THE HYPERPLANE SECTIONS OF
NORMAL VARIETIES

BY

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Introduction. The study of normal varieties was initiated by O. Zariski in the course of his investigation into the resolution of singularities of an algebraic variety. The intrinsic geometrical properties of normal varieties, uncovered in that investigation show, however, that we have in them a class of varieties demanding study for its own sake. Another very good, but in one sense probably more transient, reason for the study of normal varieties is that as yet we are not assured of the existence of a model free from singularities for any given field of algebraic functions, and in fact a greater knowledge of normal varieties may be a prerequisite for the resolution of singularities of arbitrary varieties.

Below we direct attention to the question whether, or to what extent, the hyperplane sections of a normal variety[2] are themselves normal. Quite generally, if P is a property of irreducible varieties, we may ask whether the hyperplane sections of a variety with property P share this property. In particular, we may raise this question for the property P of being irreducible. For curves, it is clear that the hyperplane sections will for the most part be reducible, so we shall confine the question to varieties of dimension $r \geq 2$. For varieties of dimension $r \geq 2$, it is still clear that not all the hyperplane sections will, in general, be irreducible: for example, consider a (suitable) cone; the hyperplane sections through the vertex will be reducible. This example leads us to reformulate the question. The hyperplanes of a projective space in themselves form a projective space, the dual space S_n' : we shall say that *almost all* hyperplanes have the property P, if the hyperplanes not having the property P lie on (though they need not fill out) a proper algebraic subvariety of S_n'. Even if now it turned out to be false that almost all hyperplane sections of an irreducible variety are themselves irreducible, we would not consider the original question on normal varieties as closed, but would reformulate it in local terms; it turns out, however, that they are irreducible almost always (Theorem 12), and therefore it is possible to deal with the

[2] A variety V/k is said to be normal if the quotient ring of each of its points is integrally closed. The variety V/k will be normal if and only if the ring of nonhomogeneous coördinates of V/k is integrally closed for every choice of the hyperplane at infinity.

problem in the large. The main result established below is that *almost all hyperplane sections of a normal variety are themselves normal.* Here we must add that the ground-field will be assumed (largely for expository purposes) infinite; the coefficients will be restricted to k, though the points of the variety will be allowed coördinates in a universal domain over k. The main result mentioned is first established under the assumption that the field of rational functions on the given variety V/k is separably generated (Theorem 7); this restriction is removed in §4 (Theorem 7'). We also formulate and prove it for k finite, under the restriction that V/k be absolutely irreducible (Theorem 16).

A normal variety is free of $(r-1)$-dimensional singularities [15; Theorem 11, p. 280](3), and so the first step is to prove that almost all hyperplane sections of a normal variety are free of $(r-2)$-dimensional singularities. Our statement and result (Theorem 2) are not merely geometric, but ideal-theoretic: it precedes the theorem on the irreducibility of almost all hyperplane sections, and contributes to its proof. (Actually, only the freedom from $(r-1)$-dimensional singularities, and not normality, is needed in order to prove that almost all hyperplane sections are free of $(r-2)$-dimensional singularities.)

Having proved that almost all hyperplane sections are irreducible and free of $(r-2)$-dimensional singularities, we are at the center of the problem. In proceeding, we consider only the affine part of space for various choices of the hyperplane at infinity, but this is clearly sufficient. We then characterize varieties normal in the affine space by two well known properties. The first is the absence of $(r-1)$-dimensional singularities, as mentioned above. The second is an arithmetic condition: namely, if $\xi_1, \cdots, \xi_n$ are the nonhomogeneous coördinates of the general point of an r-dimensional variety defined over a ground-field k, then $k[\xi_1, \cdots, \xi_n]$ is integrally closed, and the second condition referred to is the theorem to the effect that any principal ideal in such ring ($\neq(0)$) is unmixed $(r-1)$-dimensional (see [5; Criteria 1 and 2, p. 104]). These two conditions are sufficient (Theorem 3). To proceed, we prove first that the general hyperplane section of V/k, that is, the section by the hyperplane $u_0 + u_1 x_1 + \cdots + u_n x_n = 0$, where the u's are indeterminates and $k(u)$ is the new ground-field, is normal (Lemma 3). We then specialize the parameters u: $u \to a$, obtaining almost always an irreducible hyperplane section H_a free of $(r-2)$-dimensional singularities. If this section is not normal, then some element in its ring of nonhomogeneous coordinates is mixed, in fact any element in the conductor of that ring will be mixed. One therefore attempts to find an element $D(u, x)$, say, in $k(u)[x_1, \cdots, x_n]$ such that almost always $D(a, \eta)$ is unmixed in $k[\eta_1, \cdots, \eta_n]$ and in its conductor, where $(\eta_1, \cdots, \eta_n)$ is a general point of H_a, and in this one is successful.

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(3) Numbers in brackets refer to the bibliography at the end of the paper.

The proof is completed upon showing that almost always an unmixed ideal specializes to an unmixed ideal. Krull [6] has a theorem to this effect for one parameter, nor is the proof for n parameters essentially different: it is only a question of a correct formulation and of *repeating* the proofs—induction on n fails. Krull's proof, however, depends on results of G. Hermann [4] which are not explicitly specified, and we therefore thought it might prove useful to present explicitly exactly as much as we require for the above: this is done in the Appendix. We have also so formulated the results, at least the main one, so that they are applicable also if the ground-field k is finite (Theorem 7 of the Appendix).

The main question dealt with in this paper was raised in a joint paper by O. Zariski and H. T. Muhly[4], and was specifically called to my attention by Professor Zariski. I should like at this point to thank him for doing so, and also for various remarks he made to me in the course of this investigation.

1. **The singularities of the hyperplane sections.** Let k be an infinite field, to be taken as ground-field, and let P be a property which can be asserted or denied for each point $(a_0, a_1, \cdots, a_n)$, $a_i \in k$, either of an affine space A_{n+1} or of a projective space P_n over k. We shall say that P holds for almost all points of A_{n+1} or of P_n if it holds for all points $(a_0, a_1, \cdots, a_n)$, $a_i \in k$, except perhaps those lying on a proper algebraic subvariety of A_{n+1} or of P_n; that is, the set of points E for which the property P holds should contain the complement of a proper algebraic subvariety. In particular, if P holds almost always, then it holds for ar least one point.

In the case, for example, of hyperplanes $a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0$, $a_i \in k$, where the parameters $(a_0, \cdots, a_n)$ are clearly to be considered homogeneous, and where, say, P is a property which can be asserted or denied of each *point* $(a_0, \cdots, a_n)$, $a_i \in k$, of the (dual) *projective space P_n/k*, in proving that P holds for almost all hyperplanes, it is nonetheless clearly sufficient to represent the hyperplane by the point $(a_0, \cdots, a_n)$ in *affine space A_{n+1}* and prove that P holds for almost all points in A_{n+1}. (We agree that P does not hold for the point $(0, \cdots, 0)$.)

Note that when referring to "almost all points," we are considering only "rational" points, that is, points with coördinates in the ground-field k. On the other hand, the points of a variety V/k are allowed, in what follows, to have coördinates in a universal domain over k.

THEOREM 1. *Let V/k be an irreducible r-dimensional variety, and let $\mathfrak{A}$ be its homogeneous ideal*[5] *in $k[x_0, \cdots, x_n]$. Then for almost all hyperplanes $a_0x_0 + \cdots + a_nx_n = 0$, $a_i \in k$, it is true that any singular zero of the ideal $(\mathfrak{A}, a_0x_0 + \cdots + a_nx_n)$ is also singular for the ideal $\mathfrak{A}$. (The coördinates of the*

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[4] See the announcement entitled *Hilbert's characteristic function and the arithmetic genus of an algebraic variety* in Bull. Amer. Math. Soc. vol. 54 (1948) p. 1077.

[5] The same proof with slight modifications holds for an arbitrary homogeneous ideal $\mathfrak{A}$.

zero are not restricted to k.)

Proof. Let $f_1, \cdots, f_m$ be a basis for the ideal $\mathfrak{A}$ and let $(f_1, \cdots, f_m)$ be the mixed Jacobian matrix [19] for $(f_1, \cdots, f_m)$. We recall the definition of this matrix. In the case of characteristic 0, it is the classical Jacobian matrix $\|\partial f_i / \partial x_j\|$; we shall take i as the column index. In the case of characteristic $p \neq 0$, this matrix is augmented with derivatives of the f_i with respect to certain parameters occurring in the coefficients of the f_i. Namely, let k_1 be the field obtained by adjoining to k^p—k, the ground-field—the various coefficients of the f_i; and let $z_1, \cdots, z_s$ be a p-independent basis of k_1 / k^p, that is, any element in k can be written uniquely as a polynomial in the z_i, of degree less than p in each z_i, with coefficients in k^p. The mixed Jacobian matrix is

$$
\left\| \begin{array}{c} \partial f_i / \partial x_j \\ \partial f_i / \partial z_k \end{array} \right\|.
$$

Necessary and sufficient that the zero P of the ideal $\mathfrak{A}$ be a simple zero of $\mathfrak{A}$ is that the rank of this matrix evaluated at P be $n-r$ [19; Theorem 11, p. 39]. One sees then immediately that a point P is a simple zero of $(f_1, \cdots, f_m)$ if and only if it is a simple zero of $(x_0 f_1, \cdots, x_n f_1, f_2, \cdots, f_m)$; that is, we may replace f_1 by $x_0 f_1, \cdots, x_n f_1$ (changing the ideal $\mathfrak{A}$ at the same time). Thus we may assume, in investigating the simplicity of the zeroes of $\mathfrak{A}$, that $\mathfrak{A}$ has a basis of forms of like degree, say δ.

In writing down the mixed Jacobian matrix for $(\mathfrak{A}, l)$, where $l = a_0 x_0 + \cdots + a_n x_n$, we have a different field k_1 to take into consideration: this new field k_1 contains the old, and so we keep the parameters $z_1, \cdots, z_s$ and adjoin the new ones. As a consequence, the mixed Jacobian matrix for $(\mathfrak{A}, l)$ contains that for $\mathfrak{A}$ as a submatrix. One sees that here also $\mathfrak{A}$ may be assumed to have a basis of forms of like degree δ.

Let J be the mixed Jacobian matrix of $\mathfrak{A}$, J' that of $(\mathfrak{A}, l)$, constructed as above. Let $l = a_0 x_0 + \cdots + a_n x_n$, and we shall suppose that $l = 0$ does not contain the variety V; so the $(a_0, \cdots, a_n)$ vary over the dual space of $(x_0, \cdots, x_n)$ except possibly on a proper subvariety. The ideal $(\mathfrak{A}, l)$ is then $(r-1)$-dimensional, in fact, each component of $(\mathfrak{A}, l)$ is $(r-1)$-dimensional. A zero P of $(\mathfrak{A}, l)$ is a singular zero if and only if rank $J'_P < n-r+1$. Consider now the points (hyperplanes) a which carry a singular point $P = P_a$ which is simple for $\mathfrak{A}$. Then rank $J_P = n-r$, and since J is a submatrix of J' we have:

$$
\text{rank } J_P = \text{rank } J'_P = n - r.
$$

Let U be the *set* of points (hyperplanes) a for which this condition obtains. We want to show that the $a \in U$ are subject to proper algebraic conditions.

Let $\xi_0, \cdots, \xi_n$ be the homogeneous coördinates of a general point of V/k; d.t. $k(\xi_0, \cdots, \xi_n)/k = r+1$. Consider some $n-r$ of the f_i (degree

$f_i = \delta$), say $f_1, \cdots, f_{n-r}$ (necessarily $m \geq n - r$). Let $\tau_1, \cdots, \tau_{n-r}$ be such that just one of them, say τ_1, equals 1, while the others, $\tau_2, \cdots, \tau_{n-r}$ are indeterminates to be adjoined to $k(\xi_0, \cdots, \xi_n)$. In the field $k(\xi_0, \cdots, \xi_n, \tau_2, \cdots, \tau_{n-r})$ consider the quantities

$$(1) \qquad \zeta_i = \frac{\partial f_1}{\partial \xi_i} + \tau_2 \frac{\partial f_2}{\partial \xi_i} + \cdots + \tau_{n-r} \frac{\partial f_{n-r}}{\partial \xi_i}, \qquad i = 0, 1, \cdots, n.$$

Then $(\zeta_0, \cdots, \zeta_n)$ are the homogeneous coördinates of the general point over k of some proper subvariety of projective n-space (or are all equal to 0). In fact, since each $\partial f_j / \partial \xi_k$ ($\neq 0$) is homogeneous, of degree $\delta - 1$ in the ξ_i, it is clear that every polynomial relation over k between the ζ's is a consequence (sum) of homogeneous relations. Moreover d.t. $k(\zeta_0, \cdots, \zeta_n)/k$ $\leq$ d.t. $k(\xi_0, \cdots, \xi_n, \tau_2, \cdots, \tau_{n-r})/k = r + 1 + n - r - 1 = n$, so the variety defined by (ξ) over k is not the whole projective space (it is the empty set if all the $\xi_i = 0$). In this way we get a number of proper algebraic subvarieties, that is, according to the various choices of $n - r$ of the f_i and according to which partials in (1) get the indeterminate coefficients. Let U' be the total variety so obtained. We assert that $U \subseteq U'$.

In fact, let $(a) \in U$, and let $P = P_a$ be such that rank $J_p =$ rank $J_p' = n - r$. Then also for some $n - r$ of the f_i, say $f_1, \cdots, f_{n-r}$, we shall have:

$$\text{rank} \begin{Vmatrix} \partial f_1/\partial x_0 & & \partial f_{n-r}/\partial x_0 \\ \cdot & \cdots & \cdot \\ \cdot & & \cdot \\ \partial f_1/\partial x_n & & \partial f_{n-r}/\partial x_n \\ & * & \end{Vmatrix}_P = \text{rank} \begin{Vmatrix} \partial f_1/\partial x_0 & & \partial f_{n-r}/\partial x_0 & a_0 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \partial f_1/\partial x_n & & \partial f_{n-r}/\partial x_n & a_n \\ & * & & \end{Vmatrix}_P = n - r,$$

where the two $*$ stand for various partials which do not enter explicitly into the argument. Hence for some $t_1', \cdots, t_{n-r}'$ we shall have

$$a_i = t_1' \left(\frac{\partial f_1}{\partial x_i} \right)_P + \cdots + t_{n-r}' \left(\frac{\partial f_{n-r}}{\partial x_i} \right)_P, \qquad i = 0, 1, \cdots, n.$$

Since not all the $a_i = 0$, also some t_j', say t_1', $\neq 0$. For $b_i = a_i / t_1'$, we shall then have:

$$(2) \qquad b_i = \left(\frac{\partial f_1}{\partial x_i} \right)_P + t_2 \left(\frac{\partial f_2}{\partial x_i} \right)_P + \cdots + t_{n-r} \left(\frac{\partial f_{n-r}}{\partial x_i} \right)_P, \qquad i = 0, 1, \cdots, n,$$

where $t_i = t_i' / t_1'$. Now it is clear that $(b_0, \cdots, b_n)$ is a specialization over k of $(\zeta_0, \cdots, \zeta_n)$. Since $(a_0, \cdots, a_n)$ is a multiple of $(b_0, \cdots, b_n)$, and since $(\zeta_0, \cdots, \zeta_n)$ is a homogeneous general point, also $(a_0, \cdots, a_n)$ is a specialization over k of $(\zeta_0, \cdots, \zeta_n)$, and hence $(a_0, \cdots, a_n) \in U'$, q.e.d.

