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THE IDEAL BOUNDARIES OF COMPLETE OPEN SURFACES

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0. Introduction. It is an interesting problem to investigate compactifications of complete noncompact Riemannian manifolds. The ideal boundary of an Hadamard manifold X is defined to be the set of equivalence classes of rays in X. Here the equivalence relation between two rays in X is obtained by an asymptotic relation between them. Busemann first defined (see [Bu, Chap. 3, §22]) an asymptotic relation between two rays (which he called co-ray relation and used to define parallelism on a straight G-space). This asymptotic relation is not symmetric in general and hence the equivalence classes of rays are not defined by it. If X is an Hadamard manifold, then this asymptotic relation becomes symmetric, and makes it possible to define the ideal boundary $X(\infty)$ of X (see [EO] and [BGS]). Gromov defined in [BGS] the Tits metric on $X(\infty)$. Recently, Kause constructed an ideal boundary of an asymptotically nonnegatively curved manifold.

The purpose of the present paper is, first of all, to define the ideal boundary $M(\infty)$ with the metric d_∞ as the set of natural equivalence classes of rays in a finitely connected, oriented, complete and noncompact surface M admitting total curvature. Then we investigate the geometry on the ideal boundary in terms of the total curvature. Here the total curvature $c(M)$ of such a surface M is defined by an improper integral over M of the Gaussian curvature G:

$$c(M):=\int_M G\,dM\,,$$

where dM is the area element of M. Cohn-Vossen [Co1] proved that $c(M)\leq 2\pi\chi(M)$ if $c(M)$ exists, where $\chi(M)$ is the Euler characteristic of M. The existence of the total curvature is essential to defining our equivalence relation between rays in M. We denote the equivalence class of a ray γ by $\gamma(\infty)$. Using the total curvature, we will define the metric d_∞ of $M(\infty)$, which corresponds to the Tits metric for an Hadamard manifold.

Our main results are stated as follows.

THEOREM 2.1. *Assume that M with one end admits the total curvature. If a ray σ in M is asymptotic to a ray γ, then σ and γ are equivalent.*

THEOREM 2.4. *Assume that M with one end admits the total curvature $c(M)>-\infty$.*
(1) $2\pi\chi(M)-c(M)=0$ *if and only if $M(\infty)$ consists of a single point.*
(2) $2\pi\chi(M)-c(M)>0$ *if and only if $M(\infty)$ is isometric to a nontrivial circle with*

the total length $2\pi\chi(M)-c(M)$.

For a fixed simple closed smooth curve c and for a positive number t, we set

$$S(t):=\{p\in M\,;\,d(p,c)=t\}$$

and denote by d_t the inner distance of $S(t)$. Note that there is a closed and measurable subset E of $[0,+\infty)$ such that $S(t)$ for each $t\in[0,+\infty)-E$ is a finite union of simple closed piecewise smooth curves (see [Ha] and [ST]). Throughout this paper, all geodesics are assumed to be normal unless otherwise stated. A minimizing segment $\sigma:[0,l]\to M$ is called *a minimizing segment from* c if $d(\sigma(t),c)=t$ holds for all $t\in[0,l]$. A ray γ is called *a ray from* c if $d(\gamma(t),c)=t$ holds for all $t\geq 0$. Then we have:

THEOREM 3.3. *Assume that* M *with one end admits the total curvature. Let* c *be an arbitrarily fixed simple closed smooth curve. Then for any rays* σ *and* γ *from* c,

$$\lim_{t\to\infty}\frac{d_t(\sigma(t),\gamma(t))}{t}=d_\infty(\sigma(\infty),\gamma(\infty))\,,$$

where t *is assumed to be a number in* $[0,+\infty)-E$.

THEOREM 3.5. *Assume that* M *with one end admits the total curvature* $c(M)>-\infty$. *Let* c *be an arbitrarily fixed simple closed smooth curve. Then for any rays* σ *and* γ,

$$\lim_{t\to\infty}\frac{d_t(\sigma\cap S(t),\gamma\cap S(t))}{t}=d_\infty(\sigma(\infty),\gamma(\infty))\,,$$

where t *is assumed to be a number in* $[0,+\infty)-E$.

Note that Kasue [Ks] defined the metric of an ideal boundary by the formula in Theorem 3.5 when M is an asymptotically nonnegatively curved manifold, which always admits the total curvature provided M is two dimensional.

For a ray γ in M, we define the Busemann function $F_\gamma: M\to \boldsymbol{R}$ (see [Bu, Chap. 3, §22]) by

$$F_\gamma(x):=\lim_{t\to\infty}[t-d(x,\gamma(t))]\,.$$

In Section 4 we investigate relations between the asymptotic behavior of Busemann functions and the metric d_∞ and prove the following:

THEOREM 4.4. *Assume that* M *with one end admits the total curvature. For any rays* σ *and* γ,

$$\lim_{t\to\infty}\frac{F_\gamma\circ\sigma(t)}{t}=\cos\min\{d_\infty(\sigma(\infty),\gamma(\infty)),\pi\}\,.$$

If M is an Hadamard manifold, then Theorem 4.4 is easily proved by the L'Hospital theorem, because each Busemann function is of class C^2. However, in our case, a Busemann function is not necessarily differentiable. Therefore, we need delicate arguments as developed in Section 4. We have Corollary 4.7, which was proved earlier by Shiohama [Sh2], as a consequence of Theorem 4.4.

COROLLARY 4.7 ([Sh2]). *Assume that M with one end admits the total curvature.*
(1) *If* $2\pi\chi(M)-c(M)<\pi$, *then all Busemann functions are exhaustive.*
(2) *If* $2\pi\chi(M)-c(M)>\pi$, *then all Busemann functions are nonexhaustive.*

Here a function $f\colon M\to \boldsymbol{R}$ is said to be *exhaustive* if $f^{-1}((-\infty, a])$ is compact for all $a\in f(M)$.

