_2x^9)^3 +_G (t_0x +_G t_1x^3)^9$$

and this coefficient is equal to

$$u_1t_2^3 + t_1^9 + d$$

where d is the coefficient of x^{27} in

$$u_1(t_0x +_G t_1x^3)^3 +_G t_0^9x^9 .$$

Next we observe that Corollary 4.2 yields

$$\begin{aligned}(t_0x +_G t_1x^3)^3 \quad &\equiv \quad t_0^3x^3 + t_1^3x^9 - u_1^3t_0^6t_1^3x^{15} \\ &\quad -u_1^3t_0^3t_1^6x^{21} + u_1^6t_0^{12}t_1^3x^{21} - u_1^9t_0^{18}t_1^3x^{27} \qquad \bmod\ (x^{28})\ .\end{aligned}$$

Applying Corollary 4.2 once more and using (a) and calculating modulo (u_1^7) we obtain

$$\begin{aligned}u_1(t_0x +_G t_1x^3)^3 +_G t_0^9x^9 \quad \equiv \quad & u_1\big(t_0^3x^3 + t_1^3x^9 - u_1^3t_0^6t_1^3x^{15} - u_1^3t_0^3t_1^6x^{21}\big) + t_0^9x^9 \\ & -u_1^3\big(t_0^3x^3 + t_1^3x^9 - u_1^3t_0^6t_1^3x^{15} - u_1^3t_0^3t_1^6x^{21}\big)^2t_0^9x^9 \\ & -u_1^2\big(t_0^3x^3 + t_1^3x^9 - u_1^3t_0^6t_1^3x^{15} - u_1^3t_0^3t_1^6x^{21}\big)t_0^{18}x^{18} \\ & +u_1^6\big(t_0^3x^3 + t_1^3x^9 - u_1^3t_0^6t_1^3x^{15} - u_1^3t_0^3t_1^6x^{21}\big)^4t_0^9x^9 \quad \bmod\ (x^{28})\end{aligned}$$

and hence

$$d \equiv -u_1^3(t_1^6 - 2u_1^3t_0^9t_1^3)t_0^9 - u_1^2t_0^{18}t_1^3 + 4u_1^6t_0^{18}t_1^3 \quad \bmod\ (u_1^7)\ .$$

Therefore the coefficient of x^{27} in $[3]_G(h_g(x))$ is equal to

$$t_1^9 + u_1t_2^3 - u_1^2t_0^{18}t_1^3 - u_1^3t_0^9t_1^6 + 2u_1^6t_0^{18}t_1^3 + 4u_1^6t_0^{18}t_1^3 \qquad \bmod\ (u_1^7)$$

and (c) follows.

d) The coefficient of x^{81} in $h_g([3]_{g_*G})(x)$ is equal to the same coefficient in

$$t_0\big(g_*(u_1)x^3 +_{g_*G} x^9\big) +_G t_1\big(g_*(u_1)x^3 +_{g_*G} x^9\big)^3 +_G t_2\big(g_*(u_1)x^3 +_{g_*G} x^9\big)^9 +_G t_3(g_*(u_1)x^3)^{27}$$

which modulo (u_1^3) is equal to the same coefficient in

$$t_0\big(g_*(u_1)x^3 +_{g_*G} x^9\big) +_G t_1x^{27} +_G t_2x^{81}$$

and by Corollary 4.2 this is easily seen to be equal to t_2 modulo (u_1^2).

On the other hand the coefficient of x^{81} in $[3]_G(h_g(x))$ is equal to the same coefficient in

$$u_1(t_0x +_G t_1x^3 +_G t_2x^9 + t_3x^{27})^3 +_G (t_0x +_G t_1x^3 + t_2x^9)^9$$

and this coefficient is equal to

$$u_1t_3^3 + t_2^9 + e$$

where e is the coefficient of x^{81} in the series

$$u_1(t_0x +_G t_1x^3 +_G t_2x^9)^3 +_G (t_0x +_G t_1x^3)^9\ .$$

Now $g \equiv 1 \bmod (S^2)$ implies $t_1 \equiv 0 \bmod (u_1)$ and thus modulo (u_1^2) we find that e is also the coefficient of x^{81} in

$$u_1(t_0x +_G t_2x^9)^3 +_G t_0^9x^9$$

and by Corollary 4.2 even of the coefficient of x^{81} in

$$u_1(t_0x +_G t_2x^9)^3\ .$$

Now Lemma 4.4 below shows that the coefficient of x^{27} in $t_0x +_G t_2x^9$ is trivial modulo (u_1) (if not, either the coefficient of $x^{18}y$ or of x^9y^2 in $x +_G y$ would have to be nontrivial modulo (u_1)), hence e is trivial modulo (u_1^2) and the proof of (d) is complete. □

Lemma 4.4.

$$x +_{G_2} y \equiv x + y + \sum_{i\geq 1} P_{8i+1}(x,y) \quad \bmod\ (3, u_1)$$

where P_{8i+1} is a homogeneous polynomial of degree $8i+1$ without terms x^{8i+1} and y^{8i+1}.

Proof. It is enough to show this for the graded formal group law F_2 over $(E_2)_*$. This group law is a homogeneous series of degree -2 if x and y are given degree -2 and thus, if we write

$$x +_{G_2} y \equiv x + y + \sum_{j\geq 1} P_j(x,y) \quad \bmod\ (3, u_1)$$

with homogeneous polynomials in x and y of degree $-2j$ then the coefficients in P_j have to be in $(E_2)_{2j-2}$. Furthermore, this group law has its coefficients in the subring generated by u_1u^{-2} and u^{-8}. However, $(E_2)_*/(3,u_1) \cong \mathbb{F}_9[[u^{\pm 1}]]$ and thus $2j-2$ has to be a multiple of 16. □

Corollary 4.5. *The following equations hold in $(E_2)_0/(3)$.*

a) *Let $g = 1 + g_1S + g_2S^2 \mod (S^3)$. Then we have*

$$\begin{array}{rcll} t_1 & \equiv & g_1 + g_2^3u_1 - g_1^3u_1^2 - g_1^6u_1^3 & \mod (u_1^4) \\ t_0 & \equiv & 1 + g_1^3u_1 - g_1u_1^3 + (g_2 - g_2^3)u_1^4 + g_1^3u_1^5 + (g_1^2 + g_1^6)u_1^6 & \mod (u_1^7)\ . \end{array}$$

b) *Let $g = 1 + g_2S^2 + g_3S^3 \mod (S^4)$. Then we have*

$$\begin{array}{rcll} t_2 & \equiv & g_2 + g_3^3u_1 & \mod (u_1^2) \\ t_1 & \equiv & g_2^3u_1 + g_3u_1^4 + (g_2^3 - g_2)u_1^5 & \mod (u_1^7) \\ t_0 & \equiv & 1 + (g_2 - g_2^3)u_1^4 - g_3u_1^7 + (g_2 - g_2^3)u_1^8 & \mod (u_1^{10})\ . \end{array}$$

Proof. a) From Proposition 4.3.c we obtain

$$t_1 \equiv t_1^9 + t_2^3u_1 - t_0^{18}t_1^3u_1^2 - t_0^9t_1^6u_1^3 \mod (u_1^4)$$

and by using (18) we immediately get the formula for t_1 modulo (u_1^4). Then Proposition 4.3.b and (18) yield

$$t_0 + t_0^6(g_1 + g_2^3u_1 - g_1^3u_1^2 - g_1^6u_1^3)u_1^3 \equiv 1 + (g_1^3 + g_2u_1^3)u_1 \mod (u_1^7)$$

from which we easily get the formula for t_0 modulo (u_1^7). The formula for $g_*(u_1)$ follows now from Proposition 4.3.a .

b) From Proposition 4.3.d and (18) we immediately obtain the formula for t_2. Substituting the value for t_2 into the formula of Proposition 4.3.c and using (18) yields

$$t_1 - t_0^8t_1u_1^4 \equiv (g_2^3 + g_3u_1^3)u_1 - t_0^{18}t_1^3u_1^2 \mod (u_1^7)\ .$$

Substituting the values of t_0 modulo (u_1^7) and t_1 modulo (u_1^4) from (a) into this yields

$$t_1 - g_2^3u_1^5 \equiv (g_2^3 + g_3u_1^3)u_1 - g_2u_1^5 \mod (u_1^7)$$

from which we get the value of t_1 modulo (u_1^7). Next we substitute this value of t_1 together with the value of t_0 modulo (u_1^7) of (a) into the formula of Proposition 4.3.b and obtain

$$t_0 + \big(1 + (g_2 - g_2^4)u_1^4\big)^6\big(g_2^3u_1 + g_3u_1^4 + (g_2^3 - g_2)u_1^5\big)u_1^3 \equiv 1 + g_2u_1^4 \mod (u_1^{10})$$

from which we easily get the formula for t_0 modulo (u_1^{10}). □

The following calculation will be used repeatedly in later sections. The result is only given to the precision needed later.

Lemma 4.6. *Let $g = 1 + g_1S + g_2S^2 \mod (S^3)$ and let k be an integer. Then we have*

$$t_0(g)^k \equiv \begin{cases} 1 + g_1^3u_1 + (k'-1)g_1u_1^3 + (k'g_1^4 + g_2 - g_2^3)u_1^4 + g_1^3u_1^5 & \mod (3, u_1^6) \quad k = 3k'+1 \\ 1 - g_1^3u_1 + g_1^6u_1^2 + (k'+1)g_1u_1^3 + & \\ (g_1^4 - k'g_1^4 + g_2^3 - g_2)u_1^4 + (k'g_1^7 - g_1^3 - g_1^3g_2 + g_1^3g_2^3)u_1^5 & \mod (3, u_1^6) \quad k = 3k'+2\ . \end{cases}$$

Proof. The result follows easily from Corollary 4.5 and from

$$t_0(g)^k = \big((1 + (t_0(g) - 1))\big)^k \equiv \sum_{j=1}^{5}\binom{k}{3}(t_0(g) - 1)^j \mod (3, u_1^6)$$

by using that $\binom{k}{j} \equiv \prod_i \binom{k_i}{j_i} \mod (p)$ if $k = \sum_i k_ip^i$ and $j = \sum_i j_ip^i$ are the p-adic expansions of k and j respectively. □

Using Lemma 2.2 we finally get the following information on the action of a, b, c and d on $(E_2)_*/(3)$. We use 1 and ω^2 as a basis of $\mathbb{F}_9$ considered as an $\mathbb{F}_3$-vector space (rather then 1 and ω).

Corollary 4.7. *The action of the elements a, b, c and d on $(E_2)_*/(3)$ satisfy the formulae*

$$\begin{array}{lcll}
a_*u_1 & \equiv & u_1-(1+\omega^2)u_1^2-\omega^2u_1^3+(1-\omega^2)u_1^4-u_1^5-(1+\omega^2)u_1^6 & \mathrm{mod}\ (u_1^7)\\
b_*u_1 & \equiv & u_1+u_1^2+u_1^3-u_1^4+(1+\omega^2)u_1^5+(1-\omega^2)u_1^6 & \mathrm{mod}\ (u_1^7)\\
c_*u_1 & \equiv & u_1-\omega^2u_1^5+(-1+\omega^2)u_1^8-(1+\omega^2)u_1^9 & \mathrm{mod}\ (u_1^{11})\\
d_*u_1 & \equiv & u_1+\omega^2u_1^8 & \mathrm{mod}\ (u_1^{11})\\
 & & & \\
a_*u & \equiv & \big(1+(1+\omega^2)u_1+(-1+\omega^2)u_1^3+(1+\omega^2)u_1^5\big)u & \mathrm{mod}\ (u_1^6)\\
b_*u & \equiv & \big(1-u_1+u_1^3-\omega^2u_1^4-u_1^5\big)u & \mathrm{mod}\ (u_1^6)\\
c_*u & \equiv & \big(1+\omega^2u_1^4+(1-\omega^2)u_1^7+\omega^2u_1^8\big)u & \mathrm{mod}\ (u_1^{10})\\
d_*u & \equiv & \big(1-\omega^2u_1^7\big)u & \mathrm{mod}\ (u_1^{10})\ . \quad \square
\end{array}$$

5. The E_2-term of the algebraic spectral sequence

5.1. **The E_1-term.** We begin by giving some background on Theorem 1.1, or equivalently, on the E_1-term of the spectral sequence (2)

$$E_1^{s,t,*}=\mathrm{Ext}^t_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_s,(E_2)_*/(3))\Longrightarrow H^{s+t}(\mathbb{G}_2^1,(E_2)_*/(3))\ .$$

We note that for $s=1,2$ the module C_s is projective as $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-module and thus we have a Shapiro isomorphism

$$E_1^{s,t}=\mathrm{Ext}^t_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(\chi\uparrow_{SD_{16}}^{\mathbb{G}_2^1},(E_2)_*/(3))\cong\begin{cases}\mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\chi,(E_2)_*/(3)) & t=0\\ 0 & t>0\ .\end{cases} \tag{21}$$

The action of SD_{16} on $(E_2)_*$ is known (cf. the proof of Lemma 22 of [6] for an explicit reference) to be given by

$$\omega_*u_1=\omega^2u_1\quad\text{and}\quad\omega_*u=\omega u \tag{22}$$

and the Frobenius ϕ acts $\mathbb{Z}_3$-linearly by extending the action of Frobenius on $\mathbb{W}$ via

$$\phi_*u_1=u_1\quad\text{and}\quad\phi_*u=u\ . \tag{23}$$

This implies immediately that $(E_2)_*/(3)^{SD_{16}}$ is isomorphic to $\mathbb{F}_3[[u_1^4]][v_1,u^{\pm 8}]$ as a graded algebra and that there is an isomorphism of $(E_2)_*/(3)^{SD_{16}}$-modules

$$\mathrm{Ext}^0_{\mathbb{Z}_3[SD_{16}]}(\chi,(E_2)_*/(3))\cong\omega^2u^4\mathbb{F}_3[[u_1^4]][v_1,u^{\pm 8}]. \tag{24}$$

For $s=0,3$ we have a Shapiro isomorphism

$$E_1^{s,t}=\mathrm{Ext}^t_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(\mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]],(E_2)_*/(3))\cong H^t(G_{24},(E_2)_*/(3))\ .$$

Let G_{12} be the subgroup of G_{24} generated by the elements a and t. The calculation of the cohomology algebra $H^*(G_{12},(E_2)_*/(3))$ was deduced from that of $H^*(G_{12},(E_2)_*)$ in section 1.3 of [4]. In precisely the same way one deduces the calculation of $H^*(G_{24},(E_2)_*/(3))$ from that of $H^*(G_{12},(E_2)_*)$ which was given in section 3 of [5]. In particular there are classes

$$\begin{array}{ll}\Delta\in H^0(G_{24},(E_2)_{24}/(3)), & \alpha\in H^1(G_{24},(E_2)_4/(3))\\ \widetilde{\alpha}\in H^1(G_{24},(E_2)_{12}/(3)), & \beta\in H^2(G_{24},(E_2)_{12}/(3))\end{array}$$

and an isomorphism of algebras

$$H^*(G_{24},(E_2)_*/(3))\cong\mathbb{F}_3[[v_1^6\Delta^{-1}]][v_1,\Delta^{\pm1},\beta,\alpha,\widetilde{\alpha}]/(\alpha^2,\widetilde{\alpha}^2,v_1\alpha,v_1\widetilde{\alpha},\alpha\widetilde{\alpha}+v_1\beta)\ . \tag{25}$$

In the sequel we need some control over the elements occuring in this isomorphism (cf. section 1.3 of [4]). First we recall that α is defined as $\delta^0(v_1)$ where δ^0 is the Bockstein with respect to the short exact sequence of continuous $\mathbb{Z}_3[[\mathbb{G}_2]]$-modules

$$0\to(E_2)_*/(3)\xrightarrow{3}(E_2)_*/(9)\to(E_2)_*/(3)\to 0\ . \tag{26}$$

Similarly, $v_2:=u^{-8}$ determines an invariant in $H^0(\mathbb{G}_2,(E_2)_{16}/(3,u_1))$ and $\widetilde{\alpha}$ is defined as $\delta^1(v_2)$ where δ^1 is the Bockstein with respect to the short exact sequence of continuous $\mathbb{Z}_3[[\mathbb{G}_2]]$-modules

$$0\to\Sigma^4(E_2)_*/(3)\xrightarrow{v_1}(E_2)_*/(3)\to(E_2)_*/(3,u_1)\to 0\ . \tag{27}$$

Next β is defined to be the mod 3-reduction of $\delta^0\delta^1(v_2)$. These elements are thus defined as elements in $H^*(\mathbb{G}_2,(E_2)_*/(3))$. We denote their restriction to $H^*(G_{24},(E_2)_*/(3))$ by the same name.

