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The homotopy types of PU(3)– and PSp(2)–gauge groups

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Let G be a compact connected simple Lie group. Any principal G–bundle over S^4 is classified by an integer $k \in \mathbb{Z} \cong \pi_3(G)$, and we denote the corresponding gauge group by $\mathcal{G}_k(G)$. We prove that $\mathcal{G}_k(\mathrm{PU}(3)) \simeq \mathcal{G}_\ell(\mathrm{PU}(3))$ if and only if $(24,k)=(24,\ell)$, and $\mathcal{G}_k(\mathrm{PSp}(2)) \simeq_{(p)} \mathcal{G}_\ell(\mathrm{PSp}(2))$ for any prime p if and only if $(40,k)=(40,\ell)$, where (m,n) is the gcd of integers m,n.

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1 Introduction

Let G be a topological group, and let P be a principal G–bundle over a base X. The gauge group of P, denoted by $\mathcal{G}(P)$, is by definition the topological group of all automorphisms of P, where automorphisms of P are G–equivariant self-maps of P covering the identity map of X. We consider the following problem.

Problem 1.1 Classify the homotopy types of $\mathcal{G}(P)$ as P ranges over all principal G–bundles over X for fixed G and X.

This problem was posed by the third-named author in [11], who considered the case when $G=\mathrm{SU}(2)$ and $X=S^4$, and the most important case of the problem is when G is a compact Lie group. Let G be a compact connected simple Lie group. Then the principal G–bundle over S^4 is classified by $\pi_3(G)\cong\mathbb{Z}$. We denote by $\mathcal{G}_k(G)$ the gauge group of the principal G–bundle over S^4 classified by $k\in\mathbb{Z}\cong\pi_3(G)$. There are several classification results on $\mathcal{G}_k(G)$, and we here recall the case when G is of rank ≤ 2. The following results are proved in Kono [11], Hamanaka and Kono [5], Theriault [17]:

$$\begin{aligned}
&\mathcal{G}_k(\mathrm{SU}(2)) \simeq \mathcal{G}_\ell(\mathrm{SU}(2)) && \text{if and only if} \quad (12,k)=(12,\ell),\\
&\mathcal{G}_k(\mathrm{SU}(3)) \simeq \mathcal{G}_\ell(\mathrm{SU}(3)) && \text{if and only if} \quad (24,k)=(24,\ell),\\
&\mathcal{G}_k(\mathrm{Sp}(2)) \simeq_{(p)} \mathcal{G}_\ell(\mathrm{Sp}(2)) \text{ for any prime } p && \text{if and only if} \quad (40,k)=(40,\ell),
\end{aligned}$$

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where (m,n) denotes the gcd of integers m,n and $\simeq_{(p)}$ means a p–local homotopy equivalence. In Kishimoto, Theriault and Tsutaya [10], it is also proved that if $\mathcal{G}_k(\mathrm{G}_2)\simeq\mathcal{G}_\ell(\mathrm{G}_2)$ then $(84,k)=(84,\ell)$, and if $(168,k)=(168,\ell)$ then $\mathcal{G}_k(\mathrm{G}_2)\simeq_{(p)}\mathcal{G}_\ell(\mathrm{G}_2)$ for any prime p. Then the classification of the homotopy types or the Mislin genera of $\mathcal{G}_k(G)$ is completed when G is a simply connected compact simple Lie group of rank 2 except for G_2, and the G_2 case is almost done. The nonsimply connected compact simple Lie groups of rank 2 are $\mathrm{SO}(3),\mathrm{PU}(3),\mathrm{PSp}(2)$, and there is a classification result for $\mathrm{SO}(3)$ due to Kamiyama, Tsukuda, and the second and third authors [7]:

$$\mathcal{G}_k(\mathrm{SO}(3))\simeq\mathcal{G}_\ell(\mathrm{SO}(3)) \quad \text{if and only if} \quad (12,k)=(12,\ell).$$

In this paper, we aim to classify the homotopy types or the Mislin genera of $\mathcal{G}_k(G)$ when G is $\mathrm{PU}(3)$ and $\mathrm{PSp}(2)$, which completes the classification of $\mathcal{G}_k(G)$ when G is simple and of rank 2.

Theorem 1.2 *The following hold:*

(1) $\mathcal{G}_k(\mathrm{PU}(3))\simeq\mathcal{G}_\ell(\mathrm{PU}(3))$ *if and only if* $(24,k)=(24,\ell)$.

(2) $\mathcal{G}_k(\mathrm{PSp}(2))\simeq_{(p)}\mathcal{G}_\ell(\mathrm{PSp}(2))$ *for any prime* p *if and only if* $(40,k)=(40,\ell)$.

2 Gauge groups and Samelson products

In this section we recall a connection between gauge groups and Samelson products, and note some of its consequences. While we state the results by specializing to our special case, most of the results in this section hold in a more general setting. As in the previous section, let G be a compact connected simple Lie group and $\mathcal{G}_k(G)$ be the gauge group of the principal G–bundle over S^4 classified by $k\in\mathbb{Z}\cong\pi_3(G)$. Hereafter we denote a generator of $\pi_3(G)\cong\mathbb{Z}$ by ι. As in [4; 2] there is a homotopy equivalence

$$B\mathcal{G}_k(G)\simeq\mathrm{map}(S^4,BG;k\bar\iota), \tag{2-1}$$

where $\mathrm{map}(S^4,BG;k\bar\iota)$ is the connected component of the space of maps from S^4 to BG including $k\bar\iota$ and $\bar\iota$ is the adjoint of ι. Then the evaluation at the basepoint $\mathrm{map}(S^4,BG;k\bar\iota)\to BG$ yields a homotopy fibration sequence

$$\mathcal{G}_k(G)\xrightarrow{\rho}G\xrightarrow{\partial_k}\Omega_0^3G\to B\mathcal{G}_k(G)\to BG. \tag{2-2}$$

We identify the connecting map ∂_k.

