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The Hopf bifurcation and its stability for semilinear diffusion equations with time delay arising in ecology

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Introduction

Time delay mechanism arises in the dynamics of one or several species. One of such models is

$$\dot{U} = d\Delta U + a(1-U(t-r)/k)U, \tag{0.1}$$

where U means a population density, a, d, K and r are positive constants and $\cdot = \partial/\partial t$. In the absence of diffusion, (0.1) is well known as Volterra-Hutchinson's equation (e.g. [11, p. 94]). It is not difficult to verify that there exists a global solution of (0.1) in $(0, \infty) \times \Omega$ with initial and homogeneous Neumann boundary conditions, where Ω is a bounded domain in $\boldsymbol{R}^n$ with the smooth boundary $\partial\Omega$ (see Proposition 1.1 below). From an ecologial point of view, this boundary condition describes the situation where some population is reserved in a domain surrounded by a reflecting wall. The work by Cohen and Rosenblat [5], Lin and Kahn [9], Murray [12] and Yamada [16] related to this field should be referred.

Our interest lies in the spatio-temporal fluctuation of population density around the spatially homogeneous and postitive steady state $u(t, x) \equiv K$ caused by time delays. For this problem we study that a spatially homogeneous and temprally periodic orbit bifurcates form $u \equiv K$ as the primary bifurcation when some parameter, say r, crosses a critical value. We also discuss here a stability of the bifurcating orbit. This is done by the approach due to Chow and Mallet-Paret [4].

From both ecological and mathematical viewpoints, it is interesting to consider time delay models which exhibit spatially inhomogeneous and temporally periodic orbits bifurcating from the trivial solution as the primary bifurcation. We will show such models in the forthcoming paper.

In this paper, taking the problem of a spatially inhomogeneous bifurcating orbit into consideration, we develop a basic theory, especially the construction of a local integral manifold, and show the existence of the Hopf bifurcation for (0.1) with the homogeneous Neumann boundary condition and its stability. Here we give a remark. Every solution of the ordinary functional differential equation (shortly, OFDE) corresponding to (0.1) (i.e., $d=0$) is a solution of (0.1) with the homogeneous Neumann boundary condition and the existenec of periodic orbits for OFDEs follows from the known Hopf bifurcation theorem for OFDEs (cf. [3], [4], [7]). But even if a bifurcating orbit is stable as a solution of the OFDE, we can not say anything about a stability as the solution of the corresponding PFDE. Thus we should construct a theory for PFDEs in parallel with OFDEs. As for the Hopf bifurcation theorem for OFDEs it seems to the author that the proof in [7, Theorem 1.1, p. 246] is incomplete. We shall give a brief comment on this in Section 5.

By the change of variable $t \to rt$ in (0.1) we may assume $r=1$ from the beginning. Instead we regard a as a bifurcation parameter. Furthermore, if we change the unknown function by $U=K(1+u)$, our considering equation as a model results in

$$\dot{u} = d\Delta u - au(t-1) - au(t-1)u \tag{0.2}$$

with

$$(\partial u/\partial u)|_{\partial\Omega} = 0, \tag{0.3}$$

where $\partial/\partial n$ stands for the outer normal derivative to $\partial\Omega$.

In Section 1 as preliminaries we state a result on the existence, uniqueness, regularity and positivity of solutions for an equation including (0.1) with initial and homogeneous Neumann boundary conditions.

In Section 2 we investigate the variation of constants formula for the equation

$$\dot{u} = d\Delta u - au(t-1) + f \tag{0.4}$$

with (0.3), which is fundamental in the later discussion. We consider there the solution map which sends an initial function to the solution of the homogeneous equation of (0.4) with (0.3). These solution maps form a strongly continuous semigroup in $C=C([-1, 0]; L^p(\Omega))$ and using these maps we derive the variation of constants formula.

The considerations in Section 3 are partially done by C. C. Travis and G. F. Webb [15]. But to make our concepts clear, adding some modifications we study the spectrum of the generator of the solution map considered in Section 2, define the characteristic equations and give a decomposition theorem to C. Furthermore we study the formal adjoint equation of the homogeneous equation of (0.4) and give a decomposition theorem to the variation of constants formula which is used to construct a local integral manifold.

Section 4 is devoted to the construction of the local integral manifold which plays an essential role in the Hopf bifurcation problem.

Section 5 contains the discussions of the Hopf bifurcation theorem in [7].

Finally in Section 6 we make a remark on a stability of the Hopf bifurcation for (0.2) with (0.3) following S-N. Chow and J. Mallet-Paret [4].

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1. Preliminaries

In this section we state some results on the solution of

$$(1.1) \qquad \dot{U} = d\Delta U + (a - bU - cU(t-1))U \quad \text{in} \quad (0, \infty)\times\Omega,$$

$$(1.2) \qquad \partial U/\partial n = 0 \quad \text{on} \quad \partial\Omega,$$

$$(1.3) \qquad U(\theta) = \phi(\theta), \quad -1 \leqq \theta \leqq 0,$$

where a and c are positive constants and b is a non-negative constant. First we introduce some function spaces. Let $W^{2,p}(\Omega)$, $1<p<\infty$, be the Sobolev space of real valued L^p functions whose derivatives of order up to 2 belong to $L^p(\Omega)$ and $\|\cdot\|_{2,p}$ it norm. Put $W_N^{2,p}(\Omega)=\{u\in W^{2p}(\Omega);\ \partial u/\partial n=0 \text{ on } \partial\Omega\}$. For a Banach space H we let $C([a, b]; H)$ be the Banach space of H-valued continuous functions on $[a, b]$. Let A denote the closed operator in $L^p(\Omega)$ with dense domain $D(A)=W_N^{2,p}(\Omega)$ defined by $Au=-d\Delta u$ for $u\in D(A)$. Then $-A$ generates a holomorphic semigroup $\{e^{-tA}\}_{t\geqq 0}$ and the equation (1.1) with (1.2) is written in the following integral form:

$$(1.4) \qquad U(t) = e^{-tA}U(0) + \int_0^t e^{-(t-s)A}(a - bU(s) - cU_s(-1))U(s)ds.$$

Here we write $U_t(\theta)=U(t+\theta)$, $\theta\in[-1, 0]$, following J. K. Hale [7]. Throughout this paper we assume $n<p<\infty$. Proposition 1.1 below asserts the existence, uniqueness, regularity and positivity of solutions for (1.1) with (1.2) and (1.3). When $b\neq 0$, the proof is found in A. Schiaffino [14], and when $b=0$, we can prove the existence, uniqueness and regularity of solutions by iteration on n:

$$U^{n+1}(t) = e^{-(t-k)A}U(k) + \int_k^t e^{-(t-s)A}(a - cU_s(-1))U^n(s)ds$$

in each interval $[k, k+1]$, $k=0, 1,\dots$, with the aid of the following Lemma 1.1. The positivity of solution is, as in [13], due to the maximum principle for parabolic equations.

LEMMA 1.1 (cf. [1. Theorem 5.23, p. 115]). *Let u and v be functions in $W^{2,p}(\Omega)$. Then $uv\in{}^{2,p}(\Omega)$ and $\|uv\|_{2,p}\leqq c\|u\|_{2,p}\|v\|_{2,p}$, where c is a constant independent of u and v.*

Let C_1 denote the space $C([-1, 0]; W_N^{2,p}(\Omega))$ and $\|\cdot\|_{C_1}$ its norm.

PROPOSITION 1.1. *For any $\phi\in C_1$ there exists a unique solution $U\in C([0,$*

$\infty); W_N^{2,p}(\Omega))$ *of* (1.1) *with* (1.2) *and* (1.3) *such that* $\dot{U}\in C([0, \infty); L^p(\Omega))$. *If* $\phi\geqq 0$ *and* $\phi(0)\neq 0$, *then* $U(t)>0$ *for* $t>0$, *and furthermore if* $b\neq 0$, *then* $0< U(t, x)<\max\{\|\phi\|_{C_1}, a/b\}$.

2. The variation of constants formula

In this section we derive the variation of constants formula for the equation

$$\dot{u} = d\Delta u - au(t-1) + f \quad \text{in} \quad (\sigma, \infty)\times\Omega, \tag{2.1}$$

$$\partial u/\partial n = 0 \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

$$u_\sigma(\theta) = \phi(\theta), \quad -1 \leqq \theta \leqq 0. \tag{2.3}$$

As before we write (2.1) with (2.2) in the integral form

$$u(t) = e^{-(t-\sigma)A}u(\sigma) - a\int_\sigma^t e^{-(t-s)A}u_s(-1)ds + \int_\sigma^t e^{-(t-s)A}f(s)ds \qquad \text{for} \quad t \geqq \sigma. \tag{2.4}$$

Let $C=C([-1, 0]; L^p(\Omega))$ and $L^1_{loc}([\sigma, \infty); L^p(\Omega))$ denote the space of L^p valued locally summable functions on $[\sigma, \infty)$. It is easy to see by step by step method that for any $f\in L^1_{loc}([\sigma, \infty); L^p(\Omega))$ and any $\phi\in C$ there exists a unique solution u of (2.4) with (2.3) such that $u_t\in C$ for all $t\geqq\sigma$ and u satisfies

$$\|u_t\|_C \leqq Ke^{K(t-\sigma)}(\|\phi\|_C + \int_\sigma^t \|f(s)\|_p ds), \tag{2.5}$$

where K is a constant independent of ϕ, f and t, and $\|\cdot\|_C$ is the norm of C. Denoting by $u(\sigma, \phi, f)(t, x)$ (or $u(\sigma, \phi, f)$) the solution of (2.4) with (2.3), we define the solution map $T(t, \sigma)$ of C to itself by

$$T(t, \sigma)\phi = u_t(\sigma, \phi, 0).$$

PROPOSITION 2.1. $\{T(t, \sigma)\}_{t\geqq\sigma}$ *forms a strongly continuous semigroup in* C *and* $T(t, \sigma)$ *is compact for each* $t>1+\sigma$.

PROOF. It is obvious from the existence and uniquencess of solutions for (2.4) with (2.3) that $T(\sigma, \sigma)=I$ and the semigroup property $T(t, s)T(s, \sigma)=T(t, \sigma)$, $t\geqq s\geqq\sigma$, holds. The boundendness and the strong continuity of $T(t, \sigma)$ follows from (2.5). The compactness of $T(t, \sigma)$, $t>1+\sigma$, is due to C. C. Travis and G. F. Webb [15, Proposition 2.4]. Thus the proof is complete.

Since the equation (2.1) with $f=0$ is autonomous, we may denote $T(t, \sigma)$ by $T(t-\sigma)$. We also note that $T(t)$ is uniquely extended to a bounded linear operator from the space of piecewise continuous functions on $[-1, 0]$ with values in $L^p(\Omega)$ to C. Now we show the following variation of constants formula.

THEOREM 2.1. *Let $u(\sigma, \phi, f)$ be the solution of* (2.4) *with* (2.3) *for any $f \in L^1_{loc}([\sigma, \infty); L^p(\Omega))$ and any $\phi \in C$. Then*

$$u_t(\sigma, \phi, f) = T(t-\sigma)\phi + \int_\sigma^t T(t-s)X_0 f(s)ds, \tag{2.6}$$

where $X_0 = X_0(\theta, x)$ is such that $X_0 = 0$ for $-1 \leqq \theta < 0$ and $X_0 = 1$ for $\theta = 0$.