COROLLARY 1. *The ideal* $(\mathfrak{A}, a_0 x_0 + \cdots + a_n x_n)$ *is, except possibly for an*

irrelevant component, almost always the intersection([6]) *of prime ideals of dimension* $r-1$.

Proof. By choosing successively the hyperplanes $x_i = 0$ as plane at infinity, one sees that it is sufficient to consider the question in affine space, or non-homogeneously. We suppose then x_0 replaced by 1, and have to show that $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ is almost always the intersection of prime ideals of dimension $r-1$. If V/k is normal, and for the most part this is the case we are dealing with, the unmixed character of $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ is nothing but a special case of the principal ideal theorem: however, by a theorem we shall establish later, an unmixed ideal specializes almost always to an unmixed ideal of the same dimension, whence the unmixed character of $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ follows almost always from the fact that $(\mathfrak{A}, u_0 + u_1 x_1 + \cdots + u_n x_n)$, where the u's are indeterminates and $k(u)$ is the ground-field, is prime. If, now, some $(r-1)$-dimensional primary component of $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ is not prime, then the general zero of the corresponding prime ideal $\mathfrak{P}$ is nonsimple for $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$, hence is nonsimple for $\mathfrak{A}$ also; thus $\mathfrak{P}$ represents an $(r-1)$-dimensional singular subvariety of $V = V(\mathfrak{A})$. Since V carries only a finite number of $(r-1)$-dimensional irreducible singular subvarieties, the above situation will not obtain if $a_0 + a_1 x_1 + \cdots + a_n x_n = 0$ contains none of them entirely, and this requirement places proper algebraic inequalities on the a_i.

In the following corollary, k may be finite; k', however, shall be infinite. The term "almost all" then takes on an extended meaning, namely, the coördinates a_i are taken from k', but the exceptional points $(a_0, \cdots, a_n)$ are still to lie on an algebraic variety over k.

COROLLARY 2. *If k' is separably generated over k, then for almost all hyperplanes $a_0 x_0 + \cdots + a_n x_n = 0$, $a_i \in k'$, it is true that any singular zero of the $k'[x]$-ideal $(f_1, \cdots, f_m, a_0 x_0 + \cdots + a_n x_n)$ is also singular for the $k[x]$-ideal $\mathfrak{A}$ $(= (f_1, \cdots, f_m))$.*

Proof. The field k'^p is also separably generated over k^p, so by a result of S. MacLane [2] (see also Proposition 19 quoted in footnote 9 as well as a remark on the definition of separable generation by Chevalley [1; p. 68]), $z_1, \cdots, z_s$, which are p-independent over k^p, remain such over k'^p, so that these may still be retained as the parameters in the computation of the Jacobians. Attention is also to be called to the fact that every component of $V(f_1, \cdots, f_m, a_0 + a_1 x_1 + \cdots + a_n x_n)/k'$ is $(r-1)$-dimensional almost always. Thus the argument for the theorem also holds for the corollary.

THEOREM 2. *If V/k is free of $(r-1)$-dimensional singularities, then almost always the hyperplane section of V by $a_0 + a_1 x_1 + \cdots + a_n x_n = 0$, $a_i \in k$, is free*

([6]) It is even almost always prime by Theorem 12 below.

of $(r-2)$-*dimensional singularities. Ideal-theoretically*: *if* $\mathfrak{A}$ *is free of* $(r-1)$-*dimensional singular zeroes, then almost always* $(\mathfrak{A}, a_0+a_1x_1+ \cdots +a_nx_n)$ *is free of* $(r-2)$-*dimensional singular zeroes, where* $\mathfrak{A}$ *is the prime ideal of* V/k *in* $k[x_1, \cdots, x_n]$.

Proof. This theorem is a corollary of Theorem 1.

2. The main theorem. Throughout this section we shall be assuming that almost all the hyperplane sections of the irreducible variety V/k are themselves irreducible, given that dim $V/k \geq 2$, postponing the proof to the next section. We shall merely remark at this point that, although the statement refers to projective space, it is sufficient to give the proof for affine space. For if we suppose the theorem true in the affine space, projectively this comes to saying that almost all hyperplane sections have just one component outside of a certain hyperplane H_0. Selecting a second hyperplane H_1 as hyperplane at infinity, we may also conclude that almost all hyperplane sections have just one component outside $H_0 \cap H_1$. Selecting successively $n+1$ hyperplanes $H_0, H_1, \cdots, H_n$ with empty intersection, we conclude that almost all hyperplane sections have just one component outside the empty set $H_0 \cap H_1 \cap \cdots \cap H_n$, that is, they are irreducible. In a similar way, although our assertion that almost all hyperplane sections are normal refers to projective space, we may confine ourselves to the affine space. For to say that a variety is normal in the affine space is equivalent with saying that it is locally normal at every point at finite distance. From the statement for the affine space we can then deduce that almost all hyperplane sections are locally normal except on the empty set $H_0 \cap H_1 \cap \cdots \cap H_n$, that is, they are locally normal everywhere. In the following three theorems, then, V/k will refer to an algebraic variety considered in the affine space; let $\xi_1, \cdots, \xi_n$ be nonhomogeneous coördinates of the general point of V/k and let $R = k[\xi_1, \cdots, \xi_n]$.

THEOREM 3. *The* r-*dimensional* (*affine*) *variety* V/k *is normal if and only if*:

(1) V/k *is free of* $(r-1)$ *singularities*

and

(2) *every principal ideal in the ring* R *of nonhomogeneous coördinates on* V/k *is unmixed.*

Proof. The necessity of these conditions is well known. Conversely, from (1) we see that the $(r-1)$-dimensional primary ideals belonging to any proper principal ideal (a) are symbolic powers of minimal prime ideals, so by (2), (a) is the intersection of these symbolic powers. It then follows at once that R is the intersection of the quotient rings of R with respect to its minimal prime ideals; and since these are valuation rings, R is integrally closed, q.e.d.

Suppose now that V/k is free of $(r-1)$-dimensional singularities. Under this circumstance, if V/k is not normal, then certainly there exist *mixed*

principal ideals in R. If α is an integer not in R, and $\alpha = b/c$, b, $c \in R$, then the above argument shows that (c) must be mixed otherwise $c \mid b$ in R. Since for c we may select any element of the conductor[7], we have the following:

THEOREM 4. *If V/k is free of $(r-1)$-dimensional singularities, but is not normal, then every element $(\neq 0)$ of the conductor of R generates a mixed principal ideal.*

Let $\mathfrak{A}$ be an r-dimensional prime ideal in the polynomial ring $k[x_1, \cdots, x_n]$, and let $z_i = u_{i1}x_1 + \cdots + u_{in}x_n$, $i = 1, \cdots, r+1$, be $r+1$ linear forms in the x_i with indeterminate coefficients. The ideal $k(u)[x] \cdot \mathfrak{A}$ is also r-dimensional and prime and $k(u)[x] \cdot \mathfrak{A} \cap k(u)[z]$ is clearly a prime principal ideal $(E(z_1, \cdots, z_{r+1}, u))$. We may suppose $E(z_1, \cdots, z_{r+1}, u)$ normalized so as to be a polynomial in the u_{ij}, and primitive in them, so that E is defined to within a factor in k.

DEFINITION. $E(z_1, \cdots, z_{r+1}, u)$ is called the elementary divisor form or ground-form of $\mathfrak{A}$.

The ground-form is also defined more generally for unmixed ideals; see further remarks below (preceding Theorem 11).

LEMMA 0. *Let $k[\xi_1, \cdots, \xi_n]$ be a finite integral domain, and let u be an indeterminate. Then $k[\xi]$ is integrally closed if and only if $k(u)[\xi]$ is integrally closed.*

Proof. If $\alpha(u, \xi) \in k(u, \xi)$ is integral over $k(u)[\xi]$, then for some $d(u) \in k[u]$, $d(u)\alpha(u, \xi)$ is integral over $k(\xi)[u]$, hence in $k(\xi)[u]$, that is,

$$d(u)\alpha(u, \xi) = a_0(\xi) + a_1(\xi)u + \cdots + a_m(\xi)u^m, \ a_i(\xi) \in k(\xi).$$

Replacing u by $m+1$ values λ_i from the algebraic closure $\bar{k}$ of k, we see that $a_0(\xi) + a_1(\xi)\lambda_i + \cdots + a_m(\xi)\lambda_i^m$ is integral over $\bar{k}(\xi)$, whence each $a_i(\xi)$ is integral over $\bar{k}(\xi)$, hence also integral over $k(\xi)$. Assuming $k[\xi]$ to be integrally closed, we have $a_i(\xi) \in k[\xi]$, whence $\alpha(u, \xi) \in k(u)[\xi]$, that is, $k(u)[\xi]$ is integrally closed. Conversely, if $\alpha(\xi) \in k(\xi)$ is integral over $k[\xi]$, then, assuming $k(u)[\xi]$ to be integrally closed, we have $\alpha(\xi) \in k(u)[\xi] \cap k(\xi) = k[\xi]$, q.e.d.

THEOREM 5. *Let V/k be an irreducible r-dimensional variety (considered in*

[7] The following simple proof that the integral closure of a finite integral domain $R = k[\xi_1, \cdots, \xi_n]$ is contained in, or is, a finite R-module has apparently been overlooked; the known result is due to F. K. Schmidt [11; p. 445]. Let $\bar{k}$ be the algebraic closure of k; then $\bar{k}[\xi_1, \cdots, \xi_n]$ is certainly separably generated over $\bar{k}$. Moreover only a finite number of coefficients from $\bar{k}$ are required in bringing this separable generation to expression. Hence already for a finite extension k' of k we have that $k'(\xi)$ is separably generated over k'; for a similar reason we may suppose $k'(\xi)$ to be separable over $k'(\xi_1, \cdots, \xi_r)$ and $k'[\xi]$ to be integral over $k'[\xi_1, \cdots, \xi_r]$, where $r = $ d.t. $k(\xi)/k$. Hence [13; §99] the integral closure of $R' = k'[\xi]$ is contained in a finite R'-module, hence also in a finite R-module, since k'/k is finite. A fortiori, the integral closure of R is contained in that module.

affine space) free of $(r-1)$-dimensional singularities, and let $E(z_1, \cdots, z_{r+1}, u)$ be the ground-form of its prime ideal $\mathfrak{p}$. Suppose moreover that $k(\xi_1, \cdots, \xi_n)$, where $k[\xi] = k[x]/\mathfrak{p}$, is separably generated. Then V/k is normal if and only if $(\mathfrak{p}, \partial E/\partial z_{r+1})$ is unmixed(8).

Proof. We consider $V/k(u)$ and note that V/k is normal if and only if $V/k(u)$ is normal (Lemma 0). Also if V/k is free of $(r-1)$-dimensional singularities, then so is $V/k(u)$ [17; Lemma 2b, p. 132]. Hence we may carry out our considerations over $k(u)$. Let us now pass to $R[V]$, the residue class ring mod $\mathfrak{p}$, and designate residues with bars. It is this ring which is to be examined for integral closure. Thus we are asserting that $R[V]$ is integrally closed if and only if $(\partial E/\partial z_{r+1})$ is unmixed. Now it is well known [3; (1.1), p. 297](9) that $\partial \overline{E}/\partial \bar{z}_{r+1}$ is in the conductor of $R[V]$ (the way the known statement referred to is usually formulated is that $\partial \overline{E}/\partial \bar{z}_{r+1}$ is in the conductor of $k(u)[\bar{z}_1, \cdots, \bar{z}_{r+1}]$, but since $k(u)[\bar{\xi}_1, \cdots, \bar{\xi}_n]$ is integral over $k(u)[\bar{z}_1, \cdots, \bar{z}_{r+1}]$, and since $\bar{z}_{r+1}$ is a primitive element of $k(u)(\bar{\xi})$ over $k(u)(\bar{z}_1, \cdots, \bar{z}_r)$, the element $\partial \overline{E}/\partial \bar{z}_{r+1}$ is also in the conductor of $R[V]$). The present theorem now follows at once from the previous one, q.e.d.

Before proceeding, we have to report briefly on a theorem of Krull's that we propose to apply, particularly since we need a somewhat stronger formulation of the theorem. Krull considers a polynomial ring $R = k[x_1, \cdots, x_n]$ over an arbitrary infinite field k, and the ring $R_\tau = k(\tau)[x_1, \cdots, x_n]$ over the field $k(\tau)$ obtained from k by the adjunction of a single indeterminate. Let $\mathfrak{A}$ be an ideal in R_τ, and make the substitution $\tau \to a$, $a \in k$, into all the polynomials in $\mathfrak{A}$ for which the result of substitution is not indeterminate. These polynomials generate an ideal $\overline{\mathfrak{A}}$ (which we shall also write as $\mathfrak{A}^-$), and Krull studies the relation between the ideals $\mathfrak{A}$ and $\mathfrak{A}^-$ (note that $\mathfrak{A}^- = \{g(x, a) \mid g(x, \tau) \in \mathfrak{A} \cap k[\tau, x]\}$). He proves that for "almost all" $a \in k$, that is, for at most a finite number of exceptions on a, one has $(\mathfrak{A}+\mathfrak{B})^- = \mathfrak{A}^- + \mathfrak{B}^-$, $(\mathfrak{A}\mathfrak{B})^- = \mathfrak{A}^- \cdot \mathfrak{B}^-$, $(\mathfrak{A}\cap\mathfrak{B})^- = \mathfrak{A}^- \cap \mathfrak{B}^-$, $(\mathfrak{A}:\mathfrak{B})^- = \mathfrak{A}^- : \mathfrak{B}^-$ and that if $\mathfrak{A}$ is unmixed r-dimensional, then $\mathfrak{A}^-$ is also unmixed r-dimensional almost always; also that if $\mathfrak{A} = (f_1, \cdots, f_s)$, then almost always $\mathfrak{A}^- = (f_1^-, \cdots, f_s^-)$. A perusal of his proofs shows that his theorems hold word for word if τ represents a finite set of indeterminates $\tau = (\tau_1, \cdots, \tau_m)$, and if the phrase "almost always" has the meaning which we are at present assigning it, namely, "except possibly for the points $(a_1, \cdots, a_m)$, $a_i \in k$, lying on a proper algebraic subvariety of the affine m-space over k." Also we remark that his theorems, especially the ones mentioned, also hold if R and R_τ are finite integral do-

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(8) A question of notation is involved here. Quite generally, if R is a subring of S and $\mathfrak{A}$ is an R-ideal, then we shall frequently designate with the same symbol $\mathfrak{A}$ the extended ideal $S \cdot \mathfrak{A}$.