Proposition 2.1 [18; 12] *The adjoint of the connecting map* $\partial_k\colon G\to\Omega_0^3G$ *is the Samelson product* $\langle k\bar\iota,1_G\rangle\colon S^3\wedge G\to G$.

Corollary 2.2 *The connecting map ∂_k satisfies $\partial_k = k \circ \partial_1$ for the k–power map $k\colon G \to G$.*

Proof Combine the linearity of Samelson products and Proposition 2.1. □

Then the gauge group $\mathcal{G}_k(G)$ is homotopy equivalent to the homotopy fiber of the map $k \circ \partial_1$. So we recall the following two results on maps into H–spaces composite with power maps.

Proposition 2.3 [17] *Suppose a map $f\colon X \to Y$ into an H–space Y is of order $n < \infty$. Then $(n,k) = (n,\ell)$ implies $F_k \simeq_{(p)} F_\ell$ for any prime p, where F_k is the homotopy fiber of $k \circ f$.*

Proposition 2.4 [5] *Let Y be an H–space such that $\pi_i(Y)$ is finite for any i. Suppose a map $f\colon X \to Y$ satisfies $n \circ f \simeq *$ for $n < \infty$. Then if $(n,k) = (n,\ell)$, there is a homotopy equivalence $h\colon Y \xrightarrow{\simeq} Y$ satisfying a homotopy commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{k\circ f} & Y \\ \| & & \downarrow h \\ X & \xrightarrow{\ell\circ f} & Y \end{array}$$

3 Samelson products in PU(3) and PSp(2)

This section calculates the order of the Samelson products in PU(3) and PSp(2) corresponding to the boundary map ∂_1 by Proposition 2.1.

A Samelson product in PU(3)

We determine the order of the Samelson product $\langle \iota, 1_{\mathrm{PU}(3)}\rangle$ in PU(3). For the projection homomorphism $\pi\colon \mathrm{SU}(3) \to \mathrm{PU}(3)$ we have $\pi \circ \iota = \iota$. In particular,

$$\langle \iota, 1_{\mathrm{PU}(3)}\rangle \circ (1 \wedge \pi) = \pi \circ \langle \iota, 1_{\mathrm{SU}(3)}\rangle. \tag{3-1}$$

We recall the order of the Samelson product $\langle \iota, 1_{\mathrm{SU}(3)}\rangle$.

Proposition 3.1 [5] *The Samelson product $\langle \iota, 1_{\mathrm{SU}(3)}\rangle$ is of order 24.*

Since there is a fibration $\mathrm{SU}(3) \xrightarrow{\pi} \mathrm{PU}(3) \to B\mathbb{Z}/3$ and the projection π is a homotopy equivalence at the prime p for $p \neq 3$, we get the following by (3-1).

Corollary 3.2 *The order of* $\langle \iota, 1_{\mathrm{PU}(3)} \rangle_{(p)}$ *is* 8 *if* $p = 2$ *and* 1 *if* $p > 3$.

It remains to determine the 3–primary component of the order of the Samelson product $\langle \iota, 1_{\mathrm{PU}(3)} \rangle$. For this, we employ the method of [6; 8]. We start with an algebraic lemma.

Lemma 3.3 *Let* Γ *be a group. For* $x, y, z \in \Gamma$, $[x, z] = [y, z] = 1$ *implies* $[xy, z] = 1$, *where* $[-,-]$ *denotes the commutator in* Γ.

Proof This follows from the equality $[xy, z] = x[y, z]x^{-1}[x, z]$. □

Let G be a topological group. For maps $\alpha\colon A \to G$ and $\beta\colon B \to G$, we set $\{\alpha, \beta\}$ to be the composite

$$A \times B \xrightarrow{\alpha \times \beta} G \times G \xrightarrow{\gamma} G,$$

where γ is the commutator map in G. Then obviously the Samelson product $\langle \alpha, \beta \rangle$ is trivial if and only if $\{\alpha, \beta\}$ is trivial. So we consider the triviality of $\{\alpha, \beta\}$.

Lemma 3.4 *Let* $\pi_1\colon A \times B \to A$ *and* $\pi_2\colon A \times B \to B$ *be the projections. Then we have*

$$\{\alpha, \beta\} = [\alpha \circ \pi_1, \beta \circ \pi_2]$$

in the group of the homotopy set $[A \times B, G]$.

