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The horospherical Gauss-Bonnet type theorem in hyperbolic space

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Abstract. We introduce the notion horospherical curvatures of hypersurfaces in hyperbolic space and show that totally umbilic hypersurfaces with vanishing curvatures are only horospheres. We also show that the Gauss-Bonnet type theorem holds for the horospherical Gauss-Kronecker curvature of a closed orientable even dimensional hypersurface in hyperbolic space.

1. Introduction.

The hyperbolic Gauss map of a surface in hyperbolic space has been independently introduced by Bryant [**2**] and Epstein [**4**] in the Poincaré ball model. Kobayashi [**13**], [**14**] has also independently defined it for a hypersurface in $H^n(\boldsymbol{R}) = SO_0(n,1)/SO(n)$ and a more general setting. It is a quite useful tool for the study of mean curvature one surfaces in hyperbolic space [**2**], [**19**]. For fundamental concepts and results in this area, please refer [**2**], [**4**], [**5**], [**17**]. In [**11**] we have investigated singularities of hyperbolic Gauss maps of hypersurfaces in hyperbolic n-space $H^n_+(-1)$ by using the model in Minkowski space. We introduced the notion of hyperbolic Gauss indicatrices slightly modified the definition of hyperbolic Gauss maps. The notion of hyperbolic indicatrices is independent of the choice of the model of hyperbolic space. Using the hyperbolic Gauss indicatrix, we defined the principal hyperbolic curvatures $\bar{\kappa}^{\pm}$ and the hyperbolic Gauss-Kronecker curvature $K_h^{\pm}$ by exactly the same way as the definition of those of classical Gaussian differential geometry in Euclidean space. Totally umbilic hypersurfaces with respect to the above curvatures are equidistant hypersurfaces, hyperspheres or hyperhorospheres which are called model hypersurfaces in hyperbolic space. The hyperbolic Gauss-Kronecker curvature is a hyperbolic invariant which describes the contact of hypersurfaces with such model hypersurfaces. We remark that Kobayashi [**13**], [**14**] had already defined the notion of hyperbolic Gauss-Kronecker curvature under a different framework and studied some basic properties of it from the view point of the theory of Fourier transformations.

In this paper we introduce the principal horospherical curvature $\widetilde{\kappa}^{\pm}$ (cf., §3). This new curvature is not a hyperbolic invariant but an $SO(n)$-invariant, where we consider the canonical $SO(n)$-subgroup in the group of hyperbolic motions. However, we can show that $\widetilde{\kappa}^{\pm}(p)$ is invariant under hyperbolic motions if and only if $\widetilde{\kappa}^{\pm}(p) = 0$. We can

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also show that totally umbilic hypersurfaces with vanishing principal horospherical curvatures are hyperhorospheres. Therefore, totally umbilic hypersurfaces with hyperbolic invariant principal horospherical curvatures are only hyperhorospheres. By definition, the curvature $\widetilde{\kappa}^{\pm}(p)$ might depend on the choice of the model of hyperbolic space in Minkowski space. However, we can show that this curvature is independent of the choice of the model of hyperboilic space (cf., §3). We define a curvature $\widetilde{K}_h^{\pm}$ as the product of principal horospherical curvatures which is called the *horospherical Gauss-Kronecker curvature*. Of course the horospherical Gauss-Kronecker curvature is not a hyperbolic invariant. However it describes the contact of hypersurfaces with hyperhorospheres. We call such the geometry the "*horospherical geometry*" of hypersurfaces in hyperbolic space.

The main purpose in this paper is to study the global properties of hypersurfaces in hyperbolic space. Since the horospherical Gauss-Kronecker curvature depends on the choice of the normal direction, we need to explicitly use the normal vector of the hypersurface when dealing with global properties. Therefore, in order to define the global horospherical Gauss-Kronecker curvature $\widetilde{\mathscr{K}_h}$ (cf., §4), we shall need to assume that the hypersurface M is orientable. The main result in this paper is the following horospherical Gauss-Bonnet type theorem:

THEOREM 1.1. *If M is a closed orientable even-dimensional hypersurface in hyperbolic n-space, then*

$$\int_M \widetilde{\mathscr{K}_h} d\mathfrak{v}_M = \frac{1}{2}\gamma_{n-1}\boldsymbol{\chi}(M)$$

where $\boldsymbol{\chi}(M)$ is the Euler characteristic of M, $d\mathfrak{v}_M$ is the volume form of M and the constant γ_{n-1} is the volume of the unit $(n-1)$-sphere S^{n-1}.

We include a quick review of the local properties of the hyperbolic Gauss-Kronecker curvature in section 2. We also introduce in section 2 the concept of *de Sitter Gauss-Kronecker curvature* of a hypersurface in $H^n_+(-1)$ that will be used in section 5. In section 3 we define the notions of principal horospherical curvatures and horospherical Gauss-Kronecker curvatures. We investigate the geometric meanings of these curvatures. Theorem 1.1 is proven in section 4. Section 5 is devoted to a more detailed study of the case $n = 3$. A further consequence of the main theorem together with the generic classification of singularities of hyperbolic Gauss indicatrices is the relation between the Euler characteristics of the image of the hyperbolic Gauss indicatrix and horospherical properties on the closed surface (cf., Theorem 5.6).

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

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2. Local hyperbolic differential geometry.

We outline in this section the local differential geometry of hypersurfaces in the hyperbolic n-space developed in the previous papers [**11**], [**12**]. We adopt, for this purpose, the model of hyperbolic n-space in Minkowski $(n+1)$-space.

Let $\boldsymbol{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) \mid x_i \in \boldsymbol{R}\ (i = 0, 1, \ldots, n)\}$ be an $(n+1)$-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \ldots, y_n) \in \boldsymbol{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. We call $(\boldsymbol{R}^{n+1}, \langle , \rangle)$ *Minkowski* $(n+1)$*-space* and denote it by $\boldsymbol{R}^{n+1}_1$. We say that a non-zero vector $\boldsymbol{x} \in \boldsymbol{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \boldsymbol{R}^{n+1}_1$ and a real number c, we define *the hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \boldsymbol{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}$. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *hyperbolic* n*-space* by $H^n_+(-1) = \{\boldsymbol{x} \in \boldsymbol{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 \geq 1\}$ and *de Sitter* n*-space* by $S^n_1 = \{\boldsymbol{x} \in \boldsymbol{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$.

Given $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in \boldsymbol{R}^{n+1}_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is the canonical basis of $\boldsymbol{R}^{n+1}_1$ and $\boldsymbol{x}_i = (x^i_0, x^i_1, \ldots, x^i_n)$. We can easily show that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, \ldots, n)$.

We also define a set $LC^*_+ = \{\boldsymbol{x} = (x_0, \ldots x_n) \in LC_0 \mid x_0 > 0\}$, which is called *future lightcone* at the origin.

We now construct the local extrinsic differential geometry on hypersurfaces in $H^n_+(-1)$. Let $\boldsymbol{x} : U \longrightarrow H^n_+(-1)$ be an embedding, where $U \subset \boldsymbol{R}^{n-1}$ is an open subset. We shall identify $M = \boldsymbol{x}(U)$ and U through the embedding $\boldsymbol{x}$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv -1$, we have $\langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}(u) \rangle \equiv 0$ $(i = 1, \ldots, n-1)$, for any $u = (u_1, \ldots u_{n-1}) \in U$. Therefore, if we define

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)\|},$$

we have $\langle \boldsymbol{e}(u), \boldsymbol{x}_{u_i}(u) \rangle \equiv \langle \boldsymbol{e}(u), \boldsymbol{x}(u) \rangle \equiv 0$, $\langle \boldsymbol{e}(u), \boldsymbol{e}(u) \rangle \equiv 1$. And thus the vector $\boldsymbol{x}(u) \pm \boldsymbol{e}(u)$ is lightlike. Since $\boldsymbol{x}(u) \in H^n_+(-1)$ and $\boldsymbol{e}(u) \in S^n_1$, we have $\boldsymbol{x}(u) \pm \boldsymbol{e}(u) \in LC^*_+$ and hence we can define a map

$$\boldsymbol{L}^{\pm} : U \longrightarrow LC^*_+$$

by $\boldsymbol{L}^{\pm}(u) = \boldsymbol{x}(u) \pm \boldsymbol{e}(u)$ which is called *the hyperbolic Gauss indicatrix* (or *the lightcone dual*) of $\boldsymbol{x}$. We also define a map $\boldsymbol{E} : U \longrightarrow S^n_1$ by $\boldsymbol{E}(u) = \boldsymbol{e}(u)$ and call it *the de Sitter Gauss indicatrix* of $\boldsymbol{x}$.

In order to define the hyperbolic Gauss-Kronecker curvature and the hyperbolic mean curvature of the hypersurface $M = \boldsymbol{x}(U)$, we have shown in [**11**] that $D_v \boldsymbol{E} \in T_pM$ for any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, so that $D_v \boldsymbol{L}^{\pm} \in T_pM$. Here, D_v denotes *the* *covariant derivative* with respect to the tangent vector $\boldsymbol{v}$.