PROOF. Let $v(t) = u(\sigma, \phi, 0)(t, \cdot)$, $w_s(t) = u(0, X_0 f(s), 0)(t, \cdot)$ and

$$W(t) = \int_\sigma^t w_s(t-s)ds.$$

By definition, $w_s(\theta) = 0$ for $-1 \leqq \theta < 0$, so that $W(t) = 0$ for $\sigma - 1 \leqq t < \sigma$. If $t \geqq \sigma$, then

$$v(t) = e^{-(t-\sigma)A}\phi(0) - a\int_\sigma^t e^{-(t-s)A}v(s-1)ds.$$

On the other hand,

$$w_s(t) = e^{-tA}f(s) \qquad \text{for} \quad 0 \leqq t \leqq 1,$$

so that

$$W(t) = \int_\sigma^t e^{-(t-s)A}f(s)ds, \qquad \text{if} \quad \sigma \leqq t \leqq \sigma + 1.$$

Hence $u(t) = v(t) + W(t)$ satisfies (2.4) on $[\sigma, \sigma+1]$. For $t \geqq 1$,

$$w_s(t) = e^{-tA}f(s) - a\int_1^t e^{-(t-\tau)A}w_s(\tau-1)d\tau.$$

Hence, if $t \geqq \sigma + 1$, then

$$\begin{aligned} W(t) &= \int_\sigma^t w_s(t-s)ds \\ &= \int_\sigma^t e^{-(t-s)A}f(s)\,ds - a\int_\sigma^{t-1}\left\{\int_1^{t-s} e^{-(t-s-\tau)A}w_s(\tau-1)\,d\tau\right\}ds \\ &= \int_\sigma^t e^{-(t-s)A}f(s)\,ds - a\int_\sigma^{t-1}\left\{\int_{s+1}^t e^{-(t-\tau)A}w_s(\tau-1-s)\,d\tau\right\}ds \\ &= \int_\sigma^t e^{-(t-s)A}f(s)\,ds - a\int_{\sigma+1}^t e^{-(t-\tau)A}\left\{\int_\sigma^{\tau-1} w_s(\tau-1-s)\,ds\right\}d\tau \\ &= \int_\sigma^t e^{-(t-s)A}f(s)\,ds - a\int_\sigma^t e^{-(t-s)A}W(s-1)\,ds. \end{aligned}$$

Therefore, $u(t)=v(t)+W(t)$ satisfies (2.4) also on $[\sigma+1, \infty)$. Thus $u(t)=u(\sigma, \phi, f)(t, \cdot)$, so that

$$u_t(\sigma, \phi, f) = v_t + W_t = T(t-\sigma)\phi + \int_\sigma^t u_{t-s}(0, X_0 f(s), 0)ds$$

$$= T(t-\sigma)\phi + \int_\sigma^t T(t-s)X_0 f(s)ds.$$

The proof is complete.

3. Spectrum of the infinitesimal generator of $T(t)$

By Proposition 2.1 we know that $\{T(t)\}_{t\geqq 0}$ forms a strongly continuous semi-group in C. To begin with, studing the spectrum of the infinitesimal generator of $T(t)$ and defining the characteristic equation for (2.1) with (2.2), we give a decomposition theorem for C.

Let B (or $B(a)$ if it is necessary to specify a) be the infinitesimal generator of $T(t)$ and $D(B)$ the domain of B. When $-1\leqq\theta<0$, the equation $[T(t)\phi](\theta)=\phi(t+\theta)$ holds for sufficiently small $t>0$ and any $\phi\in D(B)$. Then we have

$$[B\phi](\theta) = \lim_{t\to 0}([T(t)\phi](\theta)-\phi(\theta))/t = \dot{\phi}(\theta).$$

Next consider the point $\theta=0$. Since $B\phi\in C$ for $\phi\in D(B)$, $B\phi(0)=\dot{\phi}(0)$. On the other hand, since

$$[T(t)\phi](0) = e^{-tA}\phi(0) - a\int_0^t e^{-(t-s)A}\phi_s(-1)ds \qquad \text{for} \quad 0<t<1,$$

it follows that

$$[B\phi](0) = \lim_{t\to 0}([T(t)\phi](0)-\phi(0))/t = -A\phi(0) - a\phi(-1).$$

Hence we have $B\phi=\dot{\phi}$ for $\phi\in D(B)$ and

$$D(B) = \{\phi\in C;\ \dot{\phi}\in C,\ \phi(0)\in D(A),\ \dot{\phi}(0) = -A\phi(0) - a\phi(-1)\}.$$

Let λ be a complex number and $\Delta(\lambda)$ a linear map of $D(A)$ to $L^p(\Omega)$ defined by $\Delta(\lambda)\alpha=\lambda\alpha+A\alpha+ae^{-\lambda}\alpha$, $\alpha\in D(A)$.

PROPOSITION 3.1. *Let $\sigma(B)$ be the spectrum of B. Then λ belongs to $\sigma(B)$ if and only if*

(3.1) $$\Delta(\lambda)\alpha = 0 \quad \textit{for some} \quad \alpha(\neq 0)\in D(A).$$

PROOF. Let $\rho(B)$ be the resolvent set of B. It is enough to show that

(3.2) $$\rho(B) = \{\lambda\in \boldsymbol{C};\ \Delta(\lambda)\alpha \neq 0 \quad \text{for all} \quad \alpha\in D(A)\backslash\{0\}\}$$

First we note that by the Riesz-Schauder theory [17, pp. 283–285] the condition in the right hand side of (3.2) is equivalent to the invertibility of $\Delta(\lambda)$ in $L^p(\Omega)$. Suppose that $\lambda \in \rho(B)$. Then for any $\psi \in C$ there exists a unique $\phi \in D(B)$ such that

$$\dot{\phi}(\theta) - \lambda\phi(\theta) = \psi(\theta), \; -1 \leqq \theta \leqq 0, \tag{3.3}$$

or equivalently,

$$\phi(\theta) = e^{\lambda\theta}\phi(0) + \int_0^\theta e^{\lambda(-\xi)}\psi(\xi)d\xi. \tag{3.4}$$

Let $\alpha = \phi(0)$. Then by the definition of B, $\alpha \in D(A)$ and

$$\lambda\alpha + \psi(0) = \dot{\phi}(0) = -A\alpha - a\phi(-1). \tag{3.5}$$

Hence we have

$$\Delta(\lambda)\alpha = -\psi(0) + ae^{-\lambda}\alpha - a\phi(-1) = -\psi(0) + a\int_{-1}^0 e^{-(1+\xi)}\psi(\xi)d\xi. \tag{3.6}$$

Since the right hand side of (3.6) covers $L^p(\Omega)$ when ψ varies in C, it follows that $\Delta(\lambda)$ is invertible in $L^p(\Omega)$.

Conversely, suppose $\Delta(\lambda)$ is invertible in $L^p(\Omega)$. Then, for any $\psi \in C$ there exists a unique $\alpha \in D(A)$ satisfying (3.6). Define $\phi(\theta)$ by (3.4) with $\phi(0) = \alpha$. Since ϕ satisfies (3.3) and (3.5), $\phi \in D(B)$, which implies $\lambda \in \rho(B)$. This completes the proof.

Let $\{\xi_j\}$, $0 = \xi_0 < \xi_1 \leqq \xi_2 \leqq \cdots \to \infty$, be the set of eigenvalues for $-\Delta$ with the homogeneous Neumann boundary condition in $L^p(\Omega)$. Then (3.1) holds if and only if λ satisfies $\lambda + ae^{-\lambda} + d\xi_l = 0$ for some l. In what follows we call

$$\lambda + ae^{-\lambda} + d\xi_j = 0, \quad j = 0, 1, \ldots, \tag{3.7}$$

the characteristic equations and their roots λ the characteristic roots. Since $\sigma(B)$ consists of the characteristic roots, it equals the point spectrum $P\sigma(B)$. Furthermore for any $\omega \in \boldsymbol{R}$ $\{\lambda \in \sigma(B) \colon \operatorname{Re}\lambda \geqq \omega\}$ is a finite set. To see this we prepare the following

LEMMA 3.1. *All roots of the equation* $\lambda + \gamma e^{-\lambda} + \delta = 0$, γ *and* δ *being real, have negative real parts if and only if*

$$\delta > -1, \tag{3.8}$$

$$\gamma + \delta > 0, \tag{3.9}$$

$$\gamma < \zeta \sin\zeta - \delta\cos\zeta, \tag{3.10}$$

where ζ *is the root of* $\zeta = -\delta\tan\zeta$, $0 < \zeta < \pi$ *if* $\delta \neq 0$, *and* $\zeta = \pi/2$ *if* $\delta = 0$. *Further-*

more if we denote the right hand side of (3.10) *by* $G(\delta)$, *then* $\pi/2 < G(\delta)$ *for* $0 < \delta < \infty$ *and* $G(\delta) \to \infty$ *as* $\delta \to \infty$.

PROOF. As for the first assertion of this lemma we refer to [7, Theorem A.5, p. 339]. We prove the second assertion. Since $\zeta = -\delta \tan \zeta$, $0 < \zeta < \pi$, it follows that ζ is differentiable with respect to δ and $\zeta' = -(\sin \zeta \cos \zeta)(\cos^2 \zeta + \delta)^{-1}$. Hence we have

$$\begin{aligned} G'(\delta) &= \zeta' \sin \zeta + \zeta'\zeta \cos \zeta - \cos \zeta + \delta\zeta' \sin \zeta = \zeta' \sin \zeta - \cos \zeta \\ &= -(\sin^2 \zeta \cos \zeta)(\cos^2 \zeta + \delta)^{-1} - \cos \zeta. \end{aligned}$$

When $\delta > 0$, the solutions of $\zeta = -\delta \tan \zeta$ such that $0 < \zeta < \pi$ are in $(\pi/2, \pi)$, and therefore we see that $G'(\delta) > 0$ for $0 < \delta < \infty$. It follows immediately that $G(\delta) \to \infty$ as $\delta \to \infty$. On the other hand, $G(\delta)$ tends to $\pi/2$ as $\delta \to +0$. Thus we have $G(\delta) > \pi/2$ for $0 < \delta < \infty$, which completes the proof.

We proceed to show that $\{\lambda \in \sigma(B);\ \mathrm{Re}\,\lambda \geqq \omega\}$ is a finite set. If we change the variable by $\zeta = \lambda - \omega$, then (3.7) results in

$$\zeta + \omega + ae^{-\omega}e^{-\zeta} + d\xi_j = 0,\ j = 0, 1, \dots . \tag{3.11}$$

Take γ and δ in Lemma 3.1 as $\gamma = ae^{-\omega}$ and $\delta = \omega + d\xi_j$. Since $G(\delta)$ in Lemma 3.1 tends to infinity as $\delta \to \infty$, the conditions (3.8), (3.9), (3.10) hold for large j. Hence Lemma 3.1 implies that the set

$$\{\lambda;\ \lambda + ae^{-\lambda} + d\xi_j = 0 \quad \text{and} \quad \mathrm{Re}\,\lambda \geqq \omega\}$$

is empty for large j. On the other hand, the number of roots with $\mathrm{Re}\,\lambda \geqq \omega$ for each equation (3.7) is clearly finite. Thus our assertion holds.

Let $P_\lambda(B)$ be the generalized eigensapce for a given λ in $\sigma(B)$ and $N(B-\lambda I)^k$ the null space of $(B-\lambda I)^k$. Then by virtue of [17, Theorem 3, p. 229] we have the following

PROPOSITION 3.2. *For any* $\lambda \in \sigma(B)$ *the dimension of* $P_\lambda(B)$ *is finite and there exists an integer k such that*

$$P_\lambda(B) = N(B-\lambda I)^k \quad \textit{and} \quad C = N(B-\lambda I)^k \oplus R(B-\lambda I)^k.$$

Let l be the dimension of $P_\lambda(B)$ and $\Phi_\lambda = (\phi_1, \dots, \phi_l)$ a basis of $P_\lambda(B)$. Since B commutes with $(B-\lambda I)^k$, we have $BP_\lambda(B) \subset P_\lambda(B)$. This yields that there exists an $l \times l$ constant matrix M_λ such that $B\Phi_\lambda = \Phi_\lambda M_\lambda$. Since $B\Phi_\lambda = d\Phi_\lambda/d\theta$, we have

$$\Phi_\lambda(\theta) = \Phi_\lambda(0)e^{M_\lambda \theta}, \quad -1 \leqq \theta \leqq 0.$$

Since $\phi_j(0) \in D(A)$, it follows that $\phi_j \in C_1$, $j = 1, \dots, l$.