Proof For the diagonal map $\Delta\colon A \times B \to (A \times B) \times (A \times B)$, we have $(\pi_1 \times \pi_2) \circ \Delta = 1_{A \times B}$, so

$$[\alpha \circ \pi_1, \beta \circ \pi_2] = \gamma \circ (\alpha \circ \pi_1 \times \beta \circ \pi_2) \circ \Delta = \gamma \circ (\alpha \times \beta) \circ (\pi_1 \times \pi_2) \circ \Delta = \{\alpha, \beta\},$$

completing the proof. □

Recall from [9] that there is a 3–local homotopy equivalence

$$\mathrm{PU}(3) \simeq_{(3)} L \times S^3,$$

where L is the lens space $S^5/(\mathbb{Z}/3)$. Then for the inclusion $\alpha\colon L_{(3)} \to \mathrm{PU}(3)_{(3)}$ we may assume that the composite

$$L_{(3)} \times S^3_{(3)} \xrightarrow{\alpha \times \iota_{(3)}} \mathrm{PU}(3)_{(3)} \times \mathrm{PU}(3)_{(3)} \to \mathrm{PU}(3)_{(3)}$$

is the identity map, where the last arrow is the multiplication. Notice that this composite is equal to the product $(\alpha \circ \pi_1) \cdot (\iota_{(3)} \circ \pi_2)$ in the group $[L_{(3)} \times S^3_{(3)}, \mathrm{PU}(3)_{(3)}]$, where $\pi_1\colon L_{(3)} \times S^3_{(3)} \to L_{(3)}$ and $\pi_2\colon L_{(3)} \times S^3_{(3)} \to S^3_{(3)}$ are the projections.

Proposition 3.5 *The Samelson product $\langle k\iota, 1_{\mathrm{PU}(3)}\rangle_{(3)}$ is trivial if and only if $\langle k\iota_{(3)}, \alpha\rangle$ and $\langle k\iota, \iota\rangle_{(3)}$ are trivial.*

Proof If $\langle k\iota_{(3)}, \alpha\rangle$ and $\langle k\iota, \iota\rangle_{(3)}$ are trivial, so are $\langle k\iota_{(3)}, \alpha\circ\pi_1\rangle = \langle k\iota_{(3)}, \alpha\rangle\circ(1\wedge\pi_1)$ and $\langle k\iota_{(3)}, \iota_{(3)}\circ\pi_2\rangle = \langle k\iota, \iota\rangle_{(3)}\circ(1\wedge\pi_2)$, implying $\{k\iota_{(3)}, \alpha\circ\pi_1\}$ and $\{k\iota_{(3)}, \iota_{(3)}\circ\pi_2\}$ are trivial. Then by Lemma 3.3 and 3.4, $\{k\iota, 1_{\mathrm{PU}(3)}\}_{(3)}$ is trivial, implying so is $\langle k\iota, 1_{\mathrm{PU}(3)}\rangle_{(3)}$. The converse direction obviously holds. □

Then in order to get an upper bound for the order of $\langle k\iota, 1_{\mathrm{PU}(3)}\rangle_{(3)}$ we calculate the homotopy set $[\Sigma^3 L, \mathrm{PU}(3)]_{(3)}$ and $\pi_6(\mathrm{PU}(3))_{(3)}$.

Proposition 3.6 $[\Sigma^3 L, \mathrm{PU}(3)]_{(3)} \cong (\mathbb{Z}/3)^3$ *and* $\pi_6(\mathrm{PU}(3))_{(3)} \cong \mathbb{Z}/3$.

Proof Let P^n be the Moore space $S^{n-1}\cup_3 e^n$. By [14] there is a 3–local homotopy equivalence $\Sigma L \simeq_{(3)} P^3 \vee P^5 \vee S^6$, so there is a group isomorphism

$$[\Sigma^3 L, \mathrm{PU}(3)]_{(3)} \cong [P^5, \mathrm{PU}(3)]_{(3)} \times [P^7, \mathrm{PU}(3)]_{(3)} \times \pi_8(\mathrm{PU}(3))_{(3)}.$$

Since the projection $\mathrm{SU}(3)\to\mathrm{PU}(3)$ is an isomorphism in the homotopy groups of dimensions ≥ 2, it follows from [15] that

$$\pi_i(\mathrm{PU}(3))_{(3)} \cong \begin{cases} \mathbb{Z}_{(3)}, & i=5, \\ \mathbb{Z}/3, & i=6,8, \\ 0, & i=4,7. \end{cases}$$

The cofibration $S^{n-1}\xrightarrow{3} S^{n-1}\to P^n$ induces an exact sequence

$$\pi_n(\mathrm{PU}(3))_{(3)} \xrightarrow{3} \pi_n(\mathrm{PU}(3))_{(3)} \to [P^n, \mathrm{PU}(3)]_{(3)} \to \pi_{n-1}(\mathrm{PU}(3))_{(3)} \xrightarrow{3} \pi_{n-1}(\mathrm{PU}(3))_{(3)}.$$

Then we get $[P^5, \mathrm{PU}(3)]_{(3)} \cong [P^7, \mathrm{PU}(3)]_{(3)} \cong \mathbb{Z}/3$. □

Hence we immediately obtain:

Corollary 3.7 *The order of $\langle \iota, 1_{\mathrm{PU}(3)}\rangle_{(3)}$ is at most* 3.

We next consider the lower bound for the order of $\langle \iota, 1_{\mathrm{PU}(3)}\rangle_{(3)}$. Since

$$\pi_*\colon [S^3\wedge\mathrm{SU}(3), \mathrm{SU}(3)] \to [S^3\wedge\mathrm{SU}(3), \mathrm{PU}(3)]$$

is an isomorphism, $\pi \circ \langle \iota, 1_{\mathrm{SU}(3)} \rangle_{(3)}$ is nontrivial by Proposition 3.1. Then for (3-1), we get that $\langle \iota, 1_{\mathrm{PU}(3)} \rangle_{(3)}$ is nontrivial. Thus we obtain:

Proposition 3.8 *The order of $\langle \iota, 1_{\mathrm{PU}(3)} \rangle_{(3)}$ is at least 3.*

Then the order of $\langle \iota, 1_{\mathrm{PU}(3)} \rangle$ is determined by Corollary 3.2 and 3.7, and Proposition 3.8.