Note that there is a manifold M with $2\pi\chi(M)-c(M)=\pi$ such that some Busemann function of M is exhaustive and another is nonexhaustive (see [Sh2]). However, when the Gaussian curvature of M is nonnegative outside some compact subset of M, we see the behavior of the values of a Busemann function along a ray as follows:

THEOREM 4.9. *Assume that* $2\pi\chi(M)-c(M)=\pi$ *and that there exists a compact subset K of M such that the Gaussian curvature G of M is nonnegative outside K. If* $d_\infty(\sigma(\infty), \gamma(\infty))=\pi/2$ *holds for rays* σ *and* γ *in M, then there exists a positive number* t_0 *such that* $F_\gamma\circ\sigma$ *is monotone nonincreasing on* $[t_0, +\infty)$.

Theorems 4.7 and 4.9 imply Corollary 4.10. Shiohama [Sh1] proved this when the Gaussian curvature of M is nonnegative everywhere.

COROLLARY 4.10. *Assume that M with one end admits the total curvature. If the Gaussian curvature G is nonnegative outside some compact subset of M, then we have*:
(1) $2\pi\chi(M)-c(M)<\pi$ *if and only if all Busemann functions are exhaustive.*
(2) $2\pi\chi(M)-c(M)\geq\pi$ *if and only if all Busemann functions are nonexhaustive.*

In Section 5, we discuss the case where M has more than one end. We will define the ideal boundary $M(\infty)$ for such a manifold M and extend results in Sections 1, 2, 3 and 4 to this case.

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1. Equivalence classes of rays. For a moment we assume that M is finitely connected with one end and admits the total curvature.

For any domain D of M bounded by piecewise smooth curves $c_1, \cdots, c_n$ each of which is parametrized positively by the arc length relative to D, we denote by $\kappa(D)$ the sum of the curvature integrals of $c_1, \cdots, c_n$ and of the outer angles at the vertices of D. Then the following (1.1), (1.2), (1.3) and (1.4) are known to hold:

(1.1) $\kappa(D)=-\kappa(M-D)$.

(1.2) If D is bounded, then $c(D)=2\pi\chi(D)-\kappa(D)$.

(1.3) Assume that the boundary ∂D of D consists of a curve c homeomorphic to a line such that $c|(-\infty, a]$ and $c|[b, +\infty)$ are geodesics for some $a, b \in \boldsymbol{R}$. Then

$$c(D) \leq 2\pi\chi(D)-\pi-\kappa(D).$$

(1.4) If $d_D(c(t), c(-t)) \geq 2t-r$ holds for all $t \geq 0$ and for some constant $r \geq 0$ in (1.3), then

$$c(D) \leq 2\pi\chi(D)-2\pi-\kappa(D),$$

where d_D is the inner distance on the closure $\mathrm{cl}(D)$ of D induced from the Riemannian metric of M.

(1.1) is obvious. (1.2) follows from the Gauss-Bonnet theorem, while (1.3) and (1.4) follow from Cohn-Vossen [Co2].

For any rays σ and γ, let $\alpha: [0, l] \to M$ be a piecewise smooth curve from $\sigma(a)$ to $\gamma(b)$ such that $\sigma([a, +\infty)) \cup \alpha([0, l]) \cup \gamma([b, +\infty))$ bounds two unbounded domains D and $M-D$ of M, where α is assumed to be parametrized positively relative to D. We set

$$L(\sigma, \gamma) := 2\pi\chi(D)-\pi-\kappa(D)-c(D).$$

In the special case where there exists a $t_0 \in \boldsymbol{R}$ such that $\sigma(t_0+t)=\gamma(t)$ for all $t \geq |t_0|$, we cannot get such a curve α. In this case we set $L(\sigma, \gamma) := 0$ and $L(\gamma, \sigma) := 0$. Then the $L(\sigma, \gamma)$ has the following properties:

(1.5) $L(\sigma, \gamma)$ does not depend on the choice of the curve α.

(1.6) $L(\sigma, \gamma) \geq 0$.

(1.7) $L(\sigma, \gamma)+L(\gamma, \sigma)=2\pi\chi(M)-c(M)$. Otherwise there exists a $t_0 \in \boldsymbol{R}$ such that $\sigma(t_0+t)=\gamma(t)$ for all $t \geq |t_0|$.

(1.5) follows from the Gauss-Bonnet theorem. (1.6) is an immediate consequence of (1.3). (1.7) is obvious.

Since M has only one end, there is a compact domain K of M such that $\mathrm{cl}(M-K)$ is a closed half cylinder bounded by a simple closed smooth curve. Following Busemann (see [Bu, Chap. 5, §43]) we call this closed half cylinder a (closed) *tube* of M.

For any rays σ and γ we choose a simple closed smooth curve c bounding a closed tube U of M in such a way that

(a) c intersects σ (resp., γ) at a unique point $\sigma(t_\sigma)$ (resp., $\gamma(t_\gamma)$),

(b) $\measuredangle(\dot{\sigma}(t_\sigma), \dot{c}) = \measuredangle(\dot{\gamma}(t_\gamma), \dot{c}) = \pi/2$,

(c) $\sigma([t_\sigma, +\infty))$ does not intersect $\gamma([t_\gamma, +\infty))$.

Note that σ and γ are not necessarily rays from c. Let $I(\sigma, \gamma)$ be a closed subarc of c from $\sigma(t_\sigma)$ to $\gamma(t_\gamma)$ with respect to the positive parameter of c relative to U, and let $D(\sigma, \gamma) \subset U$ be a domain homeomorphic to a closed half plane bounded by $\sigma([t_\sigma, +\infty)) \cup I(\sigma, \gamma) \cup \gamma([t_\gamma, +\infty))$. $I(\sigma, \gamma)$ is often identified with the closed interval $c^{-1}(I(\sigma, \gamma))$. Then by the definition of $L(\sigma, \gamma)$, (1.8) $$L(\sigma, \gamma) = -c(D(\sigma, \gamma)) - \int_{I(\sigma,\gamma)} \kappa ds\,.$$

where κ denotes the geodesic curvature of c.

