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THE HOPF ALGEBRA OF FINITE TOPOLOGIES AND MOULD COMPOSITION

by Frédéric FAUVET, Loïc FOISSY & Dominique MANCHON

Abstract. — We exhibit an internal coproduct on the Hopf algebra of finite topologies recently defined by the second author, C. Malvenuto and F. Patras, dual to the composition of "quasi-ormoulds", which are the natural version of J. Ecalle's moulds in this setting. All these results are displayed in the linear species formalism.

Résumé. — Nous mettons en évidence un coproduit interne sur l'algèbre de Hopf des topologies finies introduite récemment par C. Malvenuto, F. Patras et le second auteur. Ce coproduit est dual de la composition des "quasi-ormoules", version naturelle des moules, selon la terminologie de J. Ecalle, dans ce contexte.

1. Introduction

The study of finite topological spaces was initiated by Alexandroff in 1937 [4], and revived at several periods since then, using the natural bijection, recalled below, which exists between these spaces and finite sets endowed with a quasi–order. In [16], the topic was reexamined through the angle of Hopf algebraic techniques, which have proved quite pervasive in algebraic combinatorics in recent years. A number of so–called combinatorial Hopf algebras (graded and linearly spanned by combinatorial objects) are now of constant use in many parts of mathematics, with frequent occurences of the Hopf algebras of shuffles and quasishuffles, quasi-symmetric functions QSym [19], non commutative symmetric functions, Connes–Kreimer, Malvenuto–Reutenauer, word quasisymmetric functions WQSym, etc [8, 17, 18, 19, 20, 21]. This type of machinery to study finite spaces was implemented in the article [16], with the introduction of a commutative Hopf algebra $\mathcal{H}$ based on (isomorphism classes of) quasi–posets. These constructions were investigated further in the article [15] (see

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also [13, 14]) with in particular the description of a non commutative and non cocommutative Hopf algebra $\mathcal{H}_{\mathbf{T}}$ based on labelled quasi–posets. In the present text we show that both $\mathcal{H}$ and $\mathcal{H}_{\mathbf{T}}$ can be endowed with a second coproduct, which is degree–preserving and as such called internal. The construction of the coproduct is non–trivial and is in fact achieved within the formalism of linear species.

In [15], a family of natural morphisms from $\mathcal{H}_{\mathbf{T}}$ to WQSym was also constructed, based on the classical concept of linear extensions [28]. In the present text, we show that one of these morphisms also respects the internal coproduct. Once again, this is realized at the level of species: we construct a morphism $L : \mathbb{T} \twoheadrightarrow \mathbb{SC}$ where $\mathbb{T}$ is the linear species of finite topological spaces and $\mathbb{SC}$ is the linear species of set *compositions* (or *ordered partitions*). This morphism specializes to maps from the Hopf algebras of quasi–posets $\mathcal{H}$ and $\mathcal{H}_{\mathbf{T}}$ onto QSym and WQSym, respecting the products and both the external and internal products.

The internal coproduct on the species $\mathbb{T}$ would certainly have been very difficult to find by simple guess but it is in fact directly inspired by an operation known in J. Ecalle's mould calculus [9, 11, 12, 24] as *mould composition*. A mould is a collection $M^\bullet = \{M^\omega\}$ of elements of some commutative algebra $\mathcal{A}$, indexed by finite sequences $\omega = (\omega_1, \dots, \omega_r)$ of elements of a set Ω; equivalently, it is an $\mathcal{A}$–valued function on the set of words $\omega_1 \dots \omega_r$ in the alphabet Ω. In what follows, the alphabet is in fact the underlying set of an additive semi–group, a typical example in the applications being the set of positive integers $\Omega = \mathbb{N}_{>0}$. We can already notice that from the outset, moulds involve combinatorial objects which are both labelled and decorated: the labels form a finite sequence $[n] = \{1, \dots, n\}$ and the decorations belong to ω. When the values of a mould $M^\bullet$ are in fact independent from any set Ω, the mould $M^\bullet$ is said to be *of constant type*.

In the context in which they originated, namely the classification of dynamical sytems, moulds naturally appear matched with dual objects, named comoulds, in expansions of the following form:

$$F = \sum M^\omega B_\omega = \sum_{r \geqslant 0} \sum_{\omega = (\omega_1, \dots, \omega_r)} M^\omega B_\omega$$

A comould $B_\bullet = \{B_\omega\}$ is a collection, indexed by sequences ω as above, of elements of some bialgebra $(\mathcal{B}, +, ., \sigma)$, and such expansions, known as mould–comould *contractions*, make sense in the completed algebra spanned by the B_ω, with respect to the gradings given by the length of sequences (other gradings may be relevant). In most situations, the B_ω are products

of some building blocks B_ω ($\omega \in \Omega$) : $B_\omega = B_{\omega_r} \dots B_{\omega_1}$; the building blocks themselves are abstracted from the dynamical system under study and are mapped to ordinary differential operators acting on spaces of formal series, through some evaluation morphism ([9, 12]).

Accordingly, these expansions can be realized as elements of completions of huge linear spaces of operators, typically $\mathrm{End}(\mathbb{C}[\![x]\!])$, and as such they are naturally endowed with a linear structure and two non–linear operations, a product $\times$ and a composition product $\circ$. Indeed, for a given comould $B_\bullet$ and two moulds $M^\bullet$ and $N^\bullet$, the product of the operators associated respectively to $N^\bullet$ and $M^\bullet$ can be expanded as a contraction with $B_\bullet$, yielding a new mould $P^\bullet = M^\bullet \times N^\bullet$:

$$\left(\sum N^\omega B_\omega\right)\left(\sum M^\omega B_\omega\right) = \sum P^\omega B_\omega = \sum_{r\geqslant 0} \sum_{\omega=(\omega_1,\dots,\omega_r)} P^\omega B_\omega$$

and the formula giving the components of the product mould $P_\bullet$ is as follows:

$$P^{(\omega_1,\dots,\omega_r)} = \sum M^{(\omega_1,\dots,\omega_i)} N^{(\omega_{i+1},\dots,\omega_r)}$$

The product is obviously associative, non commutative in general, and distributive over the sum.

But besides this product of operators, we can also use some given mould M *to change the alphabet* B_ω and this will give us the composition $\circ$ of moulds, $B_\bullet \longrightarrow C_\bullet$ with:

$$C_{\omega_0} = \sum_{\|\omega\|=\omega_0} M^\omega B_\omega$$

where the norm of the sequence ω is by definition $\|(\omega_1, \dots, \omega_r)\| = \omega_1 + \dots + \omega_r$.

Performing this natural change of alphabets successively with two moulds $M^\bullet$ and $N^\bullet$, amounts to a change of alphabet with respect to a mould $Q^\bullet = M^\bullet \circ N^\bullet$ which is given by:

$$Q^{(\omega_1,\dots,\omega_r)} = \sum M^{(\|\omega^1\|,\dots\|\omega^s\|)} N^{\omega^1} \dots N^{\omega^s}$$

the sum being performed over all the ways of obtaining the sequence ω by concatenation of the subsequences ω^i : $\omega = \omega^1 \dots \omega^s$.

The composition product is also associative, non commutative in general, and right–distributive over the sum and product. It is worth noticing that the operation of *mould composition* involve *compositions of integers*.

Next, to tackle difficult questions of analytic classification, J. Ecalle had been driven to reorder mould–comould contractions by a systematic use of trees ([9]), by considering so–called arborescent moulds, armoulds for

short, which are indexed by sequences with arborescent partial orders (each element has at most one antecedent) on the labelling sets $[r] = \{1, \dots, r\}$.

In this context, the product of armoulds is nothing but the convolution with respect to Connes–Kreimer coproduct, when armoulds are seen as characters of the relevant Hopf algebra on trees ([12]). There is also a natural definition of composition of armoulds (it appears in particular in [23]), related to another coproduct on the algebra of decorated forests, which is a decorated version of the coproduct introduced and studied in [7] (see also [22]) and which corresponds to the operation of substitution in the theory of B–series. This last coproduct involves suppression of edges on a given tree and a notion of quotient tree which is the one that was to be conveniently generalized to partial orders and finally to quasi–orders in the present text.

In fact, as mentioned e.g. in the paper [11], the natural operations $+, \times, \circ$ on (ordinary) moulds and armoulds can be extended to moulds associated to sequences with a general partial order, the name ormoulds being coined for such objects by J. Ecalle. An ormould $M^\sharp$, with values in the commutative algebra $\mathcal{A}$, and indexed by elements of a semi–group Ω, is a collection of elements of $\mathcal{A}$ indexed by orsequences $\omega^\sharp$, namely sequences $\omega = (\omega_1, \dots, \omega_r)$ of elements of Ω endowed with an order on the labelling set $[r]$. It is indeed possible to give ([10]) quite natural definitions for the product and composition product of ormoulds, involving the concept of "orderable partition" of a poset and the general study of ormoulds with Hopf algebraic techniques will be the object of a separate article.

All these definitions, constructions, symmetries and operations on moulds can in fact be applied to quasi–posets, yielding very natural definitions of a product and a composition product on sequences of elements with a quasi order on the labelling sets ("quasi–ormoulds"). Eventually, when the quasi–ormoulds are interpreted as characters, the product corresponds to the external coproduct on $\mathcal{H}$ and $\mathcal{H}_{\mathbf{T}}$ and the composition product yields the internal coproduct that we introduce and describe in the present text.

This paper is organized as follows: after some background material on finite topological spaces, we introduce the notion of quotient of a topology $\mathcal{T}$ on a finite set X by another topology $\mathcal{T}'$ finer than $\mathcal{T}$. The "quotient topology" $\mathcal{T}/\mathcal{T}'$ thus obtained lives on the same set. We introduce the relation $\circledcirc$ on the topologies on X defined by $\mathcal{T}' \circledcirc \mathcal{T}$ if and only if $\mathcal{T}'$ is finer than $\mathcal{T}$ and fulfills the technical condition of "$\mathcal{T}$-admissibility" given

in Definition 2.2. This enables us to give in Section 3.1 the "internal" coproduct:

$$\Gamma(\mathcal{T}) = \sum_{\mathcal{T}' \otimes \mathcal{T}} \mathcal{T}' \otimes \mathcal{T}/\mathcal{T}'. \tag{1.1}$$

and prove its coassociativity. Each $\mathbb{T}_X$, where $\mathbb{T}$ is the linear species of finite topological spaces, is thus endowed with a structure of (finite-dimensional) pointed counital coalgebra. The species $\mathbb{T}$ has also a Hopf monoid structure which we account for in Sections 3.2 and 3.3. A key result is Theorem 3.6, which states that the internal coproduct Γ and the external coproduct Δ are compatible.

In Section 4, we obtain from this result an extra internal coproduct on the two Hopf algebras $\mathcal{H}$ and $\mathcal{H}_{\mathbf{T}}$ introduced in [15, 16]. The Hopf algebra $\mathcal{H}$ is commutative and the two coproducts are compatible in it, in the sense that $(\mathcal{H}, \cdot, \Delta)$ is comodule-coalgebra on the bialgebra $(\mathcal{H}, \cdot, \Gamma)$.

Finally we define in Section 5 the set $\mathcal{L}_{\mathcal{T}}$ of linear extensions of a topology $\mathcal{T}$ on a finite set X. These are ordered partitions of the set X subject to natural compatibility conditions with respect to the topology. It is well-known that the species $\mathbb{SC}$ of set compositions admits a Hopf monoid structure and an internal coproduct, the latter being dual to the Tits product [1, 2, 5, 6]. We recall these facts in some detail and we show (Theorem 5.6) that the surjective species map $L : \mathbb{T} \twoheadrightarrow \mathbb{SC}$ defined by:

$$L(\mathcal{T}) := \sum_{C \in \mathcal{L}_{\mathcal{T}}} C$$

respects both Hopf monoid structures as well as the internal coproducts. As an application we give two surjective Hopf algebra maps $\lambda : \mathcal{H} \to \to$ QSym and $\Lambda : \mathcal{H}_{\mathbf{T}} \twoheadrightarrow$ WQSym which moreover respect the internal coproducts. The maps L, λ and Λ are analogues of the arborification map of J. Ecalle [9, 12] from the Connes-Kreimer Hopf algebra of rooted forests to the shuffle or quasi-shuffle Hopf algebra.

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2. Refinement and quotient topologies

2.1. Finite topological spaces and quasi-orders

Recall (see e.g. [29, 30]) that a topology on a finite set X is given by the family $\mathcal{T}$ of open subsets of X subject to the following three axioms:

- $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$,
- The union of open subsets is an open subset,
- The intersection of a finite number of open subsets is an open subset.

The finiteness of X allows to consider only finite unions in the second axiom, so that axioms 2 and 3 become dual to each other. In particular the dual topology is defined by

$$\overline{\mathcal{T}} := \{X \backslash Y,\, Y \in \mathcal{T}\}. \tag{2.1}$$

In other words, open subsets in $\mathcal{T}$ are closed subsets in $\overline{\mathcal{T}}$ and vice-versa. Any topology $\mathcal{T}$ on X defines a quasi-order (i.e. a reflexive transitive relation) denoted by $\leqslant_{\mathcal{T}}$ on X:

$$x \leqslant_{\mathcal{T}} y \Longleftrightarrow \text{ any open subset containing } x \text{ also contains } y. \tag{2.2}$$

Conversely, any quasi-order $\leqslant$ on X defines a topology $\mathcal{T}_{\leqslant}$ given by its *upper ideals*, i.e. subsets $Y \subset X$ such that $(y \in Y \text{ and } y \leqslant z) \Rightarrow z \in Y$. Both operations are inverse to each other:

$$\leqslant_{\mathcal{T}_{\leqslant}} = \leqslant, \qquad\qquad \mathcal{T}_{\leqslant_{\mathcal{T}}} = \mathcal{T}.$$

Hence there is a natural bijection between topologies and quasi-orders on a finite set X.