Theorem 3.9 *The order of $\langle \iota, 1_{\mathrm{PU}(3)} \rangle$ is 24.*

A Samelson product in PSp(2)

We next determine the order of the Samelson product $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle$ in $\mathrm{PSp}(2)$ for a generator $\iota\colon S^3 \to \mathrm{PSp}(2)$. As in the $\mathrm{PU}(3)$ case, for the projection homomorphism $\pi\colon \mathrm{Sp}(2) \to \mathrm{PSp}(2)$ we have $\pi \circ \iota = \iota$, so we get

$$\langle \iota, 1_{\mathrm{PSp}(2)} \rangle \circ (1 \wedge \pi) = \pi \circ \langle \iota, 1_{\mathrm{Sp}(2)} \rangle. \tag{3-2}$$

The order of $\langle \iota, 1_{\mathrm{Sp}(2)} \rangle$ is calculated in [17].

Proposition 3.10 [17] *The order of $\langle \iota, 1_{\mathrm{Sp}(2)} \rangle$ is 40.*

Then, since there is a fibration $\mathrm{Sp}(2) \to \mathrm{PSp}(2) \to B\mathbb{Z}/2$, as well as Corollary 3.2 we have:

Corollary 3.11 *The order of $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle_{(p)}$ is 5 if $p = 5$ and 1 if $p \neq 2, 5$.*

We calculate the 2–primary component of the order of $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle$. We may assume that the map $\iota\colon S^3 \to \mathrm{PSp}(2)$ is the inclusion of the subgroup $\mathrm{Sp}(1) \times 1 \subset \mathrm{PSp}(2)$. Since the centralizer of $\mathrm{Sp}(1) \times 1$ in $\mathrm{PSp}(2)$ includes $1 \times \mathrm{Sp}(1) \subset \mathrm{PSp}(2)$, the Samelson product $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle$ factors through $\Sigma^3(\mathrm{PSp}(2)/(1 \times \mathrm{Sp}(1)))$. Then for $\mathrm{PSp}(2)/(1 \times \mathrm{Sp}(1)) = \mathbb{R}P^7$, we get:

Proposition 3.12 *The Samelson product $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle$ factors through $\Sigma^3 \mathbb{R}P^7$.*

Proposition 3.13 *The order of $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle_{(2)}$ is at most 8.*

Proof We calculate the homotopy set $[\Sigma^3 \mathbb{R}P^7, \mathrm{PSp}(2)]_{(2)}$. It is shown by [16] that there is a homotopy equivalence $\Sigma^2 \mathbb{R}P^7 \simeq \Sigma^2 \mathbb{R}P^6 \vee S^9$. Then there is a group isomorphism

$$[\Sigma^3 \mathbb{R}P^7, \mathrm{PSp}(2)]_{(2)} \cong [\Sigma^3 \mathbb{R}P^6, \mathrm{PSp}(2)]_{(2)} \times \pi_{10}(\mathrm{PSp}(2))_{(2)}.$$

By [15] we have $\pi_{10}(\mathrm{PSp}(2))_{(2)} \cong \pi_{10}(\mathrm{Sp}(2))_{(2)} \cong \mathbb{Z}/8$. Since $\Sigma^3\mathbb{R}P^6$ is simply connected, $[\Sigma^3\mathbb{R}P^6, \mathrm{PSp}(2)] \cong [\Sigma^3\mathbb{R}P^6, \mathrm{Sp}(2)]$ as groups. Since the inclusion $B\mathrm{Sp}(2) \to B\mathrm{Sp}(\infty)$ is an 11–equivalence, we have

$$[\Sigma^3\mathbb{R}P^6, \mathrm{Sp}(2)] \cong [\Sigma^4\mathbb{R}P^6, B\mathrm{Sp}(2)] \cong [\Sigma^4\mathbb{R}P^6, B\mathrm{Sp}(\infty)] \cong \widetilde{\mathrm{KSp}}(\Sigma^4\mathbb{R}P^6),$$

and by Bott periodicity and [1], we get

$$\widetilde{\mathrm{KSp}}(\Sigma^4\mathbb{R}P^6) \cong \widetilde{\mathrm{KO}}(\mathbb{R}P^6) \cong \mathbb{Z}/8.$$

Then we obtain $[\Sigma^3\mathbb{R}P^7, \mathrm{PSp}(2)]_{(2)} \cong (\mathbb{Z}/8)^2$. Therefore the proof is completed by Proposition 3.12. □

By the same argument as Proposition 3.8 we obtain a lower bound for the order of $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle_{(p)}$.

Proposition 3.14 *The order of* $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle_{(2)}$ *is at least* 8.

We now obtain the order of the Samelson product $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle$.

Theorem 3.15 *The order of* $\langle \iota, 1_{\mathrm{PSp}(2)} \rangle$ *is* 40.

Proof Combine Corollary 3.11 and Proposition 3.13 and 3.14. □

4 Proof of Theorem 1.2

The PU(3) case

We can prove the only if part of Theorem 1.2(1) by using the result of [5], but we here give an alternative simpler proof. We prepare the information of the homotopy groups of $\mathrm{PU}(3)$. Let $\pi\colon \mathrm{SU}(3) \to \mathrm{PU}(3)$ be the projection. Then $\pi_*\colon \pi_i(\mathrm{SU}(3)) \to \pi_i(\mathrm{PU}(3))$ is an isomorphism for $i > 1$, so by [15] we have the following table, which will be freely used in what follows.

i	1	2	3	4	5	6	7	8	9	10	11
$\pi_i(\mathrm{PU}(3))$	$\mathbb{Z}/3$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/6$	0	$\mathbb{Z}/12$	$\mathbb{Z}/3$	$\mathbb{Z}/30$	$\mathbb{Z}/4$

Table 1: The homotopy group of PU(3).