Analogously, from the fact that

$$(d/dt)(T(t)\Phi_\lambda) = BT(t)\Phi_\lambda = T(t)\Phi_\lambda M_\lambda,$$

it follows that

$$T(t)\Phi_\lambda = \Phi_\lambda e^{M_\lambda t} = \Phi_\lambda(0)e^{M_\lambda(t+\theta)}.$$

Since $T(t)B\phi = BT(t)\phi$ for $\phi \in D(B)$, we have

$$T(t)R(B-\lambda I)^k \subset R(B-\lambda I)^k.$$

The preceding results are summarized in the following

THEOREM 3.1. *Let Λ be a finite set $\{\lambda_j \in \sigma(B); j=1,\ldots, p\}$, $\Phi_\Lambda = (\Phi_1,\ldots, \Phi_p)$ and $M_\Lambda = \operatorname{diag}(M_1,\ldots, M_p)$, where Φ_j is a basis for the generalized eigenspace of λ_j and M_j is the matrix defined by $B\Phi_j = \Phi_j M_j$, $j=1,\ldots, p$. Then the only eigenvalue of M_j is λ_j and for any vector $\boldsymbol{a}$ of the same dimension as that of Φ_Λ, the solution $T(t)\Phi_\Lambda \boldsymbol{a}$ of* (2.1) *with $f=0$,* (2.2) *and the initial value $\Phi_\Lambda \boldsymbol{a}$ at $\sigma=0$, is defined on $(-\infty, \infty)$ by the relation*

$$T(t)\Phi_\Lambda \boldsymbol{a} = \Phi_\Lambda e^{M_\Lambda t}\boldsymbol{a}, \qquad \Phi(\theta) = \Phi_\Lambda(0)e_\Lambda{}^{M_\Lambda \theta}, \quad -1 \leqq \theta \leqq 0.$$

Furthermore $P_\Lambda \equiv \oplus_{\lambda\in\Lambda} P_\lambda \subset C_1$ and there exists a subspace Q_Λ of C such that $T(t)Q_\Lambda \subset Q_\Lambda$ for all $t \geqq 0$ and $C = P_\Lambda \oplus Q_\Lambda$.

Let us say that the equation $\dot{v}(t) = -d\Delta v(t) + av(t+1)$ is the formal adjoint equation of $\dot{u}(t) = d\Delta u(t) - au(t-1)$. Putting $C^* = C([0, 1]; L^q(\Omega))$, $1/p + 1/q = 1$, we define a bilinear form $(\!(\,,\,)\!)$ by

$$(\!(\psi, \phi)\!) = \langle \psi(0), \phi(0)\rangle - a\int_{-1}^{0} \langle \psi(\xi+1), \phi(\xi)\rangle d\xi, \quad \text{for } \psi \in C^* \text{ and } \phi \in C,$$

where $\langle \cdot, \cdot \rangle$ is the duality between L^p and L^q. Let B^* be the operator, having a dense domain in C^* and a range in C^*, defined by $(\!(\psi, B\phi)\!) = (\!(B^*\psi, \phi)\!)$ for $\phi \in D(B)$ and $\psi \in D(B^*)$. Since

$$\begin{aligned}(\!(\psi, B\phi)\!) &= \langle \psi(0), \dot{\phi}(0)\rangle - a\int_{-1}^{0} \langle \psi(\xi+1), \dot{\phi}(\xi)\rangle d\xi \\ &= \langle \psi(0), -A\phi(0) - a\phi(-1)\rangle - a\langle \psi(1), \phi(0)\rangle \\ &\quad + a\langle \psi(0), \phi(-1)\rangle + a\int_{-1}^{0} \langle \dot{\psi}(\xi+1), \phi(\xi)\rangle d\xi \\ &= \langle -A^*\psi(0), \phi(0)\rangle - a\langle \psi(1), \phi(0)\rangle \\ &\quad + a\int_{-1}^{0} \langle \psi(\xi+1), \phi(\xi)\rangle d\xi = (\!(B^*\psi, \phi)\!),\end{aligned}$$

we have $B^*\psi(s)=-(d\psi/ds)(s)$, $0\leqq s\leqq 1$, $B^*\psi(0)=-A^*\psi(0)-a\psi(1)$ and $D(B^*)=\{\psi\in C^*;\ \dot\psi\in C^*,\ \psi(0)\in D(A^*),\ \dot\psi(0)=-d\Delta\psi(0)+a\psi(1)\}$. We note that A^* represents the realization of $-d\Delta$ with the homogeneous Neumann boundary condition in $L^q(\Omega)$, $D(A^*)=W_N^{2,q}(\Omega)$ and furthermore the eigenvalues of A^* are equal to those of A(cf. [2]]). Consider the formal adjoint equation

$$\dot v = -d\Delta v + av(t+1) \quad \text{in} \quad (-\infty, 0)\times\Omega \tag{3.12}$$

with

$$(\partial v/\partial n)|_{\partial\Omega} = 0 \quad \text{and} \quad v(\xi) = \psi(\xi),\ 0\leqq\xi\leqq 1, \tag{3.13}$$

or its integral form

$$v(t) = e^{tA^*}v(0) - a\int_t^0 e^{(t-s)A^*}v^s(1)ds \qquad \text{for} \quad t\leqq 0. \tag{3.14}$$

Here $v^t(\xi)$ stands for $v(t+\xi)$ for $0\leqq\xi\leqq 1$. For any $\psi\in C^*$ there exists a unique solution v of (3.14) in $C((-\infty, 0];\ L^q(\Omega))$ with the initial function ψ. Then we define the solution map $T^*(t)$ of C^* into itself by $T^*(t)\psi=v^t$, $-\infty<t\leqq 0$. As in the case of $T(t)$, the set $\{T^*(t)\}_{t\leqq 0}$ forms a strongly continuous semigroup in C^* and B^* denotes the infinitesimal generator of $T^*(t)$.

The following Theorem 3.2 is an analogue of [7, Lemmas 3.1, 3.2 and 3.4, p. 175 and p. 177], so we omit the proof.

THEOREM 3.2 (i) $\sigma(B)=\sigma(B^*)$.

(ii) *For any* $g\in C$ *the equation* $(B-\lambda I)^k\phi=g$ *admits a solution* ϕ *in* C *if and only if* $\langle\!\langle h, g\rangle\!\rangle=0$ *for all* $h\in N(B^*-\lambda I)^k$.

(iii) *Let* $\Phi_\lambda=(\phi_1,\dots,\phi_p)$ *be a basis of* $P_\lambda(B)$ *and* $\Psi_\lambda=\mathrm{col}(\psi_1,\dots,\psi_p)$ *a basis of* $P_\lambda(B^*)$. *Then the matrix* $\langle\!\langle\Psi_\lambda, \Phi_\lambda\rangle\!\rangle=\{(\psi_i, \phi_j);\ i, j=1,\dots, p\}$ *is nonsingular. Therefore, by a suitable change of basis,* $\langle\!\langle\Psi_\lambda, \Phi_\lambda\rangle\!\rangle=I$.

(iv) $\phi\in C$ *is uniquely written as* $\phi=\phi^P+\phi^Q$ *where* $\phi^P=\Phi_\lambda\langle\!\langle\Psi_\lambda, \phi\rangle\!\rangle$.

Similarly to the definition of M_λ we define M_λ^* by $B^*\Psi_\lambda=M_\lambda^*\Psi_\lambda$. We remark that $M_\lambda^*=M$ when $\langle\!\langle\Psi_\lambda, \Phi_\lambda\rangle\!\rangle=I$. In what follows we prove a decomposition theorem for (2.6). We write the decomposition of C with respect to $\Lambda=\{\lambda_j\in\sigma(B);\ j=1,\dots, p\}$ as $C=P\oplus Q$. Let Φ_Λ be a basis of P and Ψ_Λ a dual basis such that $\langle\!\langle\Psi_\Lambda, \Phi_\Lambda\rangle\!\rangle=I$.

THEOREM 3.3. *Let* u, ϕ *and* f *be as in Theorem* 2.1. *Then*

$$u_t^P = T(t-\sigma)\phi^P + \int_\sigma^t T(t-s)X_0^P f(s)ds,$$

$$u_t^Q = T(t-\sigma)\phi^Q + \int_\sigma^t T(t-s)X_0^Q f(s)ds,$$

where $X_0^P f(s) = \Phi_\Lambda \langle \Psi_\Lambda(0), f(s) \rangle$ *and* $X_0^Q f = X_0 f - X_0^P f$.

PROOF. Put $V^t = e^{-M_\Lambda t}\Psi_\Lambda(\xi)$. Then V^t is infinitely differentiable with respect to t and it together with its derivatives takes values in $D(A^\infty)(=\cap_{n=1}^\infty D(A^n))$ and $\dot{u} \in C([0, \infty); W^{-2,p}(\Omega))$. Since V^t is a matrix solution of (3.12) defined for $-\infty < t < \infty$, and

$$\langle\!\langle V^s, u_s \rangle\!\rangle = \langle V^s(0), u_s(0) \rangle - a\int_{-1}^{0} \langle V^s(\xi+1), u_s(\xi) \rangle d\xi$$
$$= \langle V(s), u(s) \rangle - a\int_{s-1}^{s} \langle V(\xi+1), u(\xi) \rangle d\xi,$$

it follows that

$$(d/ds)\langle\!\langle V^s, u_s \rangle\!\rangle = \langle \dot{V}(s), u(s) \rangle + \langle V(s), \dot{u}(s) \rangle - a\langle V^s(1), u(s) \rangle + a\langle V(s), u(-1) \rangle$$
$$= \langle -d\Delta V(s), u(s) \rangle + a\langle V^s(1), u(s) \rangle + \langle V(s), d\Delta u(s) \rangle - a\langle V(s), u_s(-1) \rangle$$
$$+ \langle V(s), f(s) \rangle - a\langle V^s(1), u(s) \rangle + a\langle V(s), u_s(-1) \rangle = \langle V(s), f(s) \rangle .$$

This yields

$$\langle\!\langle V^t, u_t \rangle\!\rangle = \langle\!\langle V^\sigma, u_\sigma \rangle\!\rangle + \int_\sigma^t \langle V(s), f(s) \rangle ds,$$

which implies

$$\langle\!\langle e^{-M_\Lambda t}\Psi_\Lambda, u_t \rangle\!\rangle = \langle\!\langle e^{-M_\Lambda \sigma}\Psi_\Lambda, \phi \rangle\!\rangle + \int_\sigma^t \langle e^{-M_\Lambda s}\Psi_\Lambda(0), f(s) \rangle ds. \tag{3.15}$$

Consequently we have

$$u_t^P = \Phi_\Lambda \langle\!\langle \Psi_\Lambda, u_t \rangle\!\rangle = \Phi_\Lambda \langle\!\langle e^{(t-\sigma)M_\Lambda}\Psi_\Lambda, \phi \rangle\!\rangle + \Phi_\Lambda \int_\sigma^t \langle e^{(t-s)M_\Lambda}\Psi_\Lambda(0), f(s) \rangle ds$$
$$= T(t-\sigma)\Phi_\Lambda \langle\!\langle \Psi_\Lambda, \phi \rangle\!\rangle + \int_\sigma^t T(t-s)\Phi_\Lambda \langle \Psi_\Lambda(0), f(s) \rangle ds$$
$$= T(t-\sigma)\phi^P + \int_\sigma^t T(t-s) X_0^P f(s) ds.$$

Here we used the fact that $T(t)\Phi_\Lambda = \Phi_\Lambda e^{M_\Lambda t}$ and put

$$X_0^P f(s) = \Phi_\Lambda \langle \Psi_\Lambda(0), f(s) \rangle .$$

Defining $X_0^Q f$ by $X_0^Q f = X_0 f - X_0^P f$, we have

$$u_t^Q = u_t - u_t^P = T(t-\sigma)\phi^Q + \int_\sigma^t T(t-s) X_0^Q f(s) ds,$$

which completes the proof.