Any quasi-order (hence any topology $\mathcal{T}$) on X gives rise to an equivalence class:

$$x \sim_{\mathcal{T}} y \Longleftrightarrow (x \leqslant_{\mathcal{T}} y \text{ and } y \leqslant_{\mathcal{T}} x). \tag{2.3}$$

This equivalence relation is trivial if and only if the quasi-order is a (partial) order, which is equivalent to the fact that the topology $\mathcal{T}$ is T_0. Any topology $\mathcal{T}$ on X defines a T_0 topology on the quotient $X/\sim_{\mathcal{T}}$, corresponding to the partial order induced by the quasi-order $\leqslant_{\mathcal{T}}$. Hence any finite topological set can be represented by the Hasse diagram of its T_0 quotient.

Figure 2.1. A finite topological space with 10 elements and 4 equivalence classes

2.2. Refinements and quotient topologies

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a finite set X. We say that $\mathcal{T}'$ is finer than $\mathcal{T}$, and we write $\mathcal{T}' \prec \mathcal{T}$, when any open subset for $\mathcal{T}$ is an open subset for $\mathcal{T}'$. This is equivalent to the fact that for any $x, y \in X$, $x \leqslant_{\mathcal{T}'} y \Rightarrow x \leqslant_{\mathcal{T}} y$.

The *quotient* $\mathcal{T}/\mathcal{T}'$ of two topologies $\mathcal{T}$ and $\mathcal{T}'$ with $\mathcal{T}' \prec \mathcal{T}$ is defined as follows: the associated quasi-order $\leqslant_{\mathcal{T}/\mathcal{T}'}$ is the transitive closure of the relation $\mathcal{R}$ defined by:

$$x \mathcal{R} y \iff (x \leqslant_{\mathcal{T}} y \text{ or } y \leqslant_{\mathcal{T}'} x). \tag{2.4}$$

Note that, contrarily to what is usually meant by "quotient topology", $\mathcal{T}/\mathcal{T}'$ is a topology on the same finite space X as the one on which $\mathcal{T}$ and $\mathcal{T}'$ are given. The definitions immediately yield compatibility of the quotient with the involution, i.e.

$$\overline{\mathcal{T}/\mathcal{T}'} = \overline{\mathcal{T}} \big/ \overline{\mathcal{T}'}. \tag{2.5}$$

Examples.

(1) If $\mathcal{D}$ is the discrete topology on X, for which any subset is open, the quasi-order $\leqslant_{\mathcal{D}}$ is nothing but $x \leqslant_{\mathcal{D}} y \Leftrightarrow x = y$, and then $\mathcal{T}/\mathcal{D} = \mathcal{T}$.

(2) For any topology $\mathcal{T}$, the quotient $\mathcal{T}/\mathcal{T}$ has the same connected components than $\mathcal{T}$, and the restriction of $\mathcal{T}/\mathcal{T}$ to any connected component is the coarse topology. In other words, for any $x, y \in X$, x and y are in the same connected component for $\mathcal{T}$ if and only if $x \leqslant_{\mathcal{T}/\mathcal{T}} y$, which is also equivalent to $x \sim_{\mathcal{T}/\mathcal{T}} y$.

LEMMA 2.1. — *Let $\mathcal{T}'' \prec \mathcal{T}' \prec \mathcal{T}$ be three topologies on X. Then $\mathcal{T}'/\mathcal{T}'' \prec \mathcal{T}/\mathcal{T}''$, and we have the following equality between topologies*

on X:

$$\mathcal{T}/\mathcal{T}' = (\mathcal{T}/\mathcal{T}'')\Big/(\mathcal{T}'/\mathcal{T}'') \tag{2.6}$$

Proof. — We compare the associated quasi-orders. The first assertion is obvious. For $x, y \in X$ we write $x\mathcal{R}y$ for ($x \leqslant_{\mathcal{T}} y$ or $y \leqslant_{\mathcal{T}'} x$), and $x\mathcal{Q}y$ for ($x \leqslant_{\mathcal{T}/\mathcal{T}''} y$ or $y \leqslant_{\mathcal{T}'/\mathcal{T}''} x$). We have $x \leqslant_{\mathcal{T}/\mathcal{T}'} y$ if and only if there exist $a_1, \dots, a_p \in X$ such that

$$x\mathcal{R}a_1\mathcal{R}\cdots\mathcal{R}a_p\mathcal{R}y.$$

On the other hand,

$$\begin{aligned} x \leqslant_{(\mathcal{T}/\mathcal{T}'')/(\mathcal{T}'/\mathcal{T}'')} y &\iff \exists b_1, \dots, b_q \in X,\ x\mathcal{Q}b_1\mathcal{Q}\cdots\mathcal{Q}b_q\mathcal{Q}y \\ &\iff \exists c_1, \dots, c_r \in X,\ x\widetilde{\mathcal{R}}c_1\widetilde{\mathcal{R}}\cdots\widetilde{\mathcal{R}}c_r\widetilde{\mathcal{R}}y, \end{aligned}$$

with

$$\begin{aligned} a\widetilde{\mathcal{R}}b &\iff (a \leqslant_{\mathcal{T}} b \text{ or } b \leqslant_{\mathcal{T}''} a) \text{ or } (b \leqslant_{\mathcal{T}'} a \text{ or } a \leqslant_{\mathcal{T}''} b) \\ &\iff a \leqslant_{\mathcal{T}} b \text{ or } b \leqslant_{\mathcal{T}'} a \\ &\iff a\mathcal{R}b. \end{aligned}$$

Hence,

$$x \leqslant_{(\mathcal{T}/\mathcal{T}'')/(\mathcal{T}'/\mathcal{T}'')} y \iff x \leqslant_{\mathcal{T}/\mathcal{T}'} y. \qquad \square$$

Definition 2.2. — *Let $\mathcal{T}' \prec \mathcal{T}$ be two topologies on X. We will say that $\mathcal{T}'$ is* $\mathcal{T}$-admissible *if*

- $\mathcal{T}'_{|_Y} = \mathcal{T}_{|_Y}$ *for any subset $Y \subset X$ connected for the topology $\mathcal{T}'$,*
- *For any $x, y \in X$, $x \sim_{\mathcal{T}/\mathcal{T}'} y \iff x \sim_{\mathcal{T}'/\mathcal{T}'} y$.*

In particular, $\mathcal{T}$ is $\mathcal{T}$-admissible. We write $\mathcal{T}' \circledcirc \mathcal{T}$ when $\mathcal{T}' \prec \mathcal{T}$ and $\mathcal{T}'$ is $\mathcal{T}$-admissible. Note that the reverse implication in the second axiom is always true for $\mathcal{T}' \prec \mathcal{T}$. It follows easily from (2.5) that $\mathcal{T}' \circledcirc \mathcal{T}$ if and only if $\overline{\mathcal{T}'} \circledcirc \overline{\mathcal{T}}$.

Lemma 2.3. — *If $\mathcal{T}' \circledcirc \mathcal{T}$, then we have for any $x, y \in X$:*

$$x \sim_{\mathcal{T}'} y \iff x \sim_{\mathcal{T}} y.$$

Proof. — The direct implication is obvious. Conversely, if $x \sim_{\mathcal{T}} y$ then $x \sim_{\mathcal{T}/\mathcal{T}'} y$, hence $x \sim_{\mathcal{T}'/\mathcal{T}'} y$, which means that x and y are in the same $\mathcal{T}'$-connected component. The restrictions of $\mathcal{T}$ and $\mathcal{T}'$ on this component coincide, hence $x \sim_{\mathcal{T}'} y$. $\square$

Lemma 2.4. — *If $\mathcal{T}' \prec \mathcal{T}$, the connected components of $\mathcal{T}/\mathcal{T}'$ are the same as those of $\mathcal{T}$.*

Proof. — The connected components of $\mathcal{T}$, resp. $\mathcal{T}/\mathcal{T}'$, are nothing but the equivalence classes for $\mathcal{T}/\mathcal{T}$, resp.$(\mathcal{T}/\mathcal{T}')/(\mathcal{T}/\mathcal{T}')$. These two topologies coincide according to Lemma 2.1. □

PROPOSITION 2.5. — *The relation $\circledcirc$ is transitive.*

Proof. — Let $\mathcal{T}'' \prec \mathcal{T}' \prec \mathcal{T}$ be three topologies on X. Suppose that $\mathcal{T}''$ is $\mathcal{T}'$-admissible, and that $\mathcal{T}'$ is $\mathcal{T}$-admissible. If $Y \subset X$ is $\mathcal{T}''$-connected, it is also $\mathcal{T}'$-connected, hence $\mathcal{T}''|_Y = \mathcal{T}'|_Y = \mathcal{T}|_Y$. Now let $x, y \in X$ with $x \sim_{\mathcal{T}/\mathcal{T}''} y$. By definition of the transitive closure, there exist $a_1, \dots, a_p$ and $b_1, \dots, b_p$ in X such that

$$x \leqslant_{\mathcal{T}} a_1,\ b_1 \leqslant_{\mathcal{T}} a_2, \dots, b_p \leqslant_{\mathcal{T}} y$$

and $a_i \geqslant_{\mathcal{T}''} b_i$ for $i = 1, \dots, p$. We also have $a_i \geqslant_{\mathcal{T}'} b_i$ for $i = 1, \dots, p$ because $\mathcal{T}'' \prec \mathcal{T}'$. Hence,

$$x \sim_{\mathcal{T}/\mathcal{T}'} a_1 \sim_{\mathcal{T}/\mathcal{T}'} b_1 \sim_{\mathcal{T}/\mathcal{T}'} \cdots \sim_{\mathcal{T}/\mathcal{T}'} a_p \sim_{\mathcal{T}/\mathcal{T}'} b_p \sim_{\mathcal{T}/\mathcal{T}'} y,$$

from which we get:

$$x \sim_{\mathcal{T}'/\mathcal{T}'} a_1 \sim_{\mathcal{T}'/\mathcal{T}'} b_1 \sim_{\mathcal{T}'/\mathcal{T}'} \cdots \sim_{\mathcal{T}'/\mathcal{T}'} a_p \sim_{\mathcal{T}'/\mathcal{T}'} b_p \sim_{\mathcal{T}'/\mathcal{T}'} y,$$

hence x and y are in the same $\mathcal{T}'$-connected component. Using that the restrictions of $\mathcal{T}$ and $\mathcal{T}'$ on this component coincide, we get $x \sim_{\mathcal{T}'/\mathcal{T}''} y$. From $\mathcal{T}' \circledcirc \mathcal{T}$ we get then $x \sim_{\mathcal{T}''/\mathcal{T}''} y$. This ends up the proof of Proposition 2.5. □

LEMMA 2.6. — *If $\mathcal{T}'' \circledcirc \mathcal{T}' \circledcirc \mathcal{T}$, then $\mathcal{T}'/\mathcal{T}'' \circledcirc \mathcal{T}/\mathcal{T}''$.*

Proof. — Let $x, y \in X$ with $x \sim_{(\mathcal{T}/\mathcal{T}'')/(\mathcal{T}'/\mathcal{T}'')} y$. Then $x \sim_{\mathcal{T}/\mathcal{T}'} y$ according to Lemma 2.1, hence $x \sim_{\mathcal{T}'/\mathcal{T}'} y$, hence $x \sim_{(\mathcal{T}'/\mathcal{T}'')/(\mathcal{T}'/\mathcal{T}'')} y$ applying Lemma 2.1 again. □

PROPOSITION 2.7. — *Let $\mathcal{T}$ and $\mathcal{T}''$ be two topologies on X. If $\mathcal{T}'' \circledcirc \mathcal{T}$, then $\mathcal{T}' \mapsto \mathcal{T}'/\mathcal{T}''$ is a bijection from the set of topologies $\mathcal{T}'$ on X such that $\mathcal{T}'' \circledcirc \mathcal{T}' \circledcirc \mathcal{T}$, onto the set of topologies $\mathcal{U}$ on X such that $\mathcal{U} \circledcirc \mathcal{T}/\mathcal{T}''$.*

Proof. — Given $\mathcal{U} \circledcirc \mathcal{T}/\mathcal{T}''$, we have to prove the existence of a unique $\mathcal{T}'$ such that $\mathcal{T}'' \circledcirc \mathcal{T}' \circledcirc \mathcal{T}$ and $\mathcal{U} = \mathcal{T}'/\mathcal{T}''$. According to Lemma 2.4, the connected components of $\mathcal{T}'$ must be those of $\mathcal{U}$. The topologies $\mathcal{T}'$ and $\mathcal{T}$ must coincide on each of these components, which uniquely defines $\mathcal{T}'$.

Let us now check $\mathcal{T}'' \circledcirc \mathcal{T}' \circledcirc \mathcal{T}$: if $x \leqslant_{\mathcal{T}'} y$, then x and y are in the same $\mathcal{T}'$-connected component, on which $\mathcal{T}$ and $\mathcal{T}'$ coincide. Hence $x \leqslant_{\mathcal{T}} y$, which means $\mathcal{T}' \prec \mathcal{T}$. Now suppose $x \leqslant_{\mathcal{T}''} y$. Then $x \leqslant_{\mathcal{T}} y$, which implies $x \leqslant_{\mathcal{T}/\mathcal{T}''} y$, which in turn implies $x \leqslant_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} y$. The latter is equivalent to $x \leqslant_{\mathcal{U}/\mathcal{U}} y$, as well as to $x \leqslant_{\mathcal{T}'/\mathcal{T}'} y$. In other words, x and y are in

the same $\mathcal{T}'$-connected component. Moreover, since $x \leqslant_{\mathcal{T}} y$ we also have $x \leqslant_{\mathcal{T}'} y$ by definition of $\mathcal{T}'$. This proves $\mathcal{T}'' \prec \mathcal{T}'$.