Proposition 4.1 $$\pi_4(\mathcal{G}_k(\mathrm{PU}(3)))_{(3)} \cong \mathbb{Z}/(3,k)$$

Proof There is an exact sequence

$$\pi_5(\mathrm{PU}(3))_{(3)} \xrightarrow{(\partial_k)_*} \pi_5(\Omega_0^3\mathrm{PU}(3))_{(3)} \to \pi_5(B\mathcal{G}_k(\mathrm{PU}(3)))_{(3)} \to \pi_5(B\mathrm{PU}(3))_{(3)}$$

associated with (2-2). From $\pi_5(B\mathrm{PU}(3))_{(3)} \cong \pi_4(\mathrm{PU}(3))_{(3)} = 0$, we have that $\pi_4(\mathcal{G}_k(\mathrm{PU}(3)))_{(3)} \cong \pi_5(B\mathcal{G}_k(\mathrm{PU}(3)))_{(3)} = \mathrm{Coker}\,(\partial_k)_*$. Let ϵ be a generator of $\pi_5(\mathrm{SU}(3)) \cong \mathbb{Z}$. By [3] the Samelson product $\langle \iota, \epsilon \rangle_{(3)}$ in $\mathrm{SU}(3)_{(3)}$ is nontrivial. Since the map $\pi_*\colon \pi_8(\mathrm{SU}(3)) \to \pi_8(\mathrm{PU}(3))$ is an isomorphism, $\pi \circ \langle \iota, \epsilon \rangle_{(3)} = \langle \iota, \pi \circ \epsilon \rangle_{(3)}$ is a generator of $\pi_8(\mathrm{PU}(3))_{(3)} \cong \mathbb{Z}/3$. Thus the proof is completed by Proposition 2.1 and Corollary 2.2. □

Proposition 4.2 $$[\mathbb{C}P^2, \mathcal{G}_k(\mathrm{PU}(3))]_{(2)} \cong \mathbb{Z}/(8,k)$$

Proof As in the proof of Proposition 4.1, we consider an exact sequence

$$\Sigma\mathbb{C}P^2, \mathrm{PU}(3)]_{(2)} \xrightarrow{(\partial_k)_*} [\Sigma\mathbb{C}P^2, \Omega_0^3\mathrm{PU}(3)]_{(2)} \\ \to [\Sigma\mathbb{C}P^2, B\mathcal{G}_k(\mathrm{PU}(3))]_{(2)} \to [\Sigma\mathbb{C}P^2, B\mathrm{PU}(3)]_{(2)}$$

associated with (2-2). It is straightforward to see by the cofibration $S^3 \to \Sigma\mathbb{C}P^2 \to S^5$ and Table 1 that $[\Sigma\mathbb{C}P^2, B\mathrm{PU}(3)]_{(2)} = 0$ and $[\Sigma\mathbb{C}P^2, \Omega_0^3\mathrm{PU}(3)]_{(2)}$ is either $\mathbb{Z}/8$ or $\mathbb{Z}/2\times\mathbb{Z}/4$. We now determine the group $[\Sigma^4\mathbb{C}P^2, \mathrm{PU}(3)]_{(2)} \cong [\Sigma\mathbb{C}P^2, \Omega_0^3\mathrm{PU}(3)]_{(2)}$. By Proposition 3.1, the order of the Samelson product $\langle \iota, 1_{\mathrm{SU}(3)} \rangle_{(2)}$ in $\mathrm{SU}(3)_{(2)}$ is 8, which is an element of $[\Sigma^3\mathrm{SU}(3), \mathrm{SU}(3)]_{(2)}$. According to [13, Lemma 2.1(1)], the top cell of $\mathrm{SU}(3)$ splits off stably and there is a homotopy equivalence $\Sigma^3\mathrm{SU}(3) \simeq \Sigma^4\mathbb{C}P^2 \vee S^{11}$. Then there is a group isomorphism

$$[\Sigma^3\mathrm{SU}(3), \mathrm{SU}(3)]_{(2)} \cong [\Sigma^4\mathbb{C}P^2, \mathrm{SU}(3)]_{(2)} \times \pi_{11}(\mathrm{SU}(3))_{(2)},$$

where $\pi_{11}(\mathrm{SU}(3))_{(2)} \cong \mathbb{Z}/4$. Then the order of $\langle \iota, \alpha \rangle$ is 8 for the canonical inclusion $\alpha\colon \Sigma\mathbb{C}P^2 \to \mathrm{SU}(3)$. Thus since $\pi_*\colon [\Sigma^4\mathbb{C}P^2, \mathrm{SU}(3)]_{(2)} \to [\Sigma^4\mathbb{C}P^2, \mathrm{PU}(3)]_{(2)}$ is an isomorphism, $\pi \circ \langle \iota, \alpha \rangle = \langle \iota, \pi \circ \alpha \rangle$ is an element of $[\Sigma^4\mathbb{C}P^2, \mathrm{PU}(3)]_{(2)}$ and is of order 8, implying that $[\Sigma^4\mathbb{C}P^2, \mathrm{PU}(3)]_{(2)} \cong \mathbb{Z}/8\{\langle \iota, \pi \circ \alpha \rangle\}$. Therefore the proof is completed by Proposition 2.1 and Corollary 2.2. □

Proof of Theorem 1.2(1) We first prove the if part of Theorem 1.2(1). By Table 1, we have $H^2(\Omega_0^3\mathrm{PU}(3); \mathbb{Z}) \cong \mathbb{Z}$, where we write its generator by $g\colon \Omega_0^3\mathrm{PU}(3) \to K(\mathbb{Z}, 2)$. By the homotopy exact sequence of the homotopy fibration (2-2) one can see that $\pi_2(B\mathcal{G}_k(\mathrm{PU}(3)))$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/3$.