4. The local integral manifold $\mathscr{M}(\alpha)$

In the Hopf bifurcation problem the existence of a two dimensional local integral manifold for the equation (0.2) with (0.3) plays a central role. If this manifold is constructed, the problem is reduced to the two dimensional case and the well-known Hopf bifurcation theorem is applied (e.g. [6], [10] and [12]). Thus in this section we construct a two dimensional local integral manifold by analogy with O. E. Lanford III [8, 10] and N. Chafee [3].

Let a_c be a critical point, i.e., the characteristic equations

$$\lambda + a_c e^{-\lambda} + d\xi_j = 0, \quad j = 0, 1,\dots,$$

have a pair of simple complex conjugate roots $\pm i\nu_0$, $\nu_0>0$, and the other characteristic roots do not lie on the imaginary axis. Then we rewrite (0.2) as

$$\begin{aligned}(4.1)\qquad \dot{u}(t) &= d\Delta u(t) - au(t-1) - au(t-1)u(t)\\ &= d\Delta u(t) - a_c u(t-1) - \{(a-a_c)u_t(-1)+au_t(-1)u_t(0)\}.\end{aligned}$$

In what follows, taking the above nonlinear term into account we consider the equation with a slightly general nonlinear term

$$(4.2)\qquad \dot{u}(t) = d\Delta u(t) - a_c u(t-1) + f(u_t, \alpha),$$

where $\alpha=a-a_c$. Here we impose assumptions on f.

ASSUMPTION. i) There exists a positive α_0 such that f is a $k+1$ ($k\geqq 1$) times continuously differentiable function in $(\phi, \alpha)\in C_1\times[-\alpha_0, \alpha_0]$ with values in $W_N^{2,p}(\Omega)$.

ii) $f(0, \alpha)=0$ for any $\alpha\in[-\alpha_0, \alpha_0]$ and $D_\phi f(0, 0)=0$, where $D_\phi f$ is the Fréchet derivative of f with respect to ϕ.

In (4.1), $f(\phi, \alpha)=-\alpha\phi(-1)-(a_c+\alpha)\phi(-1)\phi(0)$. By virtue of Lemma 1.1 this f satisfies the above assumption.

Let us now denote by $C=P_0\oplus P_1\oplus Q$ the spectral decomposition, where P_0 is the two dimensional eigenspace of $B(a_c)$ corresponding to $\{\pm i\nu_0\}$ and P_1 the generalized eigenspace corresponding to $\Lambda_1=\{\lambda\in\sigma(B(a_c)); \operatorname{Re}\lambda>0\}$. It is to be noticed that the dimension l of P_1 is finite by the discussion in Section 3. Since P_0, $P_1\subset C_1$, it follows that $C_1=P_0\oplus P_1\oplus\tilde{Q}$, where $\tilde{Q}=Q\cap C_1$. Let Φ_0 (resp. Φ_1) be a basis of P_0 (resp. P_1) and Ψ_0 (resp. Ψ_1) the dual basis of Φ_0 (resp. Φ_1) such that $\langle\!\langle\Psi_0, \Phi_0\rangle\!\rangle=I$ (resp. $\langle\!\langle\Psi_1, \Phi_1\rangle\!\rangle=I$). For $u\in C(\boldsymbol{R}; W_N^{2,p}(\Omega))$, put $x(t)=\langle\!\langle\Psi_0, u_t\rangle\!\rangle$ and $y(t)=\langle\!\langle\Psi_1, u_t\rangle\!\rangle$. Let u_t^0, u_t^1 and u_t^Q be the projections of u_t to P_0, P_1 and Q, respectively. Then we have $u_t^0=\Phi_0 x(t)$ and $u_t^1=\Phi_1 y(t)$. Scaling (4.2) by $u\to\varepsilon u$, $\alpha\to\varepsilon\alpha$, we have

(4.2)′ $$\dot{u}(t) = d\Delta u(t) - a_c u(t-1) + (1/\varepsilon)f(\varepsilon u_t, \varepsilon\alpha).$$

Let $\chi(x, \alpha)$ be an infinitely differentiable function on $\boldsymbol{R}^3=\boldsymbol{R}^2\times\boldsymbol{R}$ such that $\chi=1$ in $B(0, 1/2)$ and $\chi=0$ in $\boldsymbol{R}^3-B(0, 1)$, where $B(0, r)=\{(x, \alpha)\in\boldsymbol{R}^3;\ |x|^2+(\alpha/\alpha_0)^2\leqq r^2\}$. We write

$$g_\varepsilon(x, y, z, \alpha) = (1/\varepsilon)\chi(x, \alpha)f(\varepsilon\Phi_0 x+\varepsilon\Phi_1 y+\varepsilon z, \varepsilon\alpha),$$

which is defined in $\boldsymbol{R}^2\times\boldsymbol{R}^l\times\tilde{Q}\times\boldsymbol{R}$. Since we construct the integral manifold only in a neighborhood of $(u, \alpha)=(0, 0)$, instead of (4.2)′ we consider

(4.3) $$\dot{u}(t) = d\Delta u(t) - a_c u(t-1) + g_\varepsilon(x(t), y(t), z_t, \alpha), \quad -\infty < t < \infty,$$

where $z_t=u_t^Q$. Then by virtue of Theorem 3.3 we have

(4.4) $$u_t^0 = T_0(t-\sigma)u_\sigma^0 + \int_\sigma^t T_0(t-s)\Phi_0 X_\varepsilon(x(s), y(s), z_s, \alpha)ds,$$

(4.5) $$u_t^1 = T_1(t-\sigma)u_\sigma^1 + \int_\sigma^t T_1(t-s)\Phi_1 Y_\varepsilon(x(s), y(s), z_s, \alpha)ds,$$

(4.6) $$z_t = T_Q(t-\sigma)z_\sigma + \int_\sigma^t T_Q(t-s)Z_\varepsilon(x(s), y(s), z_s, \alpha)ds,$$

for $t\geqq\sigma$ (σ being arbitrarily chosen and fixed in $\boldsymbol{R}$), where T_0, T_1 and T_Q are the restriction of T to P_0, P_1 and Q, respectively, $X_\varepsilon(x, y, z, \alpha)=\langle\Psi_0(0), g_\varepsilon(x, y, z, \alpha)\rangle$, $Y_\varepsilon(x, y, z, \alpha)=\langle\Psi_1(0), g_\varepsilon(x, y, z, \alpha)\rangle$ and $Z_\varepsilon(x, y, z, \alpha)=X_0^Q g_\varepsilon(x, y, z, \alpha)$. Applying Ψ_0 and Ψ_1 to the both sides of (4.4) and (4.5), respectively, and differentiating them, we obtain

(4.7) $$\dot{x}(t) = M_0 x(t) + X_\varepsilon(x(t), y(t), z_t, \alpha),$$

(4.8) $$\dot{y}(t) = M_1 y(t) + Y_\varepsilon(x(t), y(t), z_t, \alpha),$$

where M_0 and M_1 are the matrix representation of $B(a_c)$ restricted to P_0 and P_1.

In what follows in order to avoid the complexity of notations we use frequently the same notation $\|\cdot\|$ to represent norms in various Banach space whenever there is no fear of confustion. Let us put

(4.9) $$\lambda_\varepsilon=\sum\sup\{\|D_x^{j_1}D_y^{j_2}D_z^{j_3}D_\alpha^{j_4}X_\varepsilon\|+\|D_x^{j_1}D_y^{j_2}D_z^{j_3}D_\alpha^{j_4}Y_\varepsilon\|+\|D_x^{j_1}D_y^{j_2}D_z^{j_3}D_\alpha^{j_4}Z_\varepsilon\|\},$$

where the sum is taken over $|j_1|+|j_2|+j_3+j_4\leqq k+1$ and the supremum is taken over $(x, \alpha)\in\boldsymbol{R}^2\times\boldsymbol{R}$, $|y|\leqq 2$ and $\|z\|_{C_1}\leqq 2$ $(z\in\tilde{Q})$. Then we can make λ_ε as small as we like by choosing $|\varepsilon|$ sufficiently small.

THEOREM 4.1. *If $|\varepsilon|$ is sufficiently small, then there exist k times continuously differentiable functions F_0 and G_0 on $\boldsymbol{R}^2\times\boldsymbol{R}$ with values in $\boldsymbol{R}^l$ and $\tilde{Q}$ respectively, satisfying the following conditions*:

i) $F_0(0, \alpha)=0$, $D_xF_0(0, 0)=0$, $G_0(0, 0)=0$, $D_xG_0(0, 0)=0$.
ii) *For any* α *let*

$$\mathscr{M}(\alpha) = \{\phi \in C_1;\ \phi = \Phi_0 x + \Phi_1 F_0(x, \alpha) + G_0(x, \alpha),\ x \in \boldsymbol{R}^2\}.$$

Then for any $\phi \in \mathscr{M}(\alpha)$, *there exists a unique solution* $u(\phi)$ *of* (4.3) *with the homogeneous Neumann boundary condition such that* $u_0=\phi$, $u_t \in \mathscr{M}(\alpha)$ *for all* $t \in (-\infty, \infty)$ *and*

$$u_t(\phi) = \Phi_0 x(t; x_0) + \Phi_1 F_0(x(t; x_0), \alpha) + G_0(x(t; x_0), \alpha), \tag{4.10}$$

where $x_0 = \langle\!\langle \Psi_0, \Phi \rangle\!\rangle$ *and* $x(t; x_0)$ *is the solution of the ordinary differential equation*

$$\dot{x}(t) = M_0 x(t) + X_\varepsilon(x(t), F_0(x(t), \alpha), G_0(x(t), \alpha), \alpha)$$

with initial value x_0 *at* $t=0$.

Furthermore if Λ_1 *is empty, then the manifold* $\mathscr{M}(\alpha)$ *is locally attractive, i.e., if the solution* u *of* (4.3) *satisfies* $|x(t)| + \|z_t\|_{C_1} \leqq 1$ *for its* x *and* z *components and all* $t \geqq 0$, *then*

$$\|u_t - \Phi_0 x(t) - G_0(x(t), \alpha\|_{C_1} \leqq Ke^{-\gamma t}\|\phi\|_{C_1}, \quad t \geqq 0$$

for small $|\varepsilon|$, *where* K *and* γ *are positive constants independent of* t *and* ϕ.