Rays σ and γ are said to be *equivalent* and denoted by $\sigma \sim \gamma$ if $L(\sigma, \gamma)=0$ or $L(\gamma, \sigma)=0$. We will show that the relation $\sim$ is an equivalence relation on the set of all rays in M. It follows that this relation is reflexive and symmetric. For any rays σ, τ and γ let c be a simple closed smooth curve bounding a tube of M and having the properties (a), (b) and (c) for rays σ, τ and γ. If $\sigma(t_\sigma)$, $\tau(t_\tau)$ and $\gamma(t_\gamma)$ lie on c in this order, then (1.8) implies

(1.9) $$L(\sigma, \tau) + L(\tau, \gamma) = L(\sigma, \gamma)\,.$$

Here $L(\sigma, \tau)$, $L(\tau, \gamma)$ and $L(\sigma, \gamma)$ are nonnegative by (1.6). Thus we observe that the relation $\sim$ is transitive.

We denote the equivalence class of a ray γ by $\gamma(\infty)$ and the set of all equivalence classes by $M(\infty)$. We assign to rays σ and γ a number $d_\infty(\sigma(\infty), \gamma(\infty))$ in $\boldsymbol{R} \cup \{+\infty\}$ by

$$d_\infty(\sigma(\infty), \gamma(\infty)) := \min\{L(\sigma, \gamma), L(\gamma, \sigma)\}\,.$$

(1.9) shows that $d_\infty(\sigma(\infty), \gamma(\infty))$ does not depend on the choice of rays σ, γ in the equivalence classes $\sigma(\infty)$, $\gamma(\infty)$, which determines the function $d_\infty: M(\infty) \times M(\infty) \to \boldsymbol{R} \cup \{+\infty\}$. This function becomes a distance function on $M(\infty)$. We will show only the triangle inequality

(*) $$d_\infty(\sigma(\infty), \gamma(\infty)) \leq d_\infty(\sigma(\infty), \tau(\infty)) + d_\infty(\tau(\infty), \gamma(\infty))\,.$$

For any rays σ, τ and γ, let c be a simple closed smooth curve bounding a tube of M and having the properties (a), (b) and (c) for three rays σ, τ and γ. Consider the case where $\sigma(t_\sigma)$, $\tau(t_\tau)$ and $\gamma(t_\gamma)$ lie on c in this order. If $L(\tau, \sigma) < L(\sigma, \tau)$, then

$$d_\infty(\sigma(\infty), \gamma(\infty)) = L(\gamma, \sigma) \leq L(\tau, \sigma) = d_\infty(\sigma(\infty), \tau(\infty))\,.$$

If $L(\gamma, \tau) < L(\tau, \gamma)$, then

$$d_\infty(\sigma(\infty), \gamma(\infty)) = L(\gamma, \sigma) \leq L(\gamma, \tau) = d_\infty(\tau(\infty), \gamma(\infty))\,.$$

If $L(\sigma, \tau) \leq L(\tau, \sigma)$ and $L(\tau, \gamma) \leq L(\gamma, \tau)$, then

$$d_\infty(\sigma(\infty), \gamma(\infty)) \leq L(\sigma, \gamma) = L(\sigma, \tau) + L(\tau, \gamma) = d_\infty(\sigma(\infty), \tau(\infty)) + d_\infty(\tau(\infty), \gamma(\infty))\,.$$

Therefore we have (*). In the other cases, we can show (*) in the same way. Thus $M(\infty)$ with d_∞ becomes a metric space.

We have the following:

(1.10) For any rays σ and γ in M we get a simple closed smooth curve c bounding a tube of M with the properties (a), (b) and (c). If $\hat{d}(\sigma(t), \gamma(t)) \geq 2t - r$ holds for all $t \geq \max\{t_\sigma, t_\gamma\}$ and for some constant $r > 0$, where $\hat{d}$ denotes the inner distance of $D(\sigma, \gamma)$, then

$$L(\sigma, \gamma) \geq \pi .$$

(1.11) For any straight line γ in M, we have

$$d_\infty(\gamma(-\infty), \gamma(\infty)) \geq \pi ,$$

where $\gamma(-\infty)$ is the class of a ray $t \mapsto \gamma(-t)$. In particular, if M contains a straight line, then $2\pi\chi(M) - c(M) \geq 2\pi$.

(1.12) Let c be a simple closed smooth curve bounding a tube of M, and let σ and γ be rays from c. If there exist no rays from c in the interior $\mathrm{int}(D(\sigma, \gamma))$ of $D(\sigma, \gamma)$, then

$$L(\sigma, \gamma) = 0 .$$

(1.10) is an immediate consequence of (1.4). (1.10) shows (1.11). (1.12) follows from Theorem A in [Sh3].

2. Total curvature and ideal boundary. We use the following fact (cf. [Co2]) in the proof of Theorem 2.1.

FACT (2.a). Let γ be a ray in M and $\{\sigma_j\colon [0, l_j] \to M\}$ be a sequence of minimizing segments such that $\{\sigma_j(0)\}$ converges and $\sigma_j(l_j) = \gamma(t_j)$ for some monotone and divergent sequence $\{t_j\}$. Then the angle between two vectors $\dot\gamma(t_j)$ and $\dot\sigma(l_j)$ tends to zero as $j \to \infty$.

A ray σ is said to be *asymptotic* to a ray γ if there exist a monotone and divergent sequence $\{t_j\}$ of positive numbers and a sequence $\{\sigma_j\colon [0, l_j] \to M\}$ of minimizing segments such that $\sigma_j(l_j) = \gamma(t_j)$ holds for each j and σ_j tends to σ as $j \to \infty$.