If $x \leqslant_{\mathcal{U}} y$, it means that x and y are in the same $\mathcal{U}$-connected component, and moreover $x \leqslant_{\mathcal{T}/\mathcal{T}''} y$, because $\mathcal{U} \otimes \mathcal{T}/\mathcal{T}''$. By definition of the transitive closure, there exist $a_1, \dots, a_p$ and $b_1, \dots, b_p$ in X such that

$$x \leqslant_{\mathcal{T}} a_1,\ b_1 \leqslant_{\mathcal{T}} a_2, \dots, b_p \leqslant_{\mathcal{T}} y \tag{2.7}$$

and $a_i \geqslant_{\mathcal{T}''} b_i$ for $i = 1, \dots, p$. In particular, $a_i \sim_{\mathcal{T}/\mathcal{T}''} b_i$, hence:

$$x \sim_{\mathcal{T}/\mathcal{T}''} a_1 \sim_{\mathcal{T}/\mathcal{T}''} b_1 \sim_{\mathcal{T}/\mathcal{T}''} a_2 \sim_{\mathcal{T}/\mathcal{T}''} \cdots \sim_{\mathcal{T}/\mathcal{T}''} b_p \sim_{\mathcal{T}/\mathcal{T}''} y$$

which immediately yields:

$$x \sim_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} a_1 \sim_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} b_1 \sim_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} a_2 \sim_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} \cdots \sim_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} b_p \sim_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} y$$

since $\mathcal{T}/\mathcal{T}'' \prec (\mathcal{T}/\mathcal{T}'')/\mathcal{U}$. Now using $\mathcal{U} \otimes \mathcal{T}/\mathcal{T}''$ again, we get

$$x \sim_{\mathcal{U}/\mathcal{U}} a_1 \sim_{\mathcal{U}/\mathcal{U}} b_1 \sim_{\mathcal{U}/\mathcal{U}} a_2 \sim_{\mathcal{U}/\mathcal{U}} \cdots \sim_{\mathcal{U}/\mathcal{U}} b_p \sim_{\mathcal{U}/\mathcal{U}} y.$$

Hence all the chain is included in the same $\mathcal{U}$-connected component. By definition of $\mathcal{T}'$ we can then rewrite (2.7) as:

$$x \leqslant_{\mathcal{T}'} a_1,\ b_1 \leqslant_{\mathcal{T}'} a_2, \dots, b_p \leqslant_{\mathcal{T}'} y \tag{2.8}$$

with $a_i \geqslant_{\mathcal{T}''} b_i$ for $i = 1, \dots, p$, which means $x \leqslant_{\mathcal{T}'/\mathcal{T}''} y$.

Conversely, if $x \leqslant_{\mathcal{T}'/\mathcal{T}''} y$, then x and y are in the same $\mathcal{U}$-component according to the definition of $\mathcal{T}'$, and (2.8) implies (2.7). Hence $x \leqslant_{\mathcal{T}/\mathcal{T}''} y$, hence $x \leqslant_{\mathcal{U}} y$. We have then:

$$\mathcal{U} = \mathcal{T}/\mathcal{T}'. \tag{2.9}$$

To finish the proof, we have to show $\mathcal{T}' \otimes \mathcal{T}$ and $\mathcal{T}'' \otimes \mathcal{T}'$. Any $\mathcal{T}'$-connected subset $Y \subset X$ is also $\mathcal{U}$-connected, hence the restrictions of $\mathcal{T}$ and $\mathcal{T}'$ on Y coincide. Similarly, the restrictions of $\mathcal{T}'$ and $\mathcal{T}''$ on any $\mathcal{T}''$-connected subset coincide. If $x \sim_{\mathcal{T}/\mathcal{T}'} y$, then $x \sim_{(\mathcal{T}/\mathcal{T}'')/(\mathcal{T}'/\mathcal{T}'')} y$, which means $x \sim_{(\mathcal{T}/\mathcal{T}'')/\mathcal{U}} y$, which in turn yields $x \sim_{\mathcal{U}/\mathcal{U}} y$, i.e. $x \sim_{\mathcal{T}'/\mathcal{T}'} y$. Hence $\mathcal{T}' \otimes \mathcal{T}$. Finally, if $x \sim_{\mathcal{T}'/\mathcal{T}''} y$, then $x \sim_{\mathcal{T}/\mathcal{T}''} y$, hence $x \sim_{\mathcal{T}''/\mathcal{T}''} y$, which yields $\mathcal{T}'' \otimes \mathcal{T}'$. This ends up the proof of Proposition 2.7. □

3. Algebraic structures on finite topologies

The collection of all finite topological spaces shows very rich algebraic features, best viewed in the linear species formalism. We describe a commutative product, an"internal" coproduct and an "external" coproduct, as well as the interactions between them.

3.1. The coalgebra species of finite topological spaces

Recall that a linear species is a contravariant functor from the category of finite sets with bijections into the category of vector spaces[(1)] (on some field K). The species $\mathbb{T}$ of topological spaces is defined as follows: $\mathbb{T}_X$ is the vector space freely generated by the topologies on X. For any bijection $\varphi : X \longrightarrow X'$, the isomorphism $\mathbb{T}_\varphi : \mathbb{T}_{X'} \longrightarrow \mathbb{T}_X$ is defined by the obvious relabelling:

$$\mathbb{T}_\varphi(\mathcal{T}) := \{\varphi^{-1}(Y),\, Y \in \mathcal{T}\}$$

for any topology $\mathcal{T}$ on X'. For any finite set X, let us introduce the coproduct Γ on $\mathbb{T}_X$ defined as follows:

$$\Gamma(\mathcal{T}) = \sum_{\mathcal{T}' \otimes \mathcal{T}} \mathcal{T}' \otimes \mathcal{T}/\mathcal{T}'. \tag{3.1}$$

Example. — If $X = E \sqcup F = A \sqcup A \sqcup C$ are two partitions of X:

$$\begin{aligned}
\Gamma(\bullet_X) &= \bullet_X \otimes \bullet_X, \\
\Gamma(\updownarrow_E^F) &= \updownarrow_E^F \otimes \bullet_X + \bullet_E\, \bullet_F \otimes \updownarrow_E^F \\
\Gamma(\bullet_E\, \bullet_F) &= \bullet_E\, \bullet_F \otimes \bullet_E\, \bullet_F \\
\Gamma({}^B\!\curlyvee_A^C) &= {}^B\!\curlyvee_A^C \otimes \bullet_X + \updownarrow_A^B \bullet_C \otimes \updownarrow_{A\sqcup B}^C + \updownarrow_A^C \bullet_B \otimes \updownarrow_{A\sqcup C}^B \\
&\qquad + \bullet_A\, \bullet_B\, \bullet_C \otimes {}^B\!\curlyvee_A^C \\
\Gamma(\updownarrow_A^{\updownarrow_B^C}) &= \updownarrow_A^{\updownarrow_B^C} \otimes \bullet_X + \updownarrow_A^B \bullet_C \otimes \updownarrow_{A\sqcup B}^C + \bullet_A\, \updownarrow_B^C \otimes \updownarrow_A^{B\sqcup C} \\
&\qquad + \bullet_A\, \bullet_B\, \bullet_C \otimes \updownarrow_A^{\updownarrow_B^C} \\
\Gamma({}_B\!\curlywedge_C^A) &= {}_B\!\curlywedge_C^A \otimes \bullet_X + \updownarrow_B^A \bullet_C \otimes \updownarrow_C^{A\sqcup B} + \updownarrow_C^A \bullet_B \otimes \updownarrow_B^{A\sqcup C} \\
&\qquad + \bullet_A\, \bullet_B\, \bullet_C \otimes {}_B\!\curlywedge_C^A \\
\Gamma(\updownarrow_A^B \bullet_C) &= \updownarrow_A^B \bullet_C \otimes \bullet_{A\sqcup B}\, \bullet_C + \bullet_A\, \bullet_B\, \bullet_C \otimes \updownarrow_A^B \bullet_C \\
\Gamma(\bullet_A\, \bullet_B\, \bullet_C) &= \bullet_A\, \bullet_B\, \bullet_C \otimes \bullet_A\, \bullet_B\, \bullet_C
\end{aligned}$$

3.1.1. The coalgebra structure

THEOREM 3.1. — *The coproduct Γ is coassociative.*

(1) Contravariance yields actions of the permutation groups on the right. It is a pure matter of convention: [1] prefers the covariant setting.

Proof. — For any topology $\mathcal{T}$ on X we have:

$$(\Gamma \otimes \mathrm{Id})\Gamma(\mathcal{T}) = \sum_{\mathcal{T}'' \circledcirc \mathcal{T}' \circledcirc \mathcal{T}} \mathcal{T}'' \otimes \mathcal{T}'/\mathcal{T}'' \otimes \mathcal{T}/\mathcal{T}', \tag{3.2}$$

whereas

$$(\mathrm{Id} \otimes \Gamma)\Gamma(\mathcal{T}) = \sum_{\mathcal{T}'' \circledcirc \mathcal{T}'} \sum_{\mathcal{U} \circledcirc \mathcal{T}'/\mathcal{T}''} \mathcal{T}'' \otimes \mathcal{U} \otimes (\mathcal{T}/\mathcal{T}'')/\mathcal{U}. \tag{3.3}$$

The result then comes from Lemmas 2.5 and 2.1, and from Proposition 2.7. □

3.1.2. Grading and counit

Let X be a finite set. Given any topology $\mathcal{T}$ on X, we introduce $d(\mathcal{T})$ as the number of equivalence classes minus the number of connected components of $\mathcal{T}$. It is easy to see that this grading makes $(\mathbb{T}_X, \Gamma)$ a finite-dimensional graded coalgebra. The degree zero topologies in $\mathbb{T}_X$ are precisely the topologies $\mathcal{T}_\mathcal{P}$ where $\mathcal{P}$ is a partition of X, defined as the product of the coarse topologies on each block of $\mathcal{P}$. In other words, $d(\mathcal{T}) = 0$ if, and only if, $\leqslant_\mathcal{T}$ is an equivalence. The maximum possible degree $|X| - 1$ is reached for connected $\mathcal{T}_0$ topologies. For any topology $\mathcal{T}$ on X, there exists a unique degree zero topology $\mathcal{T}' \circledcirc \mathcal{T}$, namely the topology $\mathcal{T}'$ such that $\leqslant_{\mathcal{T}'} = \sim_\mathcal{T}$; moreover, $\mathcal{T}/\mathcal{T}' = \mathcal{T}$. The unique topology $\mathcal{T}''$ such that $\mathcal{T}/\mathcal{T}''$ is of degree zero is $\mathcal{T}'' = \mathcal{T}$.

Lemma 3.2. — *The group-like elements in* $(\mathbb{T}_X, \Gamma)$ *are the degree zero topologies.*

Proof. — Any group-like element of $\mathbb{T}_X$ is of degree zero, and it is easy to check that any degree zero topology is group-like. A degree zero element of $\mathbb{T}_X$ is a finite linear combination $T = \sum \lambda_\mathcal{T} \mathcal{T}$ of degree zero topologies. It is group-like if and only if $\lambda_\mathcal{T} \lambda_{\mathcal{T}'} = 0$ for $\mathcal{T} \neq \mathcal{T}'$ and $\lambda_\mathcal{T}^2 = \lambda_\mathcal{T}$, and $T \neq 0$. Hence there is a degree zero topology $\mathcal{T}$ such that $\lambda_{\mathcal{T}'} = 0$ for $\mathcal{T}' \neq \mathcal{T}$ and $\lambda_\mathcal{T} = 1$, hence $T = \mathcal{T}$. □

The linear form ε_X on $\mathbb{T}_X$ defined on the basis of topologies by $\varepsilon_X(\mathcal{T}) = 1$ if $\mathcal{T}$ is group-like and $\varepsilon(\mathcal{T}) = 0$ otherwise is a counit. The homogeneous component of degree zero elements is the coradical of the coalgebra $\mathbb{T}_X$. The involution $\mathcal{T} \mapsto \overline{\mathcal{T}}$ obviously extends linearly to a coalgebra involution on $\mathbb{T}_X$. Any relabelling induces an involutive coalgebra isomorphism in a functorial way. To summarize:

Corollary 3.3. — $\mathbb{T}$ *is a species in the category of counital pointed coalgebras with involution.*

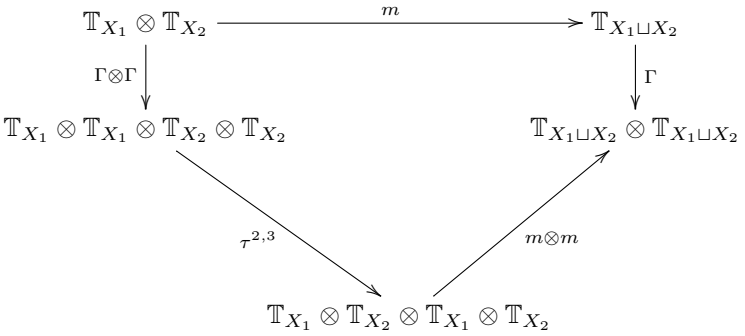
3.2. The monoid structure

A commutative monoid structure ([1, Chapter 8], [2, Paragraph 2.3]) on the species of finite topologies is defined as follows: for any pair X_1, X_2 of finite sets we introduce

$$\begin{aligned} m : \mathbb{T}_{X_1} \otimes \mathbb{T}_{X_2} &\longrightarrow \mathbb{T}_{X_1 \sqcup X_2} \\ \mathcal{T}_1 \otimes \mathcal{T}_2 &\longmapsto \mathcal{T}_1\mathcal{T}_2, \end{aligned}$$

where $\mathcal{T}_1\mathcal{T}_1$ is characterized by $Y \in \mathcal{T}_1\mathcal{T}_2$ if and only if $Y \cap X_1 \in \mathcal{T}_1$ and $Y \cap X_2 \in \mathcal{T}_2$. The notation $\sqcup$ stands for disjoint union, and the unit is given by the unique topology on the empty set.