In [**11**] we studied the geometric meaning of the hyperbolic Gauss map and the hyperbolic Gauss indicatrix of a hypersurface. A hypersurface given by the intersection of $H^n_+(-1)$ and a spacelike hyperplane, a timelike hyperplane (through the origin) or a lightlike hyperplane is respectively called a *hypersphere*, a *equidistant hypersurface* (*hyperplane*) or a *hyperhorosphere*. Then we have the following:

Proposition 2.1. *Let $\boldsymbol{x} : U \longrightarrow H^n_+(-1)$ be a hypersurface in $H^n_+(-1)$.*
(1) *If the hyperbolic Gauss indicatrix $\boldsymbol{L}^{\pm}$ is constant, then the hypersurface M is a part of a hyperhorosphere.*
(2) *If the de Sitter Gauss indicatrix $\boldsymbol{E}$ is constant, then the hypersurface M is a part of a hyperplane.*

The first assertion of the above proposition has been shown in [**11**]. The second assertion is rather easier to show and we omit its proof here.

Under the identification of U and M by the embedding $\boldsymbol{x}$, the derivative $d\boldsymbol{x}(u_0)$ can be identified to the identity mapping 1_{T_pM} on the tangent space T_pM, where $p = \boldsymbol{x}(u_0)$. Therefore, $d\boldsymbol{E}(u_0)$ can be considered as a linear transformation on the tangent space T_pM. This means that $d\boldsymbol{L}^{\pm}(u_0) = 1_{T_pM} \pm d\boldsymbol{E}(u_0)$ is also a linear transformation on the tangent space T_pM. We call the linear transformation $S^{\pm}_p = -d\boldsymbol{L}^{\pm}(u_0) : T_pM \longrightarrow T_pM$ *the hyperbolic shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We also call the linear transformation $A_p = -d\boldsymbol{E}(u_0) : T_pM \longrightarrow T_pM$ *the de Sitter shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We remark that A_p is nothing but the shape operator of M as a Riemannian submanifold of $H^n_+(-1)$. We denote the eigenvalue of $S^{\pm}_p$ by $\bar{\kappa}^{\pm}_p$ and the eigenvalue of A_p by κ_p. The relation $S^{\pm}_p = -1_{T_pM} \pm A_p$ implies that $S^{\pm}_p$ and A_p have the same eigenvectors, moreover $\bar{\kappa}^{\pm}_p = -1 \pm \kappa_p$.

We now define the notion of hyperbolic curvatures as follows: *The hyperbolic Gauss-Kronecker curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$K^{\pm}_h(u_0) = \det S^{\pm}_p.$$

The hyperbolic mean curvature of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be

$$H^{\pm}_h(u_0) = \frac{1}{n-1}\operatorname{Trace} S^{\pm}_p.$$

The *de Sitter Gauss-Kronecker curvature* is defined to be

$$K_d(u_0) = \det A_p$$

and the *de Sitter mean curvature* is

$$H_d(u_0) = \frac{1}{n-1}\operatorname{Trace} A_p.$$

We remark that the de Sitter mean curvature is actually the mean curvature of M. We, clearly, have that $H^{\pm}_h(u) = \pm H_d(u) - 1$. Surfaces with $H_d \equiv \pm 1$ represent the most important class among those with constant mean curvature in hyperbolic space. These surfaces have vanishing hyperbolic mean curvature and might thus be called *hyperbolic minimal surfaces.*

We say that a point $u \in U$ or $p = \boldsymbol{x}(u)$ is an *umbilic point* if $S_p^{\pm} = \bar{\kappa}^{\pm}(p)1_{T_pM}$. Since the eigenvectors of $S_p^{\pm}$ and A_p are the same, the above condition is equivalent to the condition $A_p = \pm\kappa(p)1_{T_pM}$. We say that $M = \boldsymbol{x}(U)$ is *totally umbilic* if all points on M are umbilic. In [**3**], Cecil and Ryan have characterized totally umbilic submanifolds by using three different functions on hyperbolic space. In [**11**] we have shown the following classification theorem on totally umbilical hypersurfaces:

PROPOSITION 2.2. *Suppose that $M = \boldsymbol{x}(U)$ is totally umbilic, then $\bar{\kappa}^{\pm}(p)$ is a constant $\bar{\kappa}^{\pm}$. Under this condition, we have the following classification*:

1) *Suppose that $\bar{\kappa}^{\pm} \neq 0$.*
   a) *If $\bar{\kappa}^{\pm} \neq -1$ and $|\bar{\kappa}^{\pm} + 1| < 1$, then M is a part of an equidistant hypersurface.*
   b) *If $\bar{\kappa}^{\pm} \neq -1$ and $|\bar{\kappa}^{\pm} + 1| > 1$, then M is a part of a hypersphere.*
   c) *If $\bar{\kappa}^{\pm} = -1$, then M is a part of a hyperplane.*
2) *If $\bar{\kappa}^{\pm} = 0$, then M is a part of a hyperhorosphere.*

It follows from the above proposition that we can classify the umbilic point as follows: Let $p = \boldsymbol{x}(u_0) \in \boldsymbol{x}(U) = M$ be an umbilic point, we say that p is *an equidistant flat point* if $\bar{\kappa}^{\pm} \neq 0$, $0 < |\bar{\kappa}^{\pm} + 1| < 1$, *a hyperspherical point* if $\bar{\kappa}^{\pm} \neq 0$, $|\bar{\kappa}^{\pm} + 1| > 1$, *a flat point* if $\bar{\kappa}^{\pm} \neq 0$, $|\bar{\kappa}^{\pm} + 1| = 0$, or *a hyperhorospherical point* if $\bar{\kappa}^{\pm} = 0$.

We establish next the hyperbolic (respectively, de Sitter) version of the Weingarten formula. Since $\boldsymbol{x}_{u_i}$ $(i = 1, \dots n-1)$ are spacelike vectors, we have the Riemannian metric (*hyperbolic first fundamental form*) given by $ds^2 = \sum_{i=1}^{n-1} g_{ij} du_i du_j$ on $M = \boldsymbol{x}(U)$, where $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u) \rangle$ and the *hyperbolic* (*respectively, de Sitter*) *second fundamental invariant* defined by $\bar{h}_{ij}^{\pm}(u) = \langle -\boldsymbol{L}_{u_i}^{\pm}(u), \boldsymbol{x}_{u_j}(u) \rangle$ (respectively, $h_{ij}(u) = -\langle \boldsymbol{e}_{u_i}(u), \boldsymbol{x}_{u_j}(u) \rangle$) for any $u \in U$. They satisfy the relation $\bar{h}_{ij}^{\pm}(u) = -g_{ij}(u) \pm h_{ij}(u)$.

PROPOSITION 2.3. *Under the above notations, we have the following formulae*:

(1) $\boldsymbol{L}_{u_i}^{\pm} = -\sum_{j=1}^{n-1} (\bar{h}^{\pm})_i^j \boldsymbol{x}_{u_j}$ (*The hyperbolic Weingarten formula*),

(2) $\boldsymbol{E}_{u_i} = -\sum_{j=1}^{n-1} (h_i^j) \boldsymbol{x}_{u_j}$ (*The de Sitter Weingarten formula*),

where $((\bar{h}^{\pm})_i^j) = (\bar{h}_{ik}^{\pm})(g^{kj})$, $(h_i^j) = (h_{ik})(g^{kj})$ *and* $(g^{kj}) = (g_{kj})^{-1}$.

PROOF. Since the hyperbolic Weingarten formula has been shown in [**11**], we only give here the proof of the de Sitter Weingarten formula.

There exist real numbers λ, μ, Γ_i^j such that $\boldsymbol{E}_{u_i} = \lambda \boldsymbol{e} + \mu \boldsymbol{x} + \sum_{j=1}^{n-1} \Gamma_i^j \boldsymbol{x}_{u_j}$. Since $\langle \boldsymbol{E}, \boldsymbol{E} \rangle = 1$, we have $0 = \langle \boldsymbol{E}_{u_i}, \boldsymbol{E} \rangle = \langle \lambda \boldsymbol{e} + \mu \boldsymbol{x}, \boldsymbol{e} \rangle = \lambda$. Therefore, $\boldsymbol{E}_{u_i} = \mu \boldsymbol{x} + \sum_{j=1}^{n-1} \cdot \Gamma_i^j \boldsymbol{x}_{u_j}$. On the other hand, it follows from the definition $-h_{i\beta} = \sum_{\alpha=1}^{n-1} \Gamma_i^{\alpha} \langle \boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_\beta} \rangle = \sum_{\alpha=1}^{n-1} \Gamma_i^{\alpha} g_{\alpha\beta}$. Hence, we have $-h_i^j = -\sum_{\beta=1}^{n-1} h_{i\beta} g^{\beta j} = \sum_{\beta=1}^{n-1} \sum_{\alpha=1}^{n-1} \Gamma_i^{\alpha} g_{\alpha\beta} g^{\beta j} = \Gamma_i^j$.

Moreover, $\langle \boldsymbol{E}, \boldsymbol{x} \rangle = 0$ and $\langle \boldsymbol{E}, \boldsymbol{x}_{u_i} \rangle = 0$, and thus $\mu = -\mu \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \langle \boldsymbol{E}_{u_i}, \boldsymbol{x} \rangle = 0$. This completes the proof of the de Sitter Weingarten formula. □

As a corollary of the above proposition, we have an explicit expression of the hyperbolic (respectively, de Sitter) Gauss-Kronecker curvature in terms of the Riemannian metric and the hyperbolic (respectively, de Sitter) second fundamental invariant.