THEOREM 2.1. *If a ray σ in M is asymptotic to a ray γ, then σ and γ are equivalent.*

PROOF. Let c be a simple closed smooth curve bounding a tube U of M and having the properties (a), (b) and (c) in Section 1 for rays σ and γ. For an $s_0 > t_\sigma$ and for a monotone and divergent sequence $\{t_j\}$ of positive numbers, if σ_j is a minimizing segment from $\sigma(s_0)$ to $\gamma(t_j)$, then

$$\lim_{j \to \infty} \sigma_j = \sigma \,|\, [s_0, \infty) , \tag{2.1.1}$$

because σ is asymptotic to γ and $\sigma(s_0)$ is an interior point of σ. Thus for all sufficiently large j, σ_j does not intersect c and then it is contained in one of the two domains $D(\sigma, \gamma)$ and $D(\gamma, \sigma)$. Without loss of generality we may assume that there exists a subsequence $\{\sigma_k\}$ of $\{\sigma_j\}$ such that $\sigma_k \subset D(\sigma, \gamma)$ for all k. Let D_k be a disk domain bounded by σ_k, $\sigma([t_\sigma, s_0])$, $I(\sigma, \gamma)$ and $\gamma([t_\gamma, t_k])$. Then $\{D_k\}$ is a monotone increasing sequence of domains with $\bigcup D_k = D(\sigma, \gamma)$. If θ_k and φ_k denote the inner angles of D_k at $\sigma(s_0)$ and $\gamma(t_k)$, respectively, then the Gauss-Bonnet theorem implies that

$$c(D_k)=\theta_k+\varphi_k-\pi-\int_{I(\sigma,\gamma)}\kappa ds$$

for all k. It follows from (2.1.1) and Fact (2.a) that θ_k tends to π, and φ_k to zero as $k\to\infty$. Hence

$$c(D(\sigma,\gamma))=\lim_{k\to\infty}c(D_k)=-\int_{I(\sigma,\gamma)}\kappa ds\,.$$

Therefore we have $L(\sigma,\gamma)=0$. This completes the proof.

LEMMA 2.2. *Let c be an arbitrarily fixed simple closed smooth curve. For any $x\in M(\infty)$ there exists a ray γ from c such that $x=\gamma(\infty)$.*

PROOF. For any $x\in M(\infty)$ there exists a ray σ such that $x=\sigma(\infty)$. Let $\{t_j\}$ be a monotone and divergent sequence of positive numbers and γ_j a minimizing segment from c to $\sigma(t_j)$. We choose a subsequence $\{\gamma_k\}$ of $\{\gamma_j\}$ converging to some ray γ from c. Since γ is asymptotic to σ, Theorem 2.1 implies $\gamma(\infty)=x$. This completes the proof.

LEMMA 2.3. *Assume that $c(M)>-\infty$. If a sequence $\{\sigma_j\}$ of rays tends to a ray σ, then $\sigma_j(\infty)$ tends to $\sigma(\infty)$.*

PROOF. First consider the case where there exists a subsequence $\{\sigma_k\}$ of $\{\sigma_j\}$ such that each σ_k intersects σ at a point $\sigma_k(s_k)=\sigma(t_k)=p_k$ and any subsequence of $\{p_k\}$ diverges. For any compact subset K of M, the minimizing property of rays implies that $\sigma_k([s_k,+\infty))$ does not intersect K for all sufficiently large k. Hence $\sigma_k([s_k,+\infty))\cup\sigma([t_k,+\infty))$ bounds two domains of M for all sufficiently large k. Let D_k be one of these two domains such that D_k does not contain $\sigma(0)$. Then for any compact set K, the domain D_k does not intersect K and is homeomorphic to a half plane if k is sufficiently large.

Note that since M admits the total curvature, Cohn-Vossen's theorem implies that $\int_M G^+dM<+\infty$. Moreover since $c(M)>-\infty$, we have $\int_M|G|dM<+\infty$. Hence for any positive ε there exists a compact set K such that

$$\int_{M-K}|G|dM<\varepsilon\,.$$

Then the inequality

$$c(D_k)\geq-\int_{D_k}|G|dM>-\varepsilon$$

holds for all sufficiently large k. If θ_k denotes the inner angle of D_k at p_k, then θ_k tends to zero by Fact (2.a). Therefore

$$d_\infty(\sigma_k(\infty),\sigma(\infty))\leq\theta_k-c(D_k)<2\varepsilon$$

for all sufficiently large K and hence $\sigma_k(\infty)$ tends to $\sigma(\infty)$.

Next consider the case where there exists a subsequence $\{\sigma_k\}$ of $\{\sigma_j\}$ such that either $\bigcup_k(\sigma([0, \infty))\cap\sigma_k([0, \infty)))$ is bounded or is empty. Then there exists a simple closed smooth curve c bounding a tube U of M such that

(2.3.1) σ (resp., σ_k) intersects c at a unique point $\sigma(t_\sigma)$ (resp., $\sigma_j(t_{\sigma_k})$),

(2.3.2) $\sigma([t_\sigma, +\infty))$ does not intersect $\sigma_j([t_{\sigma_k}, +\infty))$,

(2.3.3) $\measuredangle(\dot\sigma(t_\sigma), \dot c)=\pi/2$ holds.

Note that $\dot\sigma_k(t_{\sigma_k})$ is not necessarily perpendicular to $\dot c$. Now $\sigma([t_\sigma, +\infty))\cup\sigma_j([t_{\sigma_k}, +\infty))\cup c$ bounds two half planes D_k and $U-D_k$ in U, where $\{D_k\}$ is taken to be monotone decreasing in U (by choosing a subsequence if necessary). If θ_k denotes the inner angle of D_k at $\sigma_k(t_{\sigma_k})$, then

$$d_\infty(\sigma_k(\infty), \sigma(\infty))\leq\theta_k-\frac{\pi}{2}-c(D_k)-\int_{\mathrm{cl}(D_k)\cap c}\kappa ds \tag{2.3.4}$$

by the definition of the distance d_∞. Moreover since σ_k tends to σ,

$$\lim_{k\to\infty}\theta_k=\frac{\pi}{2}\quad\text{and}\quad\lim_{k\to\infty}\int_{\mathrm{cl}(D_k)\cap c}\kappa ds=0\,. \tag{2.3.5}$$

For any positive ε there exists a compact set K such that

$$\int_{M-K}|G|dM<\varepsilon\,.$$

Then $|c(D_k)-c(D_k\cap K)|<\varepsilon$ and $|c(D_k\cap K)|<\varepsilon$ for all sufficiently large k, since the area of $D_k\cap K$ tends to zero. Hence

$$|c(D_k)|<2\varepsilon \tag{2.3.6}$$

for all sufficiently large k. By (2.3.4), (2.3.5) and (2.3.6), this completes the proof of Lemma 2.3.