Proposition 3.4. — *The species coproduct* Γ *and the product are compatible, i.e. for any pair* X_1, X_2 *of finite sets the following diagram commutes:*



Proof. — Let $\mathcal{T}_1$, resp. $\mathcal{T}_2$ be a topology on X_1, resp. X_2. Let $\mathcal{U}_1 \circledcirc \mathcal{T}_1$ and $\mathcal{U}_2 \circledcirc \mathcal{T}_2$. Then $\mathcal{U}_1\mathcal{U}_2 \circledcirc \mathcal{T}_1\mathcal{T}_2$. Conversely, any topology $\mathcal{U}$ on $X_1 \sqcup X_2$ such that $\mathcal{U} \circledcirc \mathcal{T}_1\mathcal{T}_2$ can be written $\mathcal{U}_1\mathcal{U}_2$ with $\mathcal{U}_i = \mathcal{U}_{|_{X_i}}$ for $i = 1, 2$, and we have $\mathcal{U}_i \circledcirc \mathcal{T}_i$. We have then:

$$\begin{aligned} \Gamma(\mathcal{T}_1\mathcal{T}_2) &= \sum_{\mathcal{U} \circledcirc \mathcal{T}_1\mathcal{T}_2} \mathcal{U} \otimes (\mathcal{T}_1\mathcal{T}_2)/\mathcal{U} \\ &= \sum_{\substack{\mathcal{U}_1 \circledcirc \mathcal{T}_1 \\ \mathcal{U}_2 \circledcirc \mathcal{T}_2}} \mathcal{U}_1\mathcal{U}_2 \otimes (\mathcal{T}_1/\mathcal{U}_1)(\mathcal{T}_2/\mathcal{U}_2) \\ &= \Gamma(\mathcal{T}_1)\Gamma(\mathcal{T}_2). \end{aligned}$$

□

3.3. The external coproduct

For any topology $\mathcal{T}$ on a finite set X and for any subset $Y \subset X$, we denote by $\mathcal{T}_{|_Y}$ the restriction of $\mathcal{T}$ to Y. It is defined by:

$$\mathcal{T}_{|_Y} = \{Z \cap Y,\ Z \in \mathcal{T}\}.$$

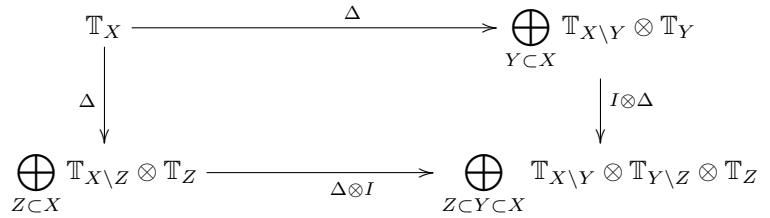
Restriction and taking quotients commute: for any subset $Y \subset X$ and for any $\mathcal{T}' \circledcirc \mathcal{T}$ we have $\mathcal{T}'_{|_Y} \circledcirc \mathcal{T}_{|_Y}$ and:

$$(\mathcal{T}/\mathcal{T}')_{|_Y} = \mathcal{T}_{|_Y}/\mathcal{T}'_{|_Y}. \tag{3.4}$$

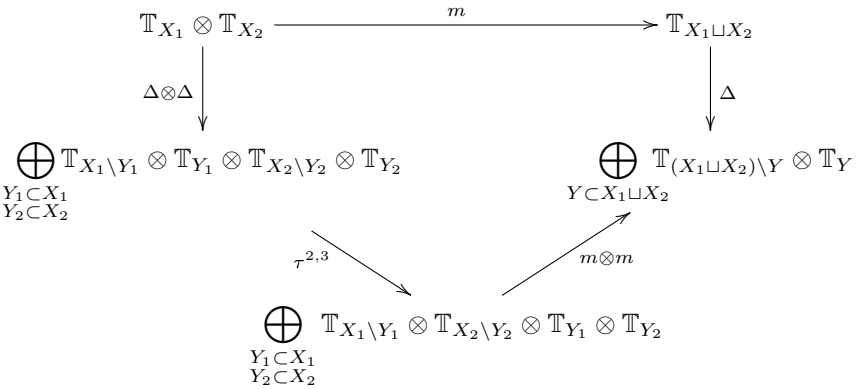
The external coproduct is defined on $\mathbb{T}_X$ as follows:

$$\begin{aligned} \Delta : \mathbb{T}_X &\longrightarrow \bigoplus_{Y \subset X} \mathbb{T}_{X\setminus Y} \otimes \mathbb{T}_Y \\ \mathcal{T} &\longmapsto \sum_{Y \in \mathcal{T}} \mathcal{T}_{|_{X\setminus Y}} \otimes \mathcal{T}_{|_Y}. \end{aligned}$$

Proposition 3.5. — *The external coproduct is coassociative and multiplicative, i.e. the two following diagrams commute:*



and



Proof. — we have:

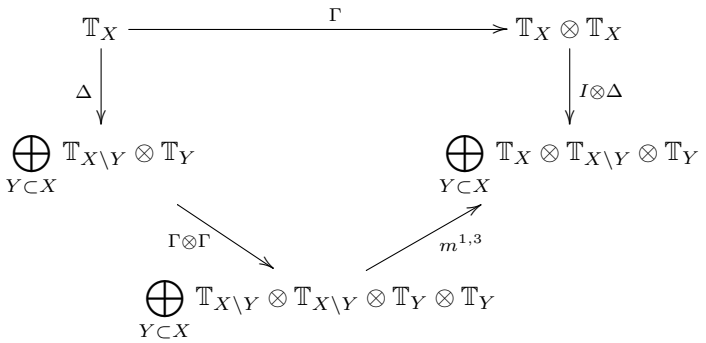
$$(\Delta \otimes I)\Delta(\mathcal{T}) = \bigoplus_{Z\in\mathcal{T},\, \widetilde{Y}\in\mathcal{T}_{|X\setminus Z}} \mathcal{T}_{|X\setminus Z\sqcup\widetilde{Y}} \otimes \mathcal{T}_{|\widetilde{Y}} \otimes \mathcal{T}_{|Z} \tag{3.5}$$

and

$$(I \otimes \Delta)\Delta(\mathcal{T}) = \bigoplus_{Y,Z\in\mathcal{T},\, Z\subset Y} \mathcal{T}_{|X\setminus Y} \otimes \mathcal{T}_{|Y\setminus Z} \otimes \mathcal{T}_{|Z} \tag{3.6}$$

Coassociativity then comes from the obvious fact that $(\widetilde{Y}, Z) \mapsto \widetilde{Y} \sqcup Z$ is a bijection from the set of pairs $(\widetilde{Y}, Z)$ with $Z \in \mathcal{T}$ and $\widetilde{Y} \in \mathcal{T}_{|X\setminus Z}$, onto the set of pairs (Y, Z) of elements of $\mathcal{T}$ subject to $Z \subset Y$. The inverse map is given by $(Y, Z) \mapsto (Y \cap X \setminus Z, Z)$. The multiplicativity property $\Delta(\mathcal{T}_1\mathcal{T}_2) = \Delta(\mathcal{T}_1)\Delta(\mathcal{T}_2)$ comes straightforwardly from the very definition of the topology $\mathcal{T}_1\mathcal{T}_2$ on the disjoint union $X_1 \sqcup X_2$. □

THEOREM 3.6. — *The internal and external coproducts are compatible, in the sense that the following diagram commutes for any finite set X:*



Proof. — For any $\mathcal{T} \in \mathbb{T}_X$ we have:

$$\begin{aligned}(I \otimes \Delta) \circ \Gamma(\mathcal{T}) &= (I \otimes \Delta) \sum_{\mathcal{U}\circledcirc\mathcal{T}} \mathcal{T} \otimes \mathcal{T}/\mathcal{U} \\ &= \sum_{\mathcal{U}\circledcirc\mathcal{T}} \sum_{Y\in\mathcal{T}/\mathcal{U}} \mathcal{U} \otimes (\mathcal{T}/\mathcal{U})_{|X\setminus Y} \otimes (\mathcal{T}/\mathcal{U})_{|Y} \\ &= \sum_{\mathcal{U}\circledcirc\mathcal{T}} \sum_{Y\in\mathcal{T}/\mathcal{U}} \mathcal{U} \otimes \mathcal{T}_{|X\setminus Y}\big/\mathcal{U}_{|X\setminus Y} \otimes \mathcal{T}_{|Y}\big/\mathcal{U}_{|Y},\end{aligned} \tag{3.7}$$

whereas

$$m^{1,3}\circ(\Gamma\otimes\Gamma)\circ\Delta(\mathcal{T}) = m^{1,3}\circ(\Gamma\otimes\Gamma)\sum_{Z\in\mathcal{T}}\mathcal{T}_{|_{X\setminus Z}}\otimes\mathcal{T}_{|_Z}$$

$$= \sum_{Z\in\mathcal{T}}\sum_{\substack{\mathcal{U}_1\otimes\mathcal{T}_{|_{X\setminus Z}}\\ \mathcal{U}_2\otimes\mathcal{T}_{|_Z}}} \mathcal{U}_1\mathcal{U}_2\otimes\mathcal{T}_{|_{X\setminus Z}}\Big/\mathcal{U}_1\otimes\mathcal{T}_{|_Z}\Big/\mathcal{U}_2. \tag{3.8}$$

Now, $Y\in\mathcal{T}/\mathcal{U}$ means that Y is a final segment for $\leqslant_{\mathcal{T}/\mathcal{U}}$, i.e. for any $y\in Y$, if $z\leqslant_{\mathcal{T}/\mathcal{U}} y$, then $z\in Y$. A fortiori $z\in Y$ if $z\leqslant_{\mathcal{U}} y$ or $y\leqslant_{\mathcal{U}} z$. Then Y is both a final and initial segment for $\leqslant_{\mathcal{U}}$, i.e. both closed and open for $\mathcal{U}$, which yields $\mathcal{U}=\mathcal{U}_1\mathcal{U}_2$, with $\mathcal{U}_1=\mathcal{U}_{|_{X\setminus Y}}$ and $\mathcal{U}_2=\mathcal{U}_{|_Y}$.

Conversely, if $\mathcal{U}=\mathcal{U}_{|_{X\setminus Y}}\mathcal{U}_{|_Y}$, then for $y\in Y$ and any $z\in X$ such that $y\leqslant_{\mathcal{U}} z$ or $z\leqslant_{\mathcal{U}} y$, we have $z\in Y$. By iteration we have $y\leqslant_{\mathcal{U}/\mathcal{U}} z\Rightarrow z\in Y$. But $\mathcal{U}\circledcirc\mathcal{T}$, hence $y\leqslant_{\mathcal{T}/\mathcal{U}} z\Rightarrow z\in Y$, which means $Y\in\mathcal{T}/\mathcal{U}$. This proves that (3.7) and (3.8) coincide. □

4. Two commutative bialgebra structures

Consider the graded vector space:

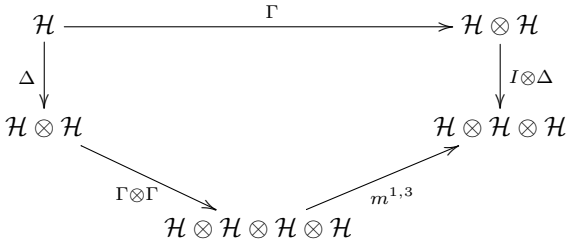
$$\mathcal{H}=\overline{\mathcal{K}}(\mathbb{T})=\bigoplus_{n\geqslant 0}\mathcal{H}_n, \tag{4.1}$$

where $\mathcal{H}_0=k.\mathbf{1}$, and where $\mathcal{H}_n$ is the linear span of topologies on $\{1,\dots,n\}$ when $n\geqslant 1$, modulo the action of the symmetric group S_n. The vector space $\mathcal{H}$ can be seen as the quotient of the species $\mathbb{T}$ by the "forget the labels" equivalence relation: $\mathcal{T}\sim\mathcal{T}'$ if $\mathcal{T}$ (resp. $\mathcal{T}'$) is a topology on a finite set X (resp. X'), such that there is a bijection from X onto X' which is a homeomorphism with respect to both topologies. This equivalence relation is compatible with the product and both coproducts introduced in Section 3, giving rise to a product $\cdot$ and two coproducts Γ and Δ on $\mathcal{H}$, the first coproduct being internal to each $\mathcal{H}_n$. The functor $\overline{\mathcal{K}}$ from linear species to graded vector spaces thus obtained is intensively studied in [1, Chapter 15] under the name "bosonic Fock functor". This naturally leads to the following:

THEOREM 4.1. — *The graded vector space $\mathcal{H}$ is endowed with the following algebraic structures:*

- *$(\mathcal{H},\cdot,\Delta)$ is a commutative graded connected Hopf algebra.*

- $(\mathcal{H}, \cdot, \Gamma)$ *is a commutative bialgebra, graded by the degree* d *introduced at the end of § 3.1.*
- $(\mathcal{H}, \cdot, \Delta)$ *is a comodule-coalgebra on* $(\mathcal{H}, \cdot, \Gamma)$. *More precisely the following diagram of unital algebra morphisms commutes:*



Remark 4.2. — The Hopf algebra of finite topologies $\mathcal{H}_{\mathbf{T}}$ of [15] can be seen as $\mathcal{K}(\mathbb{T})$ where $\mathcal{K}$ is the "full Fock functor" of [1, Chapter 15]. It is closely related to the Hopf algebra $\mathcal{H}$ above, but the product is noncommutative due to renumbering. In fact, $\mathbf{T}_n$ stands for the set of topologies on $[n] = \{1, \ldots, n\}$, and $\mathbf{T}$ is the (disjoint) union of the $\mathbf{T}_n$'s for $n \geqslant 0$. For $\mathcal{T} \in \mathbf{T}_n$ and $\mathcal{T}' \in \mathbf{T}_{n'}$, the product $\mathcal{T}\mathcal{T}'$ is the topology on $[n + n']$ the open sets of which are $Y \sqcup (Y' + n)$, where $Y \in \mathcal{T}$ and $Y' \in \mathcal{T}'$. The two topologies $\mathcal{T}\mathcal{T}'$ and $\mathcal{T}'\mathcal{T}$ are not equal, though homeomeorphic. The "joint" product $\downarrow$, for which the open sets of $\mathcal{T} \downarrow \mathcal{T}'$ are the open sets Y' of $\mathcal{T}'$ and the sets $Y \sqcup \{n + 1, \ldots, n + n'\}$ with $Y \in \mathcal{T}$, is also associative. The empty set $\emptyset$ is the common unit for both products.