(1) The case $\pi_2(B\mathcal{G}_k(\mathrm{PU}(3))) \cong \mathbb{Z}$ Let $\bar{g}\colon B\mathcal{G}_k(\mathrm{PU}(3)) \to K(\mathbb{Z}, 2)$ be a generator of $H^2(B\mathcal{G}_k(\mathrm{PU}(3)); \mathbb{Z}) \cong \mathbb{Z}$. Since the map $\pi_2(\Omega_0^3\mathrm{PU}(3)) \to \pi_2(B\mathcal{G}_k(\mathrm{PU}(3)))$ is identified with the degree 3 map $3\colon \mathbb{Z} \to \mathbb{Z}$, we may assume $3g \simeq \bar{g} \circ j$ for the canonical map $j\colon \Omega_0^3\mathrm{PU}(3) \to B\mathcal{G}_k(\mathrm{PU}(3))$. Since $3g \simeq g \circ j$ and $j \circ \partial_k$ is the composition of two consecutive maps in a homotopy fibration, the composite $(3g) \circ \partial_k$

is null homotopic, so ∂_k lifts to a map $\delta_k \colon \mathrm{PU}(3) \to F$, where F is the homotopy fiber of $3g$. Then we obtain a homotopy commutative diagram of homotopy fibrations

$$\begin{array}{ccccc}
\widehat{\mathcal{G}}_k & \longrightarrow & \mathrm{PU}(3) & \xrightarrow{\delta_k} & F \\
\downarrow & & \| & & \downarrow \\
\mathcal{G}_k(\mathrm{PU}(3)) & \longrightarrow & \mathrm{PU}(3) & \xrightarrow{\partial_k} & \Omega^3_0\mathrm{PU}(3) \\
\downarrow{\scriptstyle \Omega\bar{g}} & & \downarrow & & \downarrow{\scriptstyle 3g} \\
S^1 & \longrightarrow & * & \longrightarrow & K(\mathbb{Z},2)
\end{array}$$

where $\widehat{\mathcal{G}}_k$ is the homotopy fiber of δ_k. By definition, $\Omega\bar{g}$ is an isomorphism in π_1, so it has a right homotopy inverse, implying that there is a homotopy equivalence

$$\mathcal{G}_k(\mathrm{PU}(3)) \simeq \widehat{\mathcal{G}}_k \times S^1. \tag{4-1}$$

Since $H^1(\mathrm{PU}(3);\mathbb{Z}) = 0$, we see that $[\mathrm{PU}(3), F] \to [\mathrm{PU}(3), \Omega^3_0\mathrm{PU}(3)]$ is injective. Then by Theorem 3.9 and Corollary 2.2, we get

$$24 \circ \delta_1 \simeq * \quad \text{and} \quad \delta_k = k \circ \delta_1. \tag{4-2}$$

Now suppose $(24,k) = (24,\ell)$. It follows from Proposition 2.3 that $\pi_2(B\mathcal{G}_k(\mathrm{PU}(3))) \cong \pi_2(B\mathcal{G}_\ell(\mathrm{PU}(3)))$, implying that $\mathcal{G}_\ell(\mathrm{PU}(3))$ admits the same decomposition as (4-1). Then we compare $\widehat{\mathcal{G}}_k$ and $\widehat{\mathcal{G}}_\ell$. By definition $\pi_i(F)$ is finite for each i. Therefore by Proposition 2.4 and (4-2), $\widehat{\mathcal{G}}_k \simeq \widehat{\mathcal{G}}_\ell$, implying $\mathcal{G}_k(\mathrm{PU}(3)) \simeq \mathcal{G}_\ell(\mathrm{PU}(3))$.

(2) The case $\pi_2(B\mathcal{G}_k(\mathrm{PU}(3))) \cong \mathbb{Z} \oplus \mathbb{Z}/3$ Let $\bar{g} \colon B\mathcal{G}_k(\mathrm{PU}(3)) \to K(\mathbb{Z},2)$ be a generator of $H^2(B\mathcal{G}_k(\mathrm{PU}(3));\mathbb{Z}) \cong \mathbb{Z}$. Then we may assume $g \simeq \bar{g} \circ j$ since j is an isomorphism in H^2. Let F' be the homotopy fiber of g. For $g \circ \partial_k \simeq \bar{g} \circ j \circ \partial_k \simeq *$, ∂_k lifts to a map $\delta_k \colon \mathrm{PU}(3) \to F'$. Then as in the above case we get a homotopy commutative diagram

$$\begin{array}{ccccc}
\widehat{\mathcal{G}}'_k & \longrightarrow & \mathrm{PU}(3) & \xrightarrow{\delta'_k} & F' \\
\downarrow & & \| & & \downarrow \\
\mathcal{G}_k(\mathrm{PU}(3)) & \longrightarrow & \mathrm{PU}(3) & \xrightarrow{\partial_k} & \Omega^3_0\mathrm{PU}(3) \\
\downarrow{\scriptstyle \Omega\bar{g}} & & \downarrow & & \downarrow{\scriptstyle g} \\
S^1 & \longrightarrow & * & \longrightarrow & K(\mathbb{Z},2)
\end{array}$$

where $\widehat{\mathcal{G}}'_k$ is the homotopy fiber of δ'_k. The rest of the proof in this case is similar to the above case, so we omit it.