The relation between Δ (which lifts to an invariant of the same name in $H^0(G_{24},(E_2)_{24})$) and the classes u and u_1 is more subtle. Here we record the following result.

Proposition 5.1. $\Delta \equiv (1-\omega^2u_1^2+u_1^4)\omega^2u^{-12} \mod (3,u_1^6)$.

Proof. By (3.11) of [5] the integral lift of Δ is defined as

$$\Delta = \frac{\omega^2}{4\big(x(a_*x)(a_*(a_*x))\big)^4}$$

and by the proof of Lemma 3.1 of [5] we know $x\equiv u \mod (3,u_1)$, hence $\Delta\equiv \omega^2u^{-12} \mod (3,u_1)$. Because Δ is invariant with respect to G_{24}, in particular with respect to Q_8, we get from (22) and (23) that Δ is of the form

$$\Delta \equiv (1+\lambda_2\omega^2u_1^2+\lambda_4u_1^4)\omega^2u^{-12} \qquad \mod (3,u_1^6) \ \text{ with } \ \lambda_i\in\mathbb{F}_3\subset\mathbb{F}_9 \ .$$

Because Δ is also invariant with respect to the action of a we get from (19)

$$\Delta = a_*(\Delta) \equiv t_0(a)^{-12}(1+\lambda_2\omega^2a_*(u_1)^2+\lambda_4a_*(u_1)^4)\omega^2u^{-12} \quad \mod (3,u_1^6) \ .$$

The right hand side of this equation can be evaluated modulo (u_1^6) by using Corollary 4.5.a and Corollary 4.7. By looking at the coefficients of u_1^3 and u_1^5 in the right hand side we obtain $\lambda_2=-1$ and $\lambda_4=1$. □

5.2. **The d_1-differential.** First of all we note that all differentials are v_1-linear.

Lemma 5.2. *Let $k\not\equiv 0 \mod (3)$. Then the differential $d_1: E_1^{0,0}\to E_1^{1,0}$ satisfies*

$$d_1(\Delta^k)\equiv \begin{cases} (-1)^{m+1}\omega^2(1+u_1^4)u^{-12k} & \mod (u_1^8) \qquad k=2m+1 \\ (-1)^{m+1}m\omega^2u_1^2u^{-12k} & \mod (u_1^6) \qquad k=2m. \end{cases}$$

Proof. By Corollary 3.3 the differential is induced by the homomorphism $C_1\to C_0$ which sends e_1 to $(e-\omega)e_0$. Furthermore Proposition 5.1 and (22) give

$$\begin{array}{lcll} \Delta^k & \equiv & \big(1-k\omega^2u_1^2+ku_1^4-\binom{k}{2}u_1^4\big)\omega^{2k}u^{-12k} & \mod (u_1^6) \\ \omega_*(\Delta^k) & \equiv & \big(1+k\omega^2u_1^2+ku_1^4-\binom{k}{2}u_1^4\big)\omega^{2k-12k}u^{-12k} & \mod (u_1^6) \end{array}$$

and the result follows easily. (Note that by (24) the congruence for $d_1(\Delta^{2m+1})$ improves to a congruence modulo (u_1^8) rather than only modulo (u_1^6).) □

Proposition 5.3. *For each integer $k\neq 0$ there exists an element $\Delta_k\in E_1^{0,0,24k}$ such that*

a) $\Delta_k\equiv\Delta^k \mod (u_1^6)$
b) *the differential $d_1: E_1^{0,0}\to E_1^{1,0}$ satisfies*

$$d_1(\Delta_k)\equiv \begin{cases} (-1)^{m+1}\omega^2u^{-12k} & \mod (u_1^4) \qquad k=2m+1 \\ (-1)^{m+1}m\omega^2u_1^{4.3^n-2}u^{-12k} & \mod (u_1^{4.3^n+2}) \quad k=2.3^nm, \ m\not\equiv 0 \mod (3) \\ 0 & \qquad k=0 \ . \end{cases}$$

Proof. For $k=0$, k odd or $k=2m$ with $m\not\equiv 0 \mod (3)$ we define Δ_k to be equal to Δ^k. The formula for d_1 is then satisfied by the previous result.

If $k=2.3^nm$ with $n\geq 0$ and $m\not\equiv 0 \mod (3)$ we recursively define

$$\Delta_{3k} := \Delta_k^3 - mv_1^{3(4\cdot 3^n-2)}\Delta^{3k-2.3^n+1}. \tag{28}$$

The previous proposition gives

$$d_1(\Delta^{3k-2.3^n+1})\equiv(-1)^m\omega^2(1+u_1^4)u^{-12(3k-2.3^n+1)} \quad \mod (u_1^6) \ .$$

and by induction on n we have

$$d_1(\Delta_k)^3 \equiv \left((-1)^{m+1}m\omega^2u_1^{4.3^n-2}u^{-12k}\right)^3 \equiv (-1)^m m\omega^2u_1^{4.3^{n+1}-6}u^{-12\cdot 3k} \mod (u_1^{4.3^{n+1}+6}) .$$

Therefore v_1-linearity of d_1 yields

$$\begin{aligned} d_1(\Delta_{3k}) &= d_1(\Delta_k^3) - mv_1^{3(4.3^n-2)}d_1(\Delta_{3k-2.3^n+1}) \\ &\equiv (-1)^{m+1}\omega^2mu_1^{4.3^{n+1}-2}u^{12\cdot 3k} \qquad \mod (u_1^{4.3^{n+1}+2}) \end{aligned}$$

and the induction step is established. □

Corollary 5.4. *There is an isomorphism of $\mathbb{F}_3[v_1]$-modules $E_2^{0,0} \cong \mathbb{F}_3[v_1]$.* □

Remark By (25) the elements Δ_k form a topological basis of the continuous graded $\mathbb{F}_3[v_1]$-module $E_1^{0,0}$ (in fact, this has been implicitly used in the last corollary) and by (24) a topological basis of the continuous graded $\mathbb{F}_3[v_1]$-module $E_1^{1,0}$ can be given by any family of elements b_{2k+1}, $k \in \mathbb{Z}$, such that $b_{2k+1} \equiv \omega^2u^{-8k-4} \mod (u_1^4)$. By Proposition 5.3 we know that there are such elements $b_{2.(3m+1)+1}$ for $m \in \mathbb{Z}$ and $b_{2.3^n(3m-1)+1}$ for $n \geq 0$, $m \not\equiv 0 \mod (3)$, such that the first formula of Theorem 1.2 holds. Because the E_1-term is torsion free as a $\mathbb{F}_3[v_1]$-module and the d_1-differential is $\mathbb{F}_3[v_1]$-linear it is clear that those b_{2k+1}'s are in the kernel of the differential $d_1 : E_1^{1,0} \to E_1^{2,0}$. To complete this family to a topological basis we need to choose elements b_{2k+1} for $k = 3^{n+1}(3m+1)$ with $n \geq 0$, $m \in \mathbb{Z}$, for $k = 3^n(9m+8)$ with $n \geq 0$, $m \in \mathbb{Z}$, and for $k = 0$. Thus we are lead to concentrate on the differential on $\omega^2u^{-4(2k+1)}$ for such k. The crucial step is given by Proposition 5.6 below whose proof is quite elaborate and will be postponed to section 5.3. The proof of Proposition 5.5 will be given in section 6.

Proposition 5.5. *There exists an element $b_1 \in E_1^{1,0}$ such that $b_1 \equiv \omega^2u^{-4} \mod (u_1^4)$ and $v_1\alpha = b_1$ in $H^*(\mathbb{G}_2^1; (E_2)_*/(3))$. In particular, $d_1(b_1) = 0$.*

Proposition 5.6. *Let k be an integer such that $8k+4$ is not divisible by 3. Then the differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ satisfies*

$$\begin{aligned} d_1(\omega^2u^{8k+4}) &\equiv -(k'+k'^2)\omega^2u_1^{12}u^{8k+4} &\mod (u_1^{16}) \quad 8k+4 = 3k'+1 \\ d_1(\omega^2u^{8k+4}) &\equiv (k'-k'^2)\omega^2u_1^{8}u^{8k+4} &\mod (u_1^{12}) \quad 8k+4 = 3k'+2 . \end{aligned}$$

Proof. This will be proved in section 5.3. □

Proposition 5.7. *For each integer k there exists an element $b_{2k+1} \in E_1^{1,0,8(2k+1)}$ such that*

a) $b_{2k+1} \equiv \omega^2u^{-4(2k+1)} \mod (u_1^4)$

b) $d_1(\Delta_k) = \begin{cases} (-1)^{m+1}b_{2(3m+1)+1} & k = 2m+1, m \in \mathbb{Z} \\ (-1)^{m+1}mv_1^{4.3^n-2}b_{2.3^n(3m-1)+1} & k = 2m.3^n,\ m \not\equiv 0 \mod (3) \\ 0 & k = 0 \end{cases}$

c) *the differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ satisfies*

$$d_1(b_{2k+1}) \equiv \begin{cases} (-1)^n\omega^2u_1^{6.3^n+2}u^{-4(2k+1)} & \mod (u_1^{2.3^{n+1}+6}) \quad k = 3^{n+1}(3m+1), \quad m \in \mathbb{Z} \\ (-1)^n\omega^2u_1^{10.3^n+2}u^{-4(2k+1)} & \mod (u_1^{10.3^n+6}) \quad k = 3^n(9m+8), \quad m \in \mathbb{Z} \\ 0 & \text{otherwise} . \end{cases}$$

Proof. For $k = 3m+1$ with $m \in \mathbb{Z}$ we define b_{2k+1} to be $(-1)^{m+1}d_1(\Delta_{2m+1})$. For $k = 3^n(3m-1)$ with $n \geq 0$ and $m \not\equiv 0 \mod (3)$ we note that Proposition 5.3 shows that $d_1(\Delta_{2m.3^n})$ is divisible by $v_1^{4.3^n-2}$ and we define b_{2k+1} to be $(-1)^{m+1}mv_1^{-(4.3^n-2)}d_1(\Delta_{2m.3^n})$. For $k = 0$ we take the element given in Proposition 5.5. With these definitions (b) holds as well as the last case of (c).

For $k = 3^{n+1}(3m+1)$ resp. $k = 3^n(9m+8)$, with $n \geq 0$ and $m \in \mathbb{Z}$, we define elements b_{2k+1} by induction on n such that (a) and (c) are satisfied. In fact, for $n = 0$ we define

$$b_{2k+1} := \omega^2u^{-4(2k+1)}$$

and then Proposition 5.6 gives

$$d_1(b_{18m+7}) \equiv \omega^2u_1^8u^{-4(18m+7)} \mod (u_1^{12}), \quad d_1(b_{18m+17}) \equiv \omega^2u_1^{12}u^{-4(18m+17)} \mod (u_1^{12}) .$$

Now suppose that b_{2k+1} has already been defined for $k=3^{n+1}(3m+1)$ resp. $k=3^n(9m+8)$ with with $n\leq N$ and $m\in\mathbb{Z}$ so that (a) and (c) are satisfied. Then we observe that by Proposition 5.2 the elements

$$b_{2k+1}^3+b_{2(3k+1)+1}=b_{2k+1}^3+(-1)^{k+1}d_1(\Delta_{2k+1})\equiv\big(\omega^6+\omega^2(1+u_1^4)\big)u^{-4(6k+3)} \mod (u_1^8)$$

are divisible by v_1^4 and thus we can define

$$b_{6k+1}:=v_1^{-4}\big(b_{2k+1}^3+b_{2(3k+1)+1}\big)\ . \tag{29}$$

Then it is clear that $b_{6k+1}\equiv\omega^2u^{-4(6k+1)} \mod (u_1^4)$. Furthermore, $d_1d_1(\Delta_{2k+1})=0$ and because d_1 commutes with taking third powers and is $\mathbb{F}_3[v_1]$-linear we see that both (a) and (c) are satisfied for $k=3^{n+1}(3m+1)$ resp. $k=3^n(9m+8)$ with $n\leq N+1$ and $m\in\mathbb{Z}$ and thus the induction step is complete. □

Corollary 5.8. *There is an isomorphism of $\mathbb{F}_3[v_1]$-modules*

$$E_2^{1,0}\cong\prod_{n\geq0,m\in\mathbb{Z}\setminus3\mathbb{Z}}\mathbb{F}_3[v_1]/(v_1^{4.3^n-2})\{b_{2.3^n(3m-1)+1}\}\times\mathbb{F}_3[v_1]\{b_1\}\ .\quad\square$$

Proof. Because the elements Δ_k and b_k form a topological basis of the graded continuous $\mathbb{F}_3[v_1]$-modules $E_1^{0,0}$ and $E_1^{1,0}$ this follows immediately from Proposition 5.7. □

Remark By inspection one sees that the infinite product is finite in each bidegree and therefore it can also be identified with the direct sum.

To evaluate the homomorphism $d_1:E_1^{2,0}\to E_1^{3,0}$ we need the following result.

Lemma 5.9. *Let k be any integer. Then*

$$(e_*+a_*+(a^2)_*)(u^k)\quad\equiv\quad\big((k-k^2)\omega^2u_1^2+(k\tbinom{k}{3}+k-k^2)u_1^4\big)u^k\qquad\mod\ (3,u_1^5)\ .$$

Proof. By (19) we have $a_*(u^k)=u^kt_0(a)^k$ and $(a^2)_*(u^k)=u^kt_0(a)^ka_*(t_0(a)^k)$. Corollary 4.5 gives

$$\begin{aligned}
t_0(a)^k &\equiv \big(1+(1+\omega^2)u_1+(-1+\omega^2)u_1^3\big)^k\\
&\equiv 1+k(1+\omega^2)u_1+k(-1+\omega^2)u_1^3\\
&\quad -\tbinom{k}{2}\omega^2u_1^2-\tbinom{k}{2}(1+\omega^2)(-1+\omega^2)u_1^4+\tbinom{k}{3}(1+\omega^2)^3u_1^3-\tbinom{k}{4}u_1^4\\
&\equiv 1+k(1+\omega^2)u_1-\tbinom{k}{2}\omega^2u_1^2\\
&\quad +(\tbinom{k}{3}-k)(1-\omega^2)u_1^3-(\tbinom{k}{2}+\tbinom{k}{4})u_1^4 \qquad \mod (3,u_1^5)
\end{aligned}$$

and by Corollary 4.7 we get

$$\begin{aligned}
a_*(t_0(a)^k) &\equiv 1+k(1+\omega^2)\big(u_1-(1+\omega^2)u_1^2-\omega^2u_1^3+(1-\omega^2)u_1^4\big)\\
&\quad -\tbinom{k}{2}\omega^2\big(u_1^2+(1+\omega^2)u_1^3\big)\\
&\quad +(\tbinom{k}{3}-k)(1-\omega^2)u_1^3-(\tbinom{k}{2}+\tbinom{k}{4})u_1^4\\
&\equiv 1+k(1+\omega^2)u_1+(k-\tbinom{k}{2})\omega^2u_1^2\\
&\quad +(\tbinom{k}{3}+\tbinom{k}{2})(1-\omega^2)u_1^3-(k+\tbinom{k}{2}+\tbinom{k}{4})u_1^4 \qquad \mod (3,u_1^5)\ .
\end{aligned}$$

Finally an easy calculation (which only uses that $\tbinom{k}{2}\equiv-k(k-1)\mod(3)$ and $k^3\equiv k\mod(3)$) gives

$$\begin{aligned}
t_0(a)^ka_*(t_0(a)^k) &\equiv 1-k(1+\omega^2)u_1+(k^2-k)\omega^2u_1^2\\
&\quad +(-\tbinom{k}{3}+k)(1-\omega^2)u_1^3\\
&\quad +(\tbinom{k}{4}+\tbinom{k}{2}+k\tbinom{k}{3}+k-k^2)u_1^4 \quad \mod (3,u_1^5)
\end{aligned}$$

and the result clearly follows. □

Proposition 5.10. *For each integer k there exists an element $\overline{b}_{2k+1}\in E_1^{2,0,4(2k+1)}$ such that*

a) $\overline{b}_{2k+1}\equiv\omega^2u^{-4(2k+1)} \mod (u_1^4)$

b) $d_1(b_{2k+1})=\begin{cases}(-1)^nv_1^{6.3^n+2}\overline{b}_{3^{n+1}(6m+1)} & k=3^{n+1}(3m+1)\\ (-1)^nv_1^{10.3^n+2}\overline{b}_{3^n(18m+11)} & k=3^n(9m+8)\end{cases}$

c) *the differential* $d_1 : E_1^{2,0} \to E_1^{3,0}$ *satisfies*

$$d_1(\overline{b}_{2k+1}) \equiv \begin{cases} -u_1^2 u^{-4(2k+1)} & \bmod (u_1^4) \quad 2k+1 = 6m+1 \\ \omega^2 u_1^{4.3^n} u^{-4(2k+1)} & \bmod (u_1^{4.3^n+2}) \quad 2k+1 = 3^n(18m+17) \\ -\omega^2 u_1^{4.3^n} u^{-4(2k+1)} & \bmod (u_1^{4.3^n+2}) \quad 2k+1 = 3^n(18m+5) \\ 0 & \quad \textit{otherwise} . \end{cases}$$

Proof. For $2k+1 = 3^{n+1}(3m+1)$ resp. $2k+1 = 3^n(9m+8)$ Proposition 5.7 shows that $d_1(b_{2k+1})$ is divisible by $v_1^{6.3^n+2}$ resp. $v_1^{10.3^n+2}$ and can thus be written as $(-1)^n v_1^{6.3^n+2}\overline{b}_{3^{n+1}(6m+1)}$ resp. $(-1)^n v_1^{10.3^n+2}\overline{b}_{3^n(18m+11)}$ for unique elements $\overline{b}_{3^{n+1}(6m+1)}$ resp. $\overline{b}_{3^n(18m+11)}$ which satisfy (a) and (b).