THEOREM 2.4. *Assume that* $c(M)>-\infty$.

(1) $2\pi\chi(M)-c(M)=0$ *if and only if* $M(\infty)$ *consists of a single point.*

(2) $2\pi\chi(M)-c(M)>0$ *if and only if* $M(\infty)$ *is isometric to a nontrivial circle with the total length* $2\pi\chi(M)-c(M)$.

PROOF. (1) Let c be a fixed simple closed smooth curve bounding a tube of M. For a ray γ from c we consider the following subarc of c:

$$I_\gamma:=\bigcup\{I(\alpha, \beta);\ \alpha\text{ and }\beta\text{ are rays from }c\text{ such that }\gamma(0)\in I(\alpha, \beta)\text{ and }L(\alpha, \gamma)=L(\gamma, \beta)=0\}\,.$$

Lemma 2.3 implies that I_γ is a closed subarc of c. Note that for each ray σ from c, we have $\sigma(\infty)=\gamma(\infty)$ if and only if $\sigma(0)\in I_\gamma$. If either $I_\gamma=c$ holds or γ is the only ray from

c, then (1.7) and (1.12) show $2\pi\chi(M)-c(M)=0$, and by Lemma 2.2, $M(\infty)$ consists of a single point. Otherwise, there exist two rays γ^- and γ^+ from c such that $I_\gamma = I(\gamma^-, \gamma^+)$. In this case we have $L(\gamma^+, \gamma^-)>0$ and hence $2\pi\chi(M)-c(M)>0$. Moreover by (1.12), there is a ray σ from c in $\operatorname{int}(D(\gamma^+, \gamma^-))$, which satisfies $L(\gamma^+, \sigma)$, $L(\sigma, \gamma^-)>0$. Hence we have $\sigma(\infty)\neq\gamma(\infty)$. Thus, in particular we conclude (1).

(2) Assume that $0<2\pi\chi(M)-c(M)<+\infty$. We will prove that $M(\infty)$ is isometric to a circle with the total length $2\pi\chi(M)-c(M)$. Then the converse is clear. We use the same notation as in the proof of (1). Now, for each ray γ from c, we see that γ^- and γ^+ are defined and satisfy $(\gamma^-)^-=\gamma^-$ and $(\gamma^+)^+=\gamma^+$ because $I_\gamma=I_{\gamma^-}=I_{\gamma^+}$. For a fixed ray σ from c with $\sigma^-=\sigma$ we define the function $f_\sigma\colon M(\infty)\to[0, 2\pi\chi(M)-c(M))$ by

$$f_\sigma(\gamma(\infty)) := L(\sigma, \gamma)$$

for each ray γ from c. The definition of $M(\infty)$ and the formula (1.9) imply that the function f_σ is well-defined and is an injection. We will show that f_σ becomes a bijection. Assume that the simple closed smooth curve $c\colon [0, l]\to M$ is a unit-speed curve with length l and is parametrized positively relative to the tube. Then the restriction $c\colon [0, l)\to c([0, l))$ is a bijection. Note that for any rays τ and γ from c, we have $f_\sigma(\tau(\infty))\leq f_\sigma(\gamma(\infty))$ if $c^{-1}\circ\tau(0)\leq c^{-1}\circ\gamma(0)$.

First we will prove that $\sup f_\sigma = 2\pi\chi(M)-c(M)$. It suffices to show that $\sup f_\sigma\geq 2\pi\chi(M)-c(M)$. Suppose that $\sup f_\sigma<2\pi\chi(M)-c(M)$. We get a sequence $\{\gamma_j\}$ of rays from c such that $f_\sigma(\gamma_j(\infty))$ tends to $\sup f_\sigma$ as $j\to\infty$ and $\{c^{-1}\circ\gamma_j(0)\}$ is monotone increasing. Since the limit of a sequence of rays from c is a ray from c, γ_j tends to some ray γ from c. Now, if $L(\gamma_j, \sigma)$ tends to zero as $j\to\infty$, then $f_\sigma(\gamma_j(\infty))$ tends to $2\pi\chi(M)-c(M)$, which contradicts the assumption. Therefore we have $L(\gamma^+, \sigma)=L(\gamma, \sigma)>0$ and $f_\sigma(\gamma(\infty))=\sup f_\sigma$. Here $\operatorname{int}(D(\gamma^+, \sigma))$ does not contain any ray from c. Indeed, if there is a ray τ from c in $\operatorname{int}(D(\gamma^+, \sigma))$, then $L(\gamma^+, \tau)>0$ and hence $f_\sigma(\tau(\infty))>f_\sigma(\gamma(\infty))$, which is a contradiction. Therefore by (1.12) we conclude that $L(\gamma^+, \sigma)=0$. This is a contradiction and thus we have $\sup f_\sigma=2\pi\chi(M)-c(M)$.