For any totally ordered finite set E of cardinality n, let us denote by $\mathrm{Std} : E \to [n]$ the standardization map, i.e. the unique increasing bijection from E onto $[n]$. This map yields a bijection form $\mathcal{P}(E)$ onto $\mathcal{P}([n])$ also denoted by Std. The coproduct is defined by:

$$\Delta(\mathcal{T}) = \sum_{Y \in \mathcal{T}} \mathrm{Std}(\mathcal{T}_{|_{[n] \setminus Y}}) \otimes \mathrm{Std}(\mathcal{T}_{|_Y}). \tag{4.2}$$

Proposition 4.3 ([15] Proposition 6). — *Let* $\mathcal{H}_{\mathbf{T}}$ *be the graded vector space freely generated by the* $\mathbf{T}_n$*'s. Then*

(1) $(\mathcal{H}_{\mathbf{T}}, \cdot, \Delta)$ *is a graded Hopf algebra,*
(2) $(\mathcal{H}_{\mathbf{T}}, \downarrow, \Delta)$ *is a graded infinitesimal Hopf algebra,*
(3) *The involution* $\mathcal{T} \mapsto \overline{\mathcal{T}}$ *is a morphism for the product* $\cdot$ *and an antimorphism for the coproduct* Δ.

The internal coproduct Γ on each homogeneous component of $\mathcal{H}_{\mathbf{T}}$ does not interact so nicely with the external coproduct Δ as it does in the

commutative setting because of the shift and the standardization. Here is an example:

$$\begin{aligned}
&m^{1,3}\circ(\Gamma\otimes\Gamma)\circ\Delta(\bullet^{1,2}_{3})\\
&\qquad=\bullet^{1,2}_{3}\otimes\bullet_{1,2,3}\otimes 1+\bullet_{1,2}\bullet_{3}\otimes\bullet^{1,2}_{3}\otimes 1+\bullet^{1,2}_{3}\otimes 1\otimes\bullet_{1,2,3}\\
&\qquad\quad+\bullet_{1,2}\bullet_{3}\otimes 1\otimes\bullet^{1,2}_{3}+\bullet_{1}\bullet_{2,3}\otimes\bullet_{1}\otimes\bullet_{1,2},\\
&(Id\otimes\Delta)\circ\Gamma(\bullet^{1,2}_{3})\\
&\qquad=\bullet^{1,2}_{3}\otimes\bullet_{1,2,3}\otimes 1+\bullet_{1,2}\bullet_{3}\otimes\bullet^{1,2}_{3}\otimes 1+\bullet^{1,2}_{3}\otimes 1\otimes\bullet_{1,2,3}\\
&\qquad\quad+\bullet_{1,2}\bullet_{3}\otimes 1\otimes\bullet^{1,2}_{3}+\bullet_{1,2}\bullet_{3}\otimes\bullet_{1}\otimes\bullet_{1,2}.
\end{aligned}$$

Let us conclude this section with the description of the antipodes of $(\mathcal{H},\cdot,\Delta)$ and $(\mathcal{H}_{\mathbf{T}},\cdot,\Delta)$. Note that both $(\mathcal{H},\cdot,\Gamma)$ and $(\mathcal{H}_{\mathbf{T}},\cdot,\Gamma)$ have non invertible group-like elements, so are not Hopf algebras.

PROPOSITION 4.4. — *For all finite topology $\mathcal{T}$ on X, in $(\mathcal{H},\cdot,\Delta)$:*

$$S(\mathcal{T})=\sum_{k=0}^{n}(-1)^{k+1}\sum_{\substack{\emptyset\subsetneq X_1\subsetneq\ldots\subsetneq X_k\subsetneq X\\ X_1,\ldots,X_k\in\mathcal{T}}}\mathcal{T}_{|X\setminus X_k}\mathcal{T}_{|X_k\setminus X_{k-1}}\ldots\mathcal{T}_{|X_2\setminus X_1}\mathcal{T}_{|X_1}. \tag{4.3}$$

Proof. — We denote by S' the endomorphism of $\mathcal{H}$ defined by the right side of (4.3). Then $S'(1)=1$, so $Id*S'(1)=1$. For any nonempty $\mathcal{T}$:

$$\begin{aligned}
&(Id*S')(\mathcal{T})\\
&=S'(\mathcal{T})+\mathcal{T}+\sum_{\substack{\emptyset\subsetneq X_1\subsetneq X\\ X_1\in\mathcal{T}}}\mathcal{T}_{|X\setminus X_1}S(\mathcal{T}_{|X})\\
&=S'(\mathcal{T})+\mathcal{T}+\sum_{k=1}^{n}(-1)^{k}\sum_{\substack{\emptyset\subsetneq X_1\subsetneq\ldots\subsetneq X_k\subsetneq X\\ X_1,\ldots,X_k\in\mathcal{T}}}\mathcal{T}_{|X\setminus X_k}\mathcal{T}_{|X_k\setminus X_{k-1}}\ldots\mathcal{T}_{|X_2\setminus X_1}\mathcal{T}_{|X_1}\\
&=S'(\mathcal{T})+\sum_{k=0}^{n}(-1)^{k}\sum_{\substack{\emptyset\subsetneq X_1\subsetneq\ldots\subsetneq X_k\subsetneq X\\ X_1,\ldots,X_k\in\mathcal{T}}}\mathcal{T}_{|X\setminus X_k}\mathcal{T}_{|X_k\setminus X_{k-1}}\ldots\mathcal{T}_{|X_2\setminus X_1}\mathcal{T}_{|X_1}\\
&=0.
\end{aligned}$$

So $Id*S'(\mathcal{T})=\varepsilon(\mathcal{T})1$ for all $\mathcal{T}$: S' is indeed the antipode of $\mathcal{H}$. □

Similarly, in $(\mathcal{H}_{\mathbf{T}},\cdot,\Delta)$, if $\mathcal{T}$ is a topology on $[n]=\{1,\ldots,n\}$:

$$\begin{aligned}
S(\mathcal{T})=\sum_{k=0}^{n}(-1)^{k+1}\sum_{\substack{\emptyset\subsetneq X_1\subsetneq\ldots\subsetneq X_k\subsetneq[n]\\ X_1,\ldots,X_k\in\mathcal{T}}}&\operatorname{Std}(\mathcal{T}_{|[n]\setminus X_k})\operatorname{Std}(\mathcal{T}_{|X_k\setminus X_{k-1}})\\
&\ldots\operatorname{Std}(\mathcal{T}_{|X_2\setminus X_1})\operatorname{Std}(\mathcal{T}_{|X_1}).
\end{aligned} \tag{4.4}$$

For example:

$$S(\begin{smallmatrix}\bullet^3\\ \bullet^2\\ \bullet^1\end{smallmatrix}) = -\begin{smallmatrix}\bullet^3\\ \bullet^2\\ \bullet^1\end{smallmatrix} + \bullet_1\ \begin{smallmatrix}\bullet^3\\ \bullet^2\end{smallmatrix} + \begin{smallmatrix}\bullet^2\\ \bullet^1\end{smallmatrix}\bullet_3 - \bullet_1\bullet_2\bullet_3\ ,$$
$$S(\begin{smallmatrix}\bullet^1\\ \bullet^2\\ \bullet^3\end{smallmatrix}) = -\begin{smallmatrix}\bullet^1\\ \bullet^2\\ \bullet^3\end{smallmatrix} + \bullet_1\ \begin{smallmatrix}\bullet^2\\ \bullet^3\end{smallmatrix} + \begin{smallmatrix}\bullet^1\\ \bullet^2\end{smallmatrix}\bullet_3 - \bullet_1\bullet_2\bullet_3\ .$$

5. Linear extensions and set compositions

5.1. Two Hopf algebras on words

We first recall some facts on two well-known Hopf algebras. Let us start with the Hopf algebra of quasi-symmetric functions [19]. A presentation close to ours, including the internal coproduct, can be found in [3, Paragraph 11].

Let X be a denumerable well-ordered alphabet. A map $\rho : x \mapsto |x|$ from X into $\mathbb{N}_{>0} = \{1, 2, \ldots\}$ is a *rank* if $|x+1| \geqslant |x|$ for any $x \in X$ (where $x+1$ stands for the successor of x, i.e. the smallest element bigger that x), and if the preimage $\rho^{-1}(n)$ is finite for each $n \in \mathbb{N}_{>0}$. Let $\mathbb{Q}[\![X]\!]$ be the algebra of formal series generated by X, i.e. formal sums:

$$\sum_{P \text{ finite subset of } X} \sum_{\nu : P \to \mathbb{N}_{>0}} \lambda_{P,\nu} \prod_{x \in P} x^{\nu(x)}, \tag{5.1}$$

where the coefficients $\lambda_{P,\nu}$ belong to the base field K. The algebra $\mathbb{Q}[\![X]\!]$ is complete for the topology induced by the decreasing filtration given by the weighted valuation: to be precise, a monomial $\prod_{x\in P} x^{\nu(x)}$ is of degree $\sum_{x\in P} \nu(x)$ and of weight $\sum_{x\in P} |x|\nu(x)$, and the weighted valuation of the series (5.1) in $\mathbb{Q}[\![X]\!]$ is the minimal weight of a monomial arising with a nonvanishing coefficient. The topology on $\mathbb{Q}[\![X]\!]$ is as usual given by the distance:

$$d(f, g) = 2^{-\operatorname{val}(f-g)},$$

and one can easily show that it does not depend on the choice of the rank map, due to the fact that for any sequence $x_1, x_2, x_3, \ldots$ of distinct letters of X, we have $|x_k| \underset{k\to+\infty}{\longrightarrow} +\infty$ for any choice of rank.

A formal series $f \in \mathbb{Q}[\![X]\!]$ is quasi-symmetric if for any $x_1 < \ldots < x_k$ and $y_1 < \ldots < y_k$ in X, for any $a_1, \ldots, a_k \geqslant 1$, the coefficients of $x_1^{a_1} \ldots x_k^{a_k}$ and of $y_1^{a_1} \ldots y_k^{a_k}$ in f are equal. The subalgebra of quasi-symmetric functions on X will be denoted by $\mathrm{QSym}(X)$. For any composition $(a_1, \ldots, a_k)$, we put:

$$M_{(a_1,\ldots,a_k)}(X) = \sum_{x_1 < \ldots < x_k} x_1^{a_1} \ldots x_k^{a_k}.$$

The family $(M_c(X))$ indexed by compositions linearly spends $\mathrm{QSym}(X)$; if X is infinite, this is a basis.

Let k, l, r be three nonnegative integers. We shall denote by $\mathrm{QSh}(k,l;r)$ the set of surjections $u : [k+l] \longrightarrow\!\!\!\!\!\rightarrow [k+l-r]$, such that $u_1 < \cdots < u_k$ and $u_{k+1} < \cdots < u_{k+l-r}$. With this notation, for all compositions $(c_1, \ldots, c_k)$, $(c_{k+1}, \ldots, c_{k+l})$:

$$M_{(c_1,\ldots,c_k)}(X)M_{(c_{k+1},\ldots,c_{k+l})}(X) = \sum_{r\geqslant 0} \sum_{\sigma\in \mathrm{QSh}(k,l;r)} M_{(c_1^\sigma,\ldots,c_{k+l-r}^\sigma)}(X)$$

with $c_j^\sigma := \displaystyle\sum_{\sigma_i=j} c_i$ (this sum contains one or two terms).

Let X and Y be two denumerable well-ordered alphabets. The alphabet $X \sqcup Y$ is also well-ordered, the elements of X being smaller than the elements of Y, and the alphabet $X \times Y$ is well-ordered by the lexicographic order. If ρ_X and ρ_Y are rank maps on X and Y respectively, we can define rank maps on $X \sqcup Y$ and on $X \times Y$ as follows:

$$\rho_{X\sqcup Y}(x) = \rho_X(x), \quad \rho_{X\sqcup Y}(y) = \rho_Y(y), \quad \rho_{X\times Y}(x,y) = \rho_X(x) + \rho_Y(y)$$

for any $x \in X$ and $y \in Y$. One can identify $\mathbb{Q}[\![X \sqcup Y]\!]$ with the completed tensor product $\mathbb{Q}[\![X]\!]\widehat{\otimes}\mathbb{Q}[\![Y]\!]$ by separation of variables: the unique unital algebra morphism $\iota : \mathbb{Q}[\![X \sqcup Y]\!] \to \mathbb{Q}[\![X]\!]\widehat{\otimes}\mathbb{Q}[\![Y]\!]$ such that $\iota(x) = x \otimes 1$ and $\iota(y) = 1 \otimes y$ is an isomorphism.