(9) By a result of S. MacLane ([2]; see also the proof of Proposition 19, Chap. I, p. 17 in [14]) if $k(\xi_1, \cdots, \xi_n)$ is separably generated over k and d.t. $k(\xi)/k = r$, then for some r of the ξ_i, say $\xi_1, \cdots, \xi_r$, $k(\xi)$ is separable over $k(\xi_1, \cdots, \xi_r)$.

mains, and not merely polynomial rings, since in any case they are homomorphic images of polynomial rings, and the results can be carried over directly from the polynomial rings to R and R_r. Explicitly, we shall make use of the following slight generalization of Krull's theorem:

THEOREM OF KRULL. *Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over an infinite field k, and let $R_r = k(\tau_1, \cdots, \tau_m)[x_1, \cdots, x_n]$, where the τ_i are indeterminates. Let $\mathfrak{A}$ be an ideal in R_r, and let $\overline{\mathfrak{A}} = \{g(x, a) \mid g(x, \tau) \in k[\tau, x] \cap \mathfrak{A}\}$, where $a = (a_1, \cdots, a_n)$ and $a_i \in k$. Then almost always, if $\mathfrak{A}$ is unmixed r-dimensional, then also $\overline{\mathfrak{A}}$ is unmixed r-dimensional.*

We give a proof of this theorem in the Appendix.

We now seek the behavior of the ground-form under specialization of the parameters (indeterminates) $\tau = (\tau_1, \cdots, \tau_m)$. Theorem 6 gives the result, for which the next lemma prepares the way.

LEMMA 1. *Let $\mathfrak{A}$ be an r-dimensional prime ideal in $k(\tau)[x_1, \cdots, x_n]$ which is such that $\overline{\mathfrak{A}}$ is almost always prime and r-dimensional. If $x_1, \cdots, x_r$ are algebraically independent (over $k(\tau)$) mod $\mathfrak{A}$, then almost always they are also algebraically independent (over k) mod $\overline{\mathfrak{A}}$ and $[k(\tau)(\xi):k(\tau)(\xi_1, \cdots, \xi_r)]$ $= [k(\xi'):k(\xi', \cdots, \xi_r')]$, where $k(\tau)[\xi] = k(\tau)[x]/\mathfrak{A}$ and $k[\xi'] = k[x]/\overline{\mathfrak{A}}$.*

Proof. Let $F_i(\xi_1, \cdots, \xi_{r+i-1}, x) = 0$ be the field equation for ξ_{r+1} over $k(\tau)(\xi_1, \cdots, \xi_{r+i-1})$, where $F_i(x_1, \cdots, x_{r+i-1}, x_{r+i}) \in k(\tau)[x_1, \cdots, x_{r+1}]$, and where we may further assume that the coefficient of the highest power of x_{r+i} is in $k[x_1, \cdots, x_r]$. If upon specialization $\deg_{x_{r+i}} F_i(x_1, \cdots, x_{r+i})$ $= \deg_{x_{r+i}} \overline{F}_i(x_1, \cdots, x_{r+i})$ then x_{r+i} will be algebraic over $k[x_1, \cdots, x_{r+i-1}]$ mod $\overline{\mathfrak{A}}$, $i = 1, \cdots, n-r$, and so $x_1, \cdots, x_r$ will be algebraically independent mod $\overline{\mathfrak{A}}$. It is well known that $k(\tau)(x_1, \cdots, x_r)[x_{r+1}, \cdots, x_n]\mathfrak{A} = (F_1, \cdots, F_{n-r})$ [18; Lemma 9, p. 541]. Moreover, if $(F_1, \cdots, F_{n-r}, G_1, \cdots, G_s)$ is a basis of $\mathfrak{A}$, then there exists a polynomial $P(x_1, \cdots, x_r) \in k(\tau)[x_1, \cdots, x_r]$, $P \neq 0$, such that $P(x_1, \cdots, x_r) G_i \in k(\tau)[x] \cdot (F_1, \cdots, F_{n-r})$, $i = 1, \cdots, s$. Almost always we will have $\overline{\mathfrak{A}} = (\overline{F}_1, \cdots, \overline{G}_s)$, $\deg_{x_{r+1}} F_i = \deg_{x_{r+1}} \overline{F}_i$, $\overline{P} \neq 0$, $\overline{P}\overline{G}_i \in (\overline{F}_1, \cdots, \overline{F}_{n-r})$, and $\overline{\mathfrak{A}}$ prime, r-dimensional[10]. Hence $k(x_1, \cdots, x_r)$ $\cdot [x_{r+1}, \cdots, x_n] \cdot \overline{\mathfrak{A}} = (\overline{F}_1, \cdots, \overline{F}_{n-r})$. The present lemma now follows quite simply upon applying the following lemma.

LEMMA 2. *If $\mathfrak{p} = (H_1(x_1), H_2(x_1, x_2), \cdots, H_n(x_1, \cdots, x_n))$ is a prime 0-dimensional ideal and $H_i(x_1, \cdots, x_i)$ is monic in x_i, then $[k(\xi):k] = \prod \deg_{x_i} H_i$, where $k[\xi] = k[x]/\mathfrak{p}$.*

Proof. First we prove that $\mathfrak{p} \cap k[x_1, \cdots, x_{n-1}] = (H_1, \cdots, H_{n-1})$. For let $A(x_1, \cdots, x_{n-1}) = B_1(x_1, \cdots, x_n)H_1 + \cdots + B_{n-1}(x_1, \cdots, x_n)H_{n-1} + B_n(x_1, \cdots, x_n)H_n$. One sees, comparing coefficients, that the coefficient of

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[10] See Theorem 1 of the Appendix.

the highest power of x_n in B_n must be in $(H_1, \cdots, H_{n-1})$, hence may be removed and distributed to the first $(n-1)$ terms. In this way we may suppose $B_n = 0$, which proves the statement first made. Moreover, in $k[x_1, \cdots, x_{n-1}]$, the ideal $(H_1, \cdots, H_{n-1})$ is clearly prime and 0-dimensional. By induction $\mathfrak{p} \cap k[x_1, \cdots, x_i] = (H_1, \cdots, H_i)$ is prime and 0-dimensional. Let $A(x_1, \cdots, x_i) \in \mathfrak{p}$ be monic in x_i, and let $(\alpha_1, \cdots, \alpha_n)$ be a zero of $\mathfrak{p}$. We have $A(x_1, \cdots, x_i) = B_1 H_1 + \cdots + B_i H_i$, whence $A(\alpha_1, \cdots, \alpha_{i-1}, x_i) = B_i(\alpha_1, \cdots, \alpha_{i-1}, x_i) H_i(\alpha_1, \cdots, \alpha_{i-1}, x_i)$, and it follows that $\deg_{x_i} A \geqq \deg_{x_i} H_i$; the statement in the lemma is an immediate consequence.

THEOREM 6. *Let $\mathfrak{A}$ be an r-dimensional prime ideal in $k(\tau)[x_1, \cdots, x_n]$ which is such that $\overline{\mathfrak{A}}$ is almost always prime and r-dimensional. Assume moreover that the quotient-field of $k(\tau)[x]/\mathfrak{A}$ is separably generated. Then almost always the ground-form of $\mathfrak{A}$ specializes to the ground-form of $\overline{\mathfrak{A}}$, and almost always the quotient-field of $k[x]/\overline{\mathfrak{A}}$ is separably generated.*

Proof. Let $E(z_1, \cdots, z_{r+1}, \tau, u)$ be the ground-form of $\mathfrak{A}$, and let $\tau \rightarrow a$ give rise to the r-dimensional prime ideal $\overline{\mathfrak{A}}$. Since a separable equation goes almost always into a separable equation, it is clear that almost always the quotient-field of $k[x]/\overline{\mathfrak{A}}$ is also separably generated. Now it is well known[9] that $[k(u, \tau)(\xi):k(u, \tau)(z_1(\xi), \cdots, z_r(\xi))] = \deg_{z_{r+1}} E(z_1, \cdots, z_{r+1}, \tau, u)$ in the case that $k(u, \tau)(\xi)$ is separably generated, where $k(u, \tau)[\xi] = k(u, \tau)[x]/\mathfrak{A}$; and that case is present. A like remark holds for $\overline{\mathfrak{A}}$. By Lemma 1, the degree of the ground-form of $\overline{\mathfrak{A}}$ in z_{r+1} equals the degree of the ground-form of $\mathfrak{A}$ in z_{r+1}, and since clearly the ground-form of $\overline{\mathfrak{A}}$ is a factor of $E(z_1, \cdots, z_{r+1}, a, u)$, we see that it must actually equal $E(z_1, \cdots, z_{r+1}, a, u)$ almost always.

LEMMA 3. *Let $\mathfrak{A}$ be the prime ideal in $k[x_1, \cdots, x_n]$ of the normal variety V/k. Then $(\mathfrak{A}, u_0 + u_1 x_1 + \cdots + u_n x_n)$, where the u_i are indeterminates, is the prime ideal[10] of a normal variety over $k(u)$.*

Proof. Let $\xi_1, \cdots, \xi_n$ be the nonhomogeneous coördinates of a general point of V/k. These also define a variety over $k(u_0, u_1, \cdots, u_n)$, where the u's are indeterminates. In the ring $k(u)[\xi_1, \cdots, \xi_n]$, one verifies immediately that (l), where $l = u_0 + u_1 \xi_1 + \cdots + u_n \xi_n$, defines a prime ideal, that is, the general hyperplane section is irreducible; so $(\mathfrak{A}, u_0 + u_1 x_1 + \cdots + u_n x_n)$ is the prime ideal of this section. The field of rational functions on this hyperplane section is the quotient field of $k(u)[\xi]/(l) \cong k(u)[\eta]$. Observe that $(l) \cap k(u_1, \cdots, u_n)[\xi] = (0)$. Hence $k(u)[\xi]/(l) \cong k(u_1, \cdots, u_n, \bar{u}_0)[\xi]$, where $-\bar{u}_0 = u_1 \xi_1 + \cdots + u_n \xi_n$. Thus $k(u_1, \cdots, u_n, u_0)[\eta] \cong k(u_1, \cdots, u_n, \bar{u}_0)[\xi]$, and to study (η) over $k(u_1, \cdots, u_n, u_0)$ is the same as to study (ξ) over $k(u_1, \cdots, u_n, \bar{u}_0)$. By Lemma 0, $k(u_1, \cdots, u_n)[\xi]$ is integrally closed, and it is now immediate that also $k(u_1, \cdots, u_n, \bar{u}_0)[\xi]$ is integrally closed, q.e.d.

We shall call a prime ideal $\mathfrak{A}$ in the polynomial ring $k[x_1, \cdots, x_n]$ a

separable prime ideal if $k(\xi)$ is separably generated, where $k[\xi] \cong k[x]/\mathfrak{A}$.

LEMMA 3.5. *If $\mathfrak{A}$ is a separable prime ideal in $k[x]$, then $(\mathfrak{A}, \tau_0 + \tau_1 x_1 + \cdots + \tau_n x_n)$ is a separable prime ideal in $k(\tau)[x]$, where the τ's are indeterminates.*

Proof. In the previous lemma we have seen that the quotient field of $k(\tau)[x]/(\mathfrak{A}, \tau_0 + \tau_1 x_1 + \cdots + \tau_n x_n)$ is isomorphic to $k(\tau_1, \cdots, \tau_n, \xi_1, \cdots, \xi_n)$, where $k(\xi) = k[x]/\mathfrak{A}$, whence the present lemma is immediate.

THEOREM 7. *Let V/k be an r-dimensional normal variety over the infinite ground-field k, $r \geqq 2$, and let the field of rational functions on V/k be separably generated. Then almost all sections of V/k by hyperplanes $a_0 x_0 + \cdots + a_n x_n = 0$, $a_i \in k$, are also (irreducible and) normal.*

Proof. This is now an immediate consequence of the preceding lemmas and theorems. Namely, if $\mathfrak{A}$ is the prime ideal in $k[x_1, \cdots, x_n]$ of V/k, then $\mathfrak{p} = (\mathfrak{A}, \tau_0 + \tau_1 x_1 + \cdots + \tau_n x_n)$ is also a separable prime ideal (Lemma 3.5), and defines an irreducible and normal variety over $k(\tau)$ (Lemma 3). Therefore if $G(z_1, \cdots, z_r, \tau, u)$ is the ground-form of $\mathfrak{p}$, then the $k(\tau, u)[x]$-ideal $(\mathfrak{p}, \partial G/\partial z_r)$ is unmixed (Theorem 5). Almost always $(\bar{\mathfrak{p}}, \partial \bar{G}/\partial \bar{z}_r)$ is also unmixed, where the bar indicates a specialization of τ (Theorem of Krull). Almost always $\bar{\mathfrak{p}}$ is prime (Theorem 12; also Theorem 1 of the Appendix) and separable (Theorem 6), $\bar{G}$ is the ground-form of $\bar{\mathfrak{p}}$ (Theorem 6), $V(\bar{\mathfrak{p}})/k$ is free of $(r-2)$-dimensional singularities (Theorem 2), whence the unmixed character of $(\bar{\mathfrak{p}}, \partial \bar{G}/\partial \bar{z}_r)$ yields the normality of $V(\bar{\mathfrak{p}})/k$ (Theorem 5), q.e.d.

3. The irreducibility of the hyerplane sections. In this section we shall prove that almost all hyperplane sections of an algebraic variety V/k are irreducible. We first add the hypothesis that V/k is quasi-absolutely irreducible (see definition below), and then remove it.

The following lemma is well known [1; Proposition 6a, p. 72]:

LEMMA 4. *If the field k is algebraically closed in the field Σ and u is an indeterminate, then $k(u)$ is algebraically closed in $\Sigma(u)$.*

DEFINITION. Let the field k be a subfield of the field Σ. Then k is said to to be quasi-algebraically closed in Σ, if every quantity in Σ which is algebraic over k is purely inseparable over k.

LEMMA 5. *If k is quasi-algebraically (q.a.) closed in Σ and u is an indeterminate, then $k(u)$ is q.a. closed in $\Sigma(u)$.*

Proof. Let $\alpha(u) \in \Sigma(u)$ be separably algebraic over $k(u)$. Let k' be the algebraic closure of k in Σ: k' is purely inseparable over k, and k' is algebraically closed in Σ. By the previous lemma $\alpha(u) \in k'(u)$. Now $k'(u)$ is purely inseparable over $k(u)$. So $\alpha(u)$ is separable and purely inseparable over $k(u)$, whence $\alpha(u) \in k(u)$.