Corollary 2.4. *Under the same notations as in the above proposition, we have the following formulae*:

$$K_h^{\pm} = \frac{\det\left(\bar{h}_{ij}^{\pm}\right)}{\det(g_{\alpha\beta})}, \quad K_d = \frac{\det(h_{ij})}{\det(g_{\alpha\beta})}.$$

Proof. By the hyperbolic Weingarten formula, the representation matrix of the hyperbolic shape operator $S_p^{\pm}$ with respect to the basis $\{\boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$ is

$$\left((\bar{h}^{\pm})_i^j\right) = \left(\bar{h}_{i\beta}^{\pm}\right)(g^{\beta j}).$$

It follows from this fact that

$$K_h^{\pm} = \det S_p^{\pm} = \det\left((\bar{h}^{\pm})_i^j\right) = \det\left(\bar{h}_{i\beta}^{\pm}\right)(g^{\beta j}) = \frac{\det\left(\bar{h}_{ij}^{\pm}\right)}{\det(g_{\alpha\beta})}.$$

By Proposition 2.3, the representation matrix of de Sitter shape operator A_p is also given by (h_i^j), the formula for the de Sitter Gauss-Kronecker curvature follows. □

We also get in this context the *hyperbolic Gauss equations* as we shall see next and it will be used in section 4. Since $\boldsymbol{x}(U) = M$ is a Riemannian manifold, it makes sense to consider the *Christoffel symbols*:

$$\left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} = \frac{1}{2} \sum_m g^{km} \left\{ \frac{\partial g_{jm}}{\partial u_i} + \frac{\partial g_{im}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_m} \right\}.$$

Proposition 2.5. *Let $\boldsymbol{x} : U \longrightarrow H_+^n(-1)$ be a hypersurface. Then we have the following hyperbolic Gauss equations*:

$$\boldsymbol{x}_{u_i u_j} = \sum_k \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} \boldsymbol{x}_{u_k} + h_{ij}\boldsymbol{e} + g_{ij}\boldsymbol{x}.$$

Proof. Since $\{\boldsymbol{e}, \boldsymbol{x}, \boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$ is a pseudo-orthonormal frame of $\boldsymbol{R}_1^{n+1}$, we can write $\boldsymbol{x}_{u_i u_j} = \sum_k \Gamma_{ij}^k \boldsymbol{x}_{u_k} + \Gamma_{ij}\boldsymbol{e} + \Gamma^{ij}\boldsymbol{x}$. We now have

$$\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x}_{u_\ell} \rangle = \sum_k \Gamma_{ij}^k \langle \boldsymbol{x}_{u_k}, \boldsymbol{x}_{u_\ell} \rangle = \sum_k \Gamma_{ij}^k g_{k\ell}.$$

Since $\frac{\partial g_{i\ell}}{\partial u_j} = \langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x}_{u_\ell} \rangle + \langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_\ell u_j} \rangle$ and $\boldsymbol{x}_{u_i u_j} = \boldsymbol{x}_{u_j u_i}$, we get $\Gamma_{ij}^k = \Gamma_{ji}^k$, $\Gamma_{ij} = \Gamma_{ji}$, $\Gamma^{ij} = \Gamma^{ji}$. Then by exactly the same calculation as those applied in the case of hypersurfaces in Euclidean space, it follows $\Gamma_{ij}^k = \left\{ \begin{smallmatrix} k \\ i\ j \end{smallmatrix} \right\}$.

On the other hand, $\mathit{\Gamma}_{ij} = \langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{e} \rangle = h_{ij}$. Moreover $\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x} \rangle = -\mathit{\Gamma}^{ij}$. And since $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle = 0$, we have $\langle \boldsymbol{x}_{u_i u_j}, \boldsymbol{x} \rangle = -\langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_j} \rangle = -g_{ij}$, which implies that $\mathit{\Gamma}^{ij} = g_{ij}$. $\square$

3. Horospherical geometry in hyperbolic space.

In the previous section we reviewed the properties of hyperbolic Gauss indicatrices and hyperbolic Gauss-Kronecker curvatures. The original definition of hyperbolic Gauss maps introduced by Bryant [**2**] and Epstein [**4**] is given in the Poincaré ball model. Here, we introduce the corresponding definition in Minkowski model as follows: If $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ is a lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\tilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in S_+^{n-1} = \{\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\}.$$

We call S_+^{n-1} *the lightcone* $(n-1)$*-sphere*. We define a map

$$\widetilde{\boldsymbol{L}}^{\pm} : U \longrightarrow S_+^{n-1}$$

by $\widetilde{\boldsymbol{L}}^{\pm}(u) = \widetilde{\boldsymbol{L}^{\pm}(u)}$ and call it *the hyperbolic Gauss map* of $\boldsymbol{x}$. We remark that for $n = 3$, our definition of hyperbolic Gauss map is equivalent to the one introduced in [**2**], [**4**]. Let T_pM be the tangent space of M at p and N_pM be the pseudo-normal space of T_pM in $T_p\boldsymbol{R}_1^{n+1}$. We have the decomposition $T_p\boldsymbol{R}_1^{n+1} = T_pM \oplus N_pM$, so that we also have the Whitney sum $T\boldsymbol{R}^{n+1} = TM \oplus NM$. Therefore we have the canonical projection $\Pi : T\boldsymbol{R}^{n+1} \longrightarrow TM$. It follows that we have a linear transformation $\Pi_p \circ d\widetilde{\boldsymbol{L}}^{\pm}(u) : T_pM \longrightarrow T_pM$ for $p = \boldsymbol{x}(u)$ by the identification of U and $\boldsymbol{x}(U) = M$ via $\boldsymbol{x}$. We have the following proposition:

PROPOSITION 3.1. *Under the above notation we have the following horospherical Weingarten formula*:

$$\Pi_p \circ \widetilde{\boldsymbol{L}}_{u_i}^{\pm} = -\sum_{j=1}^{n-1} \frac{1}{\ell_0^{\pm}(u)} (\bar{h}^{\pm})_i^j \boldsymbol{x}_{u_j},$$

where $\boldsymbol{L}^{\pm}(u) = (\ell_0^{\pm}(u), \ell_1^{\pm}(u), \ldots, \ell_n^{\pm}(u))$.

PROOF. By definition, we have $\ell_0^{\pm}\widetilde{\boldsymbol{L}}^{\pm} = \boldsymbol{L}^{\pm}$. It follows that we have $\ell_0^{\pm}\widetilde{\boldsymbol{L}}_{u_i}^{\pm} = \boldsymbol{L}_{u_i}^{\pm} - \ell_{0u_i}^{\pm}\widetilde{\boldsymbol{L}}^{\pm}$. Since $\widetilde{\boldsymbol{L}}^{\pm} \in NM$ and $\boldsymbol{L}_{u_i}^{\pm} \in TM$, we have

$$\Pi_p \circ \widetilde{\boldsymbol{L}}_{u_i}^{\pm} = \frac{1}{\ell_0^{\pm}} \boldsymbol{L}_{u_i}^{\pm}.$$

By the hyperbolic Weingarten formula (Proposition 2.3), we have the desired horospherical Weingarten formula. $\square$

We call the linear transformation $\widetilde{S}_p^{\pm} = -\Pi_p \circ d\widetilde{\boldsymbol{L}}^{\pm}$ the *horospherical shape operator* of $M = \boldsymbol{x}(U)$. We also define the *principal horospherical curvature* $\widetilde{\kappa}_p^{\pm}$ as an eigenvalue of $\widetilde{S}_p^{\pm}$. By the above proposition, we have $\widetilde{\kappa}_p^{\pm} = (1/\ell_0^{\pm})\bar{\kappa}_p^{\pm}$. The *horospherical Gauss-Kronecker curvature* of $\boldsymbol{x}(U) = M$ is defined to be $\widetilde{K}_h^{\pm}(u) = \det \widetilde{S}_p^{\pm}$. It follows that we have the following relation between the horospherical Gauss-Kronecker curvature and the hyperbolic Gauss-Kronecker curvature:

$$\widetilde{K}_h^{\pm}(u) = \left(\frac{1}{\ell_0^{\pm}(u)}\right)^{n-1} K_h^{\pm}(u).$$

We say that a point $u \in U$ or $p = \boldsymbol{x}(u)$ is a *horo-umbilic point* if $\widetilde{S}_p^{\pm} = \widetilde{\kappa}^{\pm}(p)1_{T_pM}$. By the above proposition, p is a horo-umbilic point if and only if it is an umbilic point. We say that $M = \boldsymbol{x}(U)$ is *totally horo-umbilic* if all points on M are horo-umbilic as usual.

We remark that $\widetilde{\kappa}^{\pm}(p)$ is not invariant under hyperbolic motions but it is an $SO(n)$-invariant. However, we can make sense a point with vanishing horospherical principal curvature as a notion of the hyperbolic differential geometry.