We will prove that f_σ is surjective. By Lemma 2.3 and by $\sup f_\sigma=2\pi\chi(M)-c(M)$, for any number $a\in[0, 2\pi\chi(M)-c(M))$ there are two rays γ_1 and γ_2 from c such that

$$c^{-1}\circ\gamma_1(0)=\sup\{c^{-1}\circ\alpha(0);\ \alpha \text{ is a ray from } c \text{ and } f_\sigma(\alpha(\infty))\leq a\}\,,$$

$$c^{-1}\circ\gamma_2(0)=\inf\{c^{-1}\circ\alpha(0);\ \alpha \text{ is a ray from } c \text{ and } f_\sigma(\alpha(\infty))\geq a\}\,.$$

If $c^{-1}\circ\gamma_1(0)\geq c^{-1}\circ\gamma_2(0)$, then $f_\sigma(\gamma_1(\infty))=f_\sigma(\gamma_2(\infty))=a$ by the definitions of γ_1 and γ_2. If $c^{-1}\circ\gamma_1(0)<c^{-1}\circ\gamma_2(0)$, then there are no rays from c in $\operatorname{int}(D(\gamma_1, \gamma_2))$ and hence $L(\gamma_1, \gamma_2)=0$ by (1.12). Thus $f_\sigma(\gamma_1(\infty))=f_\sigma(\gamma_2(\infty))=a$.

We set $S:=\boldsymbol{R}/(2\pi\chi(M)-c(M))\boldsymbol{Z}$ and define a mapping $h\colon [0, 2\pi\chi(M)-c(M))\to S$ by

$$h(a):=a+(2\pi\chi(M)-c(M))\boldsymbol{Z}\,.$$

If d_s denotes the inner distance of S, then we have

$$d_\infty(\tau(\infty), \gamma(\infty)) = \min\{L(\tau, \gamma), L(\gamma, \tau)\} = d_s(h \circ f_\sigma(\tau(\infty)), h \circ f_\sigma(\gamma(\infty)))$$

for any rays σ and γ from c. Therefore $M(\infty)$ is isometric to S, which completes the proof.

3. Geodesic circles and the distance d_∞. Let c be a fixed simple closed smooth curve and set the geodesic circle $S(t) := \{p \in M ; d(p, c) = t\}$ for $t \geq 0$. A number $t > 0$ is said to be *exceptional* if there exists a cut point $p \in S(t)$ from c having one of the three properties: (1) p is a first focal point along some minimizing segment from c; (2) there exist more than two minimizing segments from c to p; (3) there exist exactly two minimizing segments from c to p such that the angle between the two vectors at p tangent to these minimizing segments is equal to π. Then we have the following:

(3.a) The set of all exceptional t-values is closed and of Lebesgue measure zero.

(3.b) For any non-exceptional $t > 0$, $S(t)$ consists of simple closed curves of class C^∞ except the finitely many cut points from c.

(3.c) There exists an $R > 0$ such that $S(t)$ is homeomorphic to a circle for all $t > R$.

(3.a) and (3.b) are due to Hartman (see Lemma 5.2 and Proposition 6.1 in [Ha] and also [ST]). (3.c) is due to Shiohama (Theorem B (2) in [Sh3]). In this section, we investigate the relation between the geodesic circle $S(t)$ and the distance d_∞.

LEMMA 3.1. *Let c be an arbitrarily fixed simple closed smooth curve bounding a tube of M. For any rays σ and γ from c,*

$$\lim_{t \to \infty} \frac{L(S(t) \cap D(\sigma, \gamma))}{t} = L(\sigma, \gamma),$$

where we assume that t is always non-exceptional.

PROOF. Let $z(t, s)$ be the geodesic parallel coordinates along c (see [Fi], [Ha]), where s is the arclength parameter of c and t is the distance from c. The Riemannian metric is expressed as

$$(g_{ij}) := \begin{pmatrix} 1 & 0 \\ 0 & f(t, s) \end{pmatrix}$$

and the geodesic curvature $\kappa_t(s)$ of $S(t)$ relative to $B(t) := \{p \in M ; d(p, c) \leq t\}$ is written as

$$\kappa_t(s) = \frac{1}{f(t, s)} \frac{\partial f(t, s)}{\partial s}$$

except at all cut points from c. For every non-exceptional $t > 0$, let $q_k(t)$ for $k = 1, \cdots, m(t)$ be the cut points from c on the arc $S(t) \cap L(\sigma, \gamma)$ and $\theta_k(t)$ be the inner angle at $q_k(t)$ of $B(t)$. If we set

$$\beta_k(t) := \theta_k(t) - \pi \qquad \text{and} \qquad \omega(t) := -\sum_{k=1}^{m(t)} \beta_k(t)\,,$$

then it follows in the same way as in [Fi, p. 326] that

$$\frac{d}{dt} L(S(t) \cap D(\sigma, \gamma)) = \int_{S(t) \cap D(\sigma,\gamma)} \kappa_t(s) ds + \omega(t) - \sum_{k=1}^{m(t)} \left[2 \tan \frac{\beta_k(t)}{2} - \beta_k(t) \right] \tag{3.1.1}$$

for all non-exceptional $t > 0$. The Gauss-Bonnet theorem implies

$$c(B(t) \cap D(\sigma, \gamma)) = -\int_{I(\sigma,\gamma)} \kappa ds - \int_{S(t) \cap D(\sigma,\gamma)} \kappa_t(s) ds - \omega(t)\,. \tag{3.1.2}$$

Moreover by Theorem B in [Sh3], $\sum_{k=1}^{m(t)} \beta_k(t)$ tends to zero as $t \to \infty$ and hence

$$\lim_{t \to \infty} \sum_{k=1}^{m(t)} \left[2 \tan \frac{\beta_k(t)}{2} - \beta_k(t) \right] = 0\,. \tag{3.1.3}$$

From (3.1.1), (3.1.2), (3.1.3), (1.8) and the L'Hospital theorem, we have

$$\lim_{t \to \infty} \frac{L(S(t) \cap D(\sigma, \gamma))}{t} = \lim_{t \to \infty} \frac{d}{dt} L(S(t) \cap D(\sigma, \gamma)) = L(\sigma, \gamma)\,.$$

This completes the proof.

Note that this proof is essentially contained in Shiohama's paper [Sh3]. He proved in [Sh3] and [Sh4] the formula

$$\lim_{t \to \infty} \frac{L(S(t))}{t} = 2\pi\chi(M) - c(M)\,.$$

To prove Theorem 3.3 we need the following lemma, which was stated in the proof of Theorem A in [Sh4].

