THE HYPERQUASICENTER OF A FINITE GROUP. II

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ABSTRACT. The role of Theorem B of Hall and Higman has been explained in detail to complete the proof of the fact that the hyperquasicenter is the largest supersolvably immersed subgroup. Other results included contain some sufficient conditions for super-solvability of a group and for the nontriviality of its center.

I. The proof of Theorem 1.2 in *The hyperquasicenter of a finite group*. I (Proc. Amer. Math. Soc. **26** (1970)) needs to be elaborated. One needs to consider only the case when r is positive and is a prime and induction may be applied to show that x^r is a p-group. Once this is done the proof may be completed by breaking it up into two cases: (i) $r=p$. In this case the proof may be completed by induction if $|x|$ is divisible by a prime different from p, or else one needs to consider the case when $G=\langle x\rangle\langle y\rangle$ is a p-group. In this case the proof may be completed as given in the paper by taking Z to be a subgroup of order p. (ii) $r\neq p$. In this case one needs to use Ore's theorem [**1**, p. 438, Theorem 16] namely that every quasinormal subgroup of a group is subnormal.

In this connection, it may be said that it was brought to the author's notice that the proofs of Theorems 1.2 and 1.8 needed to be modified. These imperfections are attributable to complete breakdown of communications between the author and the referee. The proof of Theorem 1.8 follows from Theorem 6.1 of Baer in [**4**], and that requires that G induces in every chief factor contained in P, a Sylow p-subgroup of the quasicenter, an abelian group of automorphisms. The verification of the fact that G does satisfy this requirement is crucially dependent on the well-known Theorem B of Hall and Higman in [**2**]. Several other results have been included without proof. They are not difficult to verify.

II.

THEOREM. *If x is a QC-element of a group G then x^r is also a QC-element of G for every integer r.*

PROOF. It is evident that it is sufficient to prove the theorem when r is positive and r is a prime. Let y be an arbitrary but fixed element of G. Then

we must prove $\langle x^r \rangle \langle y \rangle = \langle y \rangle \langle x^r \rangle$ for all r. The proof is by induction on $|G|$ and so we need only consider the case when $G = \langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$. If $|y| = p_1^{a_1} \cdot p_2^{a_2} \cdots p_n^{a_n}$ where p_i for $i = 1, 2, \cdots, n$ is a prime then $\langle y \rangle = \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_n \rangle$ where y_i is a power of y and $|y_i| = p_i^{a_i}$. Then if $\langle x^r \rangle \langle y_i \rangle = \langle y_i \rangle \langle x^r \rangle \; \forall \; i$, we will have $\langle x^r \rangle \langle y \rangle = \langle y \rangle \langle x^r \rangle$. So we need only consider the case when y is a p-element and $G = \langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$. Let $|y| = p^b$. If $|x| = p^a q$, $(p, q) = 1$ then $\langle x \rangle = \langle x_1 \rangle \times \langle x_2 \rangle$ where x_i, for $i = 1, 2$, is a power of x and $|\langle x_1 \rangle| = p^a$ and $|\langle x_2 \rangle| = q$. Since a quasinormal subgroup of a group is subnormal [1, p. 438, Theorem 16] it follows that $\langle x \rangle$ is subnormal in G. If $\langle x \rangle$ is normal in G then $\langle x^r \rangle \lhd G$ and so $\langle x^r \rangle \langle y \rangle = \langle y \rangle \langle x^r \rangle$. Otherwise we have a subnormal chain as follows: $\langle x \rangle \lhd \lhd \lhd \cdots \lhd H \lhd G$. Since $x \in H$ and any element in H can be written as $x^m y^n$ where m, n are integers it follows that $\{y^n | x^m y^n \in H\}$ is a subgroup $\langle y^d \rangle$ of $\langle y \rangle$. Hence $H \subseteq \langle x \rangle \langle y^d \rangle$ and so $H = \langle y^d \rangle \langle x \rangle = \langle x \rangle \langle y^d \rangle$ since H is a group.

Induction applies to H and it follows that $x_2 = x^{p^a}$ is a QC-element of H. We claim that $\langle x_2 \rangle \lhd H$. Otherwise $T = \langle x_2 \rangle^{y^d} \neq \langle x_2 \rangle$ is a cyclic subgroup of H of order q. But x_2 is a QC-element of H and so it follows that $\langle x_2 \rangle \cdot T = T \cdot \langle x_2 \rangle = R$ is a subgroup of H and $|R| = |T| \; |\langle x_2 \rangle| / |T \cap \langle x_2 \rangle| = q \cdot v$ where v is a divisor of q different from 1. Since $(p, q) = 1$ it follows that $(p, v) = 1$. But R being a subgroup of H, $|R|$ must divide $|H|$ which is impossible since $|H| = p^c q$ for some integer c and v does not divide p. Hence $\langle x_2 \rangle \lhd H$. Since $|x_2|$ is prime to p, every Sylow subgroup of $\langle x_2 \rangle$ is also a Sylow sugbroup of H and is therefore normal in H. This implies that $\langle x_2 \rangle$ is characteristic in H Therefore, $\langle x_2 \rangle \lhd G$ since $H \lhd G$.