PROPOSITION 3.2. *For a point $p = \boldsymbol{x}(u)$, $\widetilde{\kappa}^{\pm}(p)$ is invariant under hyperbolic motions if and only if $\widetilde{\kappa}^{\pm}(p) = 0$.*

PROOF. We have the relation $\widetilde{\kappa}^{\pm}(p) = (1/\ell_0(p))\bar{\kappa}^{\pm}(p)$ at any point $p \in M$. Here, $\ell_0(p)$ varies with hyperbolic motions of hypersurfaces. Since $\bar{\kappa}^{\pm}(p)$ is a hyperbolic invariant, it is zero if and only if $\widetilde{\kappa}^{\pm}(p)$ is a hyperbolic invariant. □

COROLLARY 3.3. *Suppose that $M = \boldsymbol{x}(U)$ is totally horo-umbilic and $\widetilde{\kappa}^{\pm}(p)$ is a hyperbolic invariant, then M is a part of a hyperhorosphere ($\widetilde{\kappa}^{\pm}(p) \equiv 0$).*

We define a family of functions

$$\widetilde{H} : U \times S_+^{n-1} \longrightarrow \boldsymbol{R}$$

by $\widetilde{H}(u, \boldsymbol{v}) = \langle \boldsymbol{x}(u), \boldsymbol{v} \rangle$. We call $\widetilde{H}$ a *lightcone height function* on $\boldsymbol{x} : U \longrightarrow H_+^n(-1)$. We denote the *Hessian matrix* of the lightcone height function $\widetilde{h}_{v_0}(u) = \widetilde{H}(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(\widetilde{h}_{v_0})(u_0)$.

We say that a point $p = \boldsymbol{x}(u_0)$ is *a (positive or negative) horo-parabolic point* of $\boldsymbol{x} : U \longrightarrow H_+^n(-1)$ if $K_h^+(u_0) = 0$ or $K_h^-(u_0) = 0$. Moreover, a point $p = \boldsymbol{x}(u_0)$ is said to be a *horospherical point* if it is umbilic and horo-parabolic.

PROPOSITION 3.4. *Let $\widetilde{H} : U \times S_+^{n-1} \longrightarrow \boldsymbol{R}$ be a lightcone height function on $\boldsymbol{x} : U \longrightarrow H_+^n(-1)$. Then*

(1) $\partial \widetilde{H}/\partial u_i(u, \boldsymbol{v}) = 0$ $(i = 1, \dots n-1)$ *if and only if* $\boldsymbol{v} = \widetilde{\boldsymbol{L}}^{\pm}(u)$.

Suppose that $\boldsymbol{v}_0 = \widetilde{\boldsymbol{L}}^{\pm}(u_0)$. Then

(2) $p = \boldsymbol{x}(u_0)$ *is a horo-parabolic point if and only if* $\det \mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = 0$.

(3) $p = \boldsymbol{x}(u_0)$ *is a horospherical point if and only if* $\mathrm{rank}\, \mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = 0$.

PROOF. (1) Since $\{\boldsymbol{x}, \boldsymbol{e}, \boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}\}$ is a basis of the vector space $T_p\boldsymbol{R}_1^{n+1}$ where $p = \boldsymbol{x}(u)$, there exist real numbers $\lambda, \mu, \xi_1, \dots, \xi_{n-1}$ such that $\boldsymbol{v} = \lambda\boldsymbol{x} + \mu\boldsymbol{e} + \xi_1\boldsymbol{x}_{u_1} +$

$\cdots+\xi_{n-1}\boldsymbol{x}_{u_{n-1}}$. Since $\partial H/\partial u_i(u,\boldsymbol{v}) = \langle \boldsymbol{x}_{u_i}, \boldsymbol{v}\rangle$, we have $0 = \langle \boldsymbol{x}_{u_i}, \boldsymbol{v}\rangle = \xi_i \langle \boldsymbol{x}_{u_i}, \boldsymbol{x}_{u_i}\rangle$. This means that the condition $\partial \widetilde{H}/\partial u_i(u,\boldsymbol{v}) = 0$ is equivalent to the condition $\boldsymbol{v} = \lambda \boldsymbol{x} + \mu \boldsymbol{e}$. Since $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$, $\mu = \pm\lambda$. Since $\boldsymbol{v} \in S^{n-1}_+$, $\lambda = 1$.

By definition, we have

$$\mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = \big(\langle \boldsymbol{x}_{u_iu_j}(u_0), \widetilde{\boldsymbol{L}}^{\pm}(u_0)\rangle\big) = \big(-\langle \boldsymbol{x}_{u_i}(u_0), \widetilde{\boldsymbol{L}}^{\pm}_{u_j}(u_0)\rangle\big).$$

By the horospherical Weingarten formula, we have

$$-\big\langle \boldsymbol{x}_{u_i}, \widetilde{\boldsymbol{L}}^{\pm}_{u_j}\big\rangle = \frac{1}{\ell_0}\sum_{\alpha=1}^{n-1}(\bar{h}^{\pm})^{\alpha}_{i}\langle \boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_j}\rangle = \frac{1}{\ell_0}\sum_{\alpha=1}^{n-1}(\bar{h}^{\pm})^{\alpha}_{i} g_{\alpha j} = \frac{1}{\ell_0}\bar{h}^{\pm}_{ij}.$$

Therefore we have

$$\widetilde{K}^{\pm}_{h}(u_0) = \frac{\det \mathrm{Hess}(\widetilde{h}_{v_0})(u_0)}{\det(g_{\alpha\beta}(u_0))}.$$

The assertion (2) follows from this formula. For the assertion (3), by the hyperbolic Weingarten formula, $p = \boldsymbol{x}(u_0)$ is an umbilic point if and only if there exists an orthogonal matrix A such that ${}^tA\big((\bar{h}^{\pm})^{\alpha}_{i}\big)A = \bar{\kappa}^{\pm}I$. Therefore, we have $\big((\bar{h}^{\pm})^{\alpha}_{i}\big) = A\bar{\kappa}^{\pm}I\,{}^tA = \bar{\kappa}^{\pm}I$, so that

$$\mathrm{Hess}(\widetilde{h}_{v_0}) = \frac{1}{\ell_0}\big(\bar{h}^{\pm}_{ij}\big) = \frac{1}{\ell_0}\big((\bar{h}^{\pm})^{\alpha}_{i}\big)(g_{\alpha j}) = \frac{1}{\ell_0}\bar{\kappa}^{\pm}(g_{ij}).$$

Thus, p is a hyperhorospherical point if and only if $\operatorname{rank}\mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = 0$. $\square$

COROLLARY 3.5. *For a point $p = \boldsymbol{x}(u_0) \in M$, the following conditions are equivalent*:

(1) *The point $p \in M$ is a horo-parabolic point (i.e., $K^{\pm}_{h}(p) = 0$).*
(2) *The point $p \in M$ is a singular point of the hyperbolic Gauss indicatrix $\boldsymbol{L}^{\pm}$.*
(3) *The point $p \in M$ is a singular point of the hyperbolic Gauss map $\widetilde{\boldsymbol{L}}^{\pm}$.*
(4) $\widetilde{K}^{\pm}_{h}(p) = 0$.
(5) $\det \mathrm{Hess}(\widetilde{h}_{v_0})(u_0) = 0$ *for* $\boldsymbol{v}_0 = \widetilde{\boldsymbol{L}}^{\pm}(u_0)$.

We now interpret the results of Proposition 3.4 and Corollary 3.5 from another view point. We consider the relationship between the contact of submanifolds with foliations and the $\mathscr{R}^{+}$-class of functions. Let X_i $(i = 1,2)$ be submanifolds of $\boldsymbol{R}^n$ with $\dim X_1 = \dim X_2$, $g_i : (X_i, \bar{x}_i) \longrightarrow (\boldsymbol{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\boldsymbol{R}^n, \bar{y}_i) \longrightarrow (\boldsymbol{R}, 0)$ be submersion germs. For a submersion germ $f : (\boldsymbol{R}^n, 0) \longrightarrow (\boldsymbol{R}, 0)$, we denote that $\mathscr{F}_f$ be the regular foliation defined by f; i.e., $\mathscr{F}_f = \{f^{-1}(c) | c \in (\boldsymbol{R}, 0)\}$. We say that *the contact of X_1 with the regular foliation $\mathscr{F}_{f_1}$* at $\bar{y}_1$ is of the *same type* as *the contact of X_2 with the regular foliation $\mathscr{F}_{f_2}$* at $\bar{y}_2$ if there is a diffeomorphism germ $\Phi : (\boldsymbol{R}^n, \bar{y}_1) \longrightarrow (\boldsymbol{R}^n, \bar{y}_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1(c)) = Y_2(c)$, where $Y_i(c) = f_i^{-1}(c)$ for each $c \in (\boldsymbol{R}, 0)$. In this case we write $K(X_1, \mathscr{F}_{f_1}; \bar{y}_1) = K(X_2, \mathscr{F}_{f_2}; \bar{y}_2)$. It is clear that in the definition

$\boldsymbol{R}^n$ could be replaced by any manifold. We apply the method of Goryunov [**6**] to the case for $\mathscr{R}^+$-equivalences among function germs, so that we have the following:

Proposition 3.6 ([**6**, Appendix]). *Let X_i $(i = 1, 2)$ be submanifolds of $\boldsymbol{R}^n$ with $\dim X_1 = \dim X_2 = n - 1$ (i.e. hypersurface), $g_i : (X_i, \bar{x}_i) \longrightarrow (\boldsymbol{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\boldsymbol{R}^n, \bar{y}_i) \longrightarrow (\boldsymbol{R}, 0)$ be submersion germs. Then $K(X_1, \mathscr{F}_{f_1}; \bar{y}_1) = K(X_2, \mathscr{F}_{f_2}; \bar{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathscr{R}^+$-equivalent (i.e., there exists a diffeomorphism germ $\phi : (X_1, \bar{x}_1) \longrightarrow (X_2, \bar{x}_2)$ such that $(f_2 \circ g_2) \circ \phi = f_1 \circ g_1$).*

On the other hand, Golubitsky and Guillemin [**7**] have given an algebraic characterization for the $\mathscr{R}^+$-equivalence among function germs. We denote $C_0^\infty(X)$ is the set of function germs $(X, 0) \longrightarrow \boldsymbol{R}$. Let J_f be the Jacobian ideal in $C_0^\infty(X)$ (i.e., $J_f = \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle_{C_0^\infty(X)}$). Let $\mathscr{R}_k(f) = C_0^\infty(X)/J_f^k$ and $\bar{f}$ be the image of f in this local ring. We say that f satisfies the *Milnor Condition* if $\dim_{\boldsymbol{R}} \mathscr{R}_1(f) < \infty$.