Before the proof we note that when $\Lambda_1 = \phi$ the y component does not appear and then (4.10) is written as

$$u_t(\phi) = \Phi_0 x(t; x_0) + G_0(x(t; x_0), \alpha).$$

PROOF. For $m \geqq 1$, let

$$S_1^m = \{F \in C^m(\boldsymbol{R}^3; \boldsymbol{R}^l);\ \|D^j F\| \leqq 1,\ |j| \leqq m,\ F(0, \alpha) = 0,\ D_x F(0, 0) = 0\},$$
$$S_Q^m = \{G \in C^m(\boldsymbol{R}^3; \tilde{Q});\ \|D^j G\| \leqq 1,\ |j| \leqq m,\ G(0, \alpha) = 0,\ D_x G(0, 0) = 0\}.$$

In what follows we shall find $F_0 \in S_1^k$ and $G_0 \in S_Q^k$ such that for the unique solution of

$$\dot{x}(t) = M_0 x(t) + X_\varepsilon(x(t), F_0(x(t), \alpha), G_0(x(t), \alpha), \alpha), \tag{4.11}$$

with $x(0)=x_0$, $x_0 \in \boldsymbol{R}^2$, the following equations (4.12) and (4.13) hold:

$$F_0(x_0, \alpha) = \int_\infty^0 e_\varepsilon^{-M_1 s} Y_\varepsilon(x(s), F_0(x(s), \alpha), G_0(x(s), \alpha), \alpha)ds, \tag{4.12}$$

$$G_0(x_0, \alpha) = \int_{-\infty}^0 T_Q(-s) Z_\varepsilon(x(s), F_0(x(s), \alpha), G_0(x(s), \alpha, \alpha)ds. \tag{4.13}$$

Let us first consider (4.11) with $x(0)=x_0$ for given $F \in S_1^{k+1}$, $G \in S_Q^{k+1}$ instead of

F_0 and G_0. Then there exists a unique solution $x(t; x_0, F, G, \alpha)$, which is $k+1$ times continuously differentiable with respect to (t, x_0, α). Next we define operators $\mathscr{K}_1$ and $\mathscr{K}_2$ on $S_{1,Q}^{k+1}=S_1^{k+1}\times S_Q^{k+1}$ by

$$\text{(4.14)} \qquad \mathscr{K}_1(F, G)(x_0, \alpha) = \int_{\infty}^{0} e^{-M_1 s} Y_\varepsilon(x(s), F(x(s), \alpha), G(x(s), \alpha), \alpha)ds,$$

$$\text{(4.15)} \qquad \mathscr{K}_2(F, G)(x_0, \alpha) = \int_{-\infty}^{0} T_Q(-s)Z_\varepsilon(x(s), F(x(s), \alpha), G(x(t), \alpha), \alpha)ds,$$

where $x(t)=x(t; x_0, F, G, \alpha)$. We shall show that $\mathscr{K}_1(F, G)\in S_Q^{k+1}$ and $\mathscr{K}_2(F, G)\in S_Q^{k+1}$. We investigate only the operator $\mathscr{K}_2$, because the case of $\mathscr{K}_1$ is similar and easier. Since the spectrum of B_Q lies on a left half plane with a positive distance from the imaginary axis and $AT(t)\phi=T(t)A\phi$ for $\phi\in C_1$, it is easily verified that $\mathscr{K}_2(F, G)$ belongs to $\tilde{Q}$ and is $k+1$ times continuously differentiable. Furthermore, there exist positive constants c and β independent of x_0, α and ε such that

$$\|\mathscr{K}_2(F, G)(x_0, \alpha)\|_{C_1} \leqq c\|Z_\varepsilon\|_{C_1}\int_{-\infty}^{0} e^{\beta s}ds \leqq (c/\beta)\lambda_\varepsilon.$$

In the sequel we use the notation c to represent variable constants independent of x_0, α and ε. Before estimating $D_{x_0}\mathscr{K}_2(F, G)$, $D_\alpha\mathscr{K}_2(F, G)$, we study magnitudes of $D_{x_0}x(t; x_0, F, G, \alpha)$ and $D_\alpha x(t; x_0, F, G, \alpha)$. Putting $x^1(t)=D_{x_0}x(t; x_0, F, G, \alpha)$, we have from (4.11) with F and G in place of F_0 and G_0,

$$\text{(4.16)} \qquad \dot{x}^1(t) = M_0x^1(t) + D_xX_{\varepsilon,F,G}(x(t))x^1(t) \quad \text{with} \quad x^1(0) = I,$$

where $x(t)=x(t; x_0, F, G, \alpha)$ and $X_{\varepsilon,F,G}(x)=X_\varepsilon(x, F(x, \alpha), G(x, \alpha), \alpha)$. From (4.16) we see

$$x^1(s) = \exp\{M_0s+\int_0^s D_xX_{\varepsilon,F,G}(x(\tau))d\tau\}.$$

Since the eigenvalues of M_0 lie on the imaginary axis and

$$\|D_xX_{\varepsilon,F,G}\| \leqq \lambda_\varepsilon,$$

it follows that

$$\text{(4.17)} \qquad \|D_{x_0}x(s)\| \leqq e^{-\lambda_\varepsilon s} \qquad \text{for any } s \text{ such that } -\infty < s \leqq 0.$$

Similarly we have

$$\|D_\alpha x(s)\| \leqq e^{-\lambda_\varepsilon s} \qquad \text{for any } s \text{ such that } -\infty < s \leqq 0.$$

Thus we have from (4.15)

$$\|D_{x_0}\mathscr{K}_2(F, G)\| \leqq c\lambda_\varepsilon\int_{-\infty}^{0} e^{(\beta-\lambda_\varepsilon)s}ds \leqq c\lambda_\varepsilon/(\beta-\lambda_\varepsilon),$$

$$\|D_\alpha \mathscr{K}_2(F, G)\| \leqq c\lambda_\varepsilon \int_{-\infty}^0 e^{(\beta-\lambda_\varepsilon)s}ds \leqq c\lambda_\varepsilon/(\beta-\lambda_\varepsilon).$$

Since $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$, it follows that

$$\|D_{x_0}\mathscr{K}_2(F, G)\| \leqq 1 \quad \text{and} \quad \|D_\alpha \mathscr{K}_2(F, G)\| \leqq 1,$$

if $|\varepsilon|$ is sufficiently small. Analogously, we obtain, for sufficiently small $|\varepsilon|$,

$$\|D^j \mathscr{K}_2(F, G)\| \leqq 1, \quad |j| \leqq k+1. \tag{4.18}$$

We remain to verify that $\mathscr{K}_2(F, G)(0, \alpha)=0$ and $(D_{x_0}\mathscr{K}_2(F, G))(0, 0)=0$. Since $g_\varepsilon(0, 0, 0, \alpha)=0$ by Assumption ii), if follows from the uniqueness of the solution of the Cauchy problem for (4.11) with F and G in place of F_0 and G_0 that $x(t; 0, F, G, \alpha)=0$, which yields $\mathscr{K}_2(F, G)\,(0, \alpha)=0$. On the other hand, since

$$\begin{aligned} &D_{x_0}Z_\varepsilon(x(t), F(x(t), \alpha), G(x(t), \alpha), \alpha) \\ &\quad = D_xZ_\varepsilon \cdot D_{x_0}x(t) + D_yZ_\varepsilon \cdot D_xF \cdot D_{x_0}x(t) + D_zZ_\varepsilon \cdot D_xG \cdot D_{x_0}x(t), \end{aligned}$$

and $x(t; 0, F, G, \alpha)=0$, it follows from Assumption ii) that $D_{x_0}\mathscr{K}_2(F, G)(0, 0)=0$. Thus we have $K_2(F, G) \in S_Q^{k+1}$. Similarly we see that $\mathscr{K}_1(F, G) \in S_1^{k+1}$.

Next, we show that for any F_1, $F_2 \in S_1^{k+1}$ and any G_1, $G_2 \in S_Q^{k+1}$

$$\begin{aligned} &\textstyle\sum_{|j|\leqq k}\{\|D^j(\mathscr{K}_1(F_1, G_1)-\mathscr{K}_1(F_2, G_2))\| + \|D^j(\mathscr{K}_2(F_1, G_1)-\mathscr{K}_2(F_2, G_2))\|\} \\ &\qquad \leqq c\lambda_\varepsilon \textstyle\sum_{|j|\leqq k}\{\|D^j(F_1-G_2)\| + \|D^j(G_1-G_2)\|\}. \end{aligned} \tag{4.19}$$

Here we note that the sum in (4.19) can not be taken over $|j| \leqq k+1$, because we would need the derivatives of order $k+2$ of X_ε, Y_ε and Z_ε when we try to estimate the derivatives of order $k+1$ in the left hand side of (4.19). To prove (4.19), we consider only $\mathscr{K}_2(F_1, G_1)-\mathscr{K}_2(F_2, G_2)$. For $i=1, 2$, let $x_i(t)=x(t; x_0, F_i, G_i, \alpha)$. Then

$$\mathscr{K}_2(F_1, G_1)(x_0, \alpha)-\mathscr{K}_2(F_2, G_2)(x_0, \alpha) = \int_{-\infty}^0 T_Q(-s)\{Z_\varepsilon^1(s)-Z_\varepsilon^2(s)\}ds, \tag{4.20}$$

where $Z_\varepsilon^i(s)=Z_\varepsilon(x_i(s), F_i(x_i(s), \alpha), G_i(x_i(s), \alpha), \alpha)$, $i=1, 2$. By the mean value theorem we have

$$\begin{aligned} Z_\varepsilon^1(s)-Z(s)_\varepsilon^2 &= D_xZ_\varepsilon \cdot \{x_1(s)-x_2(s)\} + D_yZ_\varepsilon \cdot \{F_1(x_1(s), \alpha)-F_2(x_2(s), \alpha)\} \\ &\quad + D_zZ_\varepsilon \cdot \{G_1(x_1(s), \alpha) - G_2(x_2(s), \alpha)\} \\ &= D_xZ_\varepsilon \cdot \{x_1(s)-x_2(s)\} + D_yZ_\varepsilon \cdot D_xF_1 \cdot \{x_1(s)-x_2(s)\} \\ &\quad + D_yZ_\varepsilon \cdot \{F_1(x_2(s), \alpha)-F_2(x_2(s), \alpha)\} + D_zZ_\varepsilon \cdot D_xG_1 \cdot \{x_1(s)-x_2(s)\} \\ &\quad + D_zZ_\varepsilon \cdot \{G_1(x_2(s), \alpha)-G_2(x_2(s), \alpha)\}, \end{aligned}$$

which leads to

$$\|Z_\varepsilon^1(s)-Z_\varepsilon^2(s)\|_{C_1} \leqq \lambda_\varepsilon\{|x_1(s)-x_2(s)| + \|F_1-F_2\| + \|G_1-G_2\|\}.$$

This together with (4.20) yields

$$(4.21)\quad \|\mathscr{K}_2(F_1, G_1)(x_0, \alpha)-\mathscr{K}_2(F_2, G_2)(x_0, \alpha)\|_{C_1}$$
$$\leqq \lambda_\varepsilon \int_{-\infty}^0 e^{\beta s}|x_1(s)-x_2(s)|ds + (\lambda_\varepsilon/\beta)\{\|F_1-F_2\| + \|G_1-G_2\|\}.$$

Now we estimate $x_1(s)-x_2(s)$. Since $x_i(s)$ is the solution of (4.11) with $x_i(0)=x_0$, F_i and G_i instead of F_0 and G_0 ($i=1, 2$), it follows that

$$(4.22)\quad x_1(s) - x_2(s) = \int_0^s e^{M_0(s-\tau)}\{X_\varepsilon^1(\tau)-X_\varepsilon^2(\tau)\}d\tau, \quad -\infty < s \leqq 0,$$

where $X_\varepsilon^i(s)=X_\varepsilon(x_i(s), F_i(x_i(s), \alpha), G_i(x_i(s), \alpha), \alpha)$, $i=1, 2$. By the same estimation as for $Z_\varepsilon^1(s)-Z_\varepsilon^2(s)$, we obtain

$$\|X_\varepsilon^1(s)-X_\varepsilon^2(s)\|_{C_1} \leqq \lambda_\varepsilon\{|x_1(s)-x_2(s)| + \|F_1-F_2\| + \|G_1-G_2\|\}.$$

Thus we have, from (4.22),

$$|x_1(s)-x_2(s)| \leqq \lambda_\varepsilon \int_s^0 |x_1(s)-x_2(s)|ds - \lambda_\varepsilon s\{\|F_1-F_2\|+\|G_1-G_2\|\}, \quad -\infty<s\leqq 0,$$

which yields, by Gronwall's inequality (cf. [7, Lemma 3.1, p. 15]),

$$(4.23)\quad |x_1(s)-x_2(s)| \leqq -\lambda_\varepsilon s e^{-\lambda_\varepsilon s}\{\|F_1-F_2\|+\|G_1-G_2\|\}, \quad -\infty < s \leqq 0.$$

Substituting (4.23) into (4.21), we have, for sufficiently small $|\varepsilon|$,

$$\|\mathscr{K}_2(F_1, G_1)-\mathscr{K}_2(F_2, G_2)\| \leqq c\lambda_\varepsilon\{\|F_1-F_2\| + \|G_1-G_2\|\}.$$

By repeating similar arguments we obtain (4.19).