Finally we embed $\mathbb{Q}[\![X \times Y]\!]$ into $\mathbb{Q}[\![X]\!]\widehat{\otimes}\mathbb{Q}[\![Y]\!]$ also by separation of variables, by the unique unital algebra morphism $\jmath : \mathbb{Q}[\![X \times Y]\!] \to \mathbb{Q}[\![X]\!]\widehat{\otimes}\mathbb{Q}[\![Y]\!]$ such that $\jmath(x,y) = x \otimes y$. Now we have for any composition $(c_1, \ldots, c_k)$:

$$(1)\quad \iota\big(M_{(c_1,\ldots,c_k)}(X \sqcup Y)\big) = \iota\left(\sum_{i=0}^{k} M_{(c_1,\ldots,c_i)}(X)M_{(c_{i+1},\ldots,c_k)}(Y)\right)$$
$$= \sum_{i=0}^{k} M_{(c_1,\ldots,c_i)}(X) \otimes M_{(c_{i+1},\ldots,c_k)}(Y),$$

$$(2)\quad \jmath\big(M_{(c_1,\ldots,c_k)}(X \times Y)\big)T$$
$$= \jmath\Bigg(\sum_{i_1+\cdots+i_p=k} M_{(c_1,\ldots,c_{i_1})}(Y) \ldots M_{(c_{i_1+\cdots+i_{p-1}+1},\ldots,c_{i_1+\cdots+i_p})}(Y)M_{C_1,\ldots,C_p}(X)\Bigg)$$
$$= \sum_{i_1+\cdots+i_p=k} M_{(c_1,\ldots,c_{i_1})}(Y) \ldots M_{(c_{i_1+\cdots+i_{p-1}+1},\ldots,c_{i_1+\cdots+i_p})}(Y) \otimes M_{C_1,\ldots,C_p}(X)$$

with $C_1 = c_1 + \cdots + c_{i_1}$, …, $C_p = c_{i_1+\cdots+i_{p-1}+1} + \cdots + c_{i_1+\cdots+i_p}$. With these identifications at hand, we define two coproducts on $\mathrm{QSym}(X)$ by:

$$\Delta\varphi := \varphi(X \sqcup Y),$$
$$\Gamma\varphi := \varphi(X \times Y),$$

with the shorthand notations $\varphi(X \sqcup Y) := \iota\big(\varphi(X \sqcup Y)\big)$ and $\varphi(X \times Y) := \jmath\big(\varphi(X \times Y)\big)$. We thus obtain a Hopf algebra $(\mathrm{QSym}, \cdot, \Delta)$ together with an extra internal coproduct Γ, with a basis (M_c) indexed by compositions:

$$M_{(c_1,\ldots,c_k)} M_{(c_{k+1},\ldots,c_{k+l})} = \sum_{r\geqslant 0} \sum_{\sigma\in \mathrm{QSh}(k,l;r)} M_{(c^\sigma_1,\ldots,c^\sigma_{k+l-r})},$$

$$\Delta(M_{(c_1,\ldots,c_k)}) = \sum_{i=0}^{k} M_{(c_1,\ldots,c_i)} \otimes M_{(c_{i+1},\ldots,c_k)},$$

$$\Gamma(M_{(c_1,\ldots,c_k)}) = \sum_{i_1+\cdots+i_p=k} M_{(c_1,\ldots,c_{i_1})} \ldots M_{(c_{i_1+\cdots+i_{p-1}+1},\ldots,c_{i_1+\cdots+i_p})} \otimes M_{(C_1,\ldots,C_p)}.$$

The construction of the Hopf algebra of packed words WQSym is similar. We now work in $B = \mathbb{Q}\langle\langle X\rangle\rangle$, the algebra of noncommutative formal series generated by X. Recall that a packed word is a surjective map $w : [k] \twoheadrightarrow [\max(w)]$, which we write as the word $w = w_1 \ldots w_k$. Now let $w' = x_1 \ldots x_k$ be a monomial in B, i.e. a word of length k with letters in X, and let $\mathrm{supp}\, w' \subset X$ be the support of w', i.e. the set of letters appearing in w'. There is a unique bijective, increasing map f, from $\mathrm{supp}\, w'$ to a set $[m]$. Then $\mathrm{Pack}(w')$ is the packed word $f(x_1) \ldots f(x_k)$. For any packed word w, we put:

$$M_w(X) = \sum_{\mathrm{Pack}(x_1\ldots x_k)=w} x_1 \ldots x_k \in B.$$

The subspace of B generated by these elements is a subalgebra of B, denoted by $\mathrm{WQSym}(X)$. Abstracting this, we obtain an algebra WQSym, with a basis (M_w) indexed by the set of packed words. Its product is given by:

$$M_u M_v = \sum_{r\geqslant 0} \sum_{w\in \mathrm{QSh}(\max(u),\max(v);r)} M_{w\circ(uv[\max(u)])},$$

where $v[\max(u)]$ is the word obtained from v by adding $\max(u)$ to each of its letters. The disjoint union of alphabets makes it a Hopf algebra, with the following coproduct:

$$\Delta(M_w) = \sum_{k=0}^{\max(w)} M_{w_{|\{1,\ldots,k\}}} \otimes M_{\mathrm{Pack}(w_{|\{k+1,\ldots,\max(w)\}})},$$

where for all set I, $w_{|I}$ is the word obtained by taking the letters of w belonging to I. The cartesian product of alphabets gives WQSym an internal coproduct:

$$\Gamma(M_u) = \sum_{i_1+\ldots+i_p=\max(u)} \sum_{r\geqslant 0} \sum_{v\in \mathrm{QSh}(i_1,\ldots,i_p;r)} M_{v\circ u} \otimes M_{(\underbrace{1\ldots1}_{i_1}\ldots\underbrace{p\ldots p}_{i_p})\circ u},$$

where $\mathrm{QSh}(i_1,\ldots,i_p;r)$ stands for the set of surjective maps $\sigma : [i_1+\cdots+i_p] \twoheadrightarrow [i_1+\cdots+i_p-r]$ which are increasing on each block $\{i_1+\cdots+i_q+1,\ldots,i_1+\cdots+i_{q+1}\}$. One can also write [5, Paragraph 5.2]:

$$\Gamma(M_u) = \sum_{a\wedge b=u} M_a \otimes M_b,$$

where $\wedge$ is the Tits product of the surjections viewed as set compositions of $[n]$ ([6, Paragraph 2.3], see also [5, Remark 2] and Paragraph 5.2 below).

5.2. The coalgebra species of set compositions

We now give an account of the set composition species $\mathbb{SC}$ together with its bimonoid structure in the category of coalgebra species. Applying the functors $\overline{\mathcal{K}}$ and $\mathcal{K}$ to $\mathbb{SC}$ will give QSym and WQSym respectively.

Definition 5.1. — [27] *Let X be a finite set. A set composition or an ordered partition of X is a finite sequence $(X_1,\ldots,X_k)$ of finite sets such that:*

1. *For all $1\leqslant i\leqslant k$, $X_i\neq\emptyset$.*
2. *$X = X_1\sqcup\ldots\sqcup X_k$.*

For any finite space X, the space generated by the set of set compositions of X will be denoted by $\mathbb{SC}_X$. This defines a species $\mathbb{SC}$.

The Hilbert formal series of $\mathbb{SC}$ is given by Fubini numbers, sequence A000670 of the OEIS.

We first give this species a structure of bialgebra in the category of species.

Definition 5.2.

1. *Let $Y\subseteq X$ be two finite sets and let $C=(X_1,\ldots,X_k)$ be a set composition on X. We put $I=\{i\in[k]\mid X_i\cap Y\neq\emptyset\}=\{m_1<\ldots<m_l\}$. The set composition $C_{|Y}$ of Y is:*

$$C_{|Y} = (X_{m_1}\cap Y,\ldots,X_{m_l}\cap Y).$$

For any finite sets X, Y, recall the quasi-shuffle product [1, Paragraph 10.1.6]*:*

$$\begin{cases} \mathbb{SC}_X \otimes \mathbb{SC}_Y & \longrightarrow & \mathbb{SC}_{X \sqcup Y} \\ C' \otimes C'' & \longrightarrow & C'C'' = \displaystyle\sum_{C,\, C_{|X}=C',\, C_{|Y}=C''} C. \end{cases}$$

(2) *For any finite set X, we define a coproduct:*

$$\Delta : \begin{cases} \mathbb{SC}_X & \longrightarrow & \displaystyle\bigoplus_{\substack{Y \subseteq X \\ k}} \mathbb{SC}_{X \setminus Y} \otimes \mathbb{SC}_Y \\ C = (X_1, \ldots, X_k) & \longrightarrow & \displaystyle\sum_{i=0} (X_1, \ldots, X_i) \otimes (X_{i+1}, \ldots, X_k). \end{cases}$$

(3) *For any finite set X, we define an internal coproduct Γ on $\mathbb{SC}_X$, making it a coassociative, counitary coalgebra by:*

$$\begin{aligned} &\Gamma((X_1, \ldots, X_k)) \\ &\quad = \sum_{i_1+\ldots+i_p=k} (X_1, \ldots, X_{i_1}) \ldots (X_{i_1+\ldots+i_{p-1}+1}, \ldots, X_{i_1+\ldots+i_p}) \\ &\qquad\qquad \otimes (X_1 \sqcup \ldots \sqcup X_{i_1}, \ldots, X_{i_1+\ldots+i_{p-1}+1} \sqcup \ldots \sqcup X_{i_1+\ldots+i_p}). \end{aligned}$$

Remark 5.3. — The *Tits product* ([6, Paragraph 2.3], [5, Paragraph 5.1], [2, Paragraph 1.6]) of two set compositions $C = (X_1, \ldots, X_k)$ and $C' = (X'_1, \ldots, X'_l)$ of the same finite set X is defined by:

$$C \wedge C' = (X_1 \cap X'_1, ..., X_1 \cap X'_l, X_2 \cap X'_1, ..., X_2 \cap X'_l, \ldots, X_k \cap X'_1, ..., X_k \cap X'_l).$$

The internal coproduct can then be written as:

$$\Gamma(C) = \sum_{C_1 \wedge C_2 = C} C_1 \otimes C_2.$$

Note that the Tits product is not commutative, although $C \wedge C'$ and $C' \wedge C$ define the same underlying set partition. The Tits product of two packed words v and w can also be defined as $\mathrm{pack}(v \times w)$ where $v \times w$ is the biword built up from v and w, and where the packing is taken with respect to the lexicographic order. This is a particular case of the internal product on parking functions, see [25, Section 4].

Examples. — Let A, B, C be finite, nonempty sets.

$$\begin{aligned}
(A)(B) &= (A,B)+(B,A)+(A\sqcup B).\\
(A,B)(C) &= (A,B,C)+(A,C,B)+(C,A,B)+(A,B\sqcup C)+(A\sqcup C,B).\\
(A)(B,C) &= (A,B,C)+(B,A,C)+(B,C,A)+(A\sqcup B,C)+(B,A\sqcup C);\\
\Gamma((A)) &= (A)\otimes(A).\\
\Gamma((A,B)) &= (A,B)\otimes(A\sqcup B)+(A)(B)\otimes(A,B)\\
&= (A,B)\otimes(A\sqcup B)+((A,B)+(B,A)+(A\sqcup B))\otimes(A,B).\\
\Gamma((A,B,C)) &= (A,B,C)\otimes(A\sqcup B\sqcup C)+(A,B)(C)\otimes(A\sqcup B,C)\\
&\quad+(A)(B,C)\otimes(A,B\sqcup C)+(A)(B)(C)\otimes(A,B,C)\\
&= (A,B,C)\otimes(A\sqcup B\sqcup C)\\
&\quad+((A,B,C)+(A,C,B)+(C,A,B)\\
&\qquad+(A\sqcup C,B)+(A,B\sqcup C))\otimes(A\sqcup B,C)\\
&\quad+((A,B,C)+(B,A,C)+(B,C,A)\\
&\qquad+(A\sqcup B,C)+(B,A\sqcup C))\otimes(A,B\sqcup C)\\
&\quad+\big((A,B,C)+(A,C,B)+(B,A,C)+(B,C,A)\\
&\qquad+(C,A,B)+(C,B,A)+(A\sqcup B,C)+(A\sqcup C,B)\\
&\qquad+(B\sqcup C,A)+(A,B\sqcup C)+(B,A\sqcup C)+(C,A\sqcup B)\\
&\qquad+(A\sqcup B\sqcup C)\big)\otimes(A,B,C).
\end{aligned}$$

Proposition 5.4. — *$\mathbb{SC}$ is a Hopf monoid in the category of coalgebra species.*

This is already known: the Hopf monoid structure of $\mathbb{SC}$ appears in [2, Paragraph 11.1], and the internal coproduct is dual to the Tits product. We shall recover this result from Theorem 5.6 below, which will make the coalgebra species Hopf monoid $\mathbb{SC}$ appear as a quotient of the coalgebra species Hopf monoid $\mathbb{T}$.

The counit of the coalgebra $\mathbb{SC}_X$ is given by:

$$\varepsilon(C)=\begin{cases}1 \text{ if } C=(X),\\ 0 \text{ otherwise.}\end{cases}$$

Applying the functors $\overline{\mathcal{K}}$ and $\mathcal{K}$, we obtain from $\mathbb{SC}$ two bialgebras with an internal coproduct. First, it induces a bialgebra structure on the vector space generated by the set compositions, up to a renumbering. For any set composition $C=(X_1,\dots,X_k)$, we put $type(C)=(|X_1|,\dots,|X_k|)$. If C, C' are two set compositions, C and C' are equal up to a renumbering if, and

only if, $type(C) = type(C')$. So this bialgebra has a basis (M_c), indexed by compositions, and direct computations shows this is QSym. Secondly, we restrict ourselves to sets $[n]$, $n \geqslant 0$; we identify any subset $I \subseteq [n]$ with $[|I|]$ via the unique increasing bijection. Set compositions on $[n]$ are identified with packed words of length n, via the bijection:

$$\begin{array}{rcl} \{\text{Packed words of length } n\} & \longrightarrow & \mathbb{SC}_{[n]} \\ u & \longmapsto & (u^{-1}(1), \dots, u^{-1}(\max(u))). \end{array}$$

We obtain a bialgebra with a basis indexed by packed words, which is precisely WQSym.

5.3. Linear extensions

Definition 5.5. — *Let $\mathcal{T} \in \mathbb{T}_X$ and let $C = (X_1, \dots X_k) \in \mathbb{SC}_X$. We shall say that C is a linear extension of $\mathcal{T}$ if :*

(1) *For all $i, j \in [k]$, for all $x \in X_i$, $y \in X_j$, $x <_{\mathcal{T}} y \Longrightarrow i < j$.*
(2) *For all $i, j \in [k]$, for all $x \in X_i$, $y \in X_j$, $x \sim_{\mathcal{T}} y \Longrightarrow i = j$.*

The set of linear extensions of $\mathcal{T}$ will be denoted by $\mathcal{L}_{\mathcal{T}}$.

This notion of linear extension is used in [15], where it is related to Stanley's theory of P-partitions extended to finite topologies [26].

Theorem 5.6. — *Let X be a finite set. We define:*

$$L : \begin{array}{rcl} \mathbb{T}_X & \longrightarrow & \mathbb{SC}_X \\ \mathcal{T} & \longmapsto & \displaystyle\sum_{C \in \mathcal{L}_{\mathcal{T}}} C. \end{array}$$

Then L is a surjective morphism of bialgebras in the category of coalgebra species, that is to say:

(1) *For all finite sets X, Y, for all $\mathcal{T} \in \mathbb{T}_X$, $\mathcal{T}' \in \mathbb{T}_Y$,*

$$L(\mathcal{T}\mathcal{T}') = L(\mathcal{T})L(\mathcal{T}').$$

(2) *For all finite set X, for all $\mathcal{T} \in \mathbb{T}_X$,*

$$\Delta \circ L(\mathcal{T}) = (L \otimes L) \circ \Delta(\mathcal{T}).$$

(3) *For all finite set X, for all $\mathcal{T} \in \mathbb{T}_X$,*

$$\Gamma \circ L(\mathcal{T}) = (L \otimes L) \circ \Gamma(\mathcal{T}).$$

Proof. —

First Step. — Let us prove the following lemma: if $Y \subseteq X$, $\mathcal{T} \in \mathbb{T}_X$ and $C \in \mathcal{L}_{\mathcal{T}}$, then $C_{|Y} \in \mathcal{L}_{\mathcal{T}_{|Y}}$.