Proposition 3.7 ([**7**, Proposition 4.1]). *Let f and g be germs of functions at 0 in X satisfying the Milnor condition with $df(0) = dg(0) = 0$. Then f and g are $\mathscr{R}^+$-equivalent if*

(1) *The rank and signature of the Hessians* $\operatorname{Hess}(f)(0)$ *and* $\operatorname{Hess}(g)(0)$ *are equal, and*

(2) *There is an isomorphism $\gamma : \mathscr{R}_2(f) \longrightarrow \mathscr{R}_2(g)$ such that $\gamma(\bar{f}) = \bar{g}$.*

For $\boldsymbol{v}_0 = \widetilde{\boldsymbol{L}}^\pm(u_0)$, we consider a function $\mathfrak{h}_{v_0} : H_+^n(-1) \longrightarrow \boldsymbol{R}$ defined by $\mathfrak{h}_{v_0}(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{v}_0 \rangle$. It is easy to show that $\mathfrak{h}_{v_0}$ is a submersion. Moreover we have $\mathfrak{h}_{v_0} \circ \boldsymbol{x}(u) = \widetilde{H}(u, \boldsymbol{v}_0)$. By Proposition 3.4, we have

$$\frac{\partial \mathfrak{h}_{v_0} \circ \boldsymbol{x}}{\partial u_i}(u_0) = \frac{\partial \widetilde{H}}{\partial u_i}(u_0, \boldsymbol{v}_0) = 0$$

for $i = 1, \ldots, n-1$. This means that the hyperhorosphere ${\mathfrak{h}_{v_0}}^{-1}(-1) = HS^{n-1}(\boldsymbol{v}_0, -1) = HP(\boldsymbol{v}_0, -1) \cap H_+^n(-1)$ is tangent to $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. In this case, we call $HS^{n-1}(\boldsymbol{v}_0, -1)$ a *tangent hyperhorosphere* with the center $\boldsymbol{v}_0$. We have two tangent hyperhorospheres at $p = \boldsymbol{x}(u_0)$ depending on the direction of $\boldsymbol{v}_0$. Let ε be a sufficiently small positive real number. For any $t \in I_\varepsilon = (-\varepsilon - 1, \varepsilon - 1)$, we have a hyperhorosphere

$$HS^{n-1}(\boldsymbol{v}_0, t) = HP(\boldsymbol{v}_0, t) \cap H_+^n(-1) = \mathfrak{h}_{v_0}^{-1}(t).$$

In this case $\mathscr{F}_{\mathfrak{h}_{v_0}}$ is a family of parallel hyperhorospheres around $p = \boldsymbol{x}(u_0)$ such that $\mathfrak{h}_{v_0}^{-1}(-1)$ is the tangent hyperhorosphere of M at p. Let $\boldsymbol{x}_i : (U, \bar{u}_i) \longrightarrow (H_+^n(-1), \boldsymbol{x}_i(\bar{u}_i))$ $(i = 1, 2)$ be hypersurface germs, then we have $\widetilde{h}_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \boldsymbol{x}_i(u)$. Then we have the following proposition as a corollary of Propositions 3.6 and 3.7.

Proposition 3.8. *Let $\boldsymbol{x}_i : (U, \bar{u}_i) \longrightarrow (H_+^n(-1), \boldsymbol{x}_i(\bar{u}_i))$ $(i = 1, 2)$ be hypersurface germs such that $\widetilde{h}_{i,v_i}$ satisfy the Milnor condition, where $\boldsymbol{v}_i = \widetilde{\boldsymbol{L}}^\pm(\bar{u}_i)$ are centers of the tangent hyperhorospheres of $\boldsymbol{x}_i$ respectively. Then the following conditions are equivalent*:

(1) $K(\boldsymbol{x}_1(U), \mathscr{F}_{\mathfrak{h}_{v_1}}; \boldsymbol{x}(\bar{u}_1)) = K(\boldsymbol{x}_2(U), \mathscr{F}_{\mathfrak{h}_{v_2}}; \boldsymbol{x}(\bar{u}_2))$.

(2) $\widetilde{h}_{1,v_1}$ *and* $\widetilde{h}_{2,v_2}$ *are* $\mathscr{R}^+$*- equivalent.*

(3) (a) *The rank and signature of the* $\mathrm{Hess}(\widetilde{h}_{1,v_1})(\bar{u}_1)$ *and* $\mathrm{Hess}(\widetilde{h}_{2,v_2})(\bar{u}_2)$ *are equal,*
(b) *There is an isomorphism* $\gamma : \mathscr{R}_2(\widetilde{h}_{1,v_1}) \longrightarrow \mathscr{R}_2(\widetilde{h}_{2,v_2})$ *such that* $\gamma(\overline{\widetilde{h}_{1,v_1}}) = \overline{\widetilde{h}_{2,v_2}}$.

Remarks. We can show that the hyperbolic Gauss map $\widetilde{\boldsymbol{L}}^{\pm}$ is a Lagrangian map of a certain Lagrangian submanifold $\widetilde{\mathscr{L}}^{\pm}$ in $T^*S^{n-1}_+$ whose generating family is the lightcone height function $\widetilde{H}$. For the notions and basic results of the theory of Lagrangian singularities, see [**1**]. Therefore we can apply the theory of Lagrangian singularities to the study of the contact of hypersurfaces with the parallel families of hyperhorospheres. However, arguments are almost parallel to those of section 4 and section 5 in [**12**], so that we omit the details.

In the last part of this section we show that the notion of horospherical curvatures is independent of the choice of the model of hyperbolic space. For the purpose, we introduce a smooth function on the unit tangent sphere bundle of hyperbolic space which plays the principal role of the horospherical geometry. Let $SO_0(n,1)$ be the identity component of the matrix group

$$SO(n,1) = \{g \in GL(n+1,\boldsymbol{R}) \mid gI_{n,1}\,{}^tg = I_{n,1},\},$$

where

$$I_{n,1} = \left(\begin{array}{c|c} -1 & \boldsymbol{0} \\ \hline {}^t\boldsymbol{0} & I_n \end{array}\right) \in GL(n+1,\boldsymbol{R}).$$

It is well-known that $SO_0(n,1)$ transitively acts on $H^n_+(-1)$ and the isotropic group at $p = (1,0,\dots,0)$ is $SO(n)$ which is naturally embedded in $SO_0(n,1)$. Moreover the action induces isometries on $H^n_+(-1)$.

On the other hand, we consider a submanifold $\Delta = \{(\boldsymbol{v},\boldsymbol{w}) \mid \langle \boldsymbol{v},\boldsymbol{w}\rangle = 0\}$ of $H^n_+(-1) \times S^n_1$ and the canonical projcetion $\bar{\pi} : \Delta \longrightarrow H^n_+(-1)$. Let $\pi : S(TH^n_+(-1)) \longrightarrow H^n(-1)$ be the unit tangent sphere bundle over $H^n_+(-1)$. For any $\boldsymbol{v} \in H^n_+(-1)$, we have the local (global) coordinates $(v_1,\dots,v_n)$ of $H^n_+(-1)$ such that $\boldsymbol{v} = (\sqrt{v_1^2+\cdots+v_n^2+1}, v_1,\dots,v_n)$. We can represent the tangent vector $\boldsymbol{w} = \sum_{i=1}^n w_i\partial/\partial v_i \in T_vH^n_+(-1)$ by

$$\boldsymbol{w} = \left(\frac{1}{v_0}\sum_{i=1}^n w_iv_i, w_1,\dots,w_n\right)$$

as a vector in Minkowski $(n+1)$-space. Then $\langle \boldsymbol{w},\boldsymbol{v}\rangle = \left(-\frac{1}{v_0}\sum_{i=1}^n w_iv_i\right)v_0 + \sum_{i=1}^n w_iv_i = 0$. Therefore $\boldsymbol{w} \in S(T_vH^n_+(-1))$ if and only if

$$\langle \boldsymbol{w},\boldsymbol{w}\rangle = 1 \text{ and } \langle \boldsymbol{v},\boldsymbol{w}\rangle = 0.$$