Choosing $|\varepsilon|$ sufficiently small, we have $c\lambda_\varepsilon<1$ in (4.19), which means that the map $(F, G)\in S_{1,Q}^{k+1}\to(\mathscr{K}_1(F, G), \mathscr{K}_2(F, G))\in S_{1,Q}^{k+1}$ is a strict contraction with respect to the metric in $S_{1,Q}^k$. On the other hand, it is easy to see that this map is continuous from $S_{1,Q}^k$ into itself. Consequently there exist $F_0\in S_1^k$ and $G_0\in S_Q^k$ such that (4.11), (4.12) and (4.13) hold.

Now we must show that for any $\phi\in\mathscr{M}(\alpha)$ there exists a unique solution $u(\phi)$ of (4.3) defined on $(-\infty, \infty)\times\Omega$ such that $u_0=\phi$ and $u_t\in\mathscr{M}(\alpha)$ for all $t\in(-\infty, \infty)$. To do so, put $y_0=F_0(x_0, \alpha)$ and $z_0=G_0(x_0, \alpha)$, $\bar{x}(t)=x(t; x_0, F_0, G_0, \alpha)$ for $x_0=(\!(\Psi_0, \phi)\!)$, and put $y(t)=F_0(\bar{x}(t), \alpha)$, $z_t=G_0(\bar{x}(t), \alpha)$. Then we have

$$y_0 = \int_\infty^0 e^{M_1 s} Y_\varepsilon(\bar{x}(s), F_0(\bar{x}(s), \alpha), G_0(\bar{x}(s), \alpha), \alpha)ds,$$

$$z_0 = \int_{-\infty}^{0} T_Q(-s)Z_\varepsilon(\bar{x}(s), F_0(\bar{x}(s), \alpha)G_0(\bar{x}(s), \alpha), \alpha)ds,$$

and furthermore

$$y(t) = \int_{\infty}^{0} e^{-M_1 s} Y_\varepsilon(x(s; \bar{x}(t)), F_0(x(s; \bar{x}(t)), \alpha), G_0(x(s; \bar{x}(t)), \alpha), \alpha)ds,$$

$$z_t = \int_{-\infty}^{0} T_Q(-s)Z_\varepsilon(x(s; \bar{x}(t)), F_0(x(s; \bar{x}(t)), \alpha), G_0(x(s; \bar{x}(t)), \alpha), \alpha)ds,$$

where $x(s; \bar{x}(t)) = x(s; \bar{x}(t), F_0, G_0, \alpha)$. Since the equation (4.11) is autonomous, it follows from the uniqueness of the solution of the Cauchy problem for (4.11) that $x(s; \bar{x}(t)) = x(s+t; x_0) = \bar{x}(s+t)$. Thus we have

$$y(t) = \int_{\infty}^{0} e^{-M_1 s} Y_\varepsilon(\bar{x}(s+t), F_0(\bar{x}(s+t), \alpha), G_0(\bar{x}(s+t), \alpha), \alpha)ds,$$

$$z_t = \int_{-\infty}^{0} T_Q(-s)Z_\varepsilon(\bar{x}(s+t), F_0(\bar{x}(s+t), \alpha), G_0(\bar{x}(s+t), \alpha), \alpha)ds,$$

which lead to

$$\begin{aligned} y(t) &= \int_{\infty}^{t} e^{M_1(t-s)} Y_\varepsilon(\bar{x}(s), F_0(\bar{x}(s), \alpha), G_0(\bar{x}(s), \alpha), \alpha)ds \\ &= \int_{\infty}^{\sigma} e^{M_1(t-s)} Y_\varepsilon(\bar{x}(s), F_0(\bar{x}(s), \alpha), G_0(\bar{x}(s), \alpha), \alpha)ds \\ &\quad + \int_{\sigma}^{t} e^{M_1(t-s)} Y_\varepsilon(\bar{x}(s), F_0(\bar{x}(s), \alpha), G_0(\bar{x}(s), \alpha), \alpha)ds \\ &= e^{M_1(t-\sigma)} y(\sigma) + \int_{\sigma}^{t} e^{M_1(t-s)} Y_\varepsilon(\bar{x}(s), F_0(\bar{x}(s), \alpha), G_0(\bar{x}(s), \alpha), \alpha)ds, \\ z_t &= \int_{-\infty}^{0} T_Q(-s)Z_\varepsilon(\bar{x}(s+t), F_0(\bar{x}(s+t), \alpha), G_0(\bar{x}(s+t), \alpha), \alpha)ds \\ &= T_Q(t-\sigma)z_0 + \int_{\sigma}^{t} T_Q(t-s)Z_\varepsilon(\bar{x}(s), F_0(\bar{x}(s), \alpha), G_0(\bar{x}(s), \alpha), \alpha)ds. \end{aligned}$$

Hence $u_t = \Phi_0 x(t) + \Phi_1 y(t) + z_t$ satisfies (4.5), (4.6) and $u_0 = \phi$. Since $\bar{x}(t) = \bar{x}(t; x_0)$, (4.4) is also satisfied, and so $u(\phi)(t) = u_t(0)$ is the required solution.

To prove local attractivity we prepare the following lemmas. The proof of Lemma 4.1 is elementary, so we omit the proof.

LEMMA 4.1. *Let $x(t)$ and $f(t)$ be non-negative continuous functions and a, b positive constants. If*

$$x(t) \leqq a + b\int_{s}^{t} x(\tau)d\tau + \int_{s}^{t} f(\tau)d\tau, \quad t \geqq s,$$

then

$$x(t) \leqq ae^{b(t-s)} + \int_s^t e^{b(t-\tau)} f(\tau) d\tau.$$

Similarly, if

$$x(s) \leqq a + b\int_s^t x(\tau)d\tau + \int_s^t f(\tau)d\tau, \quad t \geqq s,$$

then

$$x(s) \leqq ae^{b(t-s)} + \int_s^t e^{b(\tau-s)} f(\tau) d\tau.$$

LEMMA 4.2. *Let $\Lambda_1=\emptyset$ and $u(\phi)$ be a solution of* (4.3) *with the homogeneous Neumann boundary condition and $u_0=\phi \in C_1$ suth that its x and z components satisfy $|x(t)|+\|z_t\|_{C_1} \leqq 1$ for all $t \in [0, \infty)$. Let β be the positive constant determined in the first part of the proof of Theorem* 4.1. *Then, for sufficiently small $|\varepsilon|$, there exists a continuous function h of $(t, x, z, \alpha) \in [0, \infty) \times \boldsymbol{R}^2 \times \tilde{Q} \times \boldsymbol{R} (\equiv S)$ to $\tilde{Q}$ which is continuously differentiable with respect to x such that*

(4.24) $\|D_x^j h(t, x, z, \alpha)\| \leqq 2, \ |j| \leqq 1, \ \text{for} \ (t, x, z, \alpha) \in S \ \text{with} \ \|z\|_{C_1} \leqq 1,$

(4.25) $$h(t, 0, 0, \alpha) = 0,$$

(4.26) $$z_t(\phi) = h(t, x(t; \phi), z_0(\phi), \alpha), \quad 0 \leqq t < \infty,$$

(4.27) $$\|h(t, x_1, z_1, \alpha) - h(t, x_2, z_2, \alpha)\|_{C_1} \leqq (\lambda_\varepsilon/(\beta-\lambda_\varepsilon))e^{\delta_\varepsilon t}|x_1-x_2| + e^{-(\beta-\delta_\varepsilon)t}\|z_1-z_2\|_{C_1},$$

where $x(t; \phi) = \langle\!\langle \Psi_0, u_t(\phi) \rangle\!\rangle$, $z_t(\phi) = u_t(\phi) - \Phi_0 x(t; \Phi)$ *and* $\delta_\varepsilon = \lambda_\varepsilon + (\lambda_\varepsilon)^2/(\beta-\lambda_\varepsilon)$.

PROOF. Let us denote by S^m ($m \geqq 1$) the set of continuous functions h from S to $\tilde{Q}$ which are m times continuously differentiable with respect to x and satisfy (4.25) and

$$\|D_x^j h(t, x, z, \alpha)\| \leqq 2, \ |j| \leqq m, \quad \text{for} \quad (t, x, z, \alpha) \in S \quad \text{with} \quad \|z\|_{C_1} \leqq 1.$$

For given $t \in [0, \infty)$, $x_0 \in \boldsymbol{R}^2$ and $z_0 \in \tilde{Q}$, consider the system of equations:

(4.28) $$h(t, x_0, z_0, \alpha) = T_Q(t)z_0 + \int_0^t T_Q(t-s)Z_\varepsilon(x(s), h(s, x(s), z_0, \alpha), \alpha)ds$$

(4.29) $$\dot{x}(s) = M_0 x(s) + X_\varepsilon(x(s), h(s, x(s), z_0, \alpha), \alpha) \quad \text{with} \quad x(t) = x_0,$$

and the problem to find $h \in S^1$ such that for the unique solution of (4.29) the equation (4.28) holds. Here we note that the y component deos not appear in the arguments of X_ε and Z_ε, because $\Lambda_1=\emptyset$. Taking S^2, by the same argument as in the first part of the proof of Theorem 4.1, we can show the existence of $h \in S^1$ which satisfies (4.26) and (4.27) for sufficiently small $|\varepsilon|$. On the other hand,

$z_t(=z_t(\phi))$ and $x(t)$ $(=x(t;\ \phi))$ satisfy

$$z_t = T_Q(t)z_0 + \int_0^t T_Q(t-s)Z_\varepsilon(x(s),\ z_s,\ \alpha)ds, \tag{4.30}$$

$$\dot{x}(t) = M_0x(t) + X_\varepsilon(x(t),\ z_t,\ \phi) \quad \text{with} \quad x(0) = x_0, \tag{4.31}$$

where $x_0 = \langle\!\langle \Psi_0,\ \phi \rangle\!\rangle$ and $z_0 = \phi - \Psi_0 x_0$. Let $t \in [0,\ \infty)$ be fixed and let $\bar{x}(s)$ be the solution of (4.29) with $x_0 = x(t;\ \phi)$. Then,

$$h(t,\ \bar{x}(t),\ z_0,\ \alpha) = T_Q(t)z_0 + \int_0^t T_Q(t-s)Z_\varepsilon(\bar{x}(s),\ h(s,\ \bar{x}(s),\ z_0,\ \alpha),\ \alpha)ds, \tag{4.32}$$

$$\dot{\bar{x}}(s) = M_0\bar{x}(s) + X_\varepsilon(\bar{x}(s),\ h(s,\ \bar{x}(s),\ z_0,\ \alpha),\ \alpha) \quad \text{with} \quad \bar{x}(t) = x(t;\ \phi). \tag{4.33}$$

From (4.30) and (4.32), we obtain, as before,

$$\|z_t - h(t)\|_{C_1} \leqq \lambda_\varepsilon \int_0^t e^{-\beta(t-s)}\{|\bar{x}(s) - x(s)| + \|z_s - h(s)\|_{C_1}\}ds, \tag{4.34}$$

and from (4.31) and (4.33)

$$|\bar{x}(s) - x(s)| \leqq \lambda_\varepsilon \int_s^t \{|\bar{x}(\tau) - x(\tau)| + \|z_\tau - h(\tau)\|_{C_1}\}d\tau,$$

where $h(\tau) = h(\tau,\ \bar{x}(\tau),\ z_0,\ \alpha)$. By virtue of Lemma 4.1 we have

$$|\bar{x}(s) - x(s)| \leqq \lambda_\varepsilon \int_s^t e^{\lambda_\varepsilon(\tau - s)}\|z_\tau - h(\tau)\|_{C_1}d\tau,$$

which together with (4.34) leads to

$$\begin{aligned}\|z_t - h(t)\|_{C_1} &\leqq (\lambda_\varepsilon)^2 \int_0^t e^{-\beta(t-s)} \int_s^t e^{\lambda_\varepsilon(\tau-s)}\|z_\tau - h(\tau)\|_{C_1}d\tau ds \\ &\quad + \lambda_\varepsilon \int_0^t e^{-\beta(t-s)}\|z_s - h(s)\|_{C_1}ds \\ &\leqq (\lambda_\varepsilon + (\lambda_\varepsilon)^2/(\beta - \lambda_\varepsilon)) \int_0^t e^{-\beta(t-s)}\|z_s - h(s)\|_{C_1}ds.\end{aligned}$$

Thus

$$e^{\beta t}\|z_t - h(t)\|_{C_1} \leqq (\lambda_\varepsilon + (\lambda_\varepsilon)^2/(\beta - \lambda_\varepsilon)) \int_0^t e^{\beta t}\|z_s - h(s)\|_{C_1}ds.$$

By Gronwall's inequality we have $z_t = h(t)$, which shows (4.26).