THEOREM 8. *Let k be quasi-algebraically closed in a field $\Sigma = k\ (\xi_1, \cdots \xi_m)$, of algebraic functions, and let degree of transcendence of $\Sigma/k \geqq 2$, say, ξ_1, ξ_2 algebraically independent over k, then except for a finite number of $c \in k$, the field $k(\xi_1 + c\xi_2)$ is quasi-algebraically closed in Σ.*

Proof[11]. Let Σ' be the field of quantities in Σ which are separably algebraic over $k(\xi_1, \xi_2)$. Then Σ'/k is also finitely generated[12], and if $k(\xi_1 + c\xi_2)$ is q.a. closed in Σ', then it is also q.a. closed in Σ. Hence in the continuation of the proof we may suppose $\Sigma = \Sigma'$, in particular, therefore, that $\Sigma/k(\xi_1, \xi_2)$ is separably algebraic and finite. Let Ω_c be the field of separable quantities in Σ over $k(\xi_1 + c\xi_2)$, so that Ω_c is q.a. closed in Σ. We have

$$k(\xi_1, \xi_2) \subseteq \Omega_c(\xi_1, \xi_2) \subseteq \Sigma$$

and since Σ is a finite separable extension of $k(\xi_1, \xi_2)$, there are but a finite number of possibilities for $\Omega_c(\xi_1, \xi_2)$. Hence $\Omega_c(\xi_1, \xi_2) = \Omega_d(\xi_1, \xi_2)$ for some $c, d \in k$, $c \neq d$ (assuming k infinite). We now prove that $\Omega_c(\xi_1, \xi_2) = \Omega_d(\xi_1, \xi_2)$, $c \neq d$, implies $\Omega_c = k(\xi_1 + c\xi_2)$. Changing the notation slightly we may assume that Ω_i equals the field of separable quantities in Σ over $k(\xi_i)$, $i = 1, 2$, and that $\Omega_1(\xi_2) = \Omega_2(\xi_1)$. We had k q.a. closed in Σ, hence a fortiori in Ω_1, whence $k(\xi_2)$ is q.a. closed in $\Omega_1(\xi_2) = \Omega_2(\xi_1)$, whence $\Omega_2 \subseteq k(\xi_2) \subseteq \Omega_2$, so $\Omega_2 = k(\xi_2)$, q.e.d.

COROLLARY. *Let k and Σ be as in the theorem, and let u be an indeterminate. Then $k(u)\ (\xi_1 + u\xi_2)$ is quasi-algebraically closed in $\Sigma(u)$.*

Proof. For some integer s we shall have that $k(u)(\xi_1 + u^{p^s}\xi_2)$ is q.a. closed in $\Sigma(u)$, whence it is clear that also $k(u^{p^s})(\xi_1 + u^{p^s}\xi_2)$ is q.a. closed in $\Sigma(u)$, and then a fortiori it is q.a. closed in $\Sigma(u^{p^s})$. Thus $k(v)(\xi_1 + v\xi_2)$ is q.a. closed in $\Sigma(v)$, where v is a transcendental quantity over Σ.

DEFINITION. *V/k is said to be quasi-absolutely irreducible if k is quasi-algebraically closed in the field $k(\xi_1, \cdots, \xi_n)$ of rational functions on V/k.*

THEOREM 9. *If V/k is a quasi-absolutely irreducible variety of dimension $r \geqq 2$, then the general hyperplane section of V/k is also quasi-absolutely irreducible.*

Proof. This is a corollary to the previous theorem. Using the notation of Lemma 3.5, and keeping especially in mind the statement in the proof of that lemma that to study (η) over $k(u_1, \cdots, u_n, u_0)$ is the same as to study (ξ) over $k(u_1, \cdots, u_n, \bar{u}_0)$, we must see that $k(u_1, \cdots, u_n, u_1\xi_1 + \cdots + u_n\xi_n)$ is q.a. closed in $\Sigma(u_1, \cdots, u_n)$. By Lemma 5, $k(u_2, \cdots, u_n)$ is q.a. closed in $\Sigma(u_2, \cdots, u_n)$. Supposing now that, say, ξ_1, ξ_2 are algebraically independent, one sees also that ξ_1 and $u_2\xi_2 + \cdots + u_n\xi_n$ are algebraically inde-

[11] The main idea of this proof is due to Zariski [16; Lemma 5, p. 68].

[12] For if not, let $\xi_3, \cdots, \xi_r \in \Sigma$ be algebraically independent over Σ', where $r = \text{d.t.}\Sigma/k$; then also $\Sigma'(\xi_3, \cdots, \xi_r)/k(\xi_1, \xi_2)(\xi_3, \cdots, \xi_r)$ is not finite, a contradiction.

pendent, and now the theorem follows from the corollary to Theorem 8.

At this point we wish to interpolate some remarks relating the notion of quasi-absolutely irreducible variety to the question of the behavior of a prime ideal in a polynomial ring upon extension of the ground-field. We have dealt with this problem elsewhere[13], but will derive briefly the connection we need. Let $\mathfrak{p}$ be a prime ideal in the polynomial ring $k[x_1, \cdots, x_n]$. It is well known ([1, Proposition 9, p. 75]; see also the following Lemma 6) that $\bar{k}[x] \cdot \mathfrak{p}$ is unmixed where $\bar{k}$ is the algebraic closure of k.

DEFINITION. $\mathfrak{p}$ is said to be quasi-absolutely irreducible if $\bar{k}[x] \cdot \mathfrak{p}$ is primary, where $\bar{k}$ is the algebraic closure of k[14].

Although it is well known that $\bar{k}[x] \cdot \mathfrak{p}$ is unmixed, it apparently is not well known[15] that also $\Sigma[x] \cdot \mathfrak{A}$ is unmixed, given that $\mathfrak{A}$ is an unmixed $k[x]$-ideal, and k a subfield of Σ, it being understood that $\Sigma[x]$ is also a polynomial ring. It may therefore be of interest to include a proof at this point.

Let the field Σ be an extension of the field k; $x_1, \cdots, x_n$ indeterminates.

LEMMA 6. *If $\mathfrak{A}$ is an unmixed r-dimensional ideal in $k[x]$, then the extension of $\mathfrak{A}$ to $\Sigma[x]$ is also unmixed r-dimensional.*

Proof. Since $\Sigma[x] \cdot (\mathfrak{B} \cap \mathfrak{C}) = \Sigma[x] \cdot \mathfrak{B} \cap \Sigma[x]\mathfrak{C}$, for any ideals $\mathfrak{B}$, $\mathfrak{C}$ in $k[x]$, we may suppose $\mathfrak{A}$ to be primary. Let $\mathfrak{A}$ be primary, $\mathfrak{p}$ its associated prime; let Σ be algebraic over the pure transcendental extension Σ' of k. Then $\mathfrak{A}$ and $\mathfrak{p}$ extend to primary and associated prime in $\Sigma'[x]$, with dimension preserved. Hence we may assume Σ/k is algebraic. A general zero of $\mathfrak{p}$ over k, say it is r-dimensional, determines an r-dimensional prime ideal $\mathfrak{p}'$ in $\Sigma[x]$, which clearly lies over $\mathfrak{p}$. It is also quite clear[16] that $\mathfrak{p}'$ is a minimal prime ideal of $\Sigma[x] \cdot \mathfrak{A}$; moreover, any prime ideal of $\Sigma[x] \cdot \mathfrak{A}$ is of dimension not greater than r, so $\Sigma[x] \cdot \mathfrak{A}$ is r-dimensional. Let now $\mathfrak{p}_1'$ be any prime ideal of $\Sigma[x] \cdot \mathfrak{A}$. We claim that $\mathfrak{p}_1' \cap k[x] = \mathfrak{p}$, the inclusion $\mathfrak{p}_1' \cap k[x] \supseteq \mathfrak{p}$ being trivial: this will prove that $\mathfrak{A}$ is unmixed r-dimensional. Let[17] $\omega \in \mathfrak{p}_1' \cap k[x]$. Since $\omega \in \mathfrak{p}_1'$, we must have $\Sigma[x] \cdot \mathfrak{A} : \Sigma[x] \cdot \omega \neq \Sigma[x] \cdot \mathfrak{A}$. Let $\omega' \in \Sigma[x] \cdot \mathfrak{A} : \Sigma[x] \cdot \omega$ which is not in $\Sigma[x]\mathfrak{A}$. Let $1, \lambda_2, \lambda_3, \cdots$ be a linearly independent, possibly transfinite, basis of Σ/k, and let us write ω' in the form $\omega' = \Sigma \lambda_i \omega_i$, where $\omega_i \in k[x]$. We have $\omega'\omega = \Sigma \lambda_i \omega_i \omega \in \Sigma[x] \cdot \mathfrak{A}$, whence $\omega_i \omega \in \mathfrak{A}$. Since $\omega' \notin \Sigma[x] \cdot \mathfrak{A}$, at least one $\omega_i \notin \mathfrak{A}$, consequently $\omega \in \mathfrak{p}$, q.e.d.

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[13] As yet unpublished.

[14] It follows by Theorem 10 below and Lemma 4 that $\Sigma(x) \cdot \mathfrak{p}$ is primary for any extension Σ of k. It should be understood that the x are still to be indeterminates over Σ. To apply Theorem 10 and Lemma 4, introduce a general zero (ξ) of $k[x] \cdot \mathfrak{p}$ over k which is such that Σ and $k(\xi)$ are free over k, that is, d.t. $\Sigma(\xi)/\Sigma = $ d.t. $k(\xi)/k$.

[15] See footnote 7 on p. 134 of Krull's paper [7]. See also, however, [13; §95, parenthesis].

[16] The following argument, given for a prime ideal by Zariski [19; Lemma 9, p. 40], carries over to our primary ideal $\mathfrak{A}$.

[17] See, for examples [2; Theorem 4]. Also Theorems 2 and 3 of that paper enter into the argument, though not with the same generality.

Let $\mathfrak{p}$ be a prime ideal in the polynomial ring $k[x_1, \cdots, x_n]$.

THEOREM 10. *The prime ideal $\mathfrak{p}$ is quasi-absolutely irreducible if and only if the variety it defines over k is quasi-absolutely irreducible, that is, if and only if k is q.a. closed in the quotient field of $k[x]/\mathfrak{p}$.*

Proof. First let us suppose that k is not q.a. closed in $k(\xi)$, where $k[\xi] = k[x]/\mathfrak{p}$. Then there exists a quantity $\theta \in k(\xi)$ which is separably algebraic over k but not in k. Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, $a_i \in k$, be the irreducible polynomial over k satisfied by θ; then $n > 1$. Let $\theta = \alpha(\xi)/\beta(\xi)$, where $\alpha(x)$, $\beta(x) \in k[x]$, and $\alpha(x)\beta(x) \notin \mathfrak{p}$. We claim that $k(\theta)[x] \cdot \mathfrak{p}$ is not prime. In fact, let $f(z) = (z-\theta)g(z) = (z-\theta)(z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1})$, $b_i \in k(\theta)$. Then $f(\theta) = 0$, so $\beta^n(x)f(\alpha(x)/\beta(x)) = \alpha^n(x) + a_1 \alpha^{n-1}(x)\beta(x) + \cdots \in k[x] \cdot \mathfrak{p}$, and

$$a^n(x) + a_1 \alpha^{n-1}(x)\beta(x) + \cdots$$
$$= (\alpha(x) - \theta\beta(x)) \cdot (\alpha^{n-1}(x) + b_1 \alpha^{n-2}(x)\beta(x) + \cdots) \in k(\theta)[x] \cdot \mathfrak{p}.$$

Here the b_i are linear combinations of $1, \theta, \cdots, \theta^{n-1}$ with coefficients in k. Since $k(\theta)[x]\mathfrak{p} \cap k[x] = \mathfrak{p}$, and since $1, \theta, \cdots, \theta^{n-1}$ are also linearly independent over $k[x]$, we see that neither the first factor, $\alpha(x) - \theta\beta(x)$, nor the second is in $k(\theta)[x] \cdot \mathfrak{p}$. For if $\alpha(x) - \theta\beta(x) \in k(\theta)[x] \cdot \mathfrak{p}$, one would conclude that $\alpha(x)$ and $\beta(x)$ are in $\mathfrak{p}$, whereas neither is in $\mathfrak{p}$: similarly, if the second factor is in $k(\theta)[x] \cdot \mathfrak{p}$, one concludes, comparing coefficients, that $\alpha(\xi)/\beta(\xi)$ satisfies an equation of degree $n-1$ over k, which is not the case. Now if $k(\theta)[x] \cdot \mathfrak{p}$ is primary, let $\mathfrak{P}$ be the associated prime ideal; then both factors are in $\mathfrak{P}$. Since $z - \theta$ and $g(z)$ have no common factor, we have that $1 \in (z-\theta, g(z)) \subseteq k(\theta)[z]$, from which one derives immediately that some power of $\beta(x)$ is in $\mathfrak{P}$, hence also some power of $\beta(x)$ is in $k(\theta)[x] \cdot \mathfrak{p} \cap k[x] = \mathfrak{p}$, a contradiction.

Conversely, if k is q.a. closed in $k(\xi_1, \cdots, \xi_n)$, then $\bar{k}[x] \cdot \mathfrak{p}$ is primary. For let k' be the algebraic closure of k in $k(\xi_1, \cdots, \xi_n)$, where we are supposing, as we may, that $\bar{k}$ and $k(\xi)$ are subfields of a common field. One sees at once that $k'[x] \cdot \mathfrak{P}$ is primary: for if $\mathfrak{P}$ is a prime ideal of $k'[x] \cdot \mathfrak{p}$, then some power of $\mathfrak{P}$ is in $k[x]$, since k'/k is purely inseparable, whence this power is in $k'[x] \cdot \mathfrak{p}$; thus $k'[x] \cdot \mathfrak{p}$ and $\mathfrak{P}$ have the same radical, whence $k'[x] \cdot \mathfrak{p}$ is primary. Now $\bar{k}[x] \cdot \mathfrak{p}$ and $\bar{k}[x] \cdot \mathfrak{P}$ have the same radical, so we may suppose we are in the case $k = k'$, that is, k algebraically closed in $k(\xi_1, \cdots, \xi_n)$. If now $k(\xi_1, \cdots, \xi_n)$ is separably generated, then it is known that $\bar{k}[x] \cdot \mathfrak{p}$ is prime [14; chap. 1, Theorems 3 and 5, pp. 15 and 18]. If $k(\xi_1, \cdots, \xi_n)$ is not separably generated, then at any rate there exist integers $i_s = p^{i_s}$, $s = 1, \cdots, n$, such that $k(\xi_1^{i_1}, \cdots, \xi_n^{i_n})$ is separably generated. Let $\xi_1^{i_1}, \cdots, \xi_n^{i_n}$ determine a prime ideal $\mathfrak{p}^*$ in $k[y]$, a polynomial ring in n variables $y_1, \cdots, y_n$. Let now $f(x) \cdot g(x) \in \bar{k}[x] \cdot \mathfrak{p}$; then $f(\xi) \cdot g(\xi) = 0$, so $f(\xi) = 0$ or $g(\xi) = 0$. Let $\rho = i_1 \cdot i_2 \cdot \cdots \cdot i_n$: then $f^\rho(x) g^\rho(x) = F(x_1^{i_1}, \cdots, x_n^{i_n}) \cdot G(x_1^{i_1}, \cdots, x_n^{i_n})$. Then $F(y) \cdot G(y)$ vanishes at $\xi_1^{i_1}, \cdots, \xi_n^{i_n}$, so $F(y) \cdot G(y) \in \bar{k}[y] \cdot \mathfrak{p}^*$, whence $F(y)$ or $G(y) \in \bar{k}[y] \cdot \mathfrak{p}^*$; replacing y_s by $x_s^{i_s}$,

one sees that $F(x_1^{t_1}, \cdots, x_n^{t_n})$ or $G(x_1^{t_1}, \cdots, x_n^{t_n}) \in \bar{k}[x] \cdot \mathfrak{p}$, that is, $f^p(x_1, \cdots, x_n)$ or $g^p(x_1, \cdots, x_n) \in \bar{k}[x] \cdot \mathfrak{p}$. This proves that $\bar{k}[x] \cdot \mathfrak{p}$ is primary in view of the fact that, at any rate, $\bar{k}[x] \cdot \mathfrak{p}$ is unmixed.