Suppose now $r = p$. Since $\langle x \rangle = \langle x_1 \rangle \times \langle x_2 \rangle$ it follows that $\langle x^p \rangle = \langle x_1^p \rangle \times \langle x_2 \rangle$. If $\langle x_2 \rangle \neq 1$ then induction applies to $\bar{G} = G / \langle x_2 \rangle$ and we have $\langle \bar{x}_1^p \rangle \langle \bar{y} \rangle = \langle \bar{y} \rangle \langle \bar{x}_1^p \rangle$, where $\bar{x}_1^p = \langle x_2 \rangle \cdot x_1^p$ and $\bar{y} = \langle x_2 \rangle \cdot y$. This equation yields $\langle x_2 \rangle \langle x_1^p \rangle \langle y \rangle = \langle x_2 \rangle \langle y \rangle \langle x_1^p \rangle$, i.e. $\langle x^p \rangle \langle y \rangle = \langle y \rangle \langle x^p \rangle$. If $\langle x_2 \rangle = 1$ then $G = \langle x \rangle \langle y \rangle$ is a p-group. In this case the proof may be completed by repeating the arguments in Theorem 1.2 of [3] by taking Z to be a subgroup of order p.

Suppose $r \neq p$. If $\langle x_2 \rangle = 1$ then $\langle x \rangle$ is a p-group and $\langle x^r \rangle = \langle x \rangle$. Therefore x^r is a quasicentral element in this case. Assume therefore $\langle x_2 \rangle \neq 1$. If r is prime to q then also $\langle x^r \rangle = \langle x_1 \rangle \times \langle x_2 \rangle = \langle x \rangle$ and x^r is a quasicentral element.

Let $r = q_1$ where q_1 is a prime divisor of q. Since $\langle x^r \rangle = \langle x_1 \rangle \times \langle x_2^{q_1} \rangle$ it follows that if $\langle x_2^{q_1} \rangle \neq 1$ then induction applies to $G / \langle x_2^{q_1} \rangle$ and the proof may be completed as before. If $\langle x_2^{q_1} \rangle = 1$, i.e., $\langle x^r \rangle = \langle x_1 \rangle$ then we proceed as follows.

Since $\langle x \rangle$ is subnormal in G it follows that $\langle x_1 \rangle = \langle x^r \rangle$ is subnormal in G and so we have a chain as follows: $\langle x^r \rangle \lhd \langle x \rangle \lhd X_1 \lhd X_2 \lhd \cdots \lhd X_k = G$. Now, $\langle x^r \rangle = O_p(\langle x \rangle)$ implies that $\langle x^r \rangle \lhd X_1$ and so $\langle x^r \rangle \subseteq O_p(X_1)$. Assume $\langle x^r \rangle \subseteq O_p(X_i)$ and since $O_p(X_i) \lhd X_{i+1}$ it follows that $O_p(X_i) \subseteq O_p(X_{i+1})$. Hence $\langle x^r \rangle \subseteq O_p(X_{i+1})$ and so $\langle x^r \rangle \subseteq O_p(X_k) = O_p(G)$. Set $M = O_p(G) \langle y \rangle$. Since

$\langle x \rangle \langle y \rangle \supseteq M \supseteq \langle y \rangle$, $M = (\langle x \rangle \cap M) \langle y \rangle$. But $\langle x^r \rangle \subseteq \langle x \rangle \cap M \neq \langle x \rangle$ and therefore $\langle x^r \rangle = \langle x \rangle \cap M$. This implies that $\langle x^r \rangle \langle y \rangle = M$ is a group and the theroem is proved completely.

It is natural to ask whether the nontriviality of the quasicenter guarantees the nontriviality of the center under suitable conditions. The following theorems provide some answers.