Finally we show (4.27). Let $\bar{x}_1(s)$ and $\bar{x}_2(s)$ be solutions of

$$\dot{x}(s) = M_0x(s) + X_\varepsilon(x(s),\ h(s,\ x(s),\ z_i,\ \alpha),\ \alpha) \ \text{with} \ x(t) = x_i.$$

Then

$$h_1(t)-h_2(t) = T_Q(t)(z_1-z_2)$$
$$+\int_0^t T_Q(t-s)\{z_\varepsilon(\bar{x}_1(s), h_1(s), \alpha)-Z_\varepsilon(\bar{x}_2(s), h_2(s), \alpha)\}ds$$

where $h_i(s)=h(s, \bar{x}_i(s), z_i, \alpha)$, $i=1, 2$. Hence

$$(4.35)\quad \|h_1(t)-h_2(t)\|_{C_1} \leqq e^{-\beta t}\|z_1-z_2\|_{C_1}$$
$$+\lambda_\varepsilon\int_0^t e^{-\beta(t-s)}\{|\bar{x}_1(s)-\bar{x}_2(s)|+\|h_1(s)-h_2(s)\|_{C_1}\}ds.$$

On the other hand, from the equation

$$\bar{x}_i(s) = e^{M_0(s-t)}x_i+\int_t^s e^{M_0(s-\tau)}X_\varepsilon(\bar{x}_i(\tau), h_i(\tau), \alpha)d\tau,$$

it follows that for any s such that $0\leqq s\leqq t$

$$|\bar{x}_1(s)-\bar{x}_2(s)| \leqq |x_1-x_2|+\lambda_\varepsilon\int_s^t\{|\bar{x}_1(\tau)-\bar{x}_2(\tau)|+\|h_1(\tau)-h_2(\tau)\|_{C_1}\}d\tau.$$

By virtue of Lemma 4.1 we have

$$|\bar{x}_1(s)-\bar{x}_2(s)| \leqq e^{\lambda_\varepsilon(t-s)}|x_1-x_2|+\lambda_\varepsilon\int_s^t e^{\lambda_\varepsilon(\tau-s)}\|h_1(\tau)-h_2(\tau)\|_{C_1}d\tau.$$

By substituting this into (4.35) we obtain

$$\|h_1(t)-h_2(t)\|_{C_1} \leqq e^{-\beta t}\|z_1-z_2\|_{C_1}+\lambda_\varepsilon\int_0^t e^{-\beta(t-s)}\|h_1(s)-h_2(s)\|_{C_1}ds$$
$$+\lambda_\varepsilon|x_1-x_2|\int_0^t e^{-(\beta-\lambda_\varepsilon)(t-s)}ds+(\lambda_\varepsilon)^2\int_0^t e^{-\beta(t-s)}\int_s^t e^{\lambda_\varepsilon(\tau-s)}\|h_1(\tau)-h_2(\tau)\|_{C_1}d\tau ds$$
$$\leqq e^{-\beta t}\|z_1-z_2\|_{C_1}+(\lambda_\varepsilon/(\beta-\lambda_\varepsilon))|x_1-x_2|+\delta_\varepsilon\int_0^t e^{-\beta(t-s)}\|h_1(s)-h_2(s)\|_{C_1}ds.$$

Therefore

$$e^{\beta t}\|h_1(t)-h_2(t)\|_{C_1} \leqq \|z_1-z_2\|_{C_1}+e^{\beta t}(\lambda_\varepsilon/(\beta-\lambda_\varepsilon))|x_1-x_2|$$
$$+\delta_\varepsilon\int_0^t e^{\beta s}\|h_1(s)-h_2(s)\|_{C_1}ds.$$

By Gronwall's inequality again we have

$$e^{\beta t}\|h_1(t)-h_2(t)\|_{C_1} \leqq e^{\delta_\varepsilon t}\{\|z_1-z_2\|_{C_1}+e^{\beta t}(\lambda_\varepsilon/(\beta-\lambda_\varepsilon))|x_1-x_2|\},$$

which leads to our assertion.

PROOF OF THE LAST ASSERTION OF THEOREM 4.1. Fix any $t>0$ and put $x_1=x(t;\phi)(=\langle\!\langle\Psi_0, u_t(\phi)\rangle\!\rangle)$. Let $\phi_1\in\mathscr{M}(\alpha)$ be defined by $\phi_1=\Phi_0x_1+G_0(x_1, \alpha)$.

Since the solution of (4.3) on $\mathscr{M}(\alpha)$ is defined on $(-\infty, \infty)\times\Omega$ and unique for each initial function on $\mathscr{M}(\alpha)$, there exists a unique $\phi_2\in\mathscr{M}(\alpha)$ such that $\phi_1=u_t(\phi_2)$. Put $x_2=\langle\!\langle\Psi_0, \phi_2\rangle\!\rangle$. Then by virtue of Lemma 4.2, we see

$$z_t(\phi) = h(t, x_1, z_0(\phi), \alpha),$$

$$G_0(x_1, \alpha) = z_t(\phi_2) = h(t, x_1, z_0(\phi_2), \alpha) = h(t, x_1, G_0(x_2, \alpha), \alpha),$$

and therefore, by (4.27),

$$\text{(4.36)}\quad \begin{aligned}\|z_t(\phi)-G_0(x_1, \alpha)\|_{C_1} &\leqq e^{-(\beta-\delta_\varepsilon)t}(\|z_0(\phi)\|_{C_1}+\|G_0(x_2, \alpha)\|_{C_1})\\ &\leqq e^{-(\beta-\delta_\varepsilon)t}(\|z_0(\phi)\|_{C_1}+|x_2|).\end{aligned}$$

In what follows let us estimate $|x_2|$. Note that $x_2=\tilde{x}(-t; x_1)$, where $\tilde{x}(s; x_1)$ is the unique solution of the equation

$$\dot{x}(s) = M_0x(s) + X_\varepsilon(x(s), G_0(x(s), \alpha), \alpha) \quad \text{with} \quad x(0) = x_1.$$

Since $x_2=\tilde{x}(-t; x_1)-\tilde{x}(-t; 0)=D_{x_2}\tilde{x}\cdot x_1$, we have, by the same reasoning as in the case of (4.17),

$$\text{(4.37)}\qquad |x_2| \leqq e^{\lambda_\varepsilon t}|x_1|.$$

On the other hand, since $x(s; \phi)$ $(\equiv x(s))$ and $z_s(\phi)$ $(\equiv z_s)$ satisfy

$$\text{(4.38)}\qquad \dot{x}(s) = M_0x(s) + X_\varepsilon(x(s), z_s, \alpha) \quad \text{with} \quad x(0)=\langle\!\langle\Psi_0, \phi\rangle\!\rangle\ (\equiv x_0)$$

$$\text{(4.39)}\qquad z_s=T_Q(s)z_0(\phi) + \int_0^s T_Q(s-\tau)Z_\varepsilon(x(\tau), z\tau, \alpha)d\tau, \quad s\geqq 0,$$

it follows, as before, from (4.39) that

$$\|z_s\|_{C_1} \leqq e^{-\beta s}\|z_0(\phi)\|_{C_1} + \lambda_\varepsilon\int_0^s e^{-\beta(s-\tau)}\{|x(\tau)| + \|z_\tau\|_{C_1}\}d\tau.$$

This yields, by Gronwall's inequality,

$$\text{(4.40)}\qquad \|z_s\|_{C_1} \leqq e^{-(\beta-\lambda_\varepsilon)s}\|z_0(\phi)\|_{C_1} + \lambda_\varepsilon\int_0^s e^{-(\beta-\lambda_\varepsilon)(s-\tau)}|x(\tau)|\,d\tau.$$

By using (4.38) we have, again as before,

$$|x(t)| \leqq |x_0| + \lambda_\varepsilon\int_0^t \{|x(s)| + \|z_s\|_{C_1}\}ds.$$

Substituting (4.40) into this, we obtain

$$|x(t)| \leqq |x_0| + \lambda_\varepsilon/(\beta-\lambda_\varepsilon)\|z_0(\phi)\|_{C_1} + \delta_\varepsilon\int_0^t |x(s)|ds,$$

which yields, by Gronwall's inequality,

$$(4.41)\qquad |x(t)| \leqq e^{\delta_\varepsilon t}\{|x_0| + \lambda_\varepsilon/(\beta-\lambda_\varepsilon)\|z_0(\phi)\|_{C_1}\}.$$

Since $x_1 = x(t) = x(t;\ \phi)$, it follows from (4.36), (4.37) and (4.41) that

$$\begin{aligned}\|z_t(\phi) - G_0(x_1,\ \alpha)\|_{C_1} &\leqq e^{-(\beta-2\delta_\varepsilon-\lambda_\varepsilon)t}\{\beta/(\beta-\lambda_\varepsilon)\|z_0(\phi)\|_{C_1} + |x_0|\}\\ &\leqq Ke^{-\gamma t}\|\phi\|_{C_1},\end{aligned}$$

where K is a positive constant independent of t and ϕ, and $\gamma = \beta - 2\delta_\varepsilon - \lambda_\varepsilon$ which is positive for sufficiently small $|\varepsilon|$. The proof is complete.

5. Hopf bifurcation

In Section 4 we saw that we can construct a local integral manifold for our equation in a neighborhood of a critical point a_c and we can reduce the Hopf bifurcation problem to the two dimensional case. On the other hand, according to the Hopf bifurcation theory for a finite dimensional case (cf. [6], [13]), if the characteristic equations have a pair of complex conjugate roots $\{\lambda(a),\ \overline{\lambda(a)}\}$ in a neighborhood of a_c such that

$$(\text{H.1})\qquad \operatorname{Re}\lambda(a_c) = 0 \quad \text{and} \quad \operatorname{Im}\lambda(a_c) \neq 0,$$

$$(\text{H.2})\qquad \operatorname{Re}\lambda'(a_c) \neq 0,$$

then non-trivial periodic orbits bifurcate from the trivial solution. Thus we first study the characteristic equations

$$(3.7)\qquad \lambda + ae^{-\lambda} + d\xi_j = 0, \quad j = 0, 1, \dots.$$

We observe that a is not a critical point if $0 < a < \pi/2$ by virtue of Lemma 3.1.