So we still need to define $\overline{b}_{2k+1}$ if $2k+1$ can be written as $2k+1 = 6m+1$ with $m \in \mathbb{Z}$ or $2k+1 = 3^n(18m+11\pm 6)$ with $n \geq 0$ and $m \in \mathbb{Z}$. In those cases we define $\overline{b}_{2k+1} := \omega^2 u^{-4(2k+1)}$ and note that $-4(2k+1) \equiv 2 \bmod (3)$ if $2k+1 = 6m+1$ and that $-4(2k+1) \equiv 7 \bmod (9)$ if $2k+1 = 18m+5$ resp. $-4(2k+1) \equiv 4 \bmod (9)$ if $2k+1 = 18m+17$. Then (c) holds by Lemma 5.9 and Proposition 3.7.c, at least if we pretend that the differential is induced by the map $\partial_1^* : C_0^* \to C_1^*$ after identification of C_3 with C_0^* and of C_2 with C_1^* via the isomorphisms given by (14). In reality the differential is induced by ∂_1^* only up to the automorphisms of $E_1^{i,0}$, $i = 2, 3$, induced by the isomorphisms f_i of Proposition 3.7 and the isomorphisms of (14). However, by Proposition 3.7 these automorphisms induce the identity on $\mathrm{Tor}_0^{\mathbb{Z}_3[[S_2^1]]}(\mathbb{F}_3, C_i)$ for $i = 2, 3$. Then Corollary 4.5 shows that they induce automorphisms of $E_1^{i,0}$ as continuous graded $\mathbb{F}_3[v_1]$-modules which map $\omega^2 u^{-4(2k+1)}$ to itself modulo (u_1^4) respectively $\omega^{2k} u^{-12k}$ to itself modulo (u_1^2) and part (c) follows. □

Corollary 5.11. *There is an isomorphism of* $\mathbb{F}_3[v_1]$*-modules*

$$E_2^{2,0} \cong \prod_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} \mathbb{F}_3[v_1]/(v_1^{2.3^{n+1}+2})\{\overline{b}_{3^{n+1}(6m+1)}\} \times \prod_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} \mathbb{F}_3[v_1]/(v_1^{10.3^n+2})\{\overline{b}_{3^n(18m+11)}\} . \quad \square$$

<u>Remark</u> By inspection one sees again that the infinite product is finite in each bidegree and therefore it can also be identified with the direct sum.

Proposition 5.12. *For each integer k there exists an element $\overline{\Delta}_k \in E_1^{3,0}$ such that*

a) $\overline{\Delta}_k \equiv \Delta^k \bmod (u_1^2)$

b) *The differential* $d_1 : E_1^{2,0} \to E_1^{3,0}$ *is given by*

$$d_1(\overline{b}_{2k+1}) \equiv \begin{cases} (-1)^{m+1} v_1^2 \overline{\Delta}_{2m} & 2k+1 = 6m+1 \\ (-1)^{m+n} v_1^{4\cdot 3^n} \overline{\Delta}_{3^n(6m+5)} & 2k+1 = 3^n(18m+17) \\ (-1)^{m+n+1} v_1^{4\cdot 3^n} \overline{\Delta}_{3^n(6m+1)} & 2k+1 = 3^n(18m+5) \\ 0 & \textit{otherwise} . \end{cases}$$

Proof. Proposition 5.10 shows that $d_1(\overline{b}_{2k+1})$ is divisible by the appropriate power of v_1. The sign is then determined by comparing the coefficients of the "leading term" $u_1^2 u^{-4(2k+1)}$ resp. $\omega^2 u_1^{4.3^n} u^{-42k+1}$ in $d_1(\overline{b}_{2k+1})$ on one hand and in $v_1^2 \Delta^{2m}$ resp. $v_1^{4.3^n} \Delta^{3^n(6m+3\pm 2)}$ on the other hand. □

Corollary 5.13. *There is an isomorphism of* $\mathbb{F}_3[v_1]$*-modules*

$$E_2^{3,0} \cong \prod_{m \in \mathbb{Z}} \mathbb{F}_3[v_1]/(v_1^2)\{\overline{\Delta}_{2m}\} \times \prod_{n \geq 0, m \in \mathbb{Z}} \mathbb{F}_3[v_1]/(v_1^{4\cdot 3^n})\{\overline{\Delta}_{3^n(6m+1)}, \overline{\Delta}_{3^n(6m+5)}\} . \quad \square$$

<u>Remark</u> By inspection one sees once again that the infinite product is finite in each bidegree and it can therefore also be identified with the direct sum.

5.3. **The proof of Proposition 5.6.** By Corollary 3.5 the differential $E_1^{1,0} \to E_1^{2,0}$ is induced by the homomorphism $C_2 \to C_1,\ e_2 \mapsto n_1'$ where

$$n_1' := \frac{1}{16} \sum_{g \in SD_{16}} \chi(g^{-1}) g(n_1)\ ,$$

and by Lemma 3.6 we can take for n_1 any element of the form $n_1 = \theta e_1$ with

$$\theta := 3c(a-c)(a-b) + (e-c)^2(a-c)(a-b) - x(a-b) - y(a-c)(a-b) \tag{30}$$

and $x, y \in IK$ satisfying $e - c^3 = x(e-b) + y(e-c)$. The next result gives approximations for x and y which are sufficient for our homological calculations.

Lemma 5.14. *Let*

$$\begin{aligned} \widetilde{x} &= b^{-1}d^{-1}(e-d) - b^{-1}d^{-1}(e-b)b^{-1}c^{-1}(e-c) \\ \widetilde{y} &= b^{-1}d^{-1}(e-b)b^{-1}c^{-1}(e-b)\ . \end{aligned}$$

Then there exists $z \in IF_{\frac{5}{2}}S_2^1 + IK.IF_2S_2^1$ such that the following identity holds in IK

$$e - c^3 = \widetilde{x}(e-b) + \widetilde{y}(e-c) + z\ .$$

Proof. From Lemma 2.1 and Lemma 2.2 we deduce

$$c^3 \equiv [b^{-1}, d^{-1}] \mod F_{\frac{5}{2}}S_2^1.$$

Thus by using the elementary formulae

$$1 - [X,Y] = XY\big((1-Y^{-1})(1-X^{-1}) - (1-X^{-1})(1-Y^{-1})\big) \tag{31}$$

$$1 - XY = X(1-Y) + (1-X) \tag{32}$$

which hold in any associative algebra we obtain

$$e - c^3 \equiv b^{-1}d^{-1}\big((e-d)(e-b) - (e-b)(e-d)\big) \bmod IF_{\frac{5}{2}}S_2^1\ . \tag{33}$$

Using Lemma 2.1 and Lemma 2.2 again we get

$$d = [b,c] \equiv [b^{-1}, c^{-1}] \bmod F_2S_2^1$$

and hence we obtain from (31)

$$e - d \equiv b^{-1}c^{-1}\big((e-c)(e-b) - (e-b)(e-c)\big) \bmod IF_2S_2^1\ .$$

Substituting this into (33) gives the result. □

We will thus be interested in analyzing the action of

$$3c(a-c)(a-b) + (e-c)^2(a-c)(a-b) - \widetilde{x}(a-b) - \widetilde{y}(a-c)(a-b) \tag{34}$$

as well as in the influence of the "error term" z on the elements $\omega^2 u^{-4k+2}$. This analysis will be simplified by the following result in which I denotes the ideal IS_2^1.

Lemma 5.15. *Let $r \geq 1$ be an integer. Then we have the following inclusions of left ideals*

a) $IF_rS_2^1 \subset I^{3^r-1}(e-b) + I^{2.(3^{r-1}-1)}(e-c) + 3I \subset I^{2.3^{r-1}} + 3I$
b) $IF_{\frac{r}{2}}S_2^1 \subset I^{3^r-1}(e-b) + I^{3^r-2}(e-c) + 3I \subset I^{3^r} + 3I.$

Proof. We note that for every integer $r \geq 0$ we have an isomorphism

$$IF_{\frac{r}{2}}S_2^1 \cong \lim_{q>r} I(F_{\frac{r}{2}}S_2^1/F_{\frac{q}{2}}S_2^1)$$

and it will therefore be enough to show the corresponding statements for the corresponding ideals in the finite quotient groups $F_rS_2^1/F_qS_2^1$. Next we remark that for every finite p-group G the ideal IG is a free $\mathbb{Z}_p$-module with basis $e-g,\ g \in G - \{e\}$. Therefore it is enough to show that

$$e - g \in I^{3^r-1}(e-b) + I^{2.(3^{r-1}-1)}(e-c) + 3I \quad \text{for every } g \in F_rS_2^1/F_qS_2^1$$

resp.

$$e - g \in I^{3^r-1}(e-b) + I^{3^r-2}(e-c) + 3I \quad \text{for every } g \in F_{r+\frac{1}{2}}S_2^1/F_qS_2^1\ .$$

(By abuse of notation we do not distinguish between $g \in S_2^1$ and its image in the quotients $S_2^1/F_qS_2^1$.) Furthermore by (32) it is enough to show this for a system of multiplicative generators of $F_rS_2^1/F_qS_2^1$ resp. $F_{r+\frac{1}{2}}S_2^1/F_qS_2^1$. By Lemma 2.1 the element $c^{3^{q-1}}$ forms a basis of the one dimensional $\mathbb{F}_3$-vector space $F_qS_2^1/F_{q+\frac{1}{2}}S_2^1$, and $d^{3^{q-1}}$ and b^{3^q} form a basis of the two dimensional $\mathbb{F}_3$-vector space $F_{q+\frac{1}{2}}S_2^1/F_{q+1}S_2^1$ and therefore it is enough to consider those elements.

We have $c = [a, b]$ and $d = [b, c]$, and thus (31) shows first

$$e - c \in I^2$$

and then

$$e - d \in I^2(e - b) + I(e - c) \subset I^3 \ .$$

Furthermore $e - g^3 \equiv (e - g)^3$ mod (3) and hence, modulo (3), we obtain for any integer $r \geq 1$

$$\begin{array}{lcl} e - c^{3^{r-1}} & \equiv & (1-c)^{3^{r-1}-1}(e-c) \subset I^{2.(3^{r-1}-1)}(e-c) \\ e - b^{3^r} & \equiv & (e-b)^{3^r-1}(e-b) \subset I^{3^r-1}(e-b) \\ e - d^{3^{r-1}} & \equiv & (e-d)^{3^{r-1}-1}(e-d) \subset I^{3^r-3}(e-d) \subset I^{3^r-1}(e-b) + I^{3^r-2}(e-c) \end{array}$$

and (a) and (b) follow. □

Remark The previous lemma can in principle also be used to get better explicit approximations of the elements x and y of Lemma 5.14, at least modulo (3). For this one has to express the element $c^3[b^{-1}, d^{-1}]^{-1}$ in $F_{\frac{5}{2}}S_2^1/F_qS_2^1$ as explicit product of the elements $b^{3^{r-1}}$, $d^{3^{r-2}}$ and $c^{3^{r-1}}$ for $q \geq r \geq 3$ and then use (32) and the formulae in the proof of the previous lemma.

We will now give a qualitative description of the action of powers of I on $(E_2)_*/(3)$. The following lemma is an immediate consequence of Lemma 4.6 and of the formula $g_*(u_1^lu^k) = t_0(g)^{k+2l}u_1^lu^k$ (cf. (19) and Lemma 4.3.a).

Lemma 5.16.

a) *IS_2^1 sends the $\mathbb{F}_9[[u_1]]$-submodule of $\mathbb{F}_9[[u_1]]u^k$ generated by $u_1^lu^k$ to the submodule generated by $u_1^{l+1}u^k$.*
b) *If $k+2l \equiv 1$ mod (3) then IS_2^1 sends $u_1^lu^k$ to the additive subgroup of $u^k\mathbb{F}_9[[u_1]]$ generated by $u_1^{l+1}u^k$ and the ideal generated by $u_1^{l+3}u^k$.*
c) *If $k + 2l \equiv 0$ mod (3) then IS_2^1 sends the $\mathbb{F}_9[[u_1^3]]$-submodule generated by $u_1^lu^k$ to the $\mathbb{F}_9[[u_1^3]]$-submodule generated by $u_1^{l+3}u^k$.* □

Lemma 5.17.

a) *Let $k + 2l = 3m + 1$ and $r \geq 1$ be an integer. Then $(IS_2)^r$ sends $u_1^lu^k$ to an element of the form $(\alpha u_1^{l+3(r-1)+1} + \beta u_1^{l+3r})u^k$ modulo (u_1^{l+3r+1}) for suitable $\alpha, \beta \in \mathbb{F}_9$.*
b) *Let $k + 2l = 3m + 2$ and $r \geq 2$ be an integer. Then $(IS_2)^r$ sends $u_1^lu^k$ to an element of the form $(\gamma u_1^{l+3(r-2)+2} + \delta u_1^{l+3(r-2)+4})u^k$ modulo $(u_1^{l+3(r-2)+5})$ for suitable $\gamma, \delta \in \mathbb{F}_9$.*

Proof. a) This follows by an easy induction on r by using the previous lemma.

b) If $k+2l = 3m+2$, the previous lemmma shows that $(IS_2)^2$ sends $u_1^lu^k$ to $(\lambda u_1^{l+2} + \mu u_1^4)u^k$ modulo (u_1^{l+5}) for suitable $\lambda, \mu \in \mathbb{F}_9$. Now the result follows again by an easy induction on $r \geq 2$ by using once more the previous lemma. □

The following immediate corollary tells us that for the evaluation of the differential we should concentrate on the coefficients of u_1^8 and u_1^{10} in the case of $u^{3k'+2}$ resp. of u_1^{10} and u_1^{12} in the case of $u^{3k'+1}$. It also gives us more flexibility for approximating θ.

Corollary 5.18. *Let k' be an integer and $\vartheta \in (3, I^4)$. Then there exist $\alpha, \beta, \gamma, \delta \in \mathbb{F}_9$ which only depend on ϑ modulo $(3, I^5)$ such that we have the following congruences.*

a) $\vartheta_*(u^{3k'+1}) \equiv (\alpha u_1^{10} + \beta u_1^{12})u^{3k'+1} \quad \text{mod} \quad (3, u_1^{13})$ *for suitable $\alpha, \beta \in \mathbb{F}_9$.*

b) $\vartheta_*(u^{3k'+2}) \equiv (\gamma u_1^8 + \delta u_1^{10})u^{3k'+2} \quad \mod \ (3, u_1^{11})$ *for a suitable* $\gamma, \delta \in \mathbb{F}_9$. $\square$

The next lemma gives a simplified approximation to θ and hence to d_1.