Let $F(\tau_1, \cdots, \tau_m, x_1, \cdots, x_n) \in k[\tau_1, \cdots, \tau_m, x_1, \cdots, x_n]$, a polynomial ring, where we shall regard the τ_i as parameters. Let Ω be any field containing k, say a universal domain over k containing $\tau_1, \cdots, \tau_m$; but $x_1, \cdots, x_n$ are still understood to be indeterminates over Ω. We consider the substitution $\tau \to a$, $a = (a_1, \cdots, a_m)$, $a_i \in k$, whereupon $F(\tau, x) \to F(a, x)$. The following lemma is well known [12; p. 707].

LEMMA 7. *There is a variety U in affine m-space over k such that $F(a, x)$ is irreducible in $\bar{k}_a[x]$([18]) and of the same degree in the x_i as $F(\tau, x)$ if and only if $(a_1, \cdots, a_m) \notin U$.*

In particular therefore, if $F(\tau, x)$ is irreducible in $\bar{k}_\tau[x]$, then almost all $F(a, x)$, $a_i \in k$, are also irreducible in $\bar{k}[x]$. Note that we need both the necessity and the sufficiency to establish this.

LEMMA 8. *Let $F(\tau, x) \in k(\tau)[x]$ be a power of an irreducible polynomial in $\bar{k}_\tau[x]$. Then there exists a variety U in affine m-space over k, different from the whole space, such that $F(a, x)$ is the same power of an irreducible polynomial in $\bar{k}_a[x]$ if $(a_1, \cdots, a_m) \notin U$.*

Proof. It is clear that we may replace k by its algebraic closure, and hence suppose k to be infinite. This we do in order that we may apply a nonsingular linear homogeneous transformation over k, and thus assume that $F(\tau, x)$ is of the same degree in x_n as in all the variables. We suppose that now to be the case. We may and do assume $F(\tau, x)$ to be irreducible in $k(\tau)[x]$.

Let us write $F(\tau, x)$ in some definite way as quotient with numerator in $k[\tau, x]$ and denominator in $k[\tau]$. We exclude the values a which annihilate the denominator, and so may suppose $F(\tau, x) \in k[\tau, x]$. Now, however, we normalize $F(\tau, x)$ so that the highest power of x_n is 1; so we suppose $F(\tau, x)$ to be of the form $F(\tau, x) = G(\tau, x)/c(\tau)$, where $c(\tau) \in k[\tau]$; $G(\tau, x) \in k[\tau, x]$, and $c(\tau)$ is the coefficient of the highest power of x_n in $F(\tau, x)$.

Suppose now that $F(\tau, x) = H^q(x)$, where $H \in \bar{k}_\tau[x]$, and where, moreover, the coefficient of the highest power of x_n may and shall be assumed to be 1. It is now not difficult to see that q is a power of p; in fact, upon applying any automorphism of $\bar{k}_\tau$ over $k(\tau)$ to $H(x)$, we must have $H(x) \to H(x)$, by the unique factorization theorem and by our normalization of H. Hence the coefficients of $H(x)$ are purely inseparable over $k(\tau)$, and so $H^{p^f}(x) \in k(\tau)[x]$ for some smallest f. Were $p^f \not\equiv 0(q)$, say $p^f = aq + b$, $0 < b < q$, we would have $H^{p^f} = F^a \cdot H^b$, whence $H^b \in \bar{k}_\tau[x] \cap k(\tau, x) = k(\tau)[x]$. Similarly we have $F/H^b \in k(\tau)[x]$, and this is impossible since F is irreducible and of larger de-

([18]) We shall designate the algebraic closures of $k(\tau)$, $k(a)$ by $\bar{k}_\tau$, $\bar{k}_a$.

gree than H^b. Thus $H(x) = H(\tau^{1/p^f}, x) \in k^{1/p^f}(\tau^{1/p^f})[x]$, where $\tau^{1/p^f} = t$ $= (\tau_1^{1/p^f}, \cdots, \tau_m^{1/p^f})$. Let $f(t) \in k^{1/p^f}[t]$, $f(t) \neq 0$, such that $H(\alpha, x)$ is absolutely irreducible and of the same degree as $H(t, x)$ if $f(\alpha) \neq 0$; such f exists by the previous lemma. Then $g(\tau) = f^{p^f}(t) \in k[\tau]$ is different from zero and is such that $F(a, x)$ is the qth power of an irreducible polynomial in $\bar{k}_a[x]$ if $g(a) \neq 0$, which was to be proved.

Let $\mathfrak{A}$ be an unmixed r-dimensional ideal in the polynomial ring $k[x_1, \cdots, x_n]$. We form $r+1$ linear forms in the x's with indeterminate coefficients u_{ij}:

$$z_i = u_{i1}x_1 + \cdots + u_{in}x_n, \qquad i = 1, \cdots, r+1,$$

and consider the ideal $k(u)[x] \cdot \mathfrak{A} \cap k(u)[z_1, \cdots, z_{r+1}]$. One sees without difficulty that this is a principal ideal $(E(z_1, \cdots, z_{r+1}, u))$; if E is normalized so as to be a polynomial in the u_{ij}, and primitive in them, so that E is defined to within a factor in k, then E is the elementary divisor form or ground-form of $\mathfrak{A}$. The polynomial E is integral in any z_i over the other z's and is a polynomial (in $z_1, \cdots, z_{r+1}$) of least degree in z_{r+1}, which is in $k(u)[x] \cdot \mathfrak{A}$. If $\mathfrak{A}$ is prime, then its ground-form is irreducible; the converse is not generally true; but $\mathfrak{A}$ is primary if and only if its ground-form is a·power of an irreducible polynomial [9; Theorem 9, p. 252]. This follows at once from the fact that distinct primes have distinct ground-forms. If $\mathfrak{A}$ is prime and quasi-absolutely irreducible, that is, k is q.a. closed in the quotient-field of $k[x_1, \cdots, x_n]/\mathfrak{A}$, then also the ideal (E) is prime and quasi-absolutely irreducible, since we may suppose $k(u)[z]/(E) = k(u)[\zeta]$ to be contained in $k(u)[x]/\mathfrak{A} = k(u)[\xi]$, that is, there is an isomorphic mapping of $k(u)[\zeta]$ into $k(u)[\xi]$ which is the identity on $k(u)$, and sends ζ_i into $\Sigma u_{ij}\xi_j$.

Let V/k be a quasi-absolutely irreducible variety of dimension $r+1 \geq 2$, and let $\mathfrak{A}$ be its prime ideal in $k[x_1, \cdots, x_n]$.

THEOREM 11. *For almost all hyperplanes $a_0 + a_1x_1 + \cdots + a_nx_n = 0$, $a_i \in k$, the ideal $(\mathfrak{A}, a_0 + a_1x_1 + \cdots + a_nx_n)$ is prime and quasi-absolutely irreducible.*

Proof. For almost all $(a_0, \cdots, a_n)$, the ideal $(\mathfrak{A}, a_0 + a_1x_1 + \cdots + a_nx_n)$ is r-dimensional, and also unmixed: the unmixed character would follow immediately for V/k normal, for in that case $k[x]/\mathfrak{A}$ is integrally closed and the Principal Ideal Theorem is applicable; but it also follows more generally from the Theorem of Krull, since the general hyperplane section, that is, the ideal $k(\tau)[x] \cdot (\mathfrak{A}, \tau_0 + \tau_1x_1 + \cdots + \tau_nx_n)$, where the τ_i are indeterminates, is prime. Let $\mathfrak{p} = k(\tau)[x] \cdot (\mathfrak{A}, \tau_0 + \tau_1x_1 + \cdots + \tau_nx_n)$, and let $E(z_1, \cdots, z_{r+1}, \tau, u)$ be its ground-form. We know, then, that E is quasi-absolutely irreducible. It is clear that we shall have $E(z_1, \cdots, z_{r+1}, a, u) \in (\mathfrak{A}, a_0 + a_1x_1 + \cdots + a_nx_n)$ almost always, so that the ground-form of $(\mathfrak{A}, a_0 + a_1x_1 + \cdots + a_nx_n)$ is either $E(z_1, \cdots, z_{r+1}, a, u)$ or a factor thereof, hence in either case, the ground-form of $(\mathfrak{A}, a_0 + a_1x_1 + \cdots + a_nx_n)$ is a power of an absolutely irreducible poly-

nomial. Hence $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ is almost always primary. We proved before that $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ is almost always its own radical, and the two facts combined complete the proof.

Let V/k be an irreducible, r-dimensional variety, $r \geqq 2$, $\xi_1, \cdots, \xi_n$ the nonhomogeneous coördinates of a general point of V/k, and let $\xi_1, \cdots, \xi_n$ determine the prime ideal $\mathfrak{A}$ in $k[x_1, \cdots, x_n]$.

THEOREM 12. *For almost all hyperplanes* $a_0 + a_1 x_1 + \cdots + a_n x_n = 0$, *the ideal* $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ *is an* $(r-1)$-*dimensional prime ideal.*

Proof. Let k' be the field of quantities in $k(\xi)$ which are separable over k, so that k' is quasi-algebraically closed in $k(\xi)$. Let k'' be the least normal extension of k which contains k'; k''/k is also separable. The extension of $\mathfrak{A}$ to $k'[x_1, \cdots, x_n]$ and to $k''[x_1, \cdots, x_n]$ is each time the intersection of primes [19; Theorem 12a, p. 40]. Let $k'[x] \cdot \mathfrak{A} = \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_s$. The ideals $\mathfrak{A}_1, \cdots,$ $\mathfrak{A}_s$ are the only prime ideals in $k'[x]$ lying over $\mathfrak{A}$. Now $(\xi_1, \cdots, \xi_n)$ clearly determines a prime ideal in $k'[x]$ lying over $\mathfrak{A}$, say this is $\mathfrak{A}_1$. Then $\mathfrak{A}_1$ is quasi-absolutely prime, though the other $\mathfrak{A}_i$ need not be. The ideal $k''[x] \cdot \mathfrak{A}_1 = \mathfrak{A}_1'$ is therefore prime. We have $k''[x] \cdot \mathfrak{A} = \mathfrak{A}_1' \cap \mathfrak{A}_2' \cap \cdots \cap \mathfrak{A}_t'$, and, moreover, some automorphism of k''/k will take any $\mathfrak{A}_i'$ into any $\mathfrak{A}_j'$ [2; Theorem 8, p. 260]. Consider now $(\mathfrak{A}, a_0 + a_1 x_1 + \cdots + a_n x_n)$, which almost always is the intersection $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cdots$ of $(r-1)$-dimensional primes; we are to see that there is just one prime: $\mathfrak{P}_1 = \mathfrak{P}_2 = \cdots$. Over $\mathfrak{P}_1$ in $k''[x]$ there lies a prime ideal $\mathfrak{m}$, necessarily $(r-1)$-dimensional; it contains $k''[x] \cdot \mathfrak{A}$, hence also one of the ideals $\mathfrak{A}_i'$, say $\mathfrak{A}_s'$, so it contains $(\mathfrak{A}_s', a_0 + a_1 x_1 + \cdots + a_n x_n)$, which itself is prime, since it is the conjugate ideal of the ideal $(\mathfrak{A}_1', a_0 + \cdots + a_n x_n)$ which we know is prime (almost always). Hence $(\mathfrak{A}_s', a_0 + a_1 x_1 + \cdots + a_n x_n)$, which is an $(r-1)$-dimensional prime ideal lying over $\mathfrak{p}_1$, coincides with $\mathfrak{m}$; applying automorphisms of k''/k we have that each $(\mathfrak{A}_i', a_0 + a_1 x_1 + \cdots + a_n x_n)$ $i = 1, \cdots, t$, lies over $\mathfrak{p}_1$, and these t ideals are all the prime ideals lying over $\mathfrak{p}_1$. Likewise they are all the prime ideals lying over $\mathfrak{p}_2, \mathfrak{p}_3, \cdots$. So $\mathfrak{p}_1 = \mathfrak{p}_2 = \cdots$, q.e.d.

4. The non-separably generated case. Let $k[x]/\mathfrak{A} = k[\eta]$, where (η) is the general point of the given r-dimensional variety V/k, $r > 1$. Let $\mathfrak{A} = (F_1, \cdots, F_m)$, and from the mixed Jacobian matrix $J(F_1, \cdots, F_m)$ select some $(n-r)$-row subdeterminant $F(x)$, $x = (x_1, \cdots, x_n)$, which is not zero on V, that is, $F(\eta) \neq 0$; such a subdeterminant exists since the general point of V is a simple point of V [19; Corollary to Theorem 11, p. 39]. The hypersurface $F(x) = 0$ passes through all the singularities at finite distance of V/k without containing V; by Theorem 1 it has a like behavior, almost always, with respect to the hyperplane section $a_0 + a_1 x_1 + \cdots + a_n x_n = 0$. Suppose now that this section H_a/k is irreducible, as is the case almost always, and let $(\xi_1, \cdots, \xi_n)$ be its general point. Consider the variety defined in $(n+1)$-space by the general point $\xi_1, \cdots, \xi_n, \zeta = 1/F(\xi)$. One sees without difficulty

that this second variety is without singularities at finite distance; in fact, if $\bar{\mathfrak{p}}$ is a prime ideal in $k[\xi_1, \cdots, \xi_n, \zeta]$ and $\mathfrak{p}$ its contraction to $k[\xi_1, \cdots, \xi_n]$, then $\bar{\mathfrak{p}}$ and $\mathfrak{p}$ have the same quotient ring; $\mathfrak{p}$ represents a simple subvariety of H_a since $F(\xi)\mathfrak{p}\not\subseteq$, whence $\bar{\mathfrak{p}}$ also represents a simple subvariety.—Assume now without loss of generality that $\xi_1, \cdots, \xi_{r-1}$ are algebraically independent over k, while the other ξ's are integral over $k[\xi_1, \cdots, \xi_{r-1}]$; if we first write indeterminates for the a_i, and then specialize, we see that this remark, as well as the following, holds for almost all H_a, keeping in mind that $F(x)$ does not involve the parameters.

We now make a reduction to dimension 1 by adjoining $r-2$ of the ξ_i, $i=1, \cdots, r-1$, to the ground-field. We still indicate the (new) ground-field by k, and the general point of the 1-dimensional variety by $\xi_1, \cdots, \xi_n$. The variety originally defined by $\xi_1, \cdots, \xi_n, \zeta$ then reduces to a 1-dimensional variety free of singularities, whence $k[\xi_1, \cdots, \xi_n, \zeta]$ is integrally closed. Since the integral closure $\mathfrak{O}$ of $k[\xi_1, \cdots, \xi_n]\subseteq k[\xi_1, \cdots, \xi_n, \zeta]$ and has a finite module basis over $k[\xi_1]$ [18; sec. 2, p. 506], we find that *some power of $F(\xi)$ is in the conductor of $k[\xi_1, \cdots, \xi_n]$*. Let $F(\xi)$ satisfy the equation of integral dependence:

$$F^p + \cdots + d(\xi_1) = 0.$$

Then also some power of $d(\xi_1)$ is in the conductor of $k[\xi_1, \cdots, \xi_n]$. We seek now to show the existence of an integer M such that $(d(\xi_1))^M$ is in the conductor, uniformly for almost all H_a.