THEOREM. *If the smallest prime divisor of the order of a group divides the order of its quasicenter then G has a nontrivial center.*

LEMMA. *If $\bar{P}_m$ is a normal Sylow p_m-subgroup of a group G and p_m divides the order of the quasicenter then the quasicenter has a cyclic normal subgroup of order p_m.*

THEOREM. *If the smallest prime divisor of the order of a group G divides the order of the hyperquasicenter and the corresponding Sylow subgroup of G is normal then G has a nontrivial center.*

III. It has been shown that the hyperquasicenter $Q^*(G)$ of a group G contains the largest supersolvably immersed subgroup of G. The proof is easy. It uses induction on G and depends on a lemma that the hyperquasicenter of G/N where $N \vartriangleleft G$ and $N \subseteq Q^*(G)$ is $Q^*(G)/N$. The proof of the lemma is quite straightforward. To prove $Q^*(G)$ is indeed supersolvably immersed in G we need the help of Theorem 6.1 of Baer in [4].

THEOREM. *The hyperquasicenter $Q^*(G)$ of a group G is the largest supersolvably immersed subgroup of G.*

PROOF. The fact that every supersolvably immersed subgroup of G is contained in $Q^*(G)$ has been proved in detail in Theorem 1.8 in [3]. To prove $Q^*(G)$ is supersolvably immersed in G we proceed by induction on $|G|$. Let Q_p be a Sylow p-subgroup of the quasicenter $Q(G)$ of G and let N be a minimal normal subgroup of G contained in Q_p. It may be observed that it is enough to prove that N is supersolvably immersed in G. For, then the result will follow by induction. To prove that N is supersolvably immersed in G we shall take the help of Theorem 6.1 in [4] which states that the following are equivalent:

1. N is supersolvably immersed in G.
2. Every chief factor of G contained in N is cyclic of order a prime.
3. G induces in every chief factor contained in N an abelian group of automorphisms and $x^{p-1} \circ y = 1$ for every prime p, every p-element y in $N \cap G'$ and every element x in G with $(|xC_G(N \cap G')|, p) = 1$.

Thus we need to prove 3 and for that matter it may be noted that it is enough to prove that $G/C_G(N)$ is abelian and is of exponent dividing $p - 1$. (Note that N is elementary abelian.)

Let $G_1 = G/C$ and $T = C_G(N)$ where $C = C_G(Q_p)$. Then $C \subseteq T$ and G_1/T_1, where $T_1 = T/C$ can be regarded as the group of automorphisms induced in N by G. It may be observed that $O_p(G_1)$ is contained in T_1. For otherwise $O_p(G_1) \cdot T_1/T_1$ will be a normal p-group of automorphisms of N and considering N as a vector space over $\mathrm{GF}(p)$ we observe that since G_1/T_1 has a faithful irreducible representation on N this is impossible. Hence $O_p(G_1)$ must necessarily be contained in T_1 which implies that in order to prove G_1/T_1 is abelian and is of exponent dividing $p-1$ it is enough to show that $G_1/O_p(G_1)$ possesses the same properties.

Let $\bar{G} = G/\Phi(Q_p)$, $C^* = C_G(Q_p/\Phi(Q_p))$ and $\bar{Q}_p = Q_p/\Phi(Q_p)$. We shall now show that if $G_2 = G/C^*$ then it suffices to prove that $G_2/O_p(G_2)$ is abelian and is of exponent dividing $p-1$ which in fact implies that $G_1/O_p(G_1)$ possesses the same properties.

It may be observed that $\bar{Q}_p$ can be considered as a vector space over $\mathrm{GF}(p)$ and the group of automorphisms induced by G in $\bar{Q}_p$, i.e. $G/C^* = G_2$, can be regarded as a group of linear transformations of $\bar{Q}_p$.

Since $C_1 = C^*/C$ is a group of automorphisms that induces identity on $\bar{Q}_p$ it follows that $|C_1|$ is a power of p and hence the normal p-subgroup C_1 of G_1 is contained in $O_p(G_1)$. Also, if $O_p(G_2) = L/C_1$ then it implies that L is a normal p-subgroup of G_1 and therefore $L \subseteq O_p(G_1)$. Thus $O_p(G_2) = O_p(G_1)/C_1$ and $G_2/O_p(G_2)$ is isomorphic to $G_1/O_p(G_1)$. As a first step towards proving $G_2/O_p(G_2)$ is abelian and is of exponent dividing $p-1$ we shall show that $O_p(G_2)$ possesses the same properties.