Proposition 5.19. *Let*

$$x' = (e-d) - (e-b)(e-c) \text{ and } y' = (e-b)(e-b)$$

and define

$$\widetilde{\theta}' := \frac{1}{16} \sum_{g \in SD_{16}} \chi(g^{-1}) g\Big(3c(a-c)(a-b) + (e-c)^2(a-c)(a-b) - x'(a-b) - y'(a-c)(a-b)\Big) .$$

Then the differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ *satisfies*

$$\begin{array}{lclll} d_1(\omega^2 u^{-4(2k+1)}) & \equiv & (\widetilde{\theta}')_*(\omega^2 u^{-4(2k+1)}) & \mod (u_1^{16}) & -4(2k+1) = 3k'+1 \\ d_1(\omega^2 u^{-4(2k+1)}) & \equiv & (\widetilde{\theta}')_*(\omega^2 u^{-4(2k+1)}) & \mod (u_1^{12}) & -4(2k+1) = 3k'+2 \ . \end{array}$$

Proof. First we note that Lemma 5.15 shows that the element z of Lemma 5.14 belongs to $(I^8(e-b) + I^7(e-c)) + I.(I^8(e-b) + I^4.(e-c))$ and therefore does not contribute to the calculation of d_1 modulo the specified precision. Furthermore $\widetilde{\theta}'$ belongs to $(3, I^4)$. Now the last corollary shows that we have equality modulo (u_1^{13}) resp. (u_1^{11}) if we replace $\widetilde{x}$ and $\widetilde{y}$ from Lemma 5.14 by x' and y', and then the following lemma implies equality even modulo (u_1^{16}) resp. (u_1^{12}). $\square$

Lemma 5.20. *Let* k *and* $l \geq 0$ *be integers and* $\lambda \in \mathbb{F}_9$. *Then*

$$\frac{1}{16} \sum_{g \in SD_{16}} \chi(g^{-1}) g(\lambda u_1^l u^k) = \begin{cases} \frac{1}{2}(\lambda - \lambda^3) u_1^l u^k & k + 2l \equiv 4 \mod (8) \\ 0 & \textit{else} \ . \end{cases}$$

Proof. By (22) and (23) we have

$$\begin{array}{rcl} \sum_{g \in SD_{16}} \chi(g^{-1}) g(\lambda u_1^l u^k) & = & \sum_{j=0}^7 (-1)^j (\omega^j)_*(\lambda u_1^l u^k) + \sum_{j=0}^7 (-1)^{j+1} (\omega^j \phi)_*(\lambda u_1^l u^k) \\ & = & \big(\sum_{j=0}^7 (-1)^j \omega^{j(k+2l)}\big) \lambda u_1^l u^k - \big(\sum_{j=0}^7 (-1)^j \omega^{j(k+2l)}\big) \lambda^3 u_1^l u^k \ . \end{array}$$

Furthermore $\sum_{j=0}^7 (-1)^j \omega^{j(k+2l)} = 0$ unless $k + 2l \equiv 4 \mod (8)$ in which case it is equal to 8. The result follows. $\square$

The previous result tells us how to get the "leading term" in the differential once we know the coefficients α, β, γ and δ of Corollary 5.18 in the case of $\vartheta = \widetilde{\theta} := \sum_{i=1}^4 \widetilde{\theta}_i$ with

$$\begin{array}{l} \widetilde{\theta}_1 := 3c(a-c)(a-b) + (e-c)^2(a-c)(a-b) \\ \widetilde{\theta}_2 := -(e-d)(a-b) \\ \widetilde{\theta}_3 := (e-b)(e-c)(a-b) \\ \widetilde{\theta}_4 := -(e-b)(e-b)(a-c)(a-b) \ , \end{array}$$

and in fact α and δ will not even matter. The coefficients β and γ of Corollary 5.18 for the action of each $\widetilde{\theta}_i$ are given in the following result.

Lemma 5.21. *Let* k *be an integer not divisible by* 3. *For* $k = 3k'+1$ *there are elements* $\alpha_{i,k}, \beta_{i,k} \in \mathbb{F}_9$, $i = 1,2,3,4$, *and for* $k = 3k'+2$ *there are elements* $\gamma_{i,k}, \delta_{i,k} \in \mathbb{F}_9$, $i = 1,2,3,4$, *such that we have the following congruences*

$$(\widetilde{\theta}_i)_* u^k \equiv \begin{cases} (\alpha_{i,k} u_1^{10} + \beta_{i,k} u_1^{12}) u^k & \mod (3, u_1^{13}) \quad k = 3k'+1 \\ (\gamma_{i,k} u_1^8 + \delta_{i,k} u_1^{10}) u^k & \mod (3, u_1^{11}) \quad k = 3k'+2 \ . \end{cases}$$

Furthermore we have

$$\begin{array}{lclrlcr} \beta_{1,k} & = & 0 & \gamma_{1,k} & = & 0 \\ \beta_{2,k} & = & -(1+\omega^2) & \gamma_{2,k} & = & -(1+\omega^2) \\ \beta_{3,k} & = & (k'-1)\omega^2 & \gamma_{3,k} & = & (k'+1)\omega^2 \\ \beta_{4,k} & = & -\big(k'^2 + k' - 1 + (k'+1)\omega^2\big) & \gamma_{4,k} & = & 1 + k' - k'^2 - k'\omega^2 \end{array} .$$

Proof. The first part of the lemma is clear by Corollary 5.18. Furthermore, the case of $\beta_{i,k}$ and $\gamma_{i,k}$ follows immediately from Lemma 5.17 because by Lemma 5.15 we have $\widetilde{\theta}_1 \in I^6 + 3I$. The actual evaluation of the other $\beta_{i,k}$ and $\gamma_{i,k}$ uses Lemma 4.6 and is a lengthy but straightforward calculation whose verification we leave to the reader. Here we just note that Lemma 5.16 guarantees that for this calculation it is enough to know the action of I on $u_1^l u^k$ modulo (u_1^{l+6}). □

Proof of Proposition 5.6. This is an immediate consequence of the previous lemma, of Proposition 5.19 and Lemma 5.20. □

6. Higher differentials and extensions in the algebraic spectral sequence

In this section we will give a proof of Proposition 5.5 and we will determine the extensions and higher differentials in the algebraic spectral sequence (2) in the case of $M_* = (E_2)_*/(3)$. This spectral sequence allows for non-trivial d_2- and d_3-differentials. Their evaluation will be reduced to studying the long exact sequence in $\mathrm{Ext}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(-, M_*)$ associated to the short exact sequence (10). In fact, we have the following more general lemma.

Lemma 6.1. *Let R be a ring and $n > 0$ be an integer and let M be a left R-module. Suppose that*

$$0 \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} L \longrightarrow 0$$

is an exact complex of left R-modules such that C_i is projective for each $0 < i < n+1$.

a) *Then there is a first quadrant cohomological spectral sequence $E_r^{*,*}$, $r \geq 1$, converging to $Ext_R^*(L, M)$ with*

$$E_1^{s,t} = \mathrm{Ext}_R^t(C_s, M) \Rightarrow \mathrm{Ext}_R^{s+t}(L, M)$$

in which $E_1^{s,t} = 0$ for $0 < s < n+1$ and $t > 0$, and $E_1^{s,t} = 0$ for $t \geq 0$ and $s > n+1$.

b) *The higher differentials in this spectral sequence can be described as follows. Let N_0 be the kernel of ∂_0 and let $j : N_0 \to C_0$ denote the resulting inclusion. Then there are isomorphisms which are natural in M*

$$\mathrm{Ext}_R^t(N_0, M) \cong \begin{cases} \mathrm{Ker} : E_1^{1,0} \xrightarrow{d_1^{1,0}} E_1^{2,0} & t = 0 \\ E_2^{t+1,0} = E_{t+1}^{t+1,0} & 1 \leq t \leq n \\ E_{n+1}^{n+1,t-n} & t > n \end{cases}$$

such that the homomorphism $\mathrm{Ext}_R^t(C_0, M) \to \mathrm{Ext}_R^t(N_0, M)$ induced by j identifies with $d_{t+1}^{0,t} : E_t^{0,t} \to E_t^{t+1,0}$ if $1 \leq t \leq n$ and with $d_{n+1}^{0,t} : E_{n+1}^{0,t} \to E_{n+1}^{n+1,t-n}$ if $t > n$. (Note that by (a) these are the only potentially non-trivial differentials in this spectral sequence.)

Proof. a) The spectral sequence can be obtained as the spectral sequence of an exact couple. In fact, if N_i is the kernel of ∂_i then we have short exact sequences $0 \to N_i \to C_i \to N_{i-1} \to 0$ for $0 \leq i \leq n$ (with $N_{-1} := L$) and the long exact sequences in $\mathrm{Ext}_R^*(-, M)$ combine to give an exact couple from which the spectral sequence is derived. Projectivity of the modules C_i for $1 \leq i \leq n$ gives the vanishing results.

b) For the first statement we note that N_0 admits a projective resolution $Q_\bullet$ which is obtained from splicing the exact complex

$$0 \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} N_0 \longrightarrow 0$$

with a projective resolution of C_{n+1}. The second statement is easily seen by inspection of the higher differentials in an exact couple. We leave the details to the reader. □

Remark The higher differentials can therefore be evaluated if we know projective resolutions $Q_\bullet$ of N_0 and $P_\bullet$ of C_0 as well as a chain map $\phi : Q_\bullet \to P_\bullet$ covering j. These data can also be assembled in a double complex $T_{\bullet\bullet}$ with $T_{\bullet 0} = P_\bullet$, $T_{\bullet 1} = Q_\bullet$, vertical differentials δ_P and δ_Q and "horizontal differentials" $(-1)^n\phi_n : Q_n \to P_n$. The lemma implies that the filtration of the spectral sequence of this double complex agrees (up to reindexing) with that of the spectral sequence of the lemma. Hence extension problems in the spectral sequence of the lemma can also be studied by using the double complex.

We apply this lemma and the remark to the algebraic spectral sequence associated to the case of the exact complex (1). We will make use of explicit projective resolutions $Q_\bullet$ of N_0 and $P_\bullet$ of C_0 and a suitable chain map $\phi : Q_\bullet \to P_\bullet$ covering j. The essential step is given in the following elementary result whose proof is left to the reader.

Lemma 6.2. *Let $\overline{\chi}$ be the $\mathbb{Z}_3[Q_8]$-module whose underlying $\mathbb{Z}_3$-module is $\mathbb{Z}_3$ and on which t acts by multiplication by -1 and ψ by the identity. Then the trivial $\mathbb{Z}_3[G_{24}]$-module $\mathbb{Z}_3$ admits a projective resolution $P'_\bullet$ of period 4 of the following form*

$$\xrightarrow{a^2-a} 1\uparrow_{Q_8}^{G_{24}} \xrightarrow{e+a+a^2} 1\uparrow_{Q_8}^{G_{24}} \xrightarrow{a^2-a} \overline{\chi}\uparrow_{Q_8}^{G_{24}} \xrightarrow{e+a+a^2} \overline{\chi}\uparrow_{Q_8}^{G_{24}} \xrightarrow{a^2-a} 1\uparrow_{Q_8}^{G_{24}} \longrightarrow \mathbb{Z}_3 \ . \quad \square$$

In the sequel we work with the induced projective resolution $P_\bullet := \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[G_{24}]} P'_\bullet$ of $C_0 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[G_{24}]} \mathbb{Z}_3$ and the resolution $Q_\bullet$ of N_0 which is obtained from splicing the exact complex obtained from (1)

$$0 \to C_3 \to C_2 \to C_1 \to N_0 \to 0$$

with a projective resolution of $C_3 = C_0$, i.e. by splicing it with $P_\bullet$. The next result records all we need to know about the chain map ϕ.

Lemma 6.3. *There is a chain map $\phi : Q_\bullet \to P_\bullet$ covering the homomorphism j such that*

$$\phi_0 : Q_0 = C_1 \to 1\uparrow_{Q_8}^{\mathbb{G}_2^1} = P_0$$

sends e_1 to $(e-\omega)\widetilde{e}_0$. (Here the generator $\widetilde{e}_0 \in P_0$ is given by $e \otimes 1 \in P_0$ and we continue to denote the generator of C_1 introduced in section 3.2, and also given by $e \otimes 1$, by e_1.)

Proof. In fact, as in the proof of Corollary 3.3 we see that SD_{16} acts on $(e-\omega)\widetilde{e}_0$ via the character χ. Hence ϕ_0 is well defined and it is clear that ϕ_0 covers j. $\square$

Proof of Proposition 5.5. From its definition it is clear that $\alpha \in H^1(G_{24}, M_4)$ is a permanent cycle in the algebraic spectral sequence for $M_* = (E_2)_*/(3)$ (the restriction of the class with the same name in $H^1(\mathbb{G}_2^1, M_4)$), i.e. there are cochains $c \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_1, M_4)$ and $d \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_0, M_4)$ such that $c+d$ is a cocycle in the total complex of the double complex $\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(T_{\bullet\bullet}, M_4)$ and such that c represents $\alpha \in \mathrm{Ext}^1_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_0, M_4) = H^1(G_{24}, M_4)$. Furthermore, the cocycle c can be obtained as the mod-3 reduction of a cocycle representing $\delta^0(v_1)$ (cf. the discussion in section 5.1), i.e. we can take $c = \frac{1}{3}({a_*}^2 - a_*)v_1$ and this is known to be of the form ωu^{-2} mod (u_1) (cf. the proof of Lemma 1 in [4]). Then v_1c is a cocycle representing $v_1\alpha$. However, $v_1\alpha$ is trivial in $\mathrm{Ext}^1_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_0, M_4)$ by Theorem 1.1, and hence there exists $h \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_0, M_4)$ such that

$$v_1c = \delta_P(h) = a_*^2h - a_*h \ .$$

Because v_1c is equal to ωu_1u^{-4} mod (u_1^2), h must have the form ϵu^{-4} mod (u_1) for some unit $\epsilon \in \mathbb{F}_9$. Corollary 4.7 shows that

$$({a_*}^2 - a_*)(u^{-4}) = -(1+\omega^2)u_1u^{-4} \bmod (u_1)$$

and hence $\epsilon = \omega^2$ by (6). In the double complex $\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(T_{\bullet\bullet}, M_4)$ the cochain v_1c is therefore cohomologous to

$$-\phi_0^*(h) \equiv -(e-\omega)_*(\omega^2u^{-4}) \equiv -\omega^2u^{-4} + \omega^{-2}u^{-4} \equiv \omega^2u^{-4} \bmod (u_1)$$

and hence $v_1(c+d)$ is cohomologous to $\omega^2u^{-4} + v_1d$ mod (u_1) and this implies the proposition. $\square$

Now we turn towards the calculation of higher differentials and extensions.

Proposition 6.4.

a) *The differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ in the algebraic spectral sequence (2) for $M = (E_2)_*/(3)$ is given by*

$$d_2(\Delta_k\alpha) = \begin{cases} (-1)^{m+n+1}v_1^{6.3^n+1}\overline{b}_{3^{n+1}(6m+1)} & k = 2.3^n(3m+1) \\ (-1)^{m+n}v_1^{10.3^{n+1}+1}\overline{b}_{3^{n+1}(18m+11)} & k = 2.3^n(9m+8) \\ 0 & \textit{otherwise} \end{cases}$$

$$d_2(\Delta_k\widetilde{\alpha}) = \begin{cases} (-1)^m v_1^{11}\overline{b}_{18m+11} & k = 6m+5 \\ 0 & \textit{otherwise} . \end{cases}$$

b) *In $H^*(\mathbb{G}_2^1; (E_2)_*/(3))$ we have the following relations*

$$\begin{array}{lcl} v_1\Delta_{2.3^n(9m+2)}\alpha & = & (-1)^{m+1}b_{2.3^{n+1}(9m+2)+1} \\ v_1\Delta_{2.3^n(9m+5)}\alpha & = & (-1)^{m+1}b_{2.3^{n+1}(9m+5)+1} \\ v_1\Delta_{2k+1}\alpha & = & 0 \\ v_1\Delta_{6m+1}\widetilde{\alpha} & = & (-1)^m b_{2.(9m+2)+1} \\ v_1\Delta_{6m+3}\widetilde{\alpha} & = & (-1)^{m+1}b_{2.(9m+5)+1} \\ v_1\Delta_{2k}\widetilde{\alpha} & = & 0 . \end{array}$$

Remark We repeat that after the fact one can check that the d_2-differential is, up to sign, determined by v_1-linearity and the principle that it is non-trivial whenever it has a chance to be so.

In the proof of the proposition we make repeated use of the following lemma.