The if part of Theorem 1.2(1) follows from Proposition 4.1 and 4.2. □

The PSp(2) case

Let $\pi\colon \mathrm{Sp}(2) \to \mathrm{PSp}(2)$ be the projection. Since $\pi_*\colon \pi_i(\mathrm{Sp}(2)) \to \pi_i(\mathrm{PSp}(2))$ is an isomorphism for $i > 1$, we have the following table by [15] which will be used freely below.

i	1	2	3	4	5	6	7	8	9	10
$\pi_i(\mathrm{PSp}(2))$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}/120$

Table 2: The homotopy group of PSp(2).

Proposition 4.3 $\pi_6(\mathcal{G}_k(\mathrm{PSp}(2))) \cong \mathbb{Z}/12(10,k)$

Proof As in the proof of Proposition 4.1, we consider the exact sequence

$$\pi_7(\mathrm{PSp}(2)) \xrightarrow{(\partial_k)_*} \pi_7(\Omega_0^3\mathrm{PSp}(2)) \to \pi_7(B\mathcal{G}_k(\mathrm{PSp}(2))) \to \pi_7(B\mathrm{PSp}(2))$$

associated with (2-2). Then it follows from $\pi_7(B\mathrm{PSp}(2)) \cong \pi_6(\mathrm{PSp}(2)) = 0$ that $\pi_6(\mathcal{G}_k(\mathrm{PSp}(2))) \cong \pi_7(B\mathcal{G}_k(\mathrm{PSp}(2))) \cong \mathrm{Coker}(\partial_k)_*$. Let ϵ be a generator of $\pi_7(\mathrm{Sp}(2)) \cong \mathbb{Z}$. By [3] the order of the Samelson product $\langle \iota, \epsilon \rangle$ in $\mathrm{Sp}(2)$ is 10. Then $\langle \iota, \epsilon \rangle$ is 12 times a generator of $\pi_{10}(\mathrm{PSp}(2)) \cong \mathbb{Z}/120$. Since the map $\pi_*\colon \pi_{10}(\mathrm{Sp}(2)) \to \pi_{10}(\mathrm{PSp}(2))$ is an isomorphism, $\pi \circ \langle \iota, \epsilon \rangle = \langle \iota, \pi \circ \epsilon \rangle$ is 12 times a generator of $\pi_{10}(\mathrm{PSp}(2)) \cong \mathbb{Z}/120$. The proof is completed by Proposition 2.1 and 2.2. □

By Table 2, the composite $S^3 \xrightarrow{\iota} \mathrm{PSp}(2) \xrightarrow{\partial_k} \Omega_0^3\mathrm{PSp}(2)$ is null homotopic, so it lifts to a map $\theta\colon S^3 \to \mathcal{G}_k(\mathrm{PSp}(2))$ through the projection $\rho\colon \mathcal{G}_k(\mathrm{PSp}(2)) \to \mathrm{PSp}(2)$. Note that there are infinitely many choices of θ by $\pi_4(\Omega_0^3\mathrm{PSp}(2)) \cong \mathbb{Z}$. Let F_θ be the homotopy fiber of θ. Note that there is a homotopy fibration diagram:

$$\begin{array}{ccccccc}
F_\theta & \longrightarrow & S^3 & \xrightarrow{\theta} & \mathcal{G}_k(\mathrm{PSp}(2)) & & \\
\downarrow & & \| & & \downarrow{\scriptstyle \rho} & & \\
\Omega\mathbb{R}P^7 & \longrightarrow & S^3 & \xrightarrow{\iota} & \mathrm{PSp}(2) & \longrightarrow & \mathbb{R}P^7 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \partial_k} & & \downarrow{\scriptstyle \delta_k} \\
\Omega_0^4\mathrm{PSp}(2) & \longrightarrow & * & \longrightarrow & \Omega_0^3\mathrm{PSp}(2) & = & \Omega_0^3\mathrm{PSp}(2)
\end{array}$$

Lemma 4.4 *For any choice of* θ*, we have*

$$\pi_5(F_\theta)_{(2)} \cong \mathbb{Z}/(8,k).$$

Proof As in the previous section the boundary maps $\partial_k\colon \mathrm{Sp}(2) \to \Omega_0^3\mathrm{Sp}(2)$ and $\partial_k\colon \mathrm{PSp}(2) \to \Omega_0^3\mathrm{PSp}(2)$ factor as

$$\mathrm{Sp}(2) \xrightarrow{\text{proj}} S^7 \xrightarrow{\delta_k} \Omega_0^3\mathrm{Sp}(2) \quad \text{and} \quad \mathrm{PSp}(2) \xrightarrow{\text{proj}} \mathbb{R}P^7 \xrightarrow{\delta_k} \Omega_0^3\mathrm{PSp}(2),$$

respectively. Note that $\delta_k = k \circ \delta_1$ since $\partial_k = k \circ \partial_1$. By definition there is a commutative diagram:

$$\begin{array}{ccc} S^7 & \xrightarrow{\delta_1} & \Omega_0^3\mathrm{Sp}(2) \\ \big\downarrow{\scriptstyle \text{proj}} & & \big\downarrow{\scriptstyle \Omega^3\pi} \\ \mathbb{R}P^7 & \xrightarrow{\delta_1} & \Omega_0^3\mathrm{PSp}(2) \end{array}$$