Let $F_i(\xi_1, \cdots, \xi_{i-1}, x_i)=0$ be the irreducible equation satisfied by ξ_i over $k(\xi_1, \cdots, \xi_{i-1})$, and let $\deg_{x_i}F_i=g_i$. Let $\theta_j = \prod \xi_2^{j_2} \cdots \xi_n^{j_n}$, $0 \le j_i \le g_i-1$; the θ_j, which are $g_2 \cdot g_3 \cdot \cdots \cdot g_n=m$ in number, form a basis of $k(\xi_1, \cdots, \xi_n)/k(\xi_1)$. Clearly there exists an integer h such that the exponent of ξ_i, that is, the least integer e such that ξ_i/ξ_1^e is integral over $k[1/\xi_1]$, is not greater than h, uniformly for almost all H. Since $\exp \xi\eta \le \exp \xi + \exp \eta$, we see that a product of k of the ξ_i has an exponent not greater than kh. In particular therefore $hm = N$ is a bound from above of the $\exp \theta_j$.

Let now $\eta_1=1, \eta_2, \cdots, \eta_m$ be a normal basis [19] for the integral closure $\mathfrak{O}$ of $k[\xi_1]$ in $k(\xi_1, \cdots, \xi_n)$. Let $\omega = c_1(\xi_1)\eta_1 + \cdots + c_m(\xi_1)\eta_m$, $c_i(\xi_1) \in k[\xi_1]$. It is known [10; Theorem 9, p. 432] that:

$$\exp_x \omega = \max_{c_i(x) \ne 0} \{\exp_x c_i(x)\eta_i\} \qquad (x = \xi_1).$$

Let now $\theta_i = \sum_{j=1}^m c_{ij}(\xi_1)\eta_j$, $i=1, \cdots, m$. Then we see that

$$\exp c_{ij} \le N, \qquad \text{all } i, j,$$

$$\exp \eta_j \le N, \qquad \text{all } j.$$

---

[19] The elements $\eta_1=1, \eta_2, \cdots, \eta_{i-1}$ having been defined, η_i is an element of $\mathfrak{O}$ of least exponent such that $\eta_1, \cdots, \eta_i$ are linearly independent over $k \bmod \mathfrak{O} \cdot x$.

Thus there exists a polynomial $c(\xi_1) \neq 0$, namely det $|c_{ij}|$, of degree $\leq N^m = M$, such that $c(\xi_1)\eta_j \in k[\xi_1]\theta_1 + \cdots + k[\xi_1]\theta_m$, $j = 1, \cdots, m$. Since $\mathfrak{O} = k[\xi_1]\eta_1 + \cdots + k[\xi_1]\eta_m$, we have established the following lemma.

LEMMA 9. *There exists a polynomial $c(\xi_1) \in k[\xi_1]$ of degree $\leq N^m = M$, and different from zero, such that $c(\xi_1)\mathfrak{O} \subseteq k[\xi_1]\theta_1 + \cdots + k[\xi_1]\theta_m$.*

THEOREM 13. $[d(\xi_1)]^M$ *is in the conductor of $k[\xi_1, \cdots, \xi_n]$.*

Proof. We have $c(\xi_1)$ is in the conductor of $k[\xi_1, \cdots, \xi_n]$. Now we may suppose each irreducible factor of $c(\xi_1)$ vanishes at at least one singularity of $V(\mathfrak{p})$. Let $c_1(\xi_1)$ be an irreducible factor of $c(\xi_1)$, and let $c_1(\xi_1) \in \mathfrak{p}_0$, where $\mathfrak{p}_0$ represents a singular point of $V(\mathfrak{p})$. Also $d(\xi_1) \in \mathfrak{p}_0$. Hence $d(\xi_1) \in \mathfrak{p}_0 \cap k[\xi_1] = (c_1(\xi_1))$. Hence $[d(\xi_1)]^M \equiv 0(c(\xi_1))$, and the theorem is proved.

We have proved more, namely, that if $\delta(\xi_1) = (d(\xi_1))^M$, then $\delta(\xi_1)\mathfrak{O} \subseteq k[\xi_1] \cdot \theta_1 + \cdots + k[\xi_1] \cdot \theta_m$. If now $\mathfrak{p}$ is the $(r-1)$-dimensional prime ideal of H_a/k, then we can compute a (canonical) element of the conductor of $k[x]/\mathfrak{p} = k[\xi]$. Namely, we introduce, in accordance with F. K. Schmidt [11; p. 450], the rings $R_i = k(\xi_1, \cdots, \xi_{i-1}, \xi_{i+1}, \cdots, \xi_{r-1})[\xi_i, \xi_r, \cdots, \xi_n]$; for each ring R_i compute $\delta_i(\delta_i \in k[\xi_1, \cdots, \xi_{r-1}])$ as above, and then $\delta = \delta_1 \cdot \cdots \cdot \delta_{r-1}$ is the required element of the conductor.

By a proof entirely parallel to that of Theorem 7, we can now state:

THEOREM 7'. *Theorem 7 holds without the restriction that the field of rational functions on V/k be separably generated.*

5. **The case of a finite ground-field.** We shall need the following theorem as a lemma to Theorem 15.

THEOREM 14. *Let k' be a purely inseparable extension of k[20], let $\xi_1, \cdots, \xi_n$ determine a normal r-dimensional variety of V over k, and a variety V' free of $(r-1)$-dimensional singularities over k', and let k' and $k(\xi)$ be linearly disjoint over k. Then V'/k' is normal.*

Proof. Since $\mathfrak{O}' = k'[\xi]$ is integral over $\mathfrak{O} = k[\xi]$, the $(r-1)$-dimensional prime ideals in $\mathfrak{O}'$ contract to $(r-1)$-dimensional prime ideals in $\mathfrak{O}$, and over any $(r-1)$-dimensional prime ideal in $\mathfrak{O}$ lies an $(r-1)$-dimensional prime ideal, in fact, only one since k'/k is purely inseparable. Thus there is a 1–1 correspondence between the prime $(r-1)$-dimensional ideals of $\mathfrak{O}'$ and those of $\mathfrak{O}$, an ideal corresponding to the ideal lying over it. Since V' is free of $(r-1)$-dimensional singularities, every quotient ring $\mathfrak{O}'_{\mathfrak{p}'}$ of $\mathfrak{O}'$ with respect to a minimal prime ideal $\mathfrak{p}'$ is integrally closed. We propose to show that $\mathfrak{O}' = \cap \mathfrak{O}'_{\mathfrak{p}'}$ over the minimal prime ideals $\mathfrak{p}'$.

Let $1, \theta_1, \theta_2, \cdots$ be a, possibly transfinite, basis of k'/k. By linear disjointness, every element α of $k'(\xi)$ can be written uniquely in the form:

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[20] This hypothesis is removed in Theorem 15 below.

$$(1) \qquad \alpha = a_0(\xi) + a_1(\xi)\theta_1 + a_2(\xi)\theta_2 + \cdots, \quad a_i(\xi) \in k(\xi),$$

where, of course, only a finite number of $a_i(\xi)$ are different from zero. Let now $\alpha \in \mathfrak{O}'_{\mathfrak{p}'}$. Then $\alpha = \beta/\gamma$, β, $\gamma \in \mathfrak{O}'$, $\gamma \notin \mathfrak{p}'$. Since $\alpha = \beta\gamma^{p-1}/\gamma^p$, we may assume $\gamma \in \mathfrak{O}$; $\gamma \notin \mathfrak{p}' \cap \mathfrak{O} = \mathfrak{p}$ then follows. From the uniqueness of (1), one sees then that $a_i(\xi) \in \mathfrak{O}_{\mathfrak{p}}$, $i = 0, 1, \cdots$, and this for every minimal prime ideal $\mathfrak{p}$. Hence $a_i(\xi) \in \cap \mathfrak{O}_{\mathfrak{p}} = \mathfrak{O}$ and $\alpha \in \mathfrak{O}'$, q.e.d.

Let the quantities $\xi_1, \cdots, \xi_n$ determine a variety V/K and also V/k, where k is a subfield of K. Let dim $V/k = r$.

THEOREM 15. *If K and $k(\xi)$ are linearly disjoint over k, if V/k is normal, and V/K is without $(r-1)$-dimensional singularities, then V/K is also normal. If K/k is separably generated, it is sufficient to assume that K and $k(\xi)$ are free over k.*

Proof. It is clear that we need consider only the affine space. Let $k \subset k' \subset K$, k'/k pure transcendental and K/k' algebraic. The hypotheses carry over to k', so we may assume $k = k'$, or that K/k is algebraic. Let k'' be the field of quantities in K which are separable over k. We first prove that V/k'' is normal: the theorem then follows from Theorem 14. If $\alpha \in k''(\xi_1, \cdots, \xi_n)$ and integral over $k''[\xi_1, \cdots, \xi_n]$, then α is also in $k'''(\xi_1, \cdots, \xi_n)$ and integral over $k'''[\xi_1, \cdots, \xi_n]$ for some finite extension k''' of k, $k \subset k''' \subset k''$: hence it will be sufficient to proceed under the assumption that k''/k is finite. Let then $k'' = k(\theta)$. Let $\theta = \theta_1, \cdots, \theta_m$ be the conjugates of θ over k. Let $\omega \in k(\theta)(\xi)$ which is integral over $k(\theta)[\xi]$; ω is also integral over $k[\xi]$, since θ is algebraic (hence integral) over k. We have $\omega = \alpha_0(\xi) + a_1(\xi)\theta + \cdots + a_{m-1}(\xi)\theta^{m-1}$, $a_i(\xi) \in k(\xi)$, and wish to prove that $a_i(\xi) \in k[\xi]$, $i = 0, \cdots$, $m-1$. Clearly $\omega_i = a_0(\xi) + a_1(\xi)\theta_i + \cdots + a_{m-1}(\xi)\theta_i^{m-1}$ is also integral over $k[\xi]$, whence $\Delta \cdot a_i$ is integral over $k[\xi]$, as also a_i, since $1/\Delta$ is algebraic, hence integral, over k.

Let the prime ideal $\mathfrak{p}$ determine a normal variety V/k, where k is an arbitrary ground-field, possibly finite; let dim $V/k = r \geq 2$. Also, let us assume that V/k is quasi-absolutely irreducible.

THEOREM 16. *There is an algebraic variety U/k in projective n-space P_n/k, with $U \neq P_n$, such that if $a = (a_0, \cdots, a_n) \notin U$ and $k(a)/k$ is separably generated, then the section of $V/k(a)$ by $a_0 + a_1x_1 + \cdots + a_nx_n = 0$ is also normal.*

Proof. Let us consider for a moment points $(a_0, \cdots, a_n)$ which are algebraic over k. Let k' be the field of quantities separable over k. If $\mathfrak{p}$ is the ideal of V/k in $k[x]$, then $k'[x] \cdot \mathfrak{p}$ is prime and is the ideal of V/k' in $k'[x]$, for $k'[x] \cdot \mathfrak{p}$ is its own radical and V/k is quasi-absolutely irreducible. Moreover, the mixed Jacobian matrix for $k'[x] \cdot \mathfrak{p}$ is, or can be taken to be, the same as that for $k[x] \cdot \mathfrak{p}$. Hence also V/k' is free of $(r-1)$-dimensional singularities, whence V/k' is normal. By Theorem 7', we then have that $k'[x] \cdot (\mathfrak{p}, a_0 + a_1x_1 + \cdots + a_nx_n)$

is the ideal of a normal variety over k' for almost all $a=(a_0, \cdots, a_n)$, $a_i \in k'$, that is, but for the points (a) on a certain variety U'/k', $\dim U' < n$: instead of U'/k', we may instead take a variety U/k on which all the points of U' lie; there exists such variety U, $\dim U < n$, as one sees without difficulty. The contraction to $k(a)[x]$ of the extension of $k(a)[x] \cdot (\mathfrak{p}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ to $k'[x]$ is itself, so $k(a)[x] \cdot (\mathfrak{p}, a_0 + a_1 x_1 + \cdots + a_n x_n)$ is prime. It is also without $(r-2)$-dimensional singularities, as one sees by considering, as above, the mixed Jacobian matrix. Moreover, if $\alpha \in k(a)[x]$, we know that $k'[x]$ $\cdot (\mathfrak{p}, l, \alpha)$ is unmixed, where $l = a_0 + a_1 x_1 + \cdots + a_n x_n$. Hence also $k(a)[x]$ $\cdot (\mathfrak{p}, l, \alpha)$ is unmixed, whence the variety of $k(a)[x] \cdot (\mathfrak{p}, l)$ is normal over $k(a)$, by Theorem 3.

For points (a) of dimension > 0, we unfortunately have no way of deriving the present theorem directly from Theorem 7'. We are compelled simply to strengthen each of the theorems and lemmas which contribute to the proof of that theorem, and then the present theorem follows just as does Theorem 7'. In several places, as, for example, Lemma 1 and Theorem 6, the modification necessary in the proofs amounts to little more than a change in notation. The modification of the Theorem of Krull is carried out explicitly in Theorem 7 of the Appendix. For Theorem 1, the modification is embodied in the proof of Corollary 2. Also no difficulty stands in the way of strengthening Theorem 2 and Corollary 1 of Theorem 1 in the direction needed for the present theorem.

APPENDIX

Let k be an infinite ground-field to which we adjoin a finite number of indeterminates $\tau_1, \cdots, \tau_m$ which will serve as parameters in what follows[21]. Over the field $k(\tau) = k(\tau_1, \cdots, \tau_m)$ we consider a polynomial ring $R = k(\tau)$ $\cdot [x_1, \cdots, x_n] = k(\tau)[x]$ in the indeterminates $x_1, \cdots, x_n$. Let $\mathfrak{m}$ be an R-module contained in the linear form module over R in indeterminates $z_1, \cdots, z_s$ (or in quantities $z_1, \cdots, z_s$ linearly independent over the quotient field of R). $\mathfrak{m}$ then consists of a set of linear forms $l = f_1(x, \tau) z_1 + \cdots + f_s(x, \tau) z_s, f_i(x, \tau) \in k(\tau)[x]$ such that if $l, l' \in \mathfrak{m}$ and $a \in R$, then $l - l' \in \mathfrak{m}$ and $a \cdot l \in \mathfrak{m}$. The ideals of R are included as a special case, namely, $s = 1$.