Lemma 6.5. *Let $s > 0$, $x \in H^s(G_{24}, M_*)$ and $c \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_s, M_*)$ be a representing cocycle. Suppose that $v_1^k x = 0 \in H^s(G_{24}, M_*)$ and suppose that $h \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_{s-1}, M_*)$ satisfies $\delta_P(h) = v_1^k c$.*

a) *Then there are elements $d, d' \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_{s-1}, M_*)$ and $d'' \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_s, M_*)$ such that $\phi_{s-1}^*(h) = d' + v_1^k d$ and $\delta_Q(d') = v_1^k d''$.*

b) *If d, d' and d'' are as in (a) then*

$$j^*(x) = (-1)^s[d''] \in \mathrm{Ext}^s(N_0, M_*) .$$

c) *If d, d' and d'' are as in (a) and $d'' = 0$ then*

$$v_1^k x' = (-1)^s[d'] \in H^*(\mathbb{G}_2^1, M_*)$$

for any x' in $H^*(\mathbb{G}_2^1, M_*)$ which restricts to x. In particular, if $d' = 0$, then $v_1^k x' = 0$.

Proof. We can write $\phi_{s-1}^*(h) \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_{s-1}, M_*) \subset M_*$ as $u^t f$ for some $t \in \mathbb{Z}$ and f a power series in u_1. Then we write $f = f_0 + u_1^k f_1$ with f_0 a polynomial in u_1 of degree less than k and f_1 a power series in u_1. If we put $d' = u^t f_0$ and $d = u^{2k+t} f_1$ then the first part of (a) holds. Next we use that the double complex $\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(T_{\bullet\bullet}, M_*)$ is a double complex of torsionfree $\mathbb{F}_3[[v_1]]$-modules and $\mathbb{F}_3[[v_1]]$-linear differentials. Then we get

$$\delta_Q(d') + v_1^k\delta_Q(d) = \delta_Q(d' + v_1^k d) = \delta_Q(\phi_{s-1}^*(h)) = \phi_s^*(\delta_P(h)) = \phi_s^*(v_1^k c) \equiv 0 \bmod (v_1^k)$$

and

$$\begin{array}{rcl} v_1^k j^*(c) & = & v_1^k(-1)^s\phi_s^*(c) = (-1)^s\phi_s^*(v_1^k c) = (-1)^s\phi_s^*(\delta_P(h)) \\ & = & (-1)^s\delta_Q\phi_{s-1}^*(h) = (-1)^s\delta_Q(d' + v_1^k d) = (-1)^s v_1^k d'' + (-1)^s v_1^k\delta_Q(d) . \end{array}$$

and thus the second part of (a) and (b) follow.

If $d'' = 0$ then the equations

$$v_1^k c = \delta_P(h), \quad (-1)^{s-1}v_1^k d + (-1)^{s-1}d' = (-1)^{s-1}\phi_{s-1}^*(h)$$

show that $v_1^k(c + (-1)^{s-1}d)$ and $(-1)^s d'$ are cohomologous in the double complex. Furthermore, d' is a cycle by assumption and hence $v_1^k(c + (-1)^{s-1}d)$ is a cycle. Because the double complex is v_1-torsion free, $c + (-1)^{s-1}d$ is a cycle as well and (c) follows. □

Proof of Proposition 6.4. We begin with the case of $\Delta_k\alpha$. As in the proof of Proposition 5.5 let $c \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_0, M_4)$ be a cocycle representing $\alpha \in \mathrm{Ext}^1_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(C_0, M_4)$. Then $\Delta_k\alpha$ is represented by the cocycle $\Delta_k c$. As Δ_k is G_{24}-invariant we have

$$\Delta_k v_1 c = \delta_P(\Delta_k h)$$

where h is as in the proof of Proposition 5.5 above. In particular $h \equiv \omega^2 u^{-4}$ and hence

$$\phi_0^*(\Delta_k h) \equiv (e-\omega)_*(\omega^{2k+2}u^{-12k-4}) \equiv (1-(-1)^{-3k-1})\omega^{2k+2}u^{-12k-4} \bmod (u_1^4)\ .$$

(We note that the congruence is modulo (u_1^4) by (24)!) In particular, if k is odd we see that

$$\phi_0^*(\Delta_k h) = v_1^4 z$$

for some $z \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_0, M_*)$ and if k is even, $k = 2.3^n(3m \pm 1)$, we get

$$\phi_0^*(\Delta_k h) = (-1)^m b_{3k+1} + v_1^4 z$$

for some $z \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_0, M_*)$. Then Lemma 6.5 and Theorem 1.2 give easily the differentials and extensions on all elements $\Delta_k\alpha$ for $k \neq 0$.

In the case of $\Delta_k\widetilde{\alpha}$ the definition of $\widetilde{\alpha}$ (cf. section 5.1) shows that $\widetilde{\alpha}$ can be represented by a cocycle $\widetilde{c} \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_1, M_{12})$ such that $v_1\widetilde{c} = \delta_P(v_2)$. Because Δ_k is G_{24}-invariant we have $\delta_P(\Delta_k v_2) = \Delta_k \delta_P(v_2)$. Furthermore,

$$\Delta_k v_2 \equiv \omega^{2k}u^{-12k-8} \bmod (u_1^2)$$

and thus

$$\phi_0^*(\Delta_k v_2) \equiv (e-\omega)_*(\omega^{2k}u^{-12k-8}) \equiv (1-(-1)^{-3k-2})\omega^{2k}u^{-12k-8} \bmod (u_1^4)\ .$$

(Again we note that the congruence is modulo (u_1^4) by (24)!) In particular, if k is even we deduce that

$$\phi_0^*(\Delta_k v_2) = v_1^4 z$$

for some $z \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_0, M_*)$ and for $k = 6m + j$ with $j \in \{1, 3, 5\}$ we have

$$\phi_0^*(\Delta_k v_2) = (-1)^{m+1}\omega^{2j-2} b_{18m+3j+2} + v_1^4 z$$

for some $z \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_0, M_*)$ and as before Lemma 6.5 and Theorem 1.2 give easily the differentials and extensions on all elements $\Delta_k\widetilde{\alpha}$. □

The following result together with Proposition 5.5 and Proposition 6.4 finishes off the proof of Proposition 1.3 and Proposition 1.5.

Proposition 6.6.

a) *For $i \geq 3$ the differentials d_i in the algebraic spectral sequence (2) for the mod-3 Moore spectrum are all trivial.*
b) *For each integer $k \in \mathbb{Z}$ and $l > 0$ we have $v_1^2\Delta_k\beta^l = v_1\Delta_k\beta^l\alpha = v_1\Delta_k\beta^l\widetilde{\alpha} = 0$ in $H^*(\mathbb{G}_2^1; (E_2)_*/(3))$.*

Proof. The differential is linear with respect to the natural $\mathrm{Ext}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(\mathbb{Z}_3, \mathbb{F}_3) = H^*(\mathbb{G}_2^1, \mathbb{F}_3)$-module structure on its target and source. Hence it is enough show that the classes $\Delta_k\beta\alpha$, $\Delta_k\beta\widetilde{\alpha}$ and $\Delta_k\beta$ are d_3-cycles and to prove (b) in the case $l = 1$. In both cases we use Lemma 6.5 once again.

We begin with the case of $\Delta^k\beta$. Let c_1 be a cocycle in $\mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_2, M_{12})$ representing β. As $v_1^2\beta = 0$ in $H^2(G_{24}, M_{20})$ there exists $h_1 \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_1, M_{20})$ such that

$$v_1^2 c_1 = \delta_P(h_1) = (e + a + a^2)_* h_1\ .$$

Next we use that $Q_1 = C_2 = \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \chi$. Hence we have (cf. (24))

$$\mathrm{Hom}_{\mathbb{G}_2^1}(Q_1, M_*) \cong \omega^2 u^4 \mathbb{F}_3[[u_1^4]][v_1, u^{\pm 8}]$$

and by degree reasons we need to have $\phi_1^*(h_1) = v_1^3 z$ for some $z \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_1, M_8)$, and then Lemma 6.5 shows that β is not only the restriction of a permanent cycle (which we knew anyway), but also that $v_1^2\beta = 0$ in $H^2(\mathbb{G}_2^1, M_{20})$. Furthermore, as Δ_k is G_{24}-invariant we have

$\delta_P(\Delta_k h_1) = \Delta_k v_1^2 c_1$ and in order to apply Lemma 6.5 in the case of $\Delta_k\beta$ we need to understand $\phi_1^*(\Delta_k h_1)$. For this we recall that $Q_1 = C_2$ is a free $\mathbb{Z}_3[[S_2^1]]$-module generated by e_2. Hence we can write $\phi_1(\widetilde{e_1}) = xe_2$ for (a unique) $x \in \mathbb{Z}_3[[S_2^1]]$ and thus $\phi_1^*(\Delta_k h_1) = x_*(\Delta_k h_1)$. Furthermore, because $\Delta \equiv \omega^2 u^{-12} \mod (u_1^2)$ we have for any $g \in S_2^1$

$$g_*(\Delta_k h_1) = g_*(\Delta_k) g_*(h_1) \equiv t_0(g)^{12k} \Delta_k g_*(h_1) \equiv \Delta_k g_*(h_1) \mod (u_1^2) \ ,$$

thus

$$\phi_1^*(\Delta_k h_1) = x_*(\Delta_k h_1) \equiv \Delta_k x_*(h_1) = \Delta_k \phi_1^*(h_1) \equiv 0 \mod (u_1^2)$$

and then Lemma 6.5 shows that $\Delta_k\beta$ is a permanent cycle and in $H^2(\mathbb{G}_2^1; M)$ we have $v_1^2\Delta_k\beta = 0$.

In the case of $\Delta_k\beta\alpha$ we choose a representing cocycle $c_2 \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_3, M_{16})$ for $\beta\alpha$. Because $v_1\beta\alpha = 0$ in $H^1(G_{24}, M_{20})$ there exists $h_2 \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_2, M_{20})$ such that $v_1 c_2 = \delta_P(h_2)$. Then

$$\phi_2^*(h_2) \in \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(Q_2, M_{20}) = \mathrm{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}(P_0, M_{20}) \cong (M_{20})^{Q_8}$$

is divisible by v_1 by degree reasons (cf. (22) or Remark 3.12.3 of [5]) and then Lemma 6.5 shows that $\Delta_k\beta\alpha$ is a permanent cycle and in $H^2(\mathbb{G}_2^1, M_*)$ we have $v_1\Delta_k\beta\alpha = 0$. The case of $v_1\beta\widetilde{\alpha}$ is completely analogous. $\square$

Proposition 6.7. *If $x \in H^1(\mathbb{G}_2^1, M_*)$ is represented in $E_\infty^{1,0,*}$ then $\beta x = 0$ in $H^3(\mathbb{G}_2^1, M_*)$.*

Proof. It is enough to show $\beta b_1 = 0$ and $\beta b_{2.3^n(3m-1)+1} = 0$ whenever $m \not\equiv 0 \mod (3)$. By Proposition 5.5 we have $b_1 = v_1\alpha$ and thus $\beta b_1 = v_1\beta\alpha = 0$ by the previous proposition. Similarly, by Proposition 6.4 we have $b_{2.3^{n+1}(3m-1)+1} = \pm v_1\Delta_{2.3^n(3m-1)}\alpha$ and $b_{2(3m-1)+1} = \pm v_1\Delta_{2m-1}\widetilde{\alpha}$ and using the previous proposition once more shows $\beta b_{2.3^n(3m-1)+1} = 0$. $\square$

The following result finishes off the proof of Proposition 1.5.

Proposition 6.8. *The following relations hold in $H^*(\mathbb{G}_2^1; M_*)$*

$$\begin{array}{lcl} \beta\overline{b}_{3^{n+1}(6m+1)} & = & \pm\overline{\Delta}_{3^n(6m+1)}\widetilde{\alpha} \\ \beta\overline{b}_{3^{n+1}(18m+11)} & = & \pm\overline{\Delta}_{3^n(18m+11)}\widetilde{\alpha} \\ \beta\overline{b}_{18m+11} & = & \pm\overline{\Delta}_{6m+4}\alpha \ . \end{array}$$

Proof. First we observe that $\widetilde{\alpha} \in H^*(G_{24}, (E_2)_*/(3))$ is non-divisible by v_1 and this implies that the mod-u_1 reduction homomorphism $H^*(G_{24}, (E_2)_*/(3)) \to H^*(G_{24}, (E_2)_*/(3, u_1))$ must send $\widetilde{\alpha}$ to $\pm\omega^2 u^{-4}\alpha$ (cf. Theorem A.3). Likewise, this map must send Δ_{2k+1} to $\pm\omega^2 v_2^{3k+1} u^{-4}$, and it clearly sends α to α. Thus the proposition follows from Theorem A.3.c and naturality. $\square$

7. Passing from $\mathbb{G}_2^1$ to $\mathbb{G}_2$

Theorem 1.7 is a simple instance of a Künneth isomorphism: in fact, if we have an isomorphism of profinite 3-groups $G = F \times \mathbb{Z}_3$, and if $\mathbb{Z}_3$ acts trivially on a 3-profinite module M, then the exterior product in cohomology induces an isomorphism

$$H^*(F, M) \otimes_{\mathbb{Z}_3} \Lambda_{\mathbb{Z}_3}(\zeta) \cong H^*(F, M) \otimes_{\mathbb{Z}_3} H^*(\mathbb{Z}_3; \mathbb{Z}_3) \to H^*(\mathbb{G}_2, M) \ . \tag{35}$$

In particular this holds if $G = \mathbb{G}_2$, $F = \mathbb{G}_2^1$ and $M = (E_2)_*/(3)$ or $M = (E_2)_*/(3, u_1)$. We will need to know how the Bockstein homomorphisms δ^1 and δ^0 associated to the exact sequences (26) and (27) behave with respect to these isomorphisms.

The proof of the following lemma is a straightforward exercise with the double complex obtained from tensoring a projective resolution of the trivial $\mathbb{Z}_3[[F]]$-module $\mathbb{Z}_3$ with the projective resolution of the trivial $\mathbb{Z}_3[[\mathbb{Z}_3]]$-module $\mathbb{Z}_3$ given by

$$0 \to \mathbb{Z}_3[[T]] \xrightarrow{T} \mathbb{Z}_3[[T]] \to \mathbb{Z}_3 \to 0$$

and is left to the reader. (Here we have identified $\mathbb{Z}_3[[T]]$ with $\mathbb{Z}_3[[\mathbb{Z}_3]]$ via the continuous isomorphism which sends T to $t - e$ if t is a topological generator of $\mathbb{Z}_3$.)