We put $C = (X_1, \dots, X_k)$ and $C_{|Y} = (X_{m_1} \cap Y, \dots, X_{m_l} \cap Y) = (Y_1, \dots, Y_l)$. Let $i, j \in [l]$, $x \in Y_i$, $y \in Y_j$. If $x <_{\mathcal{T}_{|Y}} y$, then $x <_{\mathcal{T}} y$, so $m_i < m_j$, and finally $i < j$. If $x \sim_{\mathcal{T}_{|Y}} y$, then $x \sim_{\mathcal{T}} y$, so $m_i = m_j$, and finally $i = j$.

Second Step. — We prove (1). Let $\mathcal{T} \in \mathbb{T}_X$ and $\mathcal{T}' \in \mathbb{T}_Y$. Let us prove that:

$$\mathcal{L}_{\mathcal{T}\mathcal{T}'} = \{C \in \mathbb{SC}_{X \sqcup Y} \mid C_{|X} \in \mathcal{L}_{\mathcal{T}}, C_{|Y} \in \mathcal{L}_{\mathcal{T}'}\}.$$

As $\mathcal{T}\mathcal{T}'_{|X} = \mathcal{T}$ and $\mathcal{T}\mathcal{T}'_{|Y} = \mathcal{T}'$, the first step implies that inclusion $\subseteq$ holds. Moreover, if $x <_{\mathcal{T}\mathcal{T}'} y$ or $x \sim_{\mathcal{T}\mathcal{T}'} y$ in $X \sqcup Y$, then $(x, y) \in X^2$ or $(x, y) \in Y^2$, which implies the second inclusion. Consequently:

$$\begin{aligned} L(\mathcal{T}\mathcal{T}') &= \sum_{C,\, C_{|X} \in \mathcal{L}_{\mathcal{T}}, C_{|Y} \in \mathcal{L}_{\mathcal{T}'}} C \\ &= \sum_{C' \in \mathcal{L}_{\mathcal{T}}} \sum_{C'' \in \mathcal{L}_{\mathcal{T}'}} \sum_{C,\, C_{|X} = C', C_{|Y} = C''} C \\ &= \sum_{C' \in \mathcal{L}_{\mathcal{T}}} \sum_{C'' \in \mathcal{L}_{\mathcal{T}'}} C'C'' \\ &= L(\mathcal{T})L(\mathcal{T}'). \end{aligned}$$

Third Step. — We prove (2). Let $\mathcal{T}$ be a topology on a set X. We put:

$$\begin{aligned} A &= \{(Y, C_1, C_2) \mid Y \in \mathcal{T}, C_1 \in \mathcal{L}_{\mathcal{T}_{|X \setminus Y}}, C_2 \in \mathcal{L}_{\mathcal{T}_{|Y}}\}, \\ B &= \{(C, i) \mid C \in \mathcal{L}_{\mathcal{T}}, 0 \leqslant i \leqslant lg(C)\}, \end{aligned}$$

which gives:

$$\begin{aligned} (L \otimes L) \circ \Delta(\mathcal{T}) &= \sum_{(Y, C_1, C_2) \in A} C_1 \otimes C_2, \\ \Delta \circ L(\mathcal{T}) &= \sum_{((X_1, \dots, X_k), i) \in B} (X_1, \dots, X_i) \otimes (X_{i+1}, \dots, X_k). \end{aligned}$$

We define two maps:

$$\begin{array}{rccc} f : & A & \longrightarrow & B \\ & (Y, (X_1, \dots, X_k), (X_{k+1}, \dots, X_{k+l})) & \longmapsto & ((X_1, \dots, X_{k+l}), k), \end{array}$$

$$\begin{array}{rccc} g : & B & \longrightarrow & A \\ & ((X_1, \dots, X_k), i) & \longmapsto & (X_{i+1} \sqcup \dots \sqcup X_i, (X_1, \dots, X_i), (X_{i+1}, \dots, X_k)). \end{array}$$

Let us prove that f is well-defined. If $(Y, C_1, C_2) \in A$, we put $C_1 = (X_1, \dots, X_k)$, $C_2 = (X_{k+1}, \dots, X_{k+l})$, and $C = (X_1, \dots, X_{k+l})$. Let us

prove that $C \in \mathcal{L}_{\mathcal{T}\mathcal{T}'}$. Let $x \in X_i$, $y \in X_j$. If $x <_{\mathcal{T}} y$, as Y is an open set of $\mathcal{T}$, there are only three possibilities:

- $x, y \in Y$. As C_2 is a linear extension of $\mathcal{T}_{|Y}$, $i < j$.
- $x, y \in X \setminus Y$. As C_1 is a linear extension of $\mathcal{T}_{|X\setminus Y}$, $i < j$.
- $x \in X \setminus Y$ and $y \in Y$. Then $i \leqslant k < j$.

If $x \sim_{\mathcal{T}} y$, as Y is an open set of $\mathcal{T}$, so is a union of equivalence classes of $\sim_{\mathcal{T}}$, there are only two possibilities:

- $x, y \in Y$. As C_2 is a linear extension of $\mathcal{T}_{|Y}$, $i = j$.
- $x, y \in X \setminus Y$. As C_1 is a linear extension of $\mathcal{T}_{|X\setminus Y}$, $i = j$.

So $f(Y, C_1, C_2) \in B$.

Let us prove that g is well-defined. If $((X_1, \ldots, X_k), i) \in B$, we put $f((X_1, \ldots, X_k), i) = (Y, C_1, C_2)$. Y is an open set of $\mathcal{T}$: let $x \in Y$, $x \in X$, such that $x \leqslant_{\mathcal{T}} y$. We assume that $x \in X_j$, with $j \geqslant i$, and $y \in X_k$. If $x \sim_{\mathcal{T}} y$, then $j = k \geqslant i$ and $y \in Y$. If $x <_{\mathcal{T}} y$, then $i \leqslant j < k$, so $y \in Y$. Moreover, $C_1 = (X_1, \ldots, X_i) = C_{|X\setminus Y}$ and $C_2 = (X_{i+1}, \ldots, X_k) = C_{|Y}$. By the lemma of the first point, $C_1 \in \mathcal{L}_{\mathcal{T}_{|X\setminus Y}}$ and $C_2 \in \mathcal{L}_{\mathcal{T}_{|Y}}$. So $(Y, C_1, C_2) \in A$.

Moreover:

$$\begin{aligned} f \circ g((X_1, \ldots, X_k), i) &= f(X_{i+1} \sqcup \ldots \sqcup X_k, (X_1, \ldots, X_i), (X_{i+1}, \ldots, X_k)) \\ &= ((X_1, \ldots, X_k), i); \\ g \circ f(Y, C_1, C_2) &= g(C_1.C_2, lg(C_1)) \\ &= (Y, C_1, C_2). \end{aligned}$$

So f and g are bijections, inverse one from each other. Consequently:

$$\begin{aligned} (L \otimes L) \circ \Delta(\mathcal{T}) &= \sum_{(Y,C_1,C_2)\in A} C_1 \otimes C_2 \\ &= \sum_{((X_1,\ldots,X_k),i)\in B} (X_1, \ldots, X_i) \otimes (X_{i+1}, \ldots, X_k) \\ &= \Delta \circ L(\mathcal{T}). \end{aligned}$$

Fourth Step. — Let A be the set of triples $(C, (i_1, \ldots, i_p), C')$ such that:

(1) $C = (X_1, \ldots, X_k)$ and $C' = (X'_1, \ldots, X'_p)$ are set compositions of X, of respective length k and p.
(2) For all j, $i_j > 0$ and $i_1 + \ldots + i_p = k$.
(3) For all j,

$$C'_{|X_{i_1+\ldots+i_{j-1}+1}\sqcup\ldots\sqcup X_{i_1+\ldots+i_j}} = (X_{i_1+\ldots+i_{j-1}+1}, \ldots, X_{i_1+\ldots+i_j}).$$

Let B be the set of triples $(\mathcal{T}', C', C'')$ such that:

(1) $\mathcal{T}' \circledcirc \mathcal{T}$.
(2) C' is a linear extension of $\mathcal{T}'$.
(3) C'' is a linear extension of $\mathcal{T}/\mathcal{T}'$.

Then:

$$\Gamma \circ L(\mathcal{T}) = \sum_{((X_1,\dots,X_p),(i_1,\dots,i_p),C') \in A} C' \otimes (X_1 \sqcup \dots \sqcup X_{i_1}, \dots, X_{i_1+\dots+i_{p-1}+1} \sqcup \dots \sqcup X_{i_1+\dots+i_p}),$$

$$(L \otimes L) \circ \Gamma(\mathcal{T}) = \sum_{(\mathcal{T},C,C') \in B} C' \otimes C''.$$

We now prove the following lemma: if $(\mathcal{T}, C', C'') \in B$, with $C'' = (X_1'', \dots, X_q'')$, then:

$$\mathcal{T}' = \mathcal{T}_{|X_1''} \dots \mathcal{T}_{|X_q''}.$$

We first show that for all i, $\mathcal{T}'_{|X_i''} = \mathcal{T}_{|X_i''}$. Let us assume that $x, y \in X_i''$, such that $x \leqslant_{\mathcal{T}} y$. Then $x \leqslant_{\mathcal{T}/\mathcal{T}'} y$. If $x <_{\mathcal{T}/\mathcal{T}'} y$, as C'' is a linear extension of $\mathcal{T}/\mathcal{T}'$, we would have $x \in X_a''$, $y \in X_b''$, with $a < b$: this is a contradiction. So $x \sim_{\mathcal{T}/\mathcal{T}'} y$. As $\mathcal{T}' \circledcirc \mathcal{T}$, $x \sim_{\mathcal{T}'/\mathcal{T}'} y$, so x and y are in the same connected component Y of $\mathcal{T}'$. As $\mathcal{T}' \circledcirc \mathcal{T}$, $x \leqslant_{\mathcal{T}_{|Y}} y$, so $x \leqslant_{\mathcal{T}'_{|Y}} y$, so $x \leqslant_{\mathcal{T}'} y$. Conversely, if $x \leqslant_{\mathcal{T}'} y$, as $\mathcal{T}' \prec \mathcal{T}$, $x \leqslant_{\mathcal{T}} y$.

Let $x \in X_i''$, $y \in X_j''$, with $i < j$. As C'' is a linear extension of $\mathcal{T}/\mathcal{T}'$, we do not have $x \sim_{\mathcal{T}/\mathcal{T}'} y$, and, as $\mathcal{T}' \circledcirc \mathcal{T}$, we do not have $x \sim_{\mathcal{T}'/\mathcal{T}'} y$. Consequently:

$$\mathcal{T}' = \mathcal{T}'_{|X_1''} \dots \mathcal{T}'_{|X_q''} = \mathcal{T}_{|X_1''} \dots \mathcal{T}_{|X_q''}.$$

Fifth Step. — We prove (3). We define a map $f : A \longrightarrow B$ by $f(C, (i_1, \dots, i_p), C') = (\mathcal{T}', C', C'')$, where:

(1) $C'' = (X_1 \sqcup \dots \sqcup X_{i_1}, \dots, X_{i_1+\dots+i_{p-1}+1} \sqcup \dots \sqcup X_{i_1+\dots+i_p})$.
(2) $\mathcal{T}' = \mathcal{T}_{|X_1''} \dots \mathcal{T}_{|X_p''}$.

Let us prove that f is well-defined. First, $\mathcal{T}' \prec \mathcal{T}$. If $Y \subseteq X$ is connected for $\mathcal{T}'$, then necessarily there exists a i, such that $Y \subseteq X_i''$. Then $\mathcal{T}'_{|Y} = (\mathcal{T}'_{|X_i''})_{|Y} = (\mathcal{T}_{|X_i''})_{|Y} = \mathcal{T}_{|Y}$.

Let us assume that $x \sim_{\mathcal{T}/\mathcal{T}'} y$. There exists a sequence of elements of x such that:

$$x \leqslant_{\mathcal{T}} x_1 \geqslant_{\mathcal{T}'} y_1 \leqslant_{\mathcal{T}} x_2 \geqslant_{\mathcal{T}'} \dots \leqslant_{\mathcal{T}} x_r \geqslant_{\mathcal{T}'} y.$$

If $y_a \in X''_j$, as C is a linear extension of $\mathcal{T}$, necessarily $x_{a+1} \in X''_k$, with $k \geqslant j$. If $x_a \in X''_j$, as $x_a \geqslant_{\mathcal{T}'} y_a$, $y_a \in X''_j$. Consequently, if $x \in X''_i$, then $x_1, y_1, \ldots, x_r, y \in X''_i \sqcup \ldots X''_p$. By symmetry of x and y, $x, x_1, y_1, \ldots, x_r, y \in X''_i$. So, by restriction to X''_i:

$$x \leqslant_{\mathcal{T}'} x_1 \geqslant_{\mathcal{T}'} y_1 \leqslant_{\mathcal{T}'} x_2 \geqslant_{\mathcal{T}'} \ldots \leqslant_{\mathcal{T}'} x_r \geqslant_{\mathcal{T}'} y.$$

This gives $x \sim_{T'/\mathcal{T}'} y$: we finally obtain that $\mathcal{T}' \otimes \mathcal{T}$.

By the lemma of the first step, $C_{|X''_i}$ is a linear extension of $\mathcal{T}_{|X''_i}$, so, by definition of A, C' is a linear extension of $\mathcal{T}_{|X''_1} \ldots \mathcal{T}_{|X''_p} = \mathcal{T}'$.