Consider a substitution $\tau_i \to a_i \in k$, $i = 1, \cdots, m$. If l is a linear form in $R \cdot z_1 + \cdots + R \cdot z_s$ which can be written in the form $l = l_1/f(\tau)$, $l_1 = l_1(x, \tau, z)$ $\in k[\tau, x] \cdot z_1 + \cdots + k[\tau, x] \cdot z_s, f(\tau) \in k[\tau]$, and $f(a) \neq 0$, then we shall say that $l \to \bar{l} = l_1(x, a, z)/f(a)$ under the substitution $\tau \to a$. Clearly if $\bar{l}$ is defined for a given l, then $\bar{l}$ is unique. Now let $\mathfrak{m}$ be an R-module contained in $R \cdot z_1 + \cdots + R \cdot z_s$ and consider the $k[x]$-module in $k[x] \cdot z_1 + \cdots + k[x] z_s$ generated by the linear forms $\bar{l}$ arising from the linear forms $l \in \mathfrak{m}$ upon the substitution $\tau \to a$; we designate this $k[x]$-module by $\overline{\mathfrak{m}}$ (which varies with the

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[21] The theorems may be easily formulated also for finite k. This is explicitly done in Theorem 7 below for the main result.

substitution). Were we dealing with the case $s=1$ (and $z_1=1$) we would have $\overline{R}=k[x]$ for any substitution $\tau \to a$, and therefore as a mere matter of notation we shall designate $k[x]$ by $\overline{R}$ also in the case $s>1$. The ring $k[\tau, x]$ will be designated by R_τ. Let $\mathfrak{m}_\tau = \mathfrak{m} \cap (R_\tau \cdot z_1 + \cdots + R_\tau \cdot z_s)$: it is clear that $\overline{\mathfrak{m}}$ (which we shall also write as $\mathfrak{m}^-$) consists of the linear forms $\overline{l}$ arising from the linear forms $l \in \mathfrak{m}_\tau$ upon the substitution $\tau \to a$, that is, $\overline{\mathfrak{m}} = \{l(x, a, z) \mid l(x, \tau, z) \in \mathfrak{m} \cap \Sigma k[\tau, x] \cdot z_i\}$.

THEOREM 1. *There exists a basis $l_1^*, \cdots, l_t^*$ of the module $\mathfrak{m}$ such that for any substitution $\tau \to a$, the module $\overline{\mathfrak{m}}$ has $\overline{l}_1^*, \cdots, \overline{l}_t^*$ as basis. If $l_1, \cdots, l_u$ is any basis of $\mathfrak{m}$, then $\overline{\mathfrak{m}} = (\overline{l}_1, \cdots, \overline{l}_u)$ almost always, that is, there exists a polynomial $f(\tau) \in k[\tau]$, different from zero, depending on $l_1, \cdots, l_u$, such that if $f(a) \neq 0$, then $\overline{\mathfrak{m}} = (\overline{l}_1, \cdots, \overline{l}_u)$.*

Proof [22]. Since $\mathfrak{m}_\tau = \mathfrak{m} \cap (R_\tau \cdot z_1 + \cdots + R_\tau \cdot z_s)$ is a module contained in a finite module over a chain theorem ring, it has itself a finite basis, say $l_1^*, \cdots, l_t^*$; these linear forms also clearly form a basis for $\mathfrak{m}$. By definition, $(\overline{l}_1^*, \cdots, \overline{l}_t^*) \subset \overline{\mathfrak{m}}$; since $\overline{\mathfrak{m}}$ is generated by linear forms $\overline{l}$, where $l = r_1 l_1^* + \cdots + r_t l_t^*, r_i \in k[\tau, x]$, we have $\overline{\mathfrak{m}} \subset (\overline{l}_1^*, \cdots, \overline{l}_t^*)$, whence $\overline{\mathfrak{m}} = (\overline{l}_1^*, \cdots, \overline{l}_t^*)$ for any substitution $\tau \to a$. If now $l_1, \cdots, l_u$ is any basis of $\mathfrak{m}$, then $l_i^* = \sum_{j=1}^u r_{ij} l_j$, $r_{ij} \in R$, and therefore almost always $\overline{\mathfrak{m}} = (\overline{l}_1^*, \cdots, \overline{l}_t^*) \subset (\overline{l}_1, \cdots, \overline{l}_u)$. Since also $(\overline{l}_1, \cdots, \overline{l}_u) \subset \overline{\mathfrak{m}}$ almost always, we have $\overline{\mathfrak{m}} = (\overline{l}_1, \cdots, \overline{l}_u)$ almost always, q.e.d.

THEOREM 2. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be two submodules of $R \cdot z_1 + \cdots + R \cdot z_s$. Then we always have $\mathfrak{m}^- + \mathfrak{n}^- \subseteq (\mathfrak{m} + \mathfrak{n})^-$, and almost always $\mathfrak{m}^- + \mathfrak{n}^- = (\mathfrak{m} + \mathfrak{n})^-$.*

Proof. The inclusion $\mathfrak{m}^- + \mathfrak{n}^- \subseteq (\mathfrak{m} + \mathfrak{n})^-$ is trivial. For the second part of the theorem, let $l_1, \cdots, l_t$ be a basis of $\mathfrak{m}$ and $l_1', \cdots, l_u'$ a basis of $\mathfrak{n}$, so that $l_1, \cdots, l_t, l_1', \cdots, l_u'$ form a basis for $\mathfrak{m} + \mathfrak{n}$. By the previous theorem there exist nonzero polynomials $f(\tau), g(\tau), h(\tau) \in k[\tau]$ such that $\mathfrak{m}^- = (\overline{l}_1, \cdots, \overline{l}_t)$ if $f(a) \neq 0$, $\mathfrak{n}^- = (\overline{l}_1', \cdots, \overline{l}_u')$ if $g(a) \neq 0$, and $(\mathfrak{m} + \mathfrak{n})^- = (\overline{l}_1, \cdots, \overline{l}_t, \overline{l}_1', \cdots, \overline{l}_u')$ if $h(a) \neq 0$. Hence if $f(a) \cdot g(a) \cdot h(a) \neq 0$, then $(\mathfrak{m} + \mathfrak{n})^- = (\overline{l}_1, \cdots, \overline{l}_t, \overline{l}_1', \cdots, \overline{l}_u') = \mathfrak{m}^- + \mathfrak{n}^-$, q.e.d.

In the case of ideals $(s=1, z_1=1)$ $\mathfrak{A}$, $\mathfrak{B}$ we can say that always $\mathfrak{A}^- \mathfrak{B}^- \subseteq (\mathfrak{A}\mathfrak{B})^-$ and almost always $\mathfrak{A}^- \cdot \mathfrak{B}^- = (\mathfrak{A}\mathfrak{B})^-$, the proofs being quite parallel to those above.

THEOREM 3. *If $\mathfrak{A}$ is an ideal of R and $\mathfrak{A} \neq R$ then almost always $\overline{\mathfrak{A}} \neq \overline{R}$.*

Proof. The proof is by induction on $\dim R/k(\tau) = n$. First, if $n=1$, then $\mathfrak{A} = (f(x_1, \tau)), f(x_1, \tau) \in k[\tau, x_1]$, is a principle ideal and degree $x_1 f(x_1) > 0$. Clearly $f(x_1, a)$ is almost always of positive degree in x_1, and since $\overline{\mathfrak{A}} = (f(x_1, a))$ almost always, we have that $\overline{\mathfrak{A}}$ is almost always a principle ideal with a generator of positive degree, whence $\overline{\mathfrak{A}} \neq (1)$. For $n > 1$, we first remark

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[22] The argument here follows that of Krull [6; Theorem 1, p. 57].

that we may confine ourselves to zero-dimensional prime ideals, since any ideal $\mathfrak{A} \neq R$ is contained in such an ideal. We assume then that $\mathfrak{A} = \mathfrak{p}$ is a zero-dimensional prime ideal. Let $R^{(n)} = k(\tau) [x_1, \cdots, x_{n-1}]$, and let $\mathfrak{p}^{(n)} = \mathfrak{p} \cap R^{(n)}$. By induction we have that $\overline{\mathfrak{p}^{(n)}} \neq (1)$ almost always. Let $f(x_1, \cdots, x_{n-1}, x_n) \equiv 0(\mathfrak{p})$ be the congruence of least degree in x_n satisfied by x_n: here we may, and do, assume that the coefficient of the highest power of x_n is 1. It follows quite simply that $\mathfrak{p} = R \cdot \mathfrak{p}^{(n)} + (f)$. Now almost always $\bar{\mathfrak{p}} = \overline{R} \cdot \overline{\mathfrak{p}^{(n)}} + (\bar{f})$ and $\bar{f}$ has the same degree in x_n as f does. When these conditions obtain, any zero of $\overline{\mathfrak{p}^{(n)}}$ may be extended to a zero of $\bar{\mathfrak{p}}$. Since $\overline{\mathfrak{p}^{(n)}} \neq (1)$ almost always, $\overline{\mathfrak{p}^{(n)}}$ and hence also $\overline{\mathfrak{P}}$ has a zero almost always, that is, $\overline{\mathfrak{P}} \neq (1)$ almost always.

Our main object now is to prove the following theorem:

If $\mathfrak{A}$ is an unmixed r-dimensional ideal in R, then almost always $\overline{\mathfrak{A}}$ is unmixed r-dimensional

First observe that $\overline{\mathfrak{A}}$ is almost always r-dimensional or less. For if $f_i(x_1, \cdots, x_r, x_{r+i}, \tau) \in \mathfrak{A} \cap k[\tau, x]$, $i = 1, \cdots, n-r$, and $f_i \neq 0$, then always $f_i(x_1, \cdots, x_r, x_{r+i}, a) \in \overline{\mathfrak{A}}$, and almost always these yield proper algebraic relations for $x_{r+1}, x_{r+2}, \cdots, x_{r+n}$ over $k[x_1, \cdots, x_r]$ mod $\overline{\mathfrak{A}}$. Thus if $\mathfrak{A}$ is zero-dimensional, then almost always $\overline{\mathfrak{A}}$ is zero-dimensional or (-1)-dimensional, that is, $\overline{\mathfrak{A}} = (1)$. Let $g(\tau), h(\tau) \in k[\tau]$ be such that $\overline{\mathfrak{A}}$ is zero or (-1)-dimensional if $g(a) \neq 0$ and $\overline{\mathfrak{A}} \neq (1)$ if $h(a) \neq 0$. Then $g(\tau)h(\tau)$ is a polynomial $\neq 0$ such that $\overline{\mathfrak{A}}$ is zero-dimensional if $g(a) \cdot h(a) \neq 0$.

We now prove the theorem for $r = 1$, and then proceed by induction. First we need a lemma.

Let $\mathfrak{m}$ be a module over R contained in a linear form module over R, and let $l_1, \cdots, l_t$ be a basis for $\mathfrak{m}$:

$$l_i = f_{i1}z_1 + \cdots + f_{is}z_s, \qquad\qquad i = 1, \cdots, t.$$

Let $\|f_{ij}\|$ be of rank p. We shall say that the basis $l_1, \cdots, l_t$ is regular with respect to $(x_1, \cdots, x_n)$ if at least one of the p-rowed subdeterminants of $\|f_{ij}\|$ is regular in x_n, that is, the determinant shall be of (not necessarily positive) degree d in x_n and have a term in x_n of that degree with coefficient independent of $x_1, \cdots, x_{n-1}$. We shall also say that $\mathfrak{m}$ is regular with respect to $(x_1, \cdots, x_n)$ if $\mathfrak{m}$ has a basis $l_1, \cdots, l_t$ of the kind mentioned.

Let N be a positive integer and consider the $R_n = k(\tau) [x_1, \cdots, x_{n-1}]$-module $\mathfrak{n}$ generated by $l_1, \cdots, l_t, x_n l_1, \cdots, x_n l_t, \cdots, x_n^N l_1, \cdots, x_n^N l_t$. The module $\mathfrak{n}$ is contained in a linear form module: if q is an integer equal to or greater than the maximum of degree in x_n of the f_{ij} and if we place $\zeta_{i+sj} = z_i x_n^j$, $j = 1, \cdots, N+q$, then we can write:

$$\mathfrak{n} \subset R_n \zeta_1 + R_n \zeta_2 + \cdots + R_n \zeta_g, \qquad g = s + s(N + q).$$

We shall have

$$x_n^j l_i = \sum_{k=1}^{g} h_{ijk}(x_1, \cdots, x_{n-1}) \zeta_k.$$

Thus if $\mathfrak{m}$ is regular with respect to $x_1, \cdots, x_n$, we can subject $x_1, \cdots, x_{n-1}$ to a linear homogeneous nonsingular transformation so that $\mathfrak{m}$ will still be regular in the new variables $x_1', \cdots, x_{n-1}', x_n$, and $\mathfrak{n}$ will be regular with respect to $x_1', \cdots, x_{n-1}'$.

Let $S = k(\tau, x_1)[x_2, \cdots, x_n]$ and $S_n = k(\tau, x_1)[x_2, \cdots, x_{n-1}]$. We consider the module $S \cdot \mathfrak{m} \cap \sum R \cdot z_i$, which is also defined for $n = 1$, but as we shall have to consider a similarly constructed module for $(n-1)$ dimensions, we suppose $n \geq 2$; so $x_1 \neq x_2$. The module $S \cdot \mathfrak{m} \cap \sum R \cdot z_i$ consists of the linear forms l in $\sum R \cdot z_i$ for which there exists a polynomial $F(x_1) \in k(\tau)[x_1]$ such that $F(x_1) \cdot l \in \mathfrak{m}$: the set described clearly forms an R-module.—In the lemma, let $N = qt$.

LEMMA. *If $\mathfrak{m}$ is regular with respect to $x_1, \cdots, x_n$ $(n \geq 2)$, then $S \cdot \mathfrak{m}$ $\cap \sum_1^s R \cdot z_i = R \cdot (S_n \mathfrak{n} \cap \sum_1^q R_n \cdot \zeta_i) + \mathfrak{m}$, where we suppose $l_1, \cdots, l_t$ to be a given regular basis of $\mathfrak{m}$, and $N = qt$.*

Proof [23]. Let $l \in S \cdot \mathfrak{m} \cap \sum R \cdot z_i$, $l = g_1(x)z_1 + \cdots + g_s(x) \cdot z_s$. We wish to place a limitation on the degree in x_n of l or of the $g_i(x)$, mod $\mathfrak{m}$. The matrix $\|f_{ij}\|$ is of rank p, and one of its p-rowed subdeterminants is regular in x_n: say this is

$$D = \begin{vmatrix} f_{11} \cdots f_{1p} \\ \vdots \\ f_{p1} \cdots f_{pp} \end{vmatrix}.$$

One finds then in $\mathfrak{m}$ elements m_i, $i = 1, \cdots, p$, such that

$$m_i - Dz_i \in R \cdot z_{p+1} + \cdots + R \cdot z_s,$$

and hence we may suppose (replacing l by a congruent linear form mod $\mathfrak{m}$) that

$$\deg g_i(x) < \deg D, \qquad\qquad i = 1, \cdots, p,$$

where "deg" stands for the degree in x_n, as it does throughout the proof. Since $l \in S \cdot \mathfrak{m} \cap \sum R \cdot z_i$, there exists a polynomial $F(x_1) \in k(\tau)[x_1]$ such that $F(x_1) \cdot l \in \mathfrak{m}$, so that

$$F(x_1)l = a_1(x)l_1 + \cdots + a_t(x)l_t$$

and since $Dl_{p+\lambda} \in R \cdot l_1 + \cdots + R \cdot l_p$, we may and do choose the a_i such that

$$\deg a_{p+\lambda} < \deg D \leq qt.$$

Now we have

$$F(x_1)g_i = \sum_{j=1}^{t} a_j f_{ji},$$

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[23] The algorithm employed in this proof is taken from Hermann's paper [4].

$$\sum_{i=1}^{p} F(x_1)g_i F_{ki} = \sum_{i=1}^{p}\sum_{j=1}^{t} a_j f_{ji} F_{ki} = Da_k + \sum_{i=1}^{p}\sum_{j=p+1}^{t} a_j f_{ji} F_{ki}, \qquad k = 1, \cdots, p,$$

where F_{ki} is the cofactor of f_{ki} in D. Since

$$\deg F(x_1)g_i F_{ki} < \deg D + q(t-1), \qquad i = 1, \cdots, p,$$

and

$$\deg a_j f_{ji} F_{ki} < \deg D + qt, \qquad j = p+1, \cdots, s,$$

we have

$$\deg Da_k < \deg D + qt,$$

whence

$$\deg a_k < qt = N, \qquad k = 1, \cdots, s.$$

Hence $F(x_1)\cdot l \in \mathfrak{n}$, that is, $l \in S_n \cdot \mathfrak{n}$; also clearly $l \in \sum R_n \cdot \zeta_i$; hence $l \in S_n \cdot \mathfrak{n}$ $\cap \sum R \cdot \zeta_i$, and the original l is in $(S_n \mathfrak{n} \cap \sum R_n \zeta_i) + \mathfrak{m}$. The opposite inclusion involved is trivial.