Lemma 7.1. *Let $G = F \times \mathbb{Z}_3$ and H be a closed subgroup of G, $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of continuous G-modules and δ_H be the associated Bockstein in $H^*(H, -)$. If $\mathbb{Z}_3$ acts trivially on M_1 and M_3, and if we identify $H^*(G, M_i)$ with $H^*(F, M_i) \oplus H^{*-1}(F, M_i)\zeta$ for $i = 1, 3$ via (35), M_3 with $H^0(\mathbb{Z}_3, M_3)$ and M_1 with $H^1(\mathbb{Z}_3, M_1)$, then*

$$\begin{array}{lclclll} \delta_G(x) & = & \delta_F(x) & + & (-1)^n H^n(F, \delta_{\mathbb{Z}_3})(x)\zeta & & x \in H^n(F, M_3) \\ \delta_G(y\zeta) & = & & & \delta_F(y)\zeta & & y \in H^{n-1}(F, M_3) \ . \quad \square \end{array}$$

Corollary 7.2. *Let $M = (E_2)_{4k}/(3)$ resp. $M = (E_2)_{4k}/(3, u_1)$ and identify $H^*(\mathbb{G}_2, M)$ with $H^*(\mathbb{G}_2^1, M) \oplus H^{*-1}(\mathbb{G}_2^1, M)\zeta$ via (35).*

a) *For $x \in H^n(\mathbb{G}_2^1, (E_2)_{4k}/(3, u_1))$ we have $\delta^1_{\mathbb{G}_2}(x) = \delta^1_{\mathbb{G}_2^1}(x)$.*
b) *For $x \in H^n(\mathbb{G}_2^1, (E_2)_{4k}/(3))$ we have $\delta^0_{\mathbb{G}_2}(x) = \delta^0_{\mathbb{G}_2^1}(x) + (-1)^n kx\zeta$.*

In particular, if we define

$$\begin{array}{lll} \alpha(\mathbb{G}_2) := \delta^0_{\mathbb{G}_2}(v_1) & \widetilde{\alpha}(\mathbb{G}_2) := \delta^0_{\mathbb{G}_2}(v_2) & \beta(\mathbb{G}_2) := \delta^0_{\mathbb{G}_2}\delta^1_{\mathbb{G}_2}(v_2) \\ \alpha(\mathbb{G}_2^1) := \delta^0_{\mathbb{G}_2^1}(v_1) & \widetilde{\alpha}(\mathbb{G}_2^1) := \delta^0_{\mathbb{G}_2^1}(v_2) & \beta(\mathbb{G}_2^1) := \delta^0_{\mathbb{G}_2^1}\delta^1_{\mathbb{G}_2^1}(v_2) \end{array}$$

then

$$\alpha(\mathbb{G}_2) = \alpha(\mathbb{G}_2^1) - v_1\zeta, \quad \widetilde{\alpha}(\mathbb{G}_2) = \widetilde{\alpha}(\mathbb{G}_2^1), \quad \beta(\mathbb{G}_2) = \beta(\mathbb{G}_2^1) \ .$$

Proof. The central factor $\mathbb{Z}_3$ of $\mathbb{G}_2$ is generated by the element $t := 1 + 3 \in \mathbb{Z}_3^\times$. In this case t acts trivially on $(E_2)_0$ and on u via $t_*(u) = 4u$. Therefore t acts trivially on M and via multiplication by $(1+3)^{-2k} = 1 - 6k$ on $(E_2)_{4k}/(9)$. Hence $\delta_{\mathbb{Z}_3}$ is given by multiplication by $-2k \equiv k \bmod (3)$ and the result follows. $\square$

8. The Adams-Novikov spectral sequence for $L_{K(2)}V(0)$

As before we use the E_1-term of the algebraic spectral sequence (2) for $M = (E_2)_*V(0)$ to represent elements in the E_2-term of the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(0))$. However, unlike in the introduction, we do not always insist on writing elements in terms of the $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$-module generators of Theorem 1.4. This allows for simplified statements of Lemma 8.1, Corollary 8.2, Lemma 8.4 and Corollary 8.5 below.

The E_2-term satisfies $E_2^{s,t} = 0$ unless $t \equiv 0 \mod (4)$, hence its differentials d_r are trivial if $r \not\equiv 1 \bmod (4)$. Furthermore, the differentials are linear with respect to $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$, and the existence of the resolution (4) of [5] gives further restrictions on the behaviour of the Adams-Novikov differentials. In fact, they have to preserve the filtration on its E_2-term given by the algebraic resolution for $\mathbb{G}_2$, and modulo this filtration the differentials are easily determined by the differentials in the Adams-Novikov spectral sequence for $E_2^{hG_{24}} \wedge V(0)$ (cf. Theorem A.1). However, to settle the ambiguities coming from potential contributions of smaller filtration terms we need to fall back on knowledge of the differentials in $\pi_*(L_{K(2)}V(1))$ (cf. Theorem A.4).

The following Lemma records some immediate consequences of the knowledge of the d_5-differential in the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(1))$.

Lemma 8.1. *The following identities hold in $H^*(\mathbb{G}_2, (E_2)_*/(3)) \cong E_2^{*,*} \cong E_5^{*,*}$ of the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(0))$.*

a) *Let $k \not\equiv 0 \mod (3)$. Then there are constants $\epsilon_k \in \{\pm 1\}$ such that*

$$\begin{array}{lcl} d_5(\Delta_k\widetilde{\alpha}\beta) & = & \epsilon_k\Delta_{k-1}\beta^4 v_1 \\ d_5(\Delta_k\beta^2) & = & \epsilon_k\Delta_{k-1}\alpha(\mathbb{G}_2)\beta^4 \\ d_5(\Sigma^{48}\overline{\Delta}_k\widetilde{\alpha}) & = & \epsilon_k\Sigma^{48}\overline{\Delta}_{k-1}\beta^3 v_1 \\ d_5(\Sigma^{48}\overline{\Delta}_k\beta) & = & \epsilon_k\Sigma^{48}\overline{\Delta}_{k-1}\alpha(\mathbb{G}_2)\beta^3 \ . \end{array}$$

b) *Let $k \equiv 0 \mod (3)$. Then*

$$\begin{array}{lcl} d_5(\Delta_k\widetilde{\alpha}\beta) & = & 0 \\ d_5(\Delta_k\beta^2) & = & 0 \\ d_5(\Sigma^{48}\overline{\Delta}_k\widetilde{\alpha}) & = & 0 \\ d_5(\Sigma^{48}\overline{\Delta}_k\beta^2) & = & 0\ . \end{array}$$

<u>Remark</u> By identifying vector space generators in the appropriate bidegrees it is easy to see that there are unique elements $\lambda_i, \mu_i, \nu_i \in \mathbb{F}_3$ such that

$$\begin{array}{lcl} d_5(\Delta_k\widetilde{\alpha}\beta) & = & \lambda_1\Delta_{k-1}\beta^4 v_1 + \mu_1\Sigma^{48}\overline{\Delta}_{k-2}\alpha(\mathbb{G}_2^1)\beta^2 + \nu_1\Sigma^{48}\overline{\Delta}_{k-2}\beta^2 v_1\zeta \\ d_5(\Delta_k\beta^2) & = & \lambda_2\Delta_{k-1}\alpha(\mathbb{G}_2^1)\beta^4 + \mu_2\Delta_{k-1}\beta^4 v_1\zeta + \nu_2\Sigma^{48}\overline{\Delta}_{k-2}\alpha(\mathbb{G}_2^1)\beta^2\zeta \\ d_5(\Sigma^{48}\overline{\Delta}_k\widetilde{\alpha}) & = & \lambda_3\Sigma^{48}\overline{\Delta}_{k-1}\beta^3 v_1 \\ d_5(\Sigma^{48}\overline{\Delta}_k\beta) & = & \lambda_4\Sigma^{48}\overline{\Delta}_{k-1}\alpha(\mathbb{G}_2^1)\beta^3 + \mu_4\Sigma^{48}\overline{\Delta}_{k-1}\beta^3 v_1\zeta\ . \end{array}$$

Naturality and the geometric boundary theorem (cf. Theorem 2.3.4 of [10]) applied to the resolution (4) allow to determine the values of the λ_i, i.e. to show the lemma modulo elements of lower filtration. The Lemma confirms these values and also determines μ_i and ν_i; formally μ_i and ν_i can be deduced by simply replacing in the differentials for $E_2^{hG_{24}} \wedge V(0)$ the elements α by $\alpha(\mathbb{G}_2)$.

Proof. We start with $\Delta_k\widetilde{\alpha}\beta$. This is in the kernel of v_1-multiplication and must therefore (after 4-fold suspension) be in the image of the Bockstein homomorphism $\delta^1_{\mathbb{G}_2}$ in $H^*(\mathbb{G}_2, -)$ and $\delta^1_{\mathbb{G}_2^1}$ in $H^*(\mathbb{G}_2^1, -)$ associated to the short exact sequence (27). Similarly with $\Delta_k\beta v_1$. By Theorem A.3 and by degree reasons we must therefore have

$$\delta^1_{\mathbb{G}_2^1}((\omega^2u^{-4})^{3k+2}\beta) = \pm\Sigma^4\Delta_k\widetilde{\alpha}\beta, \quad \delta^1_{\mathbb{G}_2^1}((\omega^2u^{-4})^{3k+2}\alpha(\mathbb{G}_2^1)) = \pm\Sigma^4\Delta_k\beta v_1$$

and, by Corollary 7.2, we even get

$$\delta^1_{\mathbb{G}_2}((\omega^2u^{-4})^{3k+2}\beta) = \pm\Sigma^4\Delta_k\widetilde{\alpha}\beta, \quad \delta^1_{\mathbb{G}_2}((\omega^2u^{-4})^{3k+2}\alpha(\mathbb{G}_2^1)) = \pm\Sigma^4\Delta_k\beta v_1\ . \tag{36}$$

and by β-linearity $\delta^1((\omega^2u^{-4})^{3k+2}\alpha\beta^3) = \pm\Sigma^4\Delta_k\beta^4 v_1$. Then the geometric boundary theorem and Theorem A.4.b show that $\Delta_{6k}\widetilde{\alpha}\beta$ and $\Delta_{6k+3}\widetilde{\alpha}\beta$ are permanent cycles and the value of the differential in the other cases is as stated (with a suitable constant ϵ_k).

The case of $\Sigma^{48}\overline{\Delta}_k\widetilde{\alpha}\beta$ can be treated similarly but in this case it would also suffice to use the strategy described in the remark above. The sign is clearly the same as in the previous case.

The remaining two cases are deduced from what has already been established by using the Bockstein $\delta^0_{\mathbb{G}_2}$ in $H^*(\mathbb{G}_2, -)$ associated to the short exact sequence (26) and the geometric boundary theorem. By Corollary 7.2 we have

$$\begin{array}{ll} \delta^0_{\mathbb{G}_2}(\Delta_k\widetilde{\alpha}\beta) = \Delta_k\beta^2, & \delta^0_{\mathbb{G}_2}(\Sigma^{48}\overline{\Delta}_k\widetilde{\alpha}) = \Sigma^{48}\overline{\Delta}_k\beta \\ \delta^0_{\mathbb{G}_2}(\Delta_k\beta^4 v_1) = \Delta_k\alpha(\mathbb{G}_2)\beta^4, & \delta^0_{\mathbb{G}_2}(\Sigma^{48}\overline{\Delta}_k\beta^3 v_1) = \Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2)\beta^3\ . \end{array}$$

In fact, by the Corollary we only need to determine $\delta^0_{\mathbb{G}_2^1}$ and this is straightforward in the case of $\Sigma^{48}\overline{\Delta}_k\beta^3 v_1$ and $\Sigma^{48}\overline{\Delta}_k\widetilde{\alpha}\beta^3$. In the other two cases it is straightforward modulo terms of lower filtration and by degree reasons there are no error terms of lower filtration. □

Corollary 8.2. *The d_5-differential in the Adams-Novikov spectral sequence for $L_{K(2)}V(0)$ is linear with respect to $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$ and is trivial on all $\mathbb{F}_3[\beta, v_1] \otimes \Lambda(\zeta)$-module generators of $H^*(\mathbb{G}_2^1, (E_2)_*/(3))$ of Theorem 1.6 except the following:*

$$
\begin{array}{lcll}
d_5(\Delta_m\beta) & = & \pm\Delta_{m-1}\alpha(\mathbb{G}_2)\beta^3 & m\not\equiv 0 \mod (3)\\
d_5(\Delta_{2m}\widetilde{\alpha}) & = & \pm\Delta_{2m-1}\beta^3 v_1 & m\not\equiv 0 \mod (3)\\
d_5(\Delta_{6m+1}\widetilde{\alpha}) & = & \pm\Delta_{6m}\beta^3 v_1 & \\
d_5(\Delta_{6m+5}\widetilde{\alpha}\beta) & = & \pm\Delta_{6m+4}\beta^4 v_1 & \\
 & & & \\
d_5(\overline{b}_{3^{n+1}(6m+1)}) & = & \pm\Sigma^{48}\overline{\Delta}_{3^n(6m+1)-3}\beta^2 v_1 & n\geq 0\\
d_5(\overline{b}_{3^n(6m+5)}) & = & \pm\Sigma^{48}\overline{\Delta}_{3^{n-1}(6m+5)-3}\beta^2 v_1 & m\equiv 1 \mod (3),\ n\geq 2\\
 & & & \\
d_5(\Sigma^{48}\overline{\Delta}_{2m}) & = & \pm\Sigma^{48}\overline{\Delta}_{2m-1}\alpha(\mathbb{G}_2)\beta^2 & m\not\equiv 0 \mod (3)\\
d_5(\Sigma^{48}\overline{\Delta}_{3^n(6m+1)-2}) & = & \pm\Sigma^{48}\overline{\Delta}_{3^n(6m+1)-3}\alpha(\mathbb{G}_2)\beta^2 & n\geq 0\\
d_5(\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-2}) & = & \pm\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}\alpha(\mathbb{G}_2)\beta^2 & n\geq 1\\
d_5(\Sigma^{48}\overline{\Delta}_{2m}\widetilde{\alpha}) & = & \pm\Sigma^{48}\overline{\Delta}_{2m-1}\beta^3 v_1 & m\not\equiv 0 \mod (3)\\
d_5(\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-2}\widetilde{\alpha}) & = & \pm\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}\beta^3 v_1, & m\not\equiv 1 \mod (3),\ n\geq 1\ .
\end{array}
$$

Proof. Linearity with respect to $\mathbb{F}_3[\beta, v_1]\otimes\Lambda(\zeta)$ is clear. The rest is an immediate consequence of the previous lemma together with sparseness, Proposition 1.5, Theorem 1.6, and the fact that β-multiplication is injective above cohomological degree 3. □

Corollary 8.3. *E_6 is the quotient of the direct sum of cyclic $\mathbb{F}_3[\beta, v_1]\otimes\Lambda(\zeta)$-modules with the following generators and annihilating ideals*

$$
\begin{array}{lll}
1=\Delta_0 & & (\beta v_1^2, \beta^3 v_1)\\
 & & \\
\Delta_m\beta & 0\neq m\equiv 0 \mod (3) & (v_1^2, \beta^2 v_1)\\
\Delta_{6m+1}\beta v_1 & & (v_1, \beta^2)\\
\Delta_{6m+4}\beta v_1 & & (v_1, \beta^3)\\
\Delta_m\beta v_1 & m\equiv 2 \mod (3) & (v_1)\\
 & & \\
\alpha(\mathbb{G}_2^1) & & (\beta v_1, \beta^3)\\
\Delta_{2m+1}\alpha(\mathbb{G}_2^1) & m\not\equiv 2 \mod (3) & (v_1, \beta^3)\\
\Delta_{2m+1}\alpha(\mathbb{G}_2^1) & m\equiv 2 \mod (3) & (v_1)\\
\Delta_{2.3^n(3m-1)}\alpha(\mathbb{G}_2^1) & m\not\equiv 0 \mod (3),\ n\geq 1 & (v_1^{4.3^{n+1}-1}, \beta v_1, \beta^3)\\
\Delta_{2(3m-1)}\alpha(\mathbb{G}_2^1) & m\not\equiv 0 \mod (3) & (v_1^{11}, \beta v_1, \beta^4)\\
\Delta_{6m}\widetilde{\alpha} & & (v_1)\\
b_{2(9m+2)+1} & & (v_1^2, \beta)\\
\Delta_{6m+3}\widetilde{\alpha} & & (v_1^3, \beta v_1)\\
 & & \\
\Delta_{2.3^n(3m+1)}\alpha(\mathbb{G}_2^1)\beta & n\geq 1 & (v_1, \beta^2)\\
\Delta_{2(3m+1)}\alpha(\mathbb{G}_2^1)\beta & m\equiv 0 \mod (3) & (v_1)\\
\Delta_{2.3^n(3m-1)}\alpha(\mathbb{G}_2^1)\beta & n\geq 1 & (v_1, \beta^2)\\
\Delta_{2(3m-1)}\alpha(\mathbb{G}_2^1)\beta & m\equiv 0 \mod (3) & (v_1, \beta^3)\\
 & & \\
\overline{b}_{3^{n+1}(6m+1)}v_1 & n\geq 0 & (v_1^{6.3^n}, \beta)\\
\overline{b}_{3^n(6m+5)}v_1 & m\equiv 1 \mod (3),\ n\geq 2 & (v_1^{10.3^n}, \beta)\\
\overline{b}_{3^n(6m+5)} & m\equiv 1 \mod (3),\ n=0,1 & (v_1^{10.3^n+1}, \beta v_1)\\
 & & \\
\Sigma^{48}\overline{\Delta}_{3^n(6m+1)-3} & n\geq 1 & (v_1^2, \beta^2 v_1)\\
\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3} & m\not\equiv 1 \mod (3),\ n\geq 1 & (v_1^2, \beta^3 v_1)\\
\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3} & m\equiv 1 \mod (3),\ n\geq 1 & (v_1^2, \beta^2 v_1)\\
\Sigma^{48}\overline{\Delta}_{(6m+1)-3}v_1 & & (v_1, \beta^2)\\
\Sigma^{48}\overline{\Delta}_{(6m+5)-3}v_1 & & (v_1)\\
 & & \\
\Sigma^{48}\overline{\Delta}_{3^n(6m\pm 1)-2}v_1 & n\geq 1 & (v_1^{4.3^n-1}, \beta v_1, \beta^3)\\
\Sigma^{48}\overline{\Delta}_{(6m+1)-2}v_1 & & (v_1^3, \beta v_1)\\
\Sigma^{48}\overline{\Delta}_{(6m+5)-2} & & (v_1^4, \beta v_1^2, \beta^3 v_1)
\end{array}
$$