LEMMA 5.1 ([7, Lemma 4.1, p. 254]). *If $\gamma > e^{-1}$, then there exists a pair of simple complex conjugate roots $\{\lambda(\gamma),\ \overline{\lambda(\gamma)}\}$, $\lambda(\gamma) = \mu(\gamma) + i\nu(\gamma)$, which are continuous together with their first derivatives in γ and satisfy $0 < \nu(\gamma) < \pi$, $\nu(\pi/2) = \pi/2$, $\mu(\pi/2) = 0$, $\mu'(\pi/2) > 0$ and $\mu(\gamma) > 0$ for $\gamma > \pi/2$.*

By virtue of Lemma 5.1 we see that $a_c = \pi/2$, there exists a pair of simple complex conjugate roots for (3.7) (with $j = 0$) which satisfy (H.1) and (H.2) and so, it is the first critical point in $a > 0$. On the other hand, if $\gamma = a_c$ $(= \pi/2)$ and $\delta = d\xi_j$ $(j \geqq 1)$, then the conditions (3.8), (3.9) and (3.10) in Lemma 3.1 are satisfied, which implies that all roots of (3.7), except a pair of complex conjugate roots obtained above, have negative real parts. Hence, for $a_c = \pi/2$, we can apply Theorem 4.1 and obtain a local integral manifold $\mathscr{M}(\alpha) = \{\phi \in C_1;\ \phi = \Phi_0 x + G_0(x;\ \alpha),\ x \in \boldsymbol{R}^2\}$, and the Hopf bifurcation problem for the equation (0.2) with (0.3) is reduced to the equation

$$(5.1)\qquad \dot{x}(t) = M_0x(t) + X(x(t), G_0(x(t), \alpha), \alpha), \quad -\infty < t < \infty,$$

in a neighborhood of $(x(t), \alpha)=(0, 0)$, where $\alpha=a-\pi/2$, M_0 is the 2×2 matrix $[a_{ij}]$, $a_{11}=a_{22}=0$, $a_{12}=-a_{21}=\pi/2$ and

$$X(x, z, \alpha) = -\alpha\langle \Psi_0(0), \Phi_0(-1)x+z(-1)\rangle - a\langle \Psi_0(0), (\Phi_0(-1)x+z(-1))\,(\Phi_0(0)x+z(0))\rangle$$

for any $x\in \boldsymbol{R}^2$ and $z\in \tilde{Q}$ (cf. (4.1)). We write here (5.1) in the unscaled form, i.e., $\varepsilon=1$ in (4.11), and note that the y component does not appear in the argument of X in (5.1), because the characteristic equations (3.7) do not have roots with positive real parts when $a=\pi/2$. It is easy to see that the characteristic equation of the linear part of (5.1) has a pair of complex conjugate roots which satisfy (H.1) and (H.2) at $\alpha=0$, i.e., $a=a_c=\pi/2$. Thus, by the Hopf bifurcation theorem, a non-trivial periodic solution $x_\alpha(t)$ for (5.1) exists for small $\alpha>0$. Since $u(t)=u_t(0)=\Phi_0(0)x_\alpha(t)+G_0(x_\alpha(t);\alpha)|_{\theta=0}$ is a solution of (0.2) with (0.3), we have the following

THEOREM 5.1 *The equation* (0.2) *with* (0.3) *has a temporally periodic spatially homogeneous bifurcating orbit at the first bifurcation point* $a_c=\pi/2$ *from the trivial solution.*

We close this section by giving a remark on the proof of the Hopf bifurcation theorem due to J. K. Hale [7, Theorem 1.1, p. 246]. It seems to the author that his proof is incomplete because the fact "the second integral is zero" of the third line from below at p. 248 does not hold. This fact is essential in his proof. We can also show that even if "the second integral" is correctly evaluated it does not lead to his assertion

$$(5.3)\qquad \det \frac{\partial H(0, 0, 0)}{\partial(\alpha, \beta)} \neq 0.$$

Here we give an example. Consider an ordinary functional differential equation

$$\dot{x}(t) = -(\pi/2+\alpha)x(t-1) + f(x_t, \alpha),$$

where $f(0, \alpha)=0$, $D_xf(0, 0)=0$ and f satisfies a suitable regularity condition. We denote by $\{\lambda(\alpha), \overline{\lambda(\alpha)}\}$, $\lambda(\alpha)=\mu(\alpha)+i\nu(\alpha)$, a pair of simple complex conjugate roots of the corresponding characteristic equation such that $\lambda(0)=i\nu(0)=i\pi/2$ and $\mu'(0)\neq 0$ (see Lemma 5.1). As usual we decompose $C=C([-1, 0]; \boldsymbol{R})$ as $C=P_\alpha\oplus Q_\alpha$ with respect to $\{\lambda(\alpha), \overline{\lambda(\alpha)}\}$. Let Φ_α be a basis of P_α. When $\alpha=0$, we may take $\Phi_0(\theta)=(\sin(\pi/2)\theta, \cos(\pi/2)\theta)$. Let Ψ_α be the dual basis of Φ_α such that $(\!(\Psi_\alpha, \Phi_\alpha)\!)=I$. Then $\Psi_0=\mathrm{col}(\psi_1, \psi_2)$, where

$$\psi_1(\theta) = k_0\{\sin(\pi/2)\theta + (\pi/2)\cos(\pi/2)\theta\},$$

$$\psi_2(\theta) = k_0\{\cos(\pi/2)\theta - (\pi/2)\sin(\pi/2)\theta\},$$
$$k_0 = 2/(1+\pi^2/4).$$

In this case "the second integral" becomes

$$-\pi/2\int_0^4\int_{-1}^0 e^{-M(0)(s+\xi+1)}\Psi_0(0)\Phi_0(0)\, e^{M(0)(s+\xi)}\, d\xi ds = \pi k_0\begin{bmatrix} -\pi/2 & 1 \\ -1 & \pi/2 \end{bmatrix} \neq 0,$$

where $M(0)$ is the 2×2 matrix $[a_{ij}]$, $a_{11}=a_{22}=0$, $a_{12}=-a_{21}=\pi/2$, which is written as $B(0)$ in [7]. Furthermore,

$$(5.4)\qquad (\partial H/\partial\alpha)(0, 0, 0) = -\int_0^4 e^{-M(0)s}\Psi_0(0)U_s(-1)\boldsymbol{e}_1 ds$$
$$= -2/\pi\int_0^4 e^{-M(0)s}\Psi_0(0)\dot{U}(s)\boldsymbol{e}_1\, ds = -2/\pi\int_0^4 e^{-M(0)s}\Psi_0(0)\Phi_0(0)e^{M(0)s}M(0)\boldsymbol{e}_1 ds,$$

where $\boldsymbol{e}_1=\mathrm{col}(1, 0)$ and $U(s)=\Phi_0(0)e^{M(0)s}$. Here we used the fact that $\dot{U}(t)=(\pi/2)U(t-1)$. The equation (5.4) means that $(\partial H/\partial\alpha)(0, 0, 0)$ is equal to $(\partial H/\partial\beta)(0, 0, 0)$ up to a constant (cf. the 8^{th} line from below of p. 248 in [7]). Thus Hale's assertion (5.3) does not hold.

6. Stability of bifurcation orbits

In the preceding section we showed the existence of the Hopf bifurcation for the equation (0.2) with (0.3). We shall here give a brief discussion on the stability of the Hopf bifurcation. S-N. Chow and J. Mallet-Paret [4] discussed this subject for Wright's equation $\dot{x}(t)=-ax(t-1)(1+x(t))$, i.e., in the spatially homogeneous case of the equation (0.2). Their analysis remains valid for the equation (0.2) with (0.3) with a slight modification, because (i) the local integral manifold is constructed in C_1 (this is one of the assumptions in [4, p. 141]) and (ii) the orbit appearing at the first bifurcation point is spatially homogeneous and so we can take the same basis $\Phi_0(=(\sin(\pi/2)\theta, \cos(\pi/2)\theta))$ as in the case of Wright's equation. In the case where spatially inhomogeneous orbits appear at the first bifurcation point the situation becomes very complex. This analysis will be done in the forthcoming paper.

We first state the results on stability in [4, p. 125 and p. 135]. Let x, y and z be generic points in $\boldsymbol{R}^2$, $\boldsymbol{R}^l$ and $\tilde{Q}$, respectively, and ε be the same parameter as in Section 4. Then according to [4], an annulus $\mathscr{A}^*$ surrounding a periodic solution is given by

$$\mathscr{A}^*\begin{cases} (1-\gamma)r_0 < |x| < (1+\gamma)r_0, \quad \gamma \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0, \\ |y| + \|z\|_{C_1} \leqq \Omega^*|\varepsilon|, \quad \Omega^* = \text{constant}, \end{cases}$$

where r_0 is a positive constant independent of ε which is determined by the given equation with $a=a_c$. If $B(a_c)$ does not have the spectrum with positive real parts (in this case y does not appear and the integral manifold $\mathscr{M}(\alpha)$ is locally attractive), $\mathscr{A}^*$ is positively invariant under the condition $\mu'(a_c)K<0$, where $K=K^*+K^{**}$ is the constant defined in [4, p. 133]. Here "$\mathscr{A}^*$ is positively invariant" means that if solutions are in $\mathscr{A}^*$ at $t=\sigma$, then they stay in $\mathscr{A}^*$ for $t>\sigma$. The Hopf bifurcation is stable if $\mathscr{A}^*$ is positively invariant.

In what follows we shall compute the value of K in our case ($a_c=\pi/2$). We use the notations of [4, Section 9] as possible as we can. But we employ, as before, the notations B, M_p and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ instead of A, A_P and $(\cdot, \cdot)$ in [4]. Moreover, in our case we note that $a_0=b_0=\pi/2$ (as for the notations a_N, b_N, see [4, p. 148]). As in [4] we can derive $K=K^*+K^{**}$,

$$K^* = 0, \quad K^{**} = -(b_0/2)\mathrm{Im}\, g_2(2ib_0-B_Q)^{-1}X_0^Q,$$

$$g_2 = -[b_0/2(1+b_0^2)](1-ib_0)(\phi(0)i+\phi(-1)).$$

In order to determine K^{**} we must evaluate $(2ib_0-B_Q)^{-1}X_0^Q$. The determination of $\phi=(2ib_0-B_Q)^{-1}X_0^Q$ is a little different from [4, p. 151]. To caluculate, more generally,

$$\phi = (2ib_0-B)^{-1}\psi$$

we must solve

$$\dot{\phi}(\theta) = 2ib_0\phi(\theta) - \psi(\theta) \tag{6.1}$$

subject to the conditions

$$\dot{\phi}(0) = d\Delta\phi(0) - a_0\phi(-1), \tag{6.2}$$

$$(\partial\phi(0)/\partial n)|_{\partial\Omega} = 0. \tag{6.3}$$

From (6.1)

$$\phi(\theta) = e^{2ib_0\theta}\phi(0) - \int_0^\theta e^{2ib_0(\theta-s)}\psi(s)ds$$

and so

$$\phi(-1) = e^{-2ib_0}\phi(0) - \int_0^{-1} e^{-2ib_0(s+1)}\psi(s)ds.$$

This together with (6.1) and (6.2) yields

$$-d\Delta\phi(0) + (2ib_0+a_0e^{-2ib_0})\phi(0) = \psi(0) + a_0\int_0^{-1} e^{-2ib_0(s+1)}\psi(s)ds.$$

For $\psi=X_0$ we have

$$-d\Delta\phi(0) + (2ib_0+a_0e^{-2ib_0})\phi(0) = 1,$$

which together with (6.3) leads to

$$\phi(0) = 2/\pi(2i-1).$$

Thus we have

$$\phi(-1) = -\phi(0),$$

$$-(b_0/2)\,\mathrm{Im}\,[g_2(2ib_0-B)^{-1}X_0] = -\,b_0^2(3b_0-1)/10\pi(1+b_0^2)$$
$$= -\,\pi(3\pi/2-1)/40(1+\pi^2/4).$$

On the other hand, by the same calculation as in [4], we have

$$-(b_0/2)\,\mathrm{Im}\,[g_2(2ib_0-B)^{-1}X_0^P] = 0.$$

We therefore obtain

$$K = K^{**} = -\,\pi(3\pi/2-1)/40(1+\pi^2/4) < 0.$$

Since, be virtue of Lemma 5.2, $\mu'(\pi/2)>0$, we see that $K\mu'(\pi/2)<0$ and so $\mathscr{A}^*$ is positively invariant. Thus we have shown

THEOREM 6.1. *If* $|\varepsilon|$ *is small, the bifurcating orbit in Theorem* 5.1 *is stable.*

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