LEMMA 3.2 ([Sh4]). *For any simple closed smooth curve c and for any sufficiently large $R > 0$, there exists a simple closed smooth curve c_1 bounding a tube of M such that $S(t + R) = S_1(t)$ for all sufficiently large $t > 0$, where $S_1(t) := \{p \in M\,;\, d(p, c_1) = t\}$.*

Denote the inner distance of $S(t)$ by d_t. Then we have the following:

THEOREM 3.3. *Let c be an arbitrarily fixed simple closed smooth curve. Then for any rays σ and γ from c,*

$$\lim_{t \to \infty} \frac{d_t(\sigma(t), \gamma(t))}{t} = d_\infty(\sigma(\infty), \gamma(\infty))\,,$$

where t is assumed to be non-exceptional.

PROOF. If c bounds a tube of M, then Theorem 3.3 is an immediate consequence of Lemma 3.1. Otherwise, we get a simple closed smooth curve c_1 as in Lemma 3.2. Let σ_1, γ_1 be rays such that $\sigma_1(t) := \sigma(t+R)$, $\gamma_1(t) := \gamma(t+R)$. Then the triangle inequality implies that σ_1 and γ_1 are rays from c_1. If d_t^1 denotes the inner distance of $S_1(t)$, then

$$\lim_{t\to\infty} \frac{d_t(\sigma(t), \gamma(t))}{t} = \lim_{t\to\infty} \frac{d_{t+R}(\sigma(t+R), \gamma(t+R))}{t+R} = \lim_{t\to\infty} \frac{d_t^1(\sigma_1(t), \gamma_1(t))}{t+R}$$

$$= d_\infty(\sigma_1(\infty), \gamma_1(\infty)) = d_\infty(\sigma(\infty), \gamma(\infty)) .$$

This completes the proof.

LEMMA 3.4. *Assume that $c(M) > -\infty$. Let c be an arbitrarily fixed simple closed smooth curve bounding a tube of M. Then for any ray γ there exists a ray σ from c asymptotic to γ such that*

$$\lim_{t\to\infty} \frac{d_t(\sigma(t), \gamma \cap S(t))}{t} = 0 ,$$

where t is assumed to be non-exceptional.

PROOF. We set

$$T := \{t \ge 0;\ \text{there exists a ray from } c \text{ passing through } \gamma(t)\} .$$

First consider the case where T is unbounded. Let $\{t_j\}$ be a monotone and divergent sequence of numbers in T and let σ_j be a ray from c passing through $\gamma(t_j)$ such that σ_j tends to some ray σ from c. By Lemma 2.3, $\sigma_j(\infty)$ tends to $\sigma(\infty)$. The minimizing property of rays implies that the sequence $\sigma_1(0), \sigma_2(0), \sigma_3(0), \ldots$ lies on c in this order with respect to some orientation of c. Without loss of generality, we may assume that $\{D(\sigma, \sigma_j)\}$ is monotone decreasing and satisfy $\bigcap D(\sigma, \sigma_j) = \sigma([0, +\infty))$. Then $L(\sigma, \sigma_j)$ tends to zero as $j \to \infty$. Since $\gamma([t_j, +\infty)) \subset D(\sigma, \sigma_j)$, there exists a large number R_j such that for all non-exceptional $t \ge R_j$, $\gamma \cap S(t)$ is contained in $D(\sigma, \sigma_j)$ and hence

$$d_t(\sigma(t), \gamma \cap S(t)) \le L(S(t) \cap D(\sigma, \sigma_j)) .$$

Moreover by Lemma 3.1,

$$\lim_{t\to\infty} \frac{L(S(t) \cap D(\sigma, \sigma_j))}{t} = L(\sigma, \sigma_j)$$

for all j. Since $L(\sigma, \sigma_j)$ tends to zero as $j \to \infty$, the proof is completed in this case.

Next consider the case where T is bounded. Since the limit of a sequence of rays from c is a ray from c, T is a compact subset of $\boldsymbol{R}$. Let t_0 be the maximum value in T. We get a ray τ from c passing through the point $\gamma(t_0)$. Choose a ray σ from c asymptotic to γ such that there are no rays from c between σ and τ. (1.12) implies that $d_\infty(\sigma(\infty), \tau(\infty)) = 0$ and therefore without loss of generality, we may assume that $L(\sigma, \tau) = 0$ and $\gamma([t_0, +\infty)) \subset D(\sigma, \tau)$. Since γ intersects $S(t) \cap D(\sigma, \tau)$ for all large t, we have

$$d_t(\sigma(t), \gamma \cap S(t)) \leq L(S(t) \cap D(\sigma, \tau))$$

for all sufficiently large non-exceptional t. Since $L(\sigma, \tau)=0$, Lemma 3.1 implies

$$\lim_{t \to \infty} \frac{L(S(t) \cap D(\sigma, \tau))}{t} = 0 .$$

The proof is completed in this case.

Finally, if T is empty, then since the limit of a sequence of rays from c is a ray from c, there obviously exist two rays σ and τ from c such that $D(\sigma, \tau)$ contains γ and there are no rays from c in $\text{int}(D(\sigma, \tau))$, and hence $L(\sigma, \tau)=0$. Here one of the two rays σ and τ from c is asymptotic to γ. As in the above case we can prove the formula of Lemma 3.4.

Thus this completes the proof of Lemma 3.4.

THEOREM 3.5. *Assume that $c(M) > -\infty$. Let c be an arbitrarily fixed simple closed smooth curve. Then for any rays σ and γ,*

$$\lim_{t \to \infty} \frac{d_t(\sigma \cap S(t), \gamma \cap S(t))}{t} = d_\infty(\sigma(\infty), \gamma(\infty)) ,$$

where t is assumed to be non-exceptional.

PROOF. If c bounds a tube of M, then Theorem 3.5 is an immediate consequence of Theorem 3.3 and Lemma 3.4. Otherwise, using Lemma 3.2 we have Theorem 3.5 in the same way as in the proof of Theorem 3.3. This completes the proof.