We consider transformations

$$U: \quad y_i = \sum_{j=1}^{n} g_{ij} x_j, \qquad g_{ij} \in k, i = 1, \cdots, n,$$

and speak of "all" or "almost all" such linear transformations and "all" or "almost all" coördinate systems $(y_1, \cdots, y_n)$. In order to reserve the x's for current use, let $(y_1, \cdots, y_n)$ be a fixed coördinate system and consider the transformations:

$$T: \quad x_i = \sum_{k=1}^{n} c_{ik} y_k, \qquad c_{ik} \in k, i = 1, \cdots, n.$$

THEOREM 4. *There exist coördinate systems $(x_1, \cdots, x_n)$ such that*

$$\overline{S\mathfrak{m} \cap \sum R \cdot z_i} = \overline{S} \cdot \overline{\mathfrak{m}} \cap \sum \overline{R} \cdot z_i$$

holds almost always. $(\overline{S} = k(x_1)[x_2, \cdots, x_n].)$ Moreover, the c_{ij} may be selected to satisfy any given (proper) inequalities.

Proof. For $n = 1$, we have no restriction on the coördinate system. Let $\mathfrak{m} = (l_1, \cdots, l_t)$,

$$l_i = f_{i1}(x_1)\cdot z_1 + \cdots + f_{is}(x_1)\cdot z_s, \qquad i = 1, \cdots, t.$$

It is well known [13; §106, Elementarteilersatz] that we may select a new basis $z_1', \cdots, z_s'$ for the R-module $R\cdot z_1 + \cdots + R\cdot z_s$, and a new basis $l_1', \cdots, l_p'$ for $\mathfrak{m}$ such that $l_j' = g_j(x_1)z_j', j = 1, \cdots, p$. Let $z_i' = \sum a_{ij}(x_1)z_j$ and $z_k' = \sum b_{kl}(x_1)z_l$: then the matrices $\|a_{ij}\|$ and $\|b_{kl}\|$ are inverses, and al-

most always $\|\bar{a}_{ij}\|$ and $\|\bar{b}_{kl}\|$ will also be inverses. Hence $\bar{z}_1', \cdots, \bar{z}_s'$ will be a basis for $\bar{R} \cdot z_1 + \cdots + \bar{R} \cdot z_s$ almost always, and $\bar{l}_j' = \bar{g}_j(x_1)\bar{z}_j', j = 1, \cdots, p,$ will almost always be a basis for $\bar{m}$, and we now suppose these conditions to obtain: the module $(z_1', \cdots, z_p')$ maps almost always into the module $(\bar{z}_1', \cdots, \bar{z}_p')$, and we suppose this condition also to obtain. Now $S \cdot \mathfrak{m} \cap \sum R \cdot z_i = (z_1', \cdots, z_p')$, and $\bar{S} \cdot \bar{\mathfrak{m}} \cap \sum \bar{R} \cdot z_i = (\bar{z}_1', \cdots, \bar{z}_p')$, from which the theorem for $n = 1$ is immediate.

Now for the induction step: we pass from the fixed coördinates $(y_1, \cdots, y_n)$ to $(x_1, \cdots, x_n)$ in two steps. We first pass to: $x_j' = \sum_{k=1}^n b_{jk}y_k, j = 1, \cdots, n,$ $b_{jk} \in k$, in such fashion that $\mathfrak{m}$ is regular with respect to $x_1', \cdots, x_n'$; and in fact we fix a basis $(l_1, \cdots, l_t)$ of $\mathfrak{m}$ and this basis will be a basis of regularity with respect to $x_1', \cdots, x_n'$ for almost all coördinate systems $(x_1', \cdots, x_n')$. Previously we placed $l_i = \sum f_{ij} z_j$, supposed $\|f_{ij}\|$ to be of rank p, and q to be an integer equal to or greater than the maximum of the degree in x_n of the f_{ij}: if we place q to be the maximum of the degree of the f_{ij} in all the variables $x_1, \cdots, x_n$, we shall have a q which will serve in any of the coördinate systems $(x_1', \cdots, x_n')$. In the transformation from $(x_1', \cdots, x_n')$ to $(x_1, \cdots, x_n)$ we place $x_n = a_{nn}x_n$, so that the ring $R_n = k(\tau)[x_1, \cdots, x_{n-1}]$ is already determined, as are the modules $\mathfrak{n}$ and $\sum R_n \zeta_i$; but the ring $S_n = k(\tau, x_1)[x_2, \cdots, x_{n-1}]$ is not as yet determined. We now pass to: $x_i = \sum_{j=1}^n a_{ij}x_j'$, $i = 1, \cdots, n$, $a_{ij} \in k$, $a_{in} = a_{ni} = 0$ if $i \neq n$. For some choice of the a_{ij} we have by induction that

$$\overline{S \cdot \mathfrak{n} \cap \sum R_n \zeta_i} = \overline{S_n \cdot \mathfrak{n}} \cap \sum \overline{R_n \zeta_i}$$

holds almost always, and moreover the a_{ij} may be selected to satisfy any given inequalities. Applying the lemma and the theorem on the sum of two modules, we have

$$\overline{S \cdot \mathfrak{m} \cap \sum R \cdot z_i} = \overline{R \cdot \overline{S_n \mathfrak{n}} \cap \sum \overline{R_n \zeta_i}} + \overline{\mathfrak{m}} = \overline{R \cdot (\overline{S_n \cdot \mathfrak{n}} \cap \sum \overline{R_n \zeta_i})} + \overline{\mathfrak{m}}$$

$$= \overline{S \cdot \mathfrak{m}} \cap \sum \overline{R \cdot z_i},$$

almost always ($\bar{\mathfrak{m}}$ is regular almost always). It remains to see whether the $c_{ik} = \sum a_{ij}b_{jk}$ may be chosen to satisfy any given inequality $F(c_{ik}) \neq 0$. Since almost all choices of the b_{jk} are available, we can clearly so select them that $F(\sum a_{ij}b_{jk})$ does not become identically zero in the a_{ij}; and then we may select the a_{ij} so that $F(\sum a_{ij}b_{ik}) \neq 0$, that is, $F(c_{ik}) \neq 0$, q.e.d.

COROLLARY. *The theorem is satisfied if the c_{ik} are indeterminants, where the c_{ik} are tacitly adjoined to $k(\tau)$ (though the exceptional points still lie on a proper sub-variety over the original k, as is easily seen).*

Proof. It is clear that for the b_{jk} we may take indeterminates; the a_{ij} are in $k(b_{jk})$, and if $c_{ik} = \sum a_{ij}b_{jk}$, then $k(\tau, c_{ik}) = k(\tau, b_{ij})$, whence the c_{ik} are alge-

braically independent over $k(\tau)$. It is clear we may interpret the c_{ik} as indeterminates.

THEOREM 5. *If $\mathfrak{A}$ is an unmixed 1-dimensional R-ideal, then also $\overline{\mathfrak{A}}$ is almost always unmixed 1-dimensional.*

Proof. We subject the fixed coördinate system $(y_1, \cdots, y_n)$ to a general linear homogeneous transformation, that is, we understand the c_{ij} to be indeterminates. Then we shall have $S\mathfrak{A}\cap R = \mathfrak{A}$. By the previous theorem, almost always

$$\overline{S\cdot\mathfrak{A}\cap R} = \overline{S\cdot\mathfrak{A}}\cap\overline{R},$$

that is, $\overline{S}\cdot\overline{\mathfrak{A}}\cap\overline{R} = \overline{\mathfrak{A}}$ almost always, whence $\overline{\mathfrak{A}}$ has no zero-dimensional components. Since $\overline{\mathfrak{A}}$ is almost always of dimension $s \leqq 1$, we have that $\overline{\mathfrak{A}}$ is unmixed.—We may remark that the proof could be carried out keeping the c_{ij} in the field k. The c_{ij} would have to be restricted so that the previous theorem is applicable and also so that $S\cdot\mathfrak{A}\cap R = \mathfrak{A}$: one has therefore to check that $S\cdot\mathfrak{A}\cap R = \mathfrak{A}$ for some c_{ij}, and this is the case, in fact, for almost all c_{ij}.

THEOREM 6. *If $\mathfrak{A}$ is an unmixed r-dimensional R-ideal, then also $\overline{\mathfrak{A}}$ is almost always unmixed r-dimensional.*

Proof. As in the previous theorem, we subject the given fixed coördinate system to a general linear homogeneous transformation. We have $S\cdot\mathfrak{A}\cap R = \mathfrak{A}$ (we are assuming $r > 0$, since the theorem has been proved separately for $r = 0$) and

$$\overline{\mathfrak{A}} = \overline{S\mathfrak{A}\cap R} = \overline{S\cdot\mathfrak{A}}\cap\overline{R}$$

almost always. The ideal $S\cdot\mathfrak{A}$ is an unmixed $(r-1)$-dimensional S-ideal, hence almost always $\overline{S}\cdot\overline{\mathfrak{A}}$ is unmixed $(r-1)$-dimensional, by induction. The fact that $\overline{S}\cdot\overline{\mathfrak{A}}\cap\overline{R} = \overline{\mathfrak{A}}$ shows that $\overline{\mathfrak{A}}$ has no zero-dimensional prime ideals. Moreover, $\overline{\mathfrak{A}}$ is a "transformed" ideal, just as $\mathfrak{A}$ is, that is, it has a basis which is independent of the parameters c_{ij}. Hence if $\overline{\mathfrak{A}}$ has an s-dimensional prime ideal, then $\overline{S}\cdot\overline{\mathfrak{A}}$ will have an $(s-1)$-dimensional prime ideal, whence $s - 1 = r - 1$, $s = r$, that is, $\overline{\mathfrak{A}}$ is unmixed r-dimensional almost always. This completes the proof.—If one wishes to have at hand a polynomial $f(\tau) \in k[\tau]$ such that $\overline{\mathfrak{A}}$ is unmixed r-dimensional if $f(a) \neq 0$, one can obtain it by induction as follows: we have $\overline{S}\cdot\overline{\mathfrak{A}}$ is unmixed $(r-1)$-dimensional almost always, that is, there exists a polynomial $F(c_{ij}, x_1, \tau) \in k[c_{ij}, x_1, \tau]$, $F \neq 0$, such that $\overline{S}\cdot\overline{\mathfrak{A}}$ is unmixed $(r-1)$-dimensional if $F(c_{ij}, x_1, a) \neq 0$. Since the c_{ij} and x_1 are algebraically independent over k, we can write F uniquely as a polynomial in the c_{ij}, x_1 with coefficients in $k[\tau]$. Let $h(\tau) \in k[\tau]$ be one of these coefficients, and let $g(\tau) \neq 0$, $g(\tau) \in k[\tau]$ be such that $\overline{S}\cdot\overline{\mathfrak{A}}\cap\overline{R} = \overline{\mathfrak{A}}$ if $g(a) \neq 0$. Then $f(\tau) = g(\tau)h(\tau)$ satisfies the condition required.

In the following theorem, k may be finite.

THEOREM 7. *Let $k(\tau)[x]\cdot\mathfrak{A}$ be unmixed. There exists a variety U/k, not the whole space, such that if $(a_1, \cdots, a_m)\not\in U/k$, then $k(a)[x]\cdot\mathfrak{A}$ is unmixed, where the a_i may be taken from an arbitrary extension field k.*

Proof. First let us consider points (a) which are algebraic over k. By Lemma 6, $\bar{k}(\tau)[x]\cdot\mathfrak{A}$ is unmixed, so $\bar{k}[x]\cdot\bar{\mathfrak{A}}$ is unmixed almost always by the previous result, that is, there is a variety $\bar{U}/\bar{k}$, not the whole space, such that $\bar{k}[x]\cdot\bar{\mathfrak{A}}$ is unmixed if (a) is not on $\bar{U}$; instead of $\bar{U}$, we may take a variety U/k, not the whole space, on which all points of $\bar{U}$ lie. Contracting to $k(a)$, we get that $\bar{k}[x]\cdot\bar{\mathfrak{A}}\cap k(a)[x] = k(a)[x]\cdot\mathfrak{A}$ is unmixed.

Now let us consider points (a) which are of dimension 1 over k, and say a_1 is transcendental over k. Let us regard τ_1 as adjoined to the ground-field, and only $\tau_2, \cdots, \tau_m$ as parameters. By the previous paragraph, there exists a polynomial in $\tau_2, \cdots, \tau_m$ over $k(\tau_1)$, or what comes to the same thing, a polynomial $f(\tau_1, \tau_2, \cdots, \tau_m)\in k[\tau_1, \cdots, \tau_m]$, $f\neq 0$, such that $k(\tau_1, b_2, \cdots, b_m)[x]\cdot\mathfrak{A}$ is unmixed provided $f(\tau_1, b_2, \cdots, b_m)\neq 0$ and b_i are algebraic over $k(\tau_1)$. Let now (a) be a point of the kind just mentioned, and suppose $f(a_1, a_2, \cdots, a_m)\neq 0$. Clearly there exists a point $(\tau_1, b_2, \cdots, b_m)$ which is k-isomorphic with $(a_1, \cdots, a_m)$, so $k(\tau_1, b_2, \cdots, b_m)[x]\cdot\mathfrak{A}_{\tau_1, b_2, \cdots, b_m}$ is unmixed, where $\mathfrak{A} = \mathfrak{A}_{\tau_1, \tau_2, \cdots, \tau_m}$. Hence also clearly $k(a)[x]\cdot\mathfrak{A}_{a_1, \cdots, a_m}$ is unmixed. In the same way we can dispose of the points of dimension 2, 3, $\cdots$, m over k.

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