The above conditions are equivalent to the condition $(\boldsymbol{v},\boldsymbol{w}) \in \Delta$. This means that we can canonically identify $\pi : S(TH^n_+(-1)) \longrightarrow H^n_+(-1)$ with $\bar{\pi} : \Delta \longrightarrow H^n_+(-1)$. Moreover, the linear action of $SO_0(n,1)$ on $\boldsymbol{R}^{n+1}_1$ induces the canonical action on Δ (i.e., $g(\boldsymbol{v},\boldsymbol{w}) = (g\boldsymbol{v}, g\boldsymbol{w})$ for any $g \in SO_0(n,1)$). For any $(\boldsymbol{v},\boldsymbol{w}) \in \Delta$, the first component of $\boldsymbol{v} \pm \boldsymbol{w}$ is given by

$$v_0 \pm w_0 = \sqrt{v_1^2 + \cdots + v_n^2 + 1} \pm \frac{1}{\sqrt{v_1^2 + \cdots + v_n^2 + 1}} \sum_{i=1}^n v_i w_i,$$

so that it can be considered as a function on the unit tangent bundle $S(TH^n_+(-1))$. We now define a function

$$\mathscr{N}_h^{\pm} : \Delta \longrightarrow \boldsymbol{R}\ ;\ \mathscr{N}_h^{\pm}(\boldsymbol{v},\boldsymbol{w}) = \frac{1}{v_0 \pm w_0}.$$

We call $\mathscr{N}_h^{\pm}$ a *horospherical normalization function* on $H^n_+(-1)$. Since $v^2 + \cdots + v_n^2 + 1$ and $\sum_{i=1}^n v_i w_i$ are $SO(n)$-invariant functions, $\mathscr{N}_h^{\pm}$ is an $SO(n)$-invariant function. Therefore, $\mathscr{N}_h^{\pm}$ can be considered as a function on the unit tangent sphere bundle over hyperbolic space $SO_0(n,1)/SO(n)$ which is independent of the choice of the model space.

For any embedding $\boldsymbol{x} : U \longrightarrow H^n_+(-1)$, we have the unit normal vector field $\boldsymbol{e} : U \longrightarrow S^n_1$, so that $(\boldsymbol{x}(u), \boldsymbol{e}(u)) \in \Delta$ for any $u \in U$. It follows that

$$\widetilde{K}_h^{\pm}(u) = \mathscr{N}_h^{\pm}(\boldsymbol{x}(u), \boldsymbol{e}(u))^{n-1} K_h^{\pm}(u).$$

The right hand side of the above equality is independent of the choice of the model space.

4. Proof of the main result.

In this section we give the definition of global horospherical Gauss-Kronecker curvatures and a proof for the horospherical Gauss-Bonnet type theorem. Let M be a closed orientable $(n-1)$-dimensional manifold and $f : M \longrightarrow H^n_+(-1)$ an embedding. We consider the canonical projection $\pi : \boldsymbol{R}^{n+1}_1 \longrightarrow \boldsymbol{R}^n$ defined by $\pi(x_0, x_1, \ldots, x_n) = (0, x_1, \ldots, x_n)$. Then we have orientation preserving diffeomorphisms $\pi \mid H^n_+(-1) : H^n_+(-1) \longrightarrow \boldsymbol{R}^n$ and $\pi \mid S^{n-1}_+ : S^{n-1}_+ \longrightarrow S^{n-1}$.

Consider the unit normal $\boldsymbol{E}$ of $f(M)$ in $H^n_+(-1)$, then we define the hyperbolic Gauss indicatrix in the global

$$\boldsymbol{L} : M \longrightarrow LC^*_+$$

by

$$\boldsymbol{L}(p) = f(p) + \boldsymbol{E}(p).$$

The *global hyperbolic Gauss-Kronecker curvature function* $\mathscr{K}_h : M \longrightarrow \boldsymbol{R}$ is then defined in the usual way in terms of the global hyperbolic Gauss indicatrix $\boldsymbol{L}$. We also define the hyperbolic Gauss map in the global

$$\widetilde{\boldsymbol{L}} : M \longrightarrow S_+^{n-1}$$

by

$$\widetilde{\boldsymbol{L}}(p) = \widetilde{\boldsymbol{L}(p)}.$$

We now define a *global horospherical Gauss-Kronecker curvature function* $\widetilde{\mathscr{K}_h} : M \longrightarrow \boldsymbol{R}$ by

$$\widetilde{\mathscr{K}_h}(p) = \mathscr{N}_h(f(p), \boldsymbol{E}(p))^{n-1}\mathscr{K}_h(p),$$

where we simply write $\mathscr{N}_h = \mathscr{N}_h^+$.

Proposition 4.1. *Under the above notation, we have the following relation*:

$$\widetilde{\mathscr{K}_h} d\mathfrak{v}_M = \widetilde{\boldsymbol{L}}^* d\mathfrak{v}_{S_+^{n-1}},$$

where $d\mathfrak{v}_M$ *(respectively,* $d\mathfrak{v}_{S_+^{n-1}}$*) is the volume form of* M *(respectively,* S_+^{n-1}*).*

Proof. Firstly we assume that the hyperbolic Gauss map $\widetilde{\boldsymbol{L}}$ is nonsingular at a point $p = \boldsymbol{x}(u_0) \in M = \boldsymbol{x}(U)$. In this case, there exists an open neighbourhood $W \subset U$ around u_0 such that $\widetilde{\boldsymbol{L}} : W \longrightarrow S_+^{n-1}$ is an embedding. Therefore, $\widetilde{\boldsymbol{L}}_{u_1}, \dots, \widetilde{\boldsymbol{L}}_{u_{n-1}}$ is a basis of $T_z S_+^{n-1}$ at any point $z \in V = \widetilde{\boldsymbol{L}}(W)$. We denote $\widetilde{g}_{ij}$ the Riemannian metric on V and $g_{\alpha\beta}$ the Riemannian metric on W given by the restriction of the Minkowski metric. Since $\boldsymbol{L} = \ell_0 \widetilde{\boldsymbol{L}}$, we calculate that $\ell_0 \widetilde{\boldsymbol{L}}_{u_i} = \boldsymbol{L}_{u_i} - \ell_{0u_i}\widetilde{\boldsymbol{L}}$, where $\boldsymbol{L}(u) = (\ell_0(u), \ell_1(u), \dots, \ell_n(u))$. It follows that

$$\begin{aligned}
\widetilde{g}_{ij} &= \left\langle \widetilde{\boldsymbol{L}}_{u_i}, \widetilde{\boldsymbol{L}}_{u_j} \right\rangle \\
&= \left(\frac{1}{\ell_0}\right)^2 \left\langle \boldsymbol{L}_{u_i}, \boldsymbol{L}_{u_j} \right\rangle \\
&= \left(\frac{1}{\ell_0}\right)^2 \left\langle \sum_{\alpha=1}^{n-1} \bar{h}_i^\alpha \boldsymbol{x}_{u_\alpha}, \sum_{\beta=1}^{n-1} \bar{h}_i^\beta \boldsymbol{x}_{u_\beta} \right\rangle \\
&= \left(\frac{1}{\ell_0}\right)^2 \sum_{\alpha,\beta} \bar{h}_i^\alpha \bar{h}_j^\beta \langle \boldsymbol{x}_{u_\alpha}, \boldsymbol{x}_{u_\beta} \rangle \\
&= \left(\frac{1}{\ell_0}\right)^2 \sum_{\alpha,\beta} \bar{h}_i^\alpha \bar{h}_j^\beta g_{\alpha\beta}.
\end{aligned}$$

By Proposition 3.1, $\widetilde{\mathscr{K}_h} = (1/\ell_0)^{n-1} \det\left(\bar{h}_j^i\right)$, so that

$$\det\left(\widetilde{g}_{ij}\right) = \widetilde{\mathscr{K}_h}^2 \det(g_{\alpha\beta}).$$

Let us denote $(\widetilde{u}_1, \dots, \widetilde{u}_{n-1})$ as the local coordinate on V via the embedding $\widetilde{\boldsymbol{L}}$. This means that

$$\widetilde{\boldsymbol{L}}^*(d\widetilde{u}_1 \wedge \cdots \wedge d\widetilde{u}_{n-1}) = \begin{cases} du_1 \wedge \cdots \wedge du_{n-1} & \text{if } \widetilde{\mathscr{K}_h}(u) > 0 \\ -du_1 \wedge \cdots \wedge du_{n-1} & \text{if } \widetilde{\mathscr{K}_h}(u) < 0, \end{cases}$$

where $(u_1, \dots, u_{n-1})$ is the canonical coordinate on $W \subset \boldsymbol{R}^{n-1}$. Therefore we have

$$\widetilde{\mathscr{K}_h} d\mathfrak{v}_W = \widetilde{\boldsymbol{L}}^* d\mathfrak{v}_V.$$

If p is a singular point of $\widetilde{\boldsymbol{L}}$, then the both hand sides are zero. This completes the proof. $\square$

We now start to give the proof of Theorem 1.1. Consider the (Euclidean) Gauss map

$$\boldsymbol{N} : M \longrightarrow S^{n-1}$$

on $\pi \circ f(M)$.