Let us assume that $x <_{\mathcal{T}/\mathcal{T}'} y$. Let i, j such that $x \in X''_i$, $y \in X''_j$. Up to a change of $x \in X''_i$, $y \in X''_j$, we can assume that $x <_{\mathcal{T}} y$. If $i = j$, then by restriction $x <_{\mathcal{T}'} y$, so $x \sim_{\mathcal{T}'/\mathcal{T}'} y$ and finally $x \sim_{\mathcal{T}/\mathcal{T}'} y$, as $\mathcal{T}' \otimes \mathcal{T}$: this is a contradiction. Hence, $i \neq j$, and $x <_{\mathcal{T}} y$; as C is linear extension of $\mathcal{T}$, necessarily $i < j$.

Let us assume that $x \leqslant_{\mathcal{T}/\mathcal{T}'} y$. Let i, j such that $x \in X''_i$, $y \in X''_j$. By definition of $\leqslant_{\mathcal{T}/\mathcal{T}'}$, we can assume that $x \leqslant_{\mathcal{T}} y$ or $x \sim_{\mathcal{T}'} y$. In the first case, as C is a linear extension of $\mathcal{T}$, we have $x \in X_a$, $y \in X_b$, with $a \leqslant b$, so $i \leqslant j$. In the second case, $i = j$. Consequently, if $x \sim_{\mathcal{T}/\mathcal{T}'} y$, then $i \leqslant j$ and $j \leqslant i$, so $i = j$. We proved that $C'' \in \mathcal{L}_{\mathcal{T}/\mathcal{T}'}$.

We now consider the map $g : B \longrightarrow A$, defined by $g(\mathcal{T}', C', C'') = (C, (i_1, \ldots, i_p), C')$, with:

(1) $C = C'_{|X''_1} \ldots C'_{|X''_p}$, if $C'' = (X''_1, \ldots, X''_p)$.
(2) For all j, $i_j = |X''_j|$.

Let us prove that g is well-defined. Let us assume $x <_{\mathcal{T}} y$, with $x \in X''_i$, $y \in X''_j$. Let a, b such that $x \in X_a$, $y \in X_b$, if $C = (X_1, \ldots, X_k)$. If $i = j$, then by the lemma of the fourth step, $x <_{\mathcal{T}'} y$. As C' is a linear extension of $\mathcal{T}'$, $x \in C'_c$, $y \in C'_d$, with $c < d$. By definition of C, $a < b$. If $i \neq j$, then $x \leqslant_{\mathcal{T}/\mathcal{T}'} y$; as C'' is a linear extension of $\mathcal{T}/\mathcal{T}'$, $i < j$, so $a < b$. If $x \sim_{\mathcal{T}} y$, a similar argument proves that $x, y \in X_a$ for a certain a. So $C \in \mathcal{L}_{\mathcal{T}}$. Moreover, for all j:

$$\begin{aligned} C'_{|C_{i_1+\ldots+i_{j-1}+1} \sqcup \ldots \sqcup C_{i_1+\ldots+i_j}} &= C'_{|C''_j} = C_{|C''_j} \\ &= (C_{i_1+\ldots+i_{j-1}+1}, \ldots, C_{i_1+\ldots+i_j}). \end{aligned}$$

So g is well-defined. The lemma of the fourth step implies that $f \circ g = Id_B$, and by definition of A, $g \circ f = Id_A$, so f and g are bijective, inverse one

from each other. Finally:

$$\begin{aligned}(L\otimes L)\circ\Gamma(\mathcal{T}) &= \sum_{(\mathcal{T}',C',C'')\in B} C'\otimes C''\\ &= \sum_{((X_1,\dots,X_p),(i_1,\dots,i_p),C')\in A} C'\otimes (X_1\sqcup\dots\sqcup X_{i_1},\dots,\\ &\qquad\qquad X_{i_1+\dots+i_{p-1}+1}\sqcup\dots\sqcup X_{i_1+\dots+i_p})\\ &= \Gamma\circ L(\mathcal{T}).\end{aligned}$$

Last Step. — It remains to prove the surjectivity of L. Let $(X_1,\dots,X_k)$ be a set composition of X. Let $\mathcal{T}$ be the topology whose open sets are $X_i\sqcup\dots X_k$, for $1\leqslant i\leqslant k$, and $\emptyset$. Then $\mathcal{T}$ has a unique linear extension, which is C, so $L(\mathcal{T})=C$. □

Examples. — If $X=E\sqcup F=A\sqcup A\sqcup C$ are two partitions of X:

$$\begin{aligned}L(\bullet_X) &= (X),\\ L(\overset{F}{\underset{E}{\cdot\,\cdot}}) &= (E,F),\\ L(\bullet_E\ \bullet_F) &= (E,F)+(F,E)+(E\sqcup F),\\ L(\overset{B}{A}\!\!\diagdown\!\!\overset{C}{\diagup}) &= (A,B,C)+(A,C,B)+(A,B\sqcup C),\\ L(\overset{C}{\underset{A}{\cdot\,\cdot}}_B) &= (A,B,C),\\ L(_B\!\!\overset{A}{\diagup}\!\!\underset{C}{\diagdown}) &= (B,C,A)+(C,B,A)+(B\sqcup C,A),\\ L(\overset{B}{\underset{A}{\cdot\,\cdot}}\bullet_C) &= (A,B,C)+(A,C,B)+(C,A,B)+(A\sqcup C,B)+(A,B\sqcup C),\\ L(\bullet_A\ \bullet_B\ \bullet_C) &= (A,B,C)+(A,C,B)+(B,A,C)+(B,C,A)+(C,A,B)\\ &\quad+(C,B,A)+(A\sqcup B,C)+(A\sqcup C,B)+(B\sqcup C,A)\\ &\quad+(A,B\sqcup C)+(B,A\sqcup C)+(C,A\sqcup B)+(A\sqcup B\sqcup C).\end{aligned}$$

Now we consider isomorphism classes of finite topologies and set compositions. Let $\mathcal{T}$ be a topology on a finite set X, and let Z be an infinite, totally ordered alphabet. A linear extension of $\mathcal{T}$ is map $f:X\longrightarrow Z$, such that:

(1) $x<_{\mathcal{T}} y$ in $X\Longrightarrow f(x)<f(y)$.
(2) $w\sim_{\mathcal{T}} y$ in $X\Longrightarrow f(x)=f(y)$.

The set of linear extensions of $\mathcal{T}$ with values in Z is denoted by $\mathcal{L}_{\mathcal{T}}(Z)$.

THEOREM 5.7. — *Let Z be an infinite, denumerable, totally ordered alphabet. Identifying* $\mathrm{QSym}(Z)$ *and* QSym*, we define a map:*

$$\lambda : \begin{array}{rcl} \mathcal{H} & \longrightarrow & \mathrm{QSym} \\ \mathcal{T} \in \mathbb{T}_X & \longmapsto & \displaystyle\sum_{f \in \mathcal{L}_{\mathcal{T}}(Z)} \prod_{x \in X} f(x). \end{array}$$

Then λ is a Hopf algebra morphism, compatible with the internal coproducts of $\mathcal{H}$ and QSym*.*

Examples.

$$\begin{aligned}
\lambda(\bullet_a\ ) &= M_{(a)}, \\
\lambda(\overset{b}{\bullet_a}\ ) &= M_{(a,b)}, \\
\lambda(\bullet_a\ \bullet_b\ ) &= M_{(a,b)} + M_{(b,a)} + M_{(a+b)}, \\
\lambda(\overset{b}{\bullet}\!\!\diagdown\!\!\underset{a}{\bullet}^{c}\ ) &= M_{(a,b,c)} + M_{(a,c,b)} + M_{(a,b+c)}, \\
\lambda(\overset{c}{\overset{b}{\bullet_a}}\ ) &= M_{(a,b,c)}, \\
\lambda(_b\,\overset{a}{\bullet}\!\!\diagdown\!\!\underset{c}{\bullet}\ ) &= M_{(b,c,a)} + M_{(c,b,a)} + M_{(b+c,a)}, \\
\lambda(\overset{b}{\bullet_a}\ \bullet_c\ ) &= M_{(a,b,c)} + M_{(a,c,b)} + M_{(c,a,b)} + M_{(a+c,b)} + M_{(a,b+c)}, \\
\lambda(\bullet_a\ \bullet_b\ \bullet_c\ ) &= M_{(a,b,c)} + M_{(a,c,b)} + M_{(b,a,c)} + M_{(b,c,a)} + M_{(c,a,b)} + M_{(c,b,a)} \\
&\quad + M_{(a+b,c)} + M_{(a+c,b)} + M_{(b+c,a)} + M_{(a,b+c)} + M_{(b,a+c)} \\
&\quad + M_{(c,a+b)} + M_{(a+b+c)}.
\end{aligned}$$

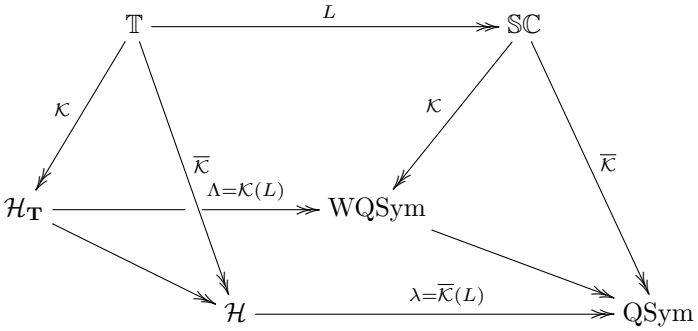
Restricting to finite topologies and set compositions on sets $[n]$, we obtain the following theorem:

THEOREM 5.8. — *Let Z be a denumerable well-ordered alphabet. Identifying* $\mathrm{WQSym}(Z)$ *and* WQSym*, we define a map:*

$$\Lambda : \begin{array}{rcl} \mathcal{H}_{\mathbf{T}} & \longrightarrow & \mathrm{WQSym} \\ \mathcal{T} \in \mathbb{T}_{[n]} & \longmapsto & \displaystyle\sum_{f \in \mathcal{L}_{\mathcal{T}}(Z)} f(1) \dots f(n). \end{array}$$

Then Λ is a Hopf algebra morphism, compatible with internal coproducts of $\mathcal{H}_{\mathbf{T}}$ and WQSym*.*

The situation is summarized by the commutative diagram below:



The triangle on the right of the diagram concerns algebraic structures well-known by now. We have shown that it is the image of a similar triangle, on the left of the diagram, which projects on the former by the explicit morphisms L, λ and Λ.

Examples.

$$\Lambda(\bullet_1) = M_{(1)},$$

$$\Lambda\left(\begin{smallmatrix}\bullet^2\\ \bullet_1\end{smallmatrix}\right) = M_{(1,2)},$$

$$\Lambda\left(\begin{smallmatrix}\bullet^1\\ \bullet_2\end{smallmatrix}\right) = M_{(2,1)},$$

$$\Lambda(\bullet_1\ \bullet_2) = M_{(1,2)} + M_{(2,1)} + M_{(1,1)},$$

$$\Lambda\left(\begin{smallmatrix}&\bullet^2\\ \bullet^3&\\ \bullet_1&\end{smallmatrix}\right) = M_{(1,2,3)} + M_{(1,3,2)} + M_{(1,2,2)},$$

$$\Lambda\left(\begin{smallmatrix}&\bullet^3\\ \bullet^1&\\ \bullet_2&\end{smallmatrix}\right) = M_{(2,1,3)} + M_{(3,1,2)} + M_{(2,1,2)},$$

$$\Lambda\left(\begin{smallmatrix}&\bullet^2\\ \bullet^1&\\ \bullet_3&\end{smallmatrix}\right) = M_{(2,3,1)} + M_{(3,2,1)} + M_{(2,2,1)},$$

$$\Lambda\left(\begin{smallmatrix}\bullet^3\\ \bullet^2\\ \bullet_1\end{smallmatrix}\right) = M_{(1,2,3)},$$

$$\Lambda\left(\begin{smallmatrix}\bullet^1\\ \bullet^3\\ \bullet_2\end{smallmatrix}\right) = M_{(3,1,2)},$$

$$\Lambda\left(\begin{smallmatrix}\bullet^2\\ \bullet^1\\ \bullet_3\end{smallmatrix}\right) = M_{(2,3,1)},$$

$$\Lambda\left(\begin{smallmatrix}&\bullet^1\\ \bullet_2&\\ &\bullet_3\end{smallmatrix}\right) = M_{(3,1,2)} + M_{(3,2,1)} + M_{(2,1,1)},$$

$$\Lambda\left(\begin{smallmatrix}&\bullet^2\\ \bullet_1&\\ &\bullet_3\end{smallmatrix}\right) = M_{(1,3,2)} + M_{(2,3,1)} + M_{(1,2,1)},$$

$$\Lambda\left(\begin{smallmatrix}&\bullet^3\\ \bullet_1&\\ &\bullet_2\end{smallmatrix}\right) = M_{(1,2,3)} + M_{(2,1,3)} + M_{(1,1,2)},$$

$$\Lambda(\begin{smallmatrix}\bullet_1^2 & \bullet_3\end{smallmatrix}) = M_{(1,2,3)} + M_{(1,3,2)} + M_{(2,3,1)} + M_{(1,2,1)} + M_{(1,2,2)},$$

$$\begin{aligned}\Lambda(\bullet_1\ \bullet_2\ \bullet_3) = {} & M_{(1,2,3)} + M_{(1,3,2)} + M_{(2,1,3)} + M_{(2,3,1)} + M_{(3,1,2)} \\ & + M_{(3,2,1)} + M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,1,1)} + M_{(1,2,2)} \\ & + M_{(2,1,2)} + M_{(2,2,1)} + M_{(1,1,1)}.\end{aligned}$$

Finally, as recalled in the text for QSym and WQSym, many interesting combinatorial Hopf algebras have polynomial realizations, in which the basis elements are realized as polynomials in an auxiliary set of commuting or non commuting variables. Such presentations have many advantages, beyond e.g. the very fast way of proving coassociativity by the doubling of alphabet trick implemented above. Polynomial realizations were recently obtained in [18] for the algebra of labelled forests and several related Hopf algebras: the extensions of the ideas of [18] to the posets and quasi–posets Hopf algebras remains to be done. The existence of a polynomial realization would be expected, as the internal coproduct of WQSym is exactly the one induced by the cartesian product of alphabets.

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