$$\begin{array}{lll}
\Sigma^{48}\overline{\Delta}_{2m-1}\alpha(\mathbb{G}_2^1) & m\not\equiv 0 \mod (3) & (v_1,\beta^3)\\
\Sigma^{48}\overline{\Delta}_{6m+5}\alpha(\mathbb{G}_2^1) & & (v_1)\\
\Sigma^{48}\overline{\Delta}_{3^n(6m+1)-3}\alpha(\mathbb{G}_2^1) & n\geq 0 & (v_1,\beta^2)\\
\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}\alpha(\mathbb{G}_2^1) & m\not\equiv 1 \mod (3),\ n\geq 1 & (v_1,\beta^3)\\
\Sigma^{48}\overline{\Delta}_{3^n(6m+5)-3}\alpha(\mathbb{G}_2^1) & m\equiv 1 \mod (3),\ n\geq 1 & (v_1,\beta^2)\\
\Sigma^{48}\overline{\Delta}_{6m}\widetilde{\alpha} & & (v_1)\\
\Sigma^{48}\overline{\Delta}_{(6m+5)-2}\widetilde{\alpha} & m\not\equiv 1 \mod (3) & (v_1)
\end{array}$$

modulo the following relations in which module generators are put into paranthesis (in order to distinguish between module multiplications and generators)

$$\begin{array}{lllll}
\beta^3(\Delta_k\alpha(\mathbb{G}_2^1)) & = & \beta^2\zeta(\Delta_k\beta v_1), & k=2(3m-1) & m\not\equiv 0 \mod (3)\\
\beta^2(\Delta_k\alpha(\mathbb{G}_2^1)\beta) & = & \beta^2\zeta(\Delta_k\beta v_1), & k=2(3m-1) & m\equiv 0 \mod (3)\\
\beta^2(\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2^1)) & = & \beta^2 v_1\zeta(\Sigma^{48}\overline{\Delta}_k), & k=2m-1 & m\not\equiv 0 \mod (3)\\
\beta^2(\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2^1) & = & \beta^2 v_1\zeta(\Sigma^{48}\overline{\Delta}_k) & k=3^n(6m+5)-3 & m\not\equiv 1 \mod (3),\ n\geq 1\ .
\end{array}$$

Proof. In principle this is a straightforward consequence of the previous corollary and Theorem 1.6. Complications arise because some of the time integers are distinguished by their residue class modulo (2), some of the time by their residue class modulo (3) and some of the time the distinction is more involved.

The most complicated case is perhaps that of the classes $\Sigma^{48}\overline{\Delta}_m$ for m even. There one uses that an even integer can be uniquely written in the form $k-3$ with k odd and an odd integer k can be uniquely written either as $3^n(6m\pm 1)$ with $n\geq 1$. This together with the previous corollary and Theorem 1.6 leads to the result for the first block of generators involving $\Sigma^{48}\overline{\Delta}_m$. The other blocks can be treated similarly. $\square$

The following Lemma records immediate consequences of the knowledge of the d_9-differential in the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(1))$.

Lemma 8.4. *The following identities hold in the E_9-term of the Adams-Novikov spectral sequence for $\pi_*(L_{K(2)}V(0))$.*

a) *Let $k\equiv 2 \mod (3)$. Then there are constants $\epsilon'_k\in\{\pm 1\}$ such that*

$$\begin{array}{lll}
d_9(\Delta_k\alpha(\mathbb{G}_2)\beta^2) & = & \epsilon'_k\Delta_{k-2}\beta^7\\
d_9(\Delta_k\beta^2 v_1) & = & \epsilon'_k\Delta_{k-2}\widetilde{\alpha}\beta^6\\
d_9(\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2)\beta) & = & \epsilon'_k\Sigma^{48}\overline{\Delta}_{k-2}\beta^6\\
d_9(\Sigma^{48}\overline{\Delta}_k\beta v_1) & = & \epsilon'_k\Sigma^{48}\overline{\Delta}_{k-2}\widetilde{\alpha}\beta^5\ .
\end{array}$$

b) *Let $k\equiv 0,1 \mod (3)$. Then*

$$\begin{array}{lll}
d_9(\Delta_k\alpha(\mathbb{G}_2)\beta^2) & = & 0\\
d_9(\Delta_k\beta^2 v_1) & = & 0\\
d_9(\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2)\beta) & = & 0\\
d_9(\Sigma^{48}\overline{\Delta}_k\beta v_1) & = & 0\ .
\end{array}$$

Remark By identifying vector space generators in the appropriate bidegrees it is easy to see that there are unique elements $\lambda_i,\mu_i,\nu_i\in\mathbb{F}_3$ such that

$$\begin{array}{lll}
d_9(\Delta_k\alpha(\mathbb{G}_2^1)\beta^2) & = & \lambda_5\Delta_{k-2}\beta^7+\mu_5\Delta_{k-2}\widetilde{\alpha}\beta^6\zeta+\nu_5\Sigma^{48}\overline{\Delta}_{k-3}\beta^5\zeta\\
d_9(\Delta_k\beta^2 v_1) & = & \lambda_6\Delta_{k-2}\widetilde{\alpha}\beta^6+\mu_6\Sigma^{48}\overline{\Delta}_{k-3}\beta^5+\nu_6\Sigma^{48}\overline{\Delta}_{k-3}\widetilde{\alpha}\beta^4\zeta\\
d_9(\Sigma^{48}\overline{\Delta}_k\alpha(\mathbb{G}_2^1)\beta) & = & \lambda_7\Sigma^{48}\overline{\Delta}_{k-2}\beta^6+\mu_7\Sigma^{48}\overline{\Delta}_{k-2}\widetilde{\alpha}\beta^5\zeta\\
d_9(\Sigma^{48}\overline{\Delta}_k\beta v_1) & = & \lambda_8\Sigma^{48}\overline{\Delta}_{k-2}\widetilde{\alpha}\beta^5\ .
\end{array}$$

As before naturality and the geometric boundary theorem applied to the resolution (4) allow to show the lemma modulo elements of lower filtration, i.e. to determine the values of the λ_i. Again the Lemma confirms these values and also determines μ_i and ν_i, and formally μ_i and ν_i can be deduced by simply replacing in the differentials for $E_2^{hG_{24}}\wedge V(0)$ the elements α by $\alpha(\mathbb{G}_2)$.

Proof. The proof resembles that of Lemma 8.1. We start with the class $\Delta_k\beta^2v_1$. From (36) and β-linearity of $\delta^1_{\mathbb{G}_2}$ we get

$$\delta^1_{\mathbb{G}_2}((\omega^2u^{-4})^{3k+2}\alpha(\mathbb{G}_2)\beta) = \pm\Sigma^4\Delta_k\beta^2v_1, \quad \delta^1_{\mathbb{G}_2}((\omega^2u^{-4})^{3k-4}\beta^6) = \pm\Sigma^4\Delta_{k-2}\widetilde{\alpha}\beta^6$$

and as before the geometric boundary theorem and Theorem A.4.c yield the value of the differential (with a suitable constant ϵ'_k).

The case of $\Sigma^{48}\overline{\Delta}_k\beta v_1$ can be treated similarly but we can also use the strategy described in the remark above. Again the sign is clearly the same as in the previous case.

The remaining two cases can once again be deduced by using the Bockstein $\delta^0_{\mathbb{G}_2}$ in $H^*(\mathbb{G}_2,-)$ associated to the short exact sequence (26). As in the proof of Lemma 8.1 we obtain

$$\begin{array}{ll}\delta^0_{\mathbb{G}_2}(\Delta_m\beta^2v_1) = \Delta_m\alpha(\mathbb{G}_2)\beta^2, & \delta^0_{\mathbb{G}_2}(\Sigma^{48}\overline{\Delta}_m\beta v_1) = \Sigma^{48}\overline{\Delta}_m\alpha(\mathbb{G}_2)\beta \\ \delta^0_{\mathbb{G}_2}(\Delta_m\widetilde{\alpha}\beta^6) = \Delta_m\beta^7, & \delta^0_{\mathbb{G}_2}(\Sigma^{48}\overline{\Delta}_m\widetilde{\alpha}\beta^5) = \Sigma^{48}\overline{\Delta}_m\beta^6\end{array}$$

and the geometric boundary theorem gives the result. □

Corollary 8.5. *The d_9-differential in the Adams-Novikov spectral sequence for $L_{K(2)}V(0)$ is linear with respect to $\mathbb{F}_3[\beta,v_1]\otimes\Lambda(\zeta)$ and is trivial on all $\mathbb{F}_3[\beta,v_1]\otimes\Lambda(\zeta)$-module generators of $E_6^{*,*}$ given in Corollary 8.3 except the following:*

$$\begin{array}{lcll}
d_9(\Delta_m\beta v_1) & = & \pm\Delta_{m-2}\widetilde{\alpha}\beta^5 & m\equiv 2 \mod (3)\\
d_9(\Delta_{6m+5}\alpha(\mathbb{G}_2^1)) & = & \pm(\Delta_{6m+3}\beta^5+\Delta_{6m+3}\widetilde{\alpha}\beta^4\zeta) & \\
d_9(\Delta_{2(3m+1)}\alpha(\mathbb{G}_2^1)\beta) & = & \pm(\Delta_{6m}\beta^6+\Delta_{6m}\widetilde{\alpha}\beta^5\zeta) & \\
d_9(\overline{b}_{18m+11}) & = & \pm(\Sigma^{48}\overline{\Delta}_{6m}\beta^4+\Sigma^{48}\overline{\Delta}_{6m}\widetilde{\alpha}\beta^3\zeta) & \\
d_9(\Sigma^{48}\overline{\Delta}_{6m+2}v_1) & = & \pm\Sigma^{48}\overline{\Delta}_{6m}\widetilde{\alpha}\beta^4 & \\
d_9(\Sigma^{48}\overline{\Delta}_{6m+5}v_1) & = & \pm\Sigma^{48}\overline{\Delta}_{6m+3}\widetilde{\alpha}\beta^4 & m\not\equiv 1 \mod (3)\\
d_9(\Sigma^{48}\overline{\Delta}_{6m+5}v_1) & = & \pm\overline{b}_{3(6m+5)}\beta^5 & m\equiv 1 \mod (3)\\
d_9(\Sigma^{48}\overline{\Delta}_{6m+5}\alpha(\mathbb{G}_2^1)) & = & \pm(\Sigma^{48}\overline{\Delta}_{6m+3}\beta^5+\Sigma^{48}\overline{\Delta}_{6m+3}\widetilde{\alpha}\beta^4\zeta) & .
\end{array}$$

Proof. Linearity with respect to $\mathbb{F}_3[\beta,v_1]\otimes\Lambda(\zeta)$ is clear. Then we note that the β-torsion classes in the E_6-term are in too low a cohomological degree to interact via d_9 and hence d_9 is trivial on them. The rest is an immediate consequence of Proposition 1.5, Corollary 8.3 and the previous Lemma, and the fact that β-multipication is injective in the relevant bidegrees. □

This finally allows us to calculate the homotopy of $\pi_*(L_{K(2)}V(0))$.

Proof of Theorem 1.8. By using the last corollary it is straightforward to verify that the E_{10}-term has the structure described in Theorem 1.8. Then we see that $E_{10}^{s,*}=0$ for $s>11$ and hence there is no room for higher differentials and we get $E_{10}=E_\infty$.

Appendix A. The Adams-Novikov spectral sequences converging towards $\pi_*(E_2^{hG_{24}}\wedge V(0))$, $\pi_*(E_2^{hG_{24}}\wedge V(1))$ and $\pi_*(L_{K(2)}V(1))$

The behaviour of the spectral sequence for $\pi_*(E_2^{hG_{24}}\wedge V(0))$ can be deduced from that for $\pi_*(E_2^{hG_{24}})$. We record this in the following result.

Theorem A.1.

a) *The differentials in the Adams-Novikov spectral sequence*

$$E_2^{s,t}\cong\mathbb{F}_3[[v_1^6\Delta^{-1}]][v_1,\Delta^{\pm1},\beta,\alpha,\widetilde{\alpha}]/(\alpha^2,\widetilde{\alpha}^2,v_1\alpha,v_1\widetilde{\alpha},\alpha\widetilde{\alpha}+v_1\beta)\Longrightarrow\pi_{t-s}(E_2^{hG_{24}}\wedge V(0))$$

are linear with respect to $\mathbb{F}_3[\Delta^{\pm3},v_1,\beta,\alpha,\widetilde{\alpha}]$. The only nontrivial differentials are d_5 and d_9. They are (redundantly) determined by

$$d_5(\Delta)=\pm\alpha\beta^2,\quad d_5(\Delta\widetilde{\alpha})=\pm\widetilde{\alpha}\alpha\beta^2=\mp v_1\beta^3$$

and

$$d_9(\Delta^2\alpha)=\pm\beta^5,\quad d_9(\Delta^2v_1)=\mp\widetilde{\alpha}\beta^4\ .$$

b) *We have an inclusion of subalgebras*

$$E_\infty^{0,*} \cong \mathbb{F}_3[[v_1^6\Delta^{-1}]][v_1, v_1\Delta, \Delta^{\pm 3}] \ .$$

In positive filtration $E_\infty^{s,t}$ has an $\mathbb{F}_3$-vector space basis given by the 16 elements which are represented on E_2 by α, $\beta\alpha$, $\Delta\alpha$, $\beta\Delta\alpha$, $\widetilde{\alpha}\alpha = \beta v_1$, $\beta\widetilde{\alpha}\alpha = \beta^2 v_1$, $\Delta\widetilde{\alpha}\alpha = \beta\Delta v_1$, $\beta\Delta\widetilde{\alpha}\alpha = \beta^2\Delta v_1$, β^j, $j = 1,2,3,4$, $\beta^k\widetilde{\alpha}$, $k = 0,1,2,3$ and their multiples by powers of $\Delta^{\pm 3}$.

Proof. This is a consequence of the behaviour of the spectral sequence converging to $\pi_*(E_2^{hG_{24}})$ (cf. [4], [5]). First one observes that every element in the transfer from $\pi_*(E_2)$ is a permanent cycle. Together with the fact that v_1 is a permant cycle as well, this shows that the only classes on the 0-line which can carry non-trivial differentials are the classes $\Delta^k v_1^\epsilon$ with $k \in \mathbb{Z}$ and $\epsilon = 0$ or $k \neq 0$ and $\epsilon = 1$. Furthermore we recall that the elements α and β are permanent cycles in the Adams-Novikov spectral sequence converging to $\pi_*(E_2^{hG_{24}})$ and $\widetilde{\alpha}$ in that for $V(0)$ detecting homotopy classes fow which we use the same names.

Now the spectral sequence for $\pi_*(E_2^{hG_{24}})$ shows $d_5(\Delta) = \pm\alpha\beta^2$ and from the fact that the spectral sequence for $\pi_*(E_2^{hG_{24}} \wedge V(0))$ is a module over that for $\pi_*(E_2^{hG_{24}})$ we deduce

$$d_5(\Delta\widetilde{\alpha}) = \pm\alpha\widetilde{\alpha}\beta^2 = \mp v_1\beta^3, \quad d_5(\Delta^k v_1) = 0, \ k \equiv 1,2 \mod (3) \ ,$$

and d_5 happens to be a derivation. In particular we obtain an isomorphism of algebras

$$E_6^{0,*} \cong \mathbb{F}_3[[v_1^6\Delta^{-1}]][\Delta^{\pm 3}, v_1, \Delta v_1, \Delta^2 v_1] \ ,$$

and in positive filtration $E_6^{s,t}$ has an $\mathbb{F}_3$-vector space basis given by the elements β^k, $k > 0$, $\beta^k v_1$, $k = 1,2$, $\beta^k\alpha$, $k = 0,1$, $\beta^k\widetilde{\alpha}$, $k \geq 0$, $\beta^k\Delta v_1$, $k = 1,2$, $\beta^k\Delta\alpha$, $k = 0,1$, $\beta^k\Delta^2 v_1$, $k > 0$ and $\beta^k\Delta^2\alpha$, $k \geq 0$ and their multiples by $\Delta^{\pm 3}$.