The proof of Theorem 1.1 is based in the following key lemma:

LEMMA 4.2. *Under the choice of a suitable direction of $\boldsymbol{N}$, $\pi \circ \widetilde{\boldsymbol{L}}$ and $\boldsymbol{N}$ are homotopic.*

PROOF. Since $\boldsymbol{E}(p)$ is normal to $f(M)$ in $H^n_+(-1)$, $\pi \circ \boldsymbol{E}(p)$ is transverse to $\pi \circ f(M)$ in $\boldsymbol{R}^n$. It follows that $\langle \pi \circ \boldsymbol{E}(p), \boldsymbol{N}(p) \rangle \neq 0$ at any $p \in M$. We choose the direction of $\boldsymbol{N}$ such that $\langle \pi \circ \boldsymbol{E}(p), \boldsymbol{N}(p) \rangle > 0$. We also denote $\overline{\pi \circ \boldsymbol{E}}(p) \in S^{n-1}$ as the unit normalization of $\pi \circ \boldsymbol{E}(p)$.

We now construct a homotopy between $\overline{\pi \circ \boldsymbol{E}}$ and $\boldsymbol{N}$. Let

$$F_1 : M \times [0,1] \longrightarrow S^{n-1}$$

be defined by

$$F_1(p,t) = \frac{t\boldsymbol{N}(p) + (1-t)\overline{\pi \circ \boldsymbol{E}}(p)}{\|t\boldsymbol{N}(p) + (1-t)\overline{\pi \circ \boldsymbol{E}(p)}\|},$$

where $\| \cdot \|$ is the Euclidean norm.

If there exists $t' \in [0,1]$ and $p' \in M$ such that $t'\boldsymbol{N}(p') + (1-t')\overline{\pi \circ \boldsymbol{E}}(p') = \boldsymbol{0}$, then we have $\boldsymbol{N}(p') = -\overline{\pi \circ \boldsymbol{E}}(p')$. This contradicts to the assumption that $\langle \pi \circ \boldsymbol{E}(p), \boldsymbol{N}(p) \rangle > 0$. Therefore F_1 is a continuous mapping satisfying $F_1(p,0) = \overline{\pi \circ \boldsymbol{E}}(p)$ and $F_1(p,1) = \boldsymbol{N}(p)$ for any $p \in M$.

We also construct a homotopy between $\pi \circ \widetilde{\boldsymbol{L}}$ and $\overline{\pi \circ \boldsymbol{E}}$. Let

$$F_2 : M \times [0,1] \longrightarrow S^{n-1}$$

be defined by

$$F_2(p,t) = \frac{t\overline{\pi \circ \boldsymbol{E}}(p) + (1-t)\pi \circ \widetilde{\boldsymbol{L}}(p)}{\|t\overline{\pi \circ \boldsymbol{E}}(p) + (1-t)\pi \circ \widetilde{\boldsymbol{L}}(p)\|}.$$

If there exists $t' \in [0,1]$ and $p' \in M$ such that $t'\pi \circ \boldsymbol{E}(p') + (1-t')\pi \circ \widetilde{\boldsymbol{L}}(p') = \boldsymbol{0}$, then we have $\overline{\pi \circ \boldsymbol{E}}(p') = -\pi \circ \widetilde{\boldsymbol{L}}(p')$. It follows that there exists a negative real number λ such that $\pi \circ \boldsymbol{L}(p') = \lambda\pi \circ \boldsymbol{E}(p')$. Therefore we have $V_{p'} = \langle \boldsymbol{E}(p'), f(p')\rangle_{\boldsymbol{R}}$ contains the kernel of π (i.e., the $\boldsymbol{e}_0$-direction). In this case V_p is Minkowski plane whose basis is $\{\pi \circ f(p'), \boldsymbol{e}_0\}$. We need the following sublemma:

Sublemma. *Let $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^2_1$ be nonzero vectors with $\langle \boldsymbol{a}, \boldsymbol{a}\rangle = -1$, $\langle \boldsymbol{a}+\boldsymbol{b}, \boldsymbol{a}+\boldsymbol{b}\rangle = 0$ and $\langle \boldsymbol{b}, \boldsymbol{b}\rangle = 1$. We assume that $a_0 > 0$, $a_1 > 0$ and $a_0 + b_0 > 0$, where $\boldsymbol{a} = (a_0, a_1)$, $\boldsymbol{b} = (b_0, b_1)$. Then we have $(a_1 + b_1)b_1 > 0$.*

Proof. Since $\langle \boldsymbol{a}, \boldsymbol{a}\rangle = -1$, we have $a_1 < a_0$. In the case when $a_1 + b_1 > 0$, we have $a_1 + b_1 = a_0 + b_0 > 0$, so that $b_0 - b_1 = a_1 - a_0 < 0$. If $b_1 < 0$, then $b_0 < b_1 < 0$. On the other hand we have $1 = -b_0^2 + b_1^2 = (b_1 - b_0)(b_0 + b_1)$, so that $b_1 - b_0 < 0$. This is a contradiction. If $a_1 + b_1 < 0$ and $b_1 > 0$, then $a_1 < -b_1 < 0$. This is also a contradiction. This completes the proof of the sublemma. □

We now apply the sublemma to our situation as $\boldsymbol{a} = f(p')$ and $\boldsymbol{b} = \boldsymbol{E}(p')$. Suppose that $\pi \circ \boldsymbol{L}(p') = \lambda\pi \circ \boldsymbol{E}(p')$ for some non-zero real number λ, then λ is positive by the sublemma. Therefore $t\overline{\pi \circ \boldsymbol{E}}(p) + (1-t)\pi \circ \widetilde{\boldsymbol{L}}(p)$ does not vanish at any point $p \in M$. It follows that F_2 is a continuous mapping satisfying that $F_2(p,0) = \pi \circ \widetilde{\boldsymbol{L}}(p)$ and $F_2(p,1) = \overline{\pi \circ \boldsymbol{E}}(p)$ for any $p \in M$.

Eventually, $\boldsymbol{N}$ and $\pi \circ \widetilde{\boldsymbol{L}}$ are homotopic. □

Since the mapping degree is a homotopy invariant, we have the following corollary (cf., [**8**], Chapter 4, §9).

Corollary 4.3. *If M is a closed orientable, even-dimensional hypersurface in $H^n_+(-1)$, then we have*

$$\deg \widetilde{\boldsymbol{L}} = \frac{1}{2}\chi(M),$$

where $\deg \widetilde{\boldsymbol{L}}$ *is the mapping degree of* $\widetilde{\boldsymbol{L}}$.

By the definition of the horospherical Gauss-Kronecker curvature $\widetilde{\mathscr{K}_h}$, we obtain:

$$\int_M \widetilde{\mathscr{K}_h} d\mathfrak{v}_M = \int_M \widetilde{\boldsymbol{L}}^* d\mathfrak{v}_{S^{n-1}_+} = \deg(\widetilde{\boldsymbol{L}}) \int_{S^{n-1}_+} d\mathfrak{v}_{S^{n-1}_+} = \deg(\widetilde{\boldsymbol{L}})\gamma_{n-1}.$$

The proof of Theorem 1.1 is now completed as a consequence of Corollary 4.3.

Remark. Since we do not assume that n is odd in Lemma 4.2, we can apply the lemma for the case $n = 2$. In this case we consider a unit speed hyperbolic plane curve

$\boldsymbol{\gamma} : S^1 \longrightarrow H^2_+(-1)$. In [**10**] we have shown the Frenet-Serre type formula:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) = -\kappa_g(s)\boldsymbol{t}(s), \end{cases}$$

Here, $\boldsymbol{t}$ is the unit tangent vector, $\boldsymbol{e}$ is the normal vector defined as in the same way as the general case and $\kappa_g(s)$ is *the geodesic curvature* of the curve γ in $H^2_+(-1)$ which is given by

$$\kappa_g(s) = \det(\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{t}'(s)).$$

In this case, $\boldsymbol{L} = \boldsymbol{\gamma} + \boldsymbol{e}$. If we fix the following parameterization of the lightlike circle:

$$S^1_+ = \{(1, \cos\theta, \sin\theta) \mid 0 \leq \theta < 2\pi\},$$

then the horospherical curvature is

$$\widetilde{\kappa_h}(s) = \mathscr{N}_h(\boldsymbol{\gamma}(s), \boldsymbol{e}(s))(\kappa_g(s) - 1).$$

Since the projection $\pi : H^1_+(-1) \longrightarrow \boldsymbol{R}^2$ is a diffeomorphism, the winding numbers of $\boldsymbol{\gamma}$ and $\pi \circ \boldsymbol{\gamma}$ are the same. Therefore we have the following formula as a corollary of Lemma 4.2:

$$\frac{1}{2\pi} \int_{S^1} \widetilde{\kappa_h} ds = W(\boldsymbol{\gamma}),$$

where $W(\boldsymbol{\gamma})$ denotes the winding number of $\boldsymbol{\gamma}$.