4. Busemann functions and the distance d_∞. For arbitrary rays σ and γ, let c be a simple closed smooth curve bounding a tube of M and having the properties (a), (b) and (c) in Section 1. We denote the inner distance function of $D(\sigma, \gamma)$ by $\hat{d}$. A curve $\alpha: [0, l] \to D(\sigma, \gamma)$ is called a *$\hat{d}$-segment* if $L(\alpha) = \hat{d}(\alpha(0), \alpha(l))$ holds. A curve $\mu: [0, +\infty) \to D(\sigma, \gamma)$ (resp., $\mu: (-\infty, +\infty) \to D(\sigma, \gamma)$) is called a *$\hat{d}$-ray* (resp., *$\hat{d}$-line*) if any subarc of μ is a $\hat{d}$-segment. Clearly σ and γ are $\hat{d}$-rays. A $\hat{d}$-ray μ is said to be *asymptotic* to a $\hat{d}$-ray ν if there exist a monotone and divergent sequence $\{t_j\}$ of positive numbers and a sequence $\{\mu_j: [0, l_j] \to D(\sigma, \gamma)\}$ of $\hat{d}$-segments such that $\mu_j(l_j) = \nu(t_j)$ holds for each j and μ_j tends to μ as $j \to \infty$. Let $\hat{F}_\gamma: D(\sigma, \gamma) \to \boldsymbol{R}$ be the function defined by

$$\hat{F}_\gamma(x) := \lim_{t \to \infty} [t - \hat{d}(x, \gamma(t))] \qquad \text{for} \quad x \in D(\sigma, \gamma) .$$

Then this is a Lipschitz function with Lipschitz constant 1, i.e.,

$$|\hat{F}_\gamma(x) - \hat{F}_\gamma(y)| \leq \hat{d}(x, y) \qquad \text{for all} \quad x, y \in D(\sigma, \gamma) ,$$

and hence this is differentiable almost everywhere.

Let γ_t for $t \geq t_\sigma$ be a $\hat{d}$-ray in $D(\sigma, \gamma)$ emanating from $\sigma(t)$ which is asymptotic to

γ. Then $\{\gamma_{t_j}\}$ converges to a $\hat{d}$-line for some monotone and divergent sequence $\{t_j\}$ if and only if $\{\gamma_{t_j}\}$ converges to a $\hat{d}$-line for every monotone and divergent sequence $\{t_j\}$. Thus either $\{\gamma_t\}$ converges, or else γ_t does not intersect a fixed compact subset K of $D(\sigma, \gamma)$ for all sufficiently large $t \geq t_\sigma$.

We have the following lemmas and theorems under these notation and definitions.

LEMMA 4.1. We have $\lim_{t\to\infty} \sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0)) = \min\{L(\sigma, \gamma), \pi\}$.

PROOF. Consider the case where γ_t tends to some $\hat{d}$-line γ_∞ as $t\to\infty$. The minimizing property of γ_∞ shows that $D(\sigma, \gamma)$ satisfies the assumption of (1.10). Hence we have $L(\sigma, \gamma) \geq \pi$. On the other hand by Fact (2.a), $\sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0))$ tends to π as $t\to\infty$.

Next consider the case where $\{\gamma_t\}$ diverges. Let $\gamma_{t,s}$ for $t \geq t_\sigma$, $s \geq t_\gamma$ be a $\hat{d}$-segment from $\sigma(t)$ to $\gamma(s)$, and $D_{t,s}$ be a domain bounded by $I(\sigma, \gamma) \cup \sigma([t_\sigma, t]) \cup \gamma_{t,s} \cup \gamma([t_\gamma, s])$. Note that $\gamma_{t,s}$ can tend to γ_t as $s\to\infty$. The Gauss-Bonnet theorem implies that

(4.1.1)
$$c(D_{t,s}) = \theta_{t,s} - \sphericalangle(\dot{\sigma}(t), \dot{\gamma}_{t,s}(0)) - \int_{I(\sigma,\gamma)} \kappa ds$$

for all sufficiently large $t \geq t_\sigma$, $s \geq t_\gamma$, where $\theta_{t,s}$ denotes the angle of $D_{t,s}$ at $\gamma(s)$. On the other hand, for any $\varepsilon > 0$ there are large numbers $t_0 \geq t_\sigma$, $s_0 \geq t_\gamma$ such that

$$|c(D_{t,s}) - c(D(\sigma, \gamma))| < \varepsilon$$

for all $t \geq t_0$, $s \geq s_0$. In particular

(4.1.2)
$$\lim_{t\to\infty} \lim_{s\to\infty} c(D_{t,s}) = c(D(\sigma, \gamma)) .$$

Moreover by Fact (2.a), $\theta_{t,s}$ tends to zero as $s\to\infty$. Thus by (4.1.1), (4.1.2) and (1.8)

$$\lim_{t\to\infty} \sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0)) = \lim_{t\to\infty} \lim_{s\to\infty} \sphericalangle(\dot{\sigma}(t), \dot{\gamma}_{t,s}(0)) = L(\sigma, \gamma) .$$

This completes the proof.

LEMMA 4.2. *Assume that $\{\gamma_t\}$ diverges. For arbitrarily given positive numbers $t_0 < t_1$ we have*

$$\cos \max_{t\in[t_0,t_1]} \sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0)) \leq \frac{\hat{F}_\gamma \circ \sigma(t_1) - \hat{F}_\gamma \circ \sigma(t_0)}{t_1 - t_0} \leq \cos \min_{t\in[t_0,t_1]} \sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0)) .$$

PROOF. We will show the inequality on the right hand side. Let $\{\varepsilon_j\}$ be a sequence of positive numbers converging to zero. Since $\hat{F}_\gamma$ is differentiable almost everywhere, Fubini's theorem shows that there is a sequence $\{\sigma_j : [t_0, t_1] \to D(\sigma, \gamma)\}$ of smooth curves such that

(4.2.1) $\hat{F}_\gamma$ is differentiable at almost all points in $\sigma_j([t_0, t_1])$ for each j,
(4