Next the Adams-Novikov spectral sequence for $\pi_*(E_2^{hG_{24}})$ implies $d_9(\beta^l\Delta^{3k+2}\alpha) = \pm\beta^{l+5}\Delta^{3k}$, hence the module structure of the spectral sequence for $\pi_*(E_2^{hG_{24}} \wedge V(0))$ with respect to that of $\pi_*(E_2^{hG_{24}})$ gives $d_9(\beta^{l+1}\Delta^{3k+2}v_1) = d_9(\beta^l\Delta^{3k+2}\widetilde{\alpha}\alpha) = \pm\beta^{l+5}\Delta^{3k}\widetilde{\alpha}$ and thus $d_9(\beta^l\Delta^{3k+2}v_1) = \pm\beta^{l+4}\Delta^{3k}\widetilde{\alpha}$. Furthermore sparseness shows that d_9 is trivial on all other classes of positive cohomological degree. In the resulting E_{10}-term we have $E_2^{s,*} = 0$ for $s > 8$ and thus there is no more room for higher differentials. Hence we get $E_{10} = E_\infty$ and the structure of E_∞ agrees with the stated result. $\square$

The following result has already been proved in [4] for the subgroup G_{12} instead of G_{24}. In fact, part (a) is an immediate consequence of the formulae (22) and (23) and the fact that the action of G_{24} on $(E_2)_*/(3,u_1)$ factors through an action of the quotient $G_{24}/\mathbb{Z}/3 = Q_8 \subset SD_{16}$. The proof of part (b) and (c) is analogous to the case of G_{12}. We leave the details to the reader.

Theorem A.2. *(cf. Theorem 9 of* [4]*)*

a) *The E_2-term of the Adams-Novikov spectral sequence converging to $\pi_*(E_2^{hG_{24}} \wedge V(1))$ is given by*

$$E_2^{*,*} \cong H^*(G_{24}, (E_2)_*/(3,u_1)) \cong \mathbb{F}_3[\omega^2 u^{\pm 4}, \beta, \alpha]/(\alpha^2) \ .$$

b) *The only non-trivial differentials in this Adams-Novikov spectral sequence are d_5 and d_9. They are determined by linearity with respect to $\mathbb{F}_3[\omega^2 u^{\pm 4}, \beta, \alpha]$ and the formulae*

$$d_5((\omega^2 u^{\pm 4})^k) = \begin{cases} 0 & k \equiv 0,1,2 \mod (9) \\ \pm(\omega^2 u^{\pm 4})^{k-3}\alpha\beta^2 & k \equiv 3,4,5,6,7,8 \mod (9) \end{cases}$$

and

$$d_9((\omega^2 u^{\pm 4})^k\alpha) = \begin{cases} 0 & k \equiv 0,1,2,3,4,5 \mod (9) \\ \pm(\omega^2 u^{\pm 4})^{k-3}\beta^5 & k \equiv 6,7,8 \mod (9) \ . \end{cases}$$

c) *There are elements in* $\pi_{8k}(E_2^{hG_{24}} \wedge V(1))$ *represented by* $(\omega^2u^{-4})^k$, $k=0,1,2$, *and in* $\pi_{8k+3}(E_2^{hG_{24}} \wedge V(1))$ *represented by* $(\omega^2u^{-4})^k\alpha$, $k=0,1,2,3,4,5$, *such that there is an isomorphism of modules over* $\mathbb{F}_9[\Delta^{\pm 3}, \beta]$

$$\begin{aligned}\pi_*(E_2^{hG_{24}} \wedge V(1)) \cong \ & \mathbb{F}_9[\Delta^{\pm 3}] \otimes \Big(\mathbb{F}_9[\beta]/(\beta^5)\{1, \omega^2u^{-4}, (\omega^2u^{-4})^2\} \oplus \\ & \oplus \mathbb{F}_9[\beta]/(\beta^2)\{\alpha, \cdots, (\omega^2u^{-4})^5\alpha\}\Big)\ .\end{aligned}$$

Next we turn towards the algebraic spectral sequence (2) in the case of $M=(E_2)_*/(3,u_1)$ and the Adams-Novikov spectral sequence converging towards $\pi_*(L_{K(2)}V(1))$. This has already been discussed in [4] and [11]. Here we merely translate those results into a form suitable for our discussion. As before v_2 is defined to be u^{-8}.

Theorem A.3. *(cf. section 4 of* [4]*)*

a) *As a module over* $\mathbb{F}_3[v_2^{\pm 1}, \beta, \alpha]/(\alpha)^2$ *the* E_1*-term of the algebraic spectral sequence (2) for* $(E_2)_*/(3,u_1)$ *is given as follows (with* β *and* α *acting trivially on* $E_1^{s,*,*}$ *if* $s=1,2$, *and the elements* e_s, $s=0,1,2,3$, *serving as module generators in tridegree* $(s,0,0)$. *)*

$$E_1^{s,*,*} \cong \begin{cases} \mathbb{F}_3[\omega^2u^{\pm 4}][\beta,\alpha]/(\alpha^2)e_s & s=0,3 \\ \omega^2u^4\mathbb{F}_3[v_2^{\pm 1}]e_s & s=1,2 \\ 0 & s>3\ . \end{cases}$$

b) *The differentials in this spectral sequence are* $\mathbb{F}_3[v_2^{\pm 1}, \beta, \alpha]$*-linear. The only non-trivial differential in this spectral sequence is* d_1 *and is determined by*

$$d_1^{0,0}(\omega^2u^{-12}e_0) = \omega^2u^{-12}e_1, \quad d_1^{1,0}=0, \quad d_1^{2,0}=0\ .$$

c) *The following* β*-extensions hold in* $H^*(\mathbb{G}_2^1, (E_2)_*/(3,u_1))$

$$\beta\omega^2u^4v_2^ke_2 = \pm v_2^k\alpha e_3\ .$$

d) $H^*(\mathbb{G}_2^1; (E_2)_*/(3,u_1))$ *is a free module over* $\mathbb{F}_3[v_2^{\pm 1}, \beta]$ *on generators*

$$e_0,\ \ \alpha e_0,\ \ \omega^2u^{-4}\alpha e_0,\ \ \omega^2u^{-4}\beta e_0,\ \ \omega^2u^{-4}e_2,\ \ e_3,\ \ \omega^2u^{-4}e_3,\ \ \omega^2u^{-4}\alpha e_3\ .$$

e) *There is an isomorphism of* $\mathbb{F}_3[v_2^{\pm 1}, \beta] \otimes \Lambda(\zeta)$*-modules (even of algebras)*

$$H^*(\mathbb{G}_2, (E_2)_*/(3,u_1)) \cong H^*(\mathbb{G}_2^1, (E_2)_*/(3,u_1)) \otimes \Lambda(\zeta)\ .$$

Proof. a) The action of $\mathbb{G}_2^1$ on $(E_2)_*/(3,u_1)$ is trivial on its Sylow-subgroup S_2^1, and on the quotient group $\mathbb{G}_2^1/S_2 \cong SD_{16}$ the action is given by the formulae in (22) and (23). With this information the calculation of the E_1-term is straightforward and is left to the reader.

b) As module over $\mathbb{F}_3[v_2^{\pm 1}, \beta, \alpha]$ the E_1-term is generated by e_s and $\omega^2u^{-12}e_s$ for $s=0,3$ and $\omega^2u^{-4}e_s$ for $s=1,2$. The map of algebraic spectral sequences (2) induced by the canonical homomorphism $(E_2)_*/(3) \to (E_2)_*/(3,u_1)$ of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$-modules sends Δ_{2k} to $v_2^{3k}e_0$, b_{2k+1} to $\omega^2u^{-4}v_2^ke_1$, $\overline{b}_{2k+1}$ to $\omega^2u^{-4}v_2^ke_2$ and $\overline{\Delta}_{2k}$ to $v_2^{3k}e_3$. This and Theorem 1.2 determine the d_1-differential and the E_2-page. The abutment of the spectral sequence is known by Corollary 19 of [4] and comparing the E_2-term with the abutment shows that the spectral sequence collapses at its E_2-term.

c) From the same corollary we know that $H^*(\mathbb{S}_2, (E_2)_*/(3,u_1))$ is free as module over $\mathbb{F}_9[\beta]$, hence $H^*(\mathbb{G}_2, (E_2)_*/(3,u_1))$ and then also $H^*(\mathbb{G}_2^1, (E_2)_*/(3,u_1))$ are free as modules over $\mathbb{F}_3[\beta]$. This requires nontrivial β-multiplications on $E_2^{2,0}$ and by degree reasons these multiplications must be as claimed.

d) This is an immediate consequence of (a), (b) and (c).

e) This is an easy consequence of the isomorphism $\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$ and the fact that the central factor $\mathbb{Z}_3$ acts trivially on $(E_2)_*/(3,u_1)$. □

Next we note that as before the existence of the resolution (4) of [5] puts additional restrictions on the Adams-Novikov differentials for $L_{K(2)}V(1)$. In fact, again by naturality and the geometric boundary theorem, these differentials can be easily read off from those for $E_2^{hG_{24}}\wedge V(1)$, at least modulo the filtration on $H^*(\mathbb{G}_2,(E_2)_*/(3,u_1))$ determined by the algebraic resolution (2). It turns out that the potential terms of lower filtration are always trivial although showing this requires a non-trivial effort which has essentially been carried out in [4].

Theorem A.4. *(cf. section 3 of* [4]*)*

a) *The only non-trivial differentials in the Adams-Novikov spectral sequence converging to $\pi_*(L_{K(2)}V(1))$ are d_5 and d_9. They are both determined by the fact that they are linear with respect to $\mathbb{F}_3[v_2^{\pm 9},\beta]\otimes\Lambda(\zeta)$ and by the following formulae in which we identify $v_2^{k+3}e_3$ with $\Sigma^{48}v_2^ke_3$ etc. .*

b) *The differential d_5 is given by*

$$\begin{array}{lcl} d_5(v_2^k\alpha e_0) & = & 0\\ d_5(v_2^k\omega^2u^{-4}\alpha e_0) & = & 0\\ d_5(v_2^k\omega^2u^{-4}e_2) & = & 0\\ d_5(\Sigma^{48}v_2^k\omega^2u^{-4}\alpha e_3) & = & 0 \end{array}$$

for all k, and

$$d_5(v_2^ke_0) = \begin{cases} 0 & k\equiv 0,1,5 \mod (9)\\ \pm v_2^{k-2}\omega^2u^{-4}\alpha\beta^2e_0 & k\equiv 2,3,4,6,7,8 \mod (9)\end{cases}$$

$$d_5(v_2^k\omega^2u^{-4}\beta e_0) = \begin{cases} 0 & k\equiv 0,4,5 \mod (9)\\ \pm v_2^{k-1}\alpha\beta^3e_0 & k\equiv 1,2,3,6,7,8 \mod (9)\end{cases}$$

$$d_5(\Sigma^{48}v_2^ke_3) = \begin{cases} 0 & k\equiv 0,1,5 \mod (9)\\ \pm\Sigma^{48}v_2^{k-2}\omega^2u^{-4}\alpha\beta^2e_3 & k\equiv 2,3,4,6,7,8 \mod (9)\end{cases}$$

$$d_5(\Sigma^{48}v_2^k\omega^2u^{-4}e_3) = \begin{cases} 0 & k\equiv 0,4,5 \mod (9)\\ \pm\Sigma^{48}v_2^{k-1}\alpha\beta^2e_3 & k\equiv 1,2,3,6,7,8 \mod (9)\ .\end{cases}$$

c) *The differential d_9 is given by*

$$\begin{array}{lclll} d_9(v_2^ke_0) & = & 0 & k\equiv 0,1,5 & \mod (9)\\ d_9(v_2^k\omega^2u^{-4}\beta e_0) & = & 0 & k\equiv 0,4,5 & \mod (9)\\ d_9(\Sigma^{48}v_2^ke_3) & = & 0 & k\equiv 0,1,5 & \mod (9)\\ d_9(\Sigma^{48}v_2^k\omega^2u^{-4}\beta e_3) & = & 0 & k\equiv 0,4,5 & \mod (9) \end{array}$$

$$d_9(v_2^k\alpha e_0) = \begin{cases} 0 & k\equiv 0,1,2,5,6,7 \mod (9)\\ \pm v_2^{k-3}\beta^5e_0 & k\equiv 3,4,8 \mod (9)\end{cases}$$

$$d_9(v_2^k\omega^2u^{-4}\alpha e_0) = \begin{cases} 0 & k\equiv 0,1,2,4,5,6 \mod (9)\\ \pm v_2^{k-3}\omega^2u^{-4}\beta^5e_0 & k\equiv 3,7,8 \mod (9)\end{cases}$$

$$d_9(v_2^{k+2}\omega^2u^{-4}e_2) = \begin{cases} 0 & k\equiv 0,1,2,5,6,7 \mod (9)\\ \pm\Sigma^{48}v_2^{k-5}\beta^4e_3 & k\equiv 3,4,8 \mod (9)\end{cases}$$

$$d_9(\Sigma^{48}v_2^k\omega^2u^{-4}\alpha e_3) = \begin{cases} 0 & k\equiv 0,1,2,4,5,6 \mod (9)\\ \pm\Sigma^{48}v_2^{k-3}\beta^5\omega^2u^{-4}e_3 & k\equiv 3,7,8 \mod (9)\ .\end{cases}$$

d) *As a module over* $P = \mathbb{F}_3[v_2^{\pm 9}, \beta] \otimes \Lambda(\zeta)$ *there is an isomorphism*

$$\begin{aligned}\pi_*(L_{K(2)}V(1)) \cong\ & P/(\beta^5)\{v_2^k e_0\}_{k=0,1,5} && \oplus P/(\beta^3)\{v_2^k \alpha e_0\}_{k=0,1,2,5,6,7} \\ & \oplus P/(\beta^4)\{v_2^k \omega^2 u^{-4}\beta e_0\}_{k=0,4,5} && \oplus P/(\beta^2)\{v_2^k \omega^2 u^{-4}\alpha e_0\}_{k=0,1,2,4,5,6} \\ & \oplus P/(\beta^4)\{\Sigma^{48} v_2^k e_3\}_{k=0,1,5} && \oplus P/(\beta^3)\{v_2^{k+2}\omega^2 u^{-4} e_2\}_{k=0,1,2,5,6,7} \\ & \oplus P/(\beta^5)\{\Sigma^{48} v_2^k \omega^2 u^{-4} e_3\}_{k=0,4,5} && \oplus P/(\beta^2)\{\Sigma^{48} v_2^k \omega^2 u^{-4}\alpha e_3\}_{k=0,1,2,4,5,6}\ .\end{aligned}$$

Proof. This is an immediate reformulation of the main theorem of [4]. We just have to use the following dictionary which translates between the $\mathbb{F}_9[v_2^{\pm 1}, \beta]$-module generators used in Corollary 19 of [4] and those of $H^*(\mathbb{S}_2^1, (E_2)_*/(3, u_1))$ of Theorem A.3[1].

By degree reasons the generators 1, α, $v_2^{\frac{1}{2}}\alpha$, $v_2^{\frac{1}{2}}\beta$, αa_{35} and $v_2^{\frac{1}{2}}\beta\alpha a_{35}$ of [5] must correspond, up to a unit in $\mathbb{F}_9$, to e_0, αe_0, $\omega^2 u^{-4}\alpha e_0$, $\omega^2 u^{-4}\beta e_0$, $\omega^2 u^{-4} v_2^2 e_2$ and $\Sigma^{48}\omega^2 u^{-4}\alpha e_3$ of Theorem A.3. The generators βa_{35} and $v_2^{\frac{1}{2}}\beta a_{35}$ are not determined (not even up to a unit) by their bidegree, but if one takes into account that they are in the kernel of the restriction map to $H^*(G_{12}; (E_2)_*/(3, u_1))$ then they are also determined up to a unit and they must therefore agree, up to a unit, with the elements $\Sigma^{48} e_3$ and $\Sigma^{48}\omega^2 u^{-4} e_3$. □

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[1] For the translation we pass from $H^*(\mathbb{G}_2^1; (E_2)_*/(3, u_1))$ to $H^*(\mathbb{S}_2^1; (E_2)_*/(3, u_1))$ in Theorem A.3 and from $H^*(\mathbb{S}_2; (E_2)_*/(3, u_1))$ to $H^*(\mathbb{S}_2^1; (E_2)_*/(3, u_1))$ in [4]. This passage is straightforward and left to the reader.