5. Surfaces in hyperbolic 3-space.

In this section we stick to the case $n = 3$. First of all we need to make some local calculations. Let $\boldsymbol{x} : U \longrightarrow H^3_+(-1)$ be a (local) surface, where $U \subset \boldsymbol{R}^2$ is an open region, and consider the Riemannian curvature tensor

$$R^\ell_{ijk} = \frac{\partial}{\partial u_k}\left\{ \begin{matrix} \ell \\ i\ j \end{matrix} \right\} - \frac{\partial}{\partial u_j}\left\{ \begin{matrix} \ell \\ i\ k \end{matrix} \right\} + \sum_m \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\}\left\{ \begin{matrix} \ell \\ m\ k \end{matrix} \right\} - \sum_m \left\{ \begin{matrix} m \\ i\ k \end{matrix} \right\}\left\{ \begin{matrix} \ell \\ m\ j \end{matrix} \right\}.$$

We also consider the tensor $R_{ijk\ell} = \sum_m g_{im} R^m_{jk\ell}$. Standard calculations, analogous to those used in the study of the classical differential geometry on surfaces in Euclidean space (cf., [**20**]), lead to the following formula.

Proposition 5.1. *Under the above notations, we have*

$$R_{ijk\ell} = h_{jk}h_{i\ell} - h_{j\ell}h_{ik} - g_{jk}g_{i\ell} + g_{j\ell}g_{ik}.$$

From Corollary 2.4 we have

$$K_d = \frac{h_{11}h_{22} - h_{21}h_{12}}{g_{11}g_{22} - g_{12}g_{21}}.$$

And thus we obtain the analogous result of the *Theorema Egregium* of Gauss for the hyperbolic case:

PROPOSITION 5.2. *Under the above notations, we have*

$$K_d = -\frac{R_{1212}}{g} + 1,$$

where $g = g_{11}g_{22} - g_{12}g_{21}$.

We remark that $-R_{1212}/g$ is the *sectional curvature* of the surface. It is denoted by K_s.

On the other hand, let κ_i $(i = 1, 2)$ be eigenvalues of (h^i_j) (i.e., de Sitter principal curvatures of the surface). We remind that $\bar{\kappa}^{\pm}_i = -1 \pm \kappa_i$, from which we deduce:

PROPOSITION 5.3. *The following relation holds*:

$$K^{\pm}_h = 1 \mp 2H_d + K_d = 2 \mp 2H_d + K_s.$$

We return to the global situation. Let M be a closed orientable 2-dimensional manifold and $f : M \longrightarrow H^3_+(-1)$ an embedding. Under the same notations as in section 4, we define a global mean curvature function $\mathscr{H}_d : M \longrightarrow \boldsymbol{R}$ by using the de Sitter Gauss map $\boldsymbol{E}$. Therefore we have the relation

$$\mathscr{K}_h = 1 - 2\mathscr{H}_d + \mathscr{K}_d = 2 - 2\mathscr{H}_d + \mathscr{K}_s,$$

where $\mathscr{K}_d$ is the global de Sitter Gauss-Kronecker curvature function and $\mathscr{K}_s$ is the global sectional curvature function. Then we obtain relations of the total curvatures on M.

THEOREM 5.4. *Let M be a closed orientable 2-dimensional manifold and $f : M \longrightarrow H^3_+(-1)$ an embedding. Then we have*

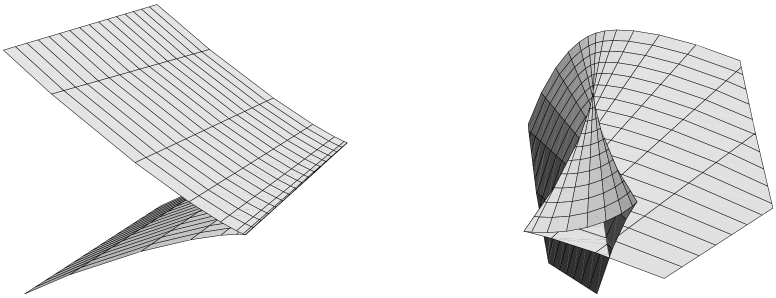
$$\int_M \mathscr{K}_h d\mathfrak{a}_M = 2A(M) - 2\int_M \mathscr{H}_d d\mathfrak{a}_M + 2\pi\boldsymbol{\chi}(M)$$

and

$$\int_M \widetilde{\mathscr{K}_h} d\mathfrak{a}_M = \int_M \mathscr{K}_h d\mathfrak{a}_M - 2A(M) + 2\int_M \mathscr{H}_d d\mathfrak{a}_M.$$

where $d\mathfrak{a}_M$ is the area form and $A(M)$ is the area of M.

PROOF. By the Gauss-Bonnet theorem on M, considered as a Riemannian manifold, we have $\int_M \mathscr{K}_s d\mathfrak{a}_M = 2\pi\boldsymbol{\chi}(M)$, so that we have the first formula. But then,



cuspidaledge swallowtail

Figure 1.

Theorem 1.1 together with the above relation imply the second formula. □

We study in the remaining of the paper some generic properties of surfaces embedded in $H^3_+(-1)$. We gave in [**11**] the following local classification of singularities for the hyperbolic Gauss indicatrix of a generic local surface in $H^3_+(-1)$.

Theorem 5.5. *Let* $\mathrm{Emb}\,(U, H^3_+(-1))$ *be the space of embeddings from an open region* $U \subset \boldsymbol{R}^2$ *into* $H^3_+(-1)$ *equipped with the Whitney* C^∞*-topology. There exists an open dense subset* $\mathscr{O} \subset \mathrm{Emb}\,(U, H^3_+(-1))$ *such that for any* $\boldsymbol{x} \in \mathscr{O}$*, the following conditions hold*:

(1) *The horo-parabolic set* $K_h^{-1}(0)$ *is a regular curve. We call such a curve the horo-parabolic curve.*

(2) *The hyperbolic Gauss indicatrix* $\boldsymbol{L}$ *along the horo-parabolic curve is locally diffeomorphic to the cuspidaledge except at isolated points. At such isolated points,* $\boldsymbol{L}$ *is locally diffeomorphic to the swallowtail.*

Here, the cuspidaledge is $C = \{(x_1, x_2, x_3) \mid {x_1}^2 = {x_2}^3\}$ *and the swallowtail is* $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *(cf., Figure* 1*).*

The proof of the theorem was given by using an appropriate jet-transversality theorem [**11**]. When considering a global embedding $f : M \longrightarrow H^3_+(-1)$, one must also pay attention to the multilocal phenomena. So we must add the double point locus, the intersection of a regular surface and the cuspidaledge and the triple point to the list of local normal forms of the singular image of hyperbolic Gauss indicatrices of generic embeddings. These follow from the multi-jet version of the above mentioned jet-transversality theorem. We also studied in [**11**] the geometric meaning of the singularities of the hyperbolic Gauss indicatrices: Given a point $p_0 \in M$ and the lightlike vector $\boldsymbol{v}_0 = \boldsymbol{L}(p_0)$, we saw that the horosphere $HS(\boldsymbol{v}_0, -1) = HP(\boldsymbol{v}_0, -1) \cap H^3_+(-1)$ is tangent to $f(M)$ at $f(p_0)$. We called $HS(\boldsymbol{v}_0, -1)$ the *tangent horosphere* of $f(M)$ at $f(p_0)$. By definition, $\boldsymbol{L}(p_1) = \boldsymbol{L}(p_2)$ if and only if $HS(\boldsymbol{v}_1, -1) = HS(\boldsymbol{v}_2, -1)$ where $\boldsymbol{v}_i = \boldsymbol{L}(p_i)$. Analogously, a triple point of the hyperbolic Gauss indicatrix of $f : M \longrightarrow H^3_+(-1)$ corresponds to a tritangent horosphere. On the other hand one of the characterizations of the swallowtail point $p_0 \in M$ of $\boldsymbol{L}$ was the following (cf., [**11**]): For any open neighbourhood U of p_0 in M, there exist two distinct points $p_1, p_2 \in U \subset M$ such that both of p_1, p_2 are not horo-parabolic points and the tangent horospheres to $f(M)$ at $f(p_1), f(p_2)$ are equal.

Remember that a point $p \in M$ is called an horo-parabolic point provided $\mathscr{K}_h(p) = 0$ which is equivalent to the condition that $\widetilde{\mathscr{K}_h}(p) = 0$ (cf., Corollary 3.5).

Denote by $T(f)$ the number of tritangent horospheres and by $SW(f)$ the number of swallowtail points of a generic embedding $f : M \longrightarrow H^3_+(-1)$. We saw in [**11**] that the image of the hyperbolic Gauss indicatrix of a hypersurface can be interpreted as a wave front set in the theory of Legendrian singularities (cf., [**1**]). Therefore, we have the following formula as a particular case of the relation obtained in [**9**] for wave fronts:

$$\boldsymbol{\chi}(\boldsymbol{L}(M)) = \boldsymbol{\chi}(M) + \frac{1}{2} SW(f) + T(f).$$

This together with Theorem 1.1 lead to the following:

THEOREM 5.6. *Given a generic embedding $f : M \longrightarrow H^3_+(-1)$, the following relation holds*:

$$\boldsymbol{\chi}(\boldsymbol{L}(M)) = \frac{1}{2\pi} \int_M \widetilde{\mathscr{K}_h} d\mathfrak{a}_M + \frac{1}{2} SW(f) + T(f).$$

This theorem tells us that the Euler number of the image of the hyperbolic Gauss indicatrix of a generic embedding can be obtained in terms of the invariants of the horospherical differential geometry.

Finally, we remark that we can also apply other formulae involving the number of swallowtails and triple points on singular surfaces in a 3-manifolds (cf., [**15**], [**16**], [**18**]) to our situation in order to get further relations among invariants of the hyperbolic differential geometry.

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