By Proposition 3.10 and $\pi_{10}(\mathrm{Sp}(2)) \cong \mathbb{Z}/120$, the map $\delta_1\colon S^7 \to \Omega_0^3\mathrm{Sp}(2)$ is a generator of $\pi_7(\Omega_0^3\mathrm{Sp}(2))_{(2)} \cong \mathbb{Z}/8$. Then since $\Omega^3\pi$ is a homotopy equivalence, the composite around the left perimeter is a generator of $\pi_7(\Omega_0^3\mathrm{PSp}(2))_{(2)} \cong \mathbb{Z}/8$. Then for $\delta_k = k \circ \delta_1$, we obtain

$$\mathrm{Coker}\,\{(\delta_k)_*\colon \pi_7(\mathbb{R}P^7)_{(2)} \to \pi_7(\Omega_0^3\mathrm{PSp}(2))_{(2)}\} \cong \mathbb{Z}/(8,k). \tag{4-3}$$

Consider the homotopy exact sequence

$$\pi_6(\Omega(\mathbb{R}P^7)) \xrightarrow{(\delta_k)_*} \pi_5(\Omega_0^5(\mathrm{PSp}(2))) \to \pi_5(F_\theta) \to \pi_5(\Omega(\mathbb{R}P^7)).$$

Then for $\pi_6(\mathbb{R}P^7) = 0$, the proof is completed by (4-3). □

Lemma 4.5 *The map* $\rho_*\colon \pi_4(\mathcal{G}_k(\mathrm{PSp}(2))) \to \pi_4(\mathrm{PSp}(2))$ *is an isomorphism, where* ρ *is the evaluation map in (2-2).*

Proof The lemma follows from the homotopy exact sequence associated with (2-2) together with Table 2. □

Proposition 4.6 *If* $\mathcal{G}_k(\mathrm{PSp}(2)) \simeq_{(2)} \mathcal{G}_\ell(\mathrm{PSp}(2))$*, then* $(8,k) = (8,\ell)$*.*

Proof Let $\theta\colon S^3 \to \mathcal{G}_k(\mathrm{PSp}(2))$ be as above. By Lemma 4.5, ρ is an isomorphism in π_4. Since $\iota\colon S^3 \to \mathrm{PSp}(2)$ is the composite of the inclusion of the 6–skeleton $\iota\colon S^3 \to \mathrm{Sp}(2)$ and the covering projection $\mathrm{Sp}(2) \to \mathrm{PSp}(2)$, $\iota\colon S^3 \to \mathrm{PSp}(2)$ is an isomorphism in π_4. So θ is also an isomorphism in π_4 for $\iota \simeq \rho \circ \theta$. Let $h\colon \mathcal{G}_k(\mathrm{PSp}(2))_{(2)} \to \mathcal{G}_\ell(\mathrm{PSp}(2))_{(2)}$ be a homotopy equivalence. Then the composite

$$S^3_{(2)} \xrightarrow{\theta_{(2)}} \mathcal{G}_k(\mathrm{PSp}(2))_{(2)} \xrightarrow{\simeq} \mathcal{G}_\ell(\mathrm{PSp}(2))_{(2)} \xrightarrow{\rho_{(2)}} \mathrm{PSp}(2)_{(2)}$$

is an isomorphism in π_4 by the above observation. Now for $\pi_4(S^3) \cong \mathbb{Z}/2$, we conclude that $\rho_{(2)} \circ h \circ \theta_{(2)} \simeq (2q+1)\iota_{(2)}$ for some integer q. The degree $2q+1$ map of $S^3_{(2)}$ is a homotopy equivalence, since we have its inverse $1/(2q+1)$. Put

$$\bar{\theta} = \frac{1}{2q+1} h \circ \theta_{(2)}.$$

Then there is a homotopy commutative diagram

$$\begin{array}{ccc} S^3_{(2)} & \xrightarrow{\theta_{(2)}} & \mathcal{G}_k(\mathrm{PSp}(2))_{(2)} \\ \simeq \Big\downarrow 2q+1 & & \simeq \Big\downarrow h \\ S^3_{(2)} & \xrightarrow{\bar{\theta}} & \mathcal{G}_\ell(\mathrm{PSp}(2))_{(2)} \end{array}$$

and $\rho_{(2)} \circ \bar{\theta} \simeq \iota_{(2)}$. Thus the homotopy fibers of $\theta_{(2)}$ and $\bar{\theta}$ are homotopy equivalent, and therefore the proof is completed by Lemma 4.4. □

Remark 4.7 We can replace PSp(2) with Sp(2) in Proposition 4.6 to yield an alternative simpler proof of [17, Proposition 5.8] at the prime 2.

Proof of Theorem 1.2(2) By Proposition 2.3 and Theorem 3.15 if $(40,k) = (40,\ell)$, then $\mathcal{G}_k(\mathrm{PSp}(2)) \simeq_{(p)} \mathcal{G}_\ell(\mathrm{PSp}(2))$ for any prime p. The converse follows from Proposition 4.3 and 4.6. □

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