From Lemma 1.4 in [3] we know that the Sylow p-subgroup of the quasicenter is generated by QC-elements and this implies that there exists QC-elements $x_1, x_2, \cdots, x_n$ such that $B = \{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n\}$ is a basis for $\bar{Q}_p$ where $\bar{x}_i = x_i \Phi(Q_p)$. Now, if y is a p'-element of G then $\langle x_i \rangle \langle y \rangle = \langle y \rangle \langle x_i \rangle$ implies that y normalises $\langle x_i \rangle$. (This follows from the fact that $\langle x_i \rangle$ is a Sylow p-subgroup of the quasicenter of $S = \langle x_i \rangle \langle y \rangle = \langle y \rangle \langle x_i \rangle$.) Therefore $\bar{y} = y\Phi(Q_p)$ normalises $\bar{x}_i$. But $\Phi(Q_p) \subseteq C_G(Q_p/\Phi(Q_p)) = C^*$ and therefore $G_2 = G/C^*$ is a homomorphic image of $\bar{G}$ under the natural homomorphism θ of $\bar{G} = G/\Phi(Q_p)$ to $G_2 = G/C^*$. Therefore, if $\hat{y}$ is the image of $\bar{y}$ under θ then $\hat{y}$ shall send every basis element $\bar{x}_i$ to $s\bar{x}_i$ (using additive notation in $\bar{Q}_p$) where $s \in \mathrm{GF}(p)$. Therefore, the matrix of $\hat{y}$ is diagonal and $\hat{y}^{p-1} = 1$ since the multiplicative group of $\mathrm{GF}(p)$ is of order $p-1$ and the product of diagonal matrices is again a diagonal matrix. Since the product of two diagonal matrices commutes it follows that the totality of p'-elements forms an abelian subgroup of exponent dividing $p-1$, i.e., $O^p(G_2)$ is abelian and is of exponent dividing $p-1$. If $p=2$ then $O^p(G_2) = 1$ and G_2 is a 2-group. But then $G_2/O_p(G_2)$ is trivially abelian and is of exponent dividing $p-1$. Thus we may assume that $p > 2$. Let $\bar{Q}_p = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_n = 0$ be a G_2 composition series for $\bar{Q}_p$ and let $K_i = C_{G_2}(V_i/V_{i+1})$, $0 \leq i \leq n-1$. It is

evident from the above discussion that $O^p(G_2/K_i)$ is abelian and is of exponent dividing $p-1$. We now show that every p-element of G_2/K_i has $(x-1)^2$ as minimum polynomial.

Let y be a p-element of G and $H_i = \langle x_i \rangle \langle y \rangle = \langle y \rangle \langle x_i \rangle$ $\forall i$. Then $H_i \cap Q_p = \langle x_i \rangle \cdot (\langle y \rangle \cap Q_p)$ and $(H_i \cap Q_p)^y \subseteq H_i^y \cap Q_p^y = H_i \cap Q_p$. Thus $H_i \cap Q_p$ is normalised by y so that $\bar{y} = y\Phi(Q_p)$ normalises $(H_i \cap Q_p)^*$. Hence the subspace $(H_i \cap Q_p)^*$ of $\bar{Q}_p$ is invariant under $\hat{y} = yC^*$. ($\hat{y}$ is not the identity transformation.) If the order of y is p^m then $\hat{y}^{p^m} = 1$, i.e. $\hat{y}^{p^m} - 1 = 0$. This implies that the linear transformation $\hat{y}$ is annulled by the polynomial $x^{p^m} - 1 = (x-1)^{p^m}$ and since the dimension of $(H_i \cap Q_p)^*$ is at most 2 it follows that the minimum polynomial of $\hat{y}$ is $(x-1)^2$ whence $(\hat{y}-1)^2 = 0$. Since every p-element of G_2/K_i is of the form $\hat{y}K_i$ where $\hat{y}$ is a p-element of G_2 it follows that every p-element of G_2/K_i has $(x-1)^2$ as minimum polynomial. Also, the Sylow 2-subgroup of G_2/K_i being contained in $O^p(G_2/K_i)$ is abelian and hence Theorem B of [2] (Hall-Higman) applies and we conclude that G_2/K_i is abelian and is of exponent dividing $p-1$. But this again implies that $G_2/\bigcap_{i=0}^{n-1} K_i$ is abelian and is of exponent dividing $p-1$, i.e. $G_2/O_p(G_2)$ is abelian and is of exponent dividing $p-1$. The theorem is therefore proved completely.

The supersolvability of the hyperquasicenter enables one to characterize supersolvable groups in certain special ways and the following theorems which are quite easy to prove may be mentioned in this connection.

THEOREM. *A group G is supersolvable if and only if, for every subgroup $S = H_1 H_2$ (H_1 and H_2 are subgroups of S), $Q^*(S) = Q^*(H_1) \cdot Q^*(H_2)$.*

THEOREM. *A subgroup G is supersolvable if and only if $H \cap Q^*(G) = Q^*(H)$ for every subgroup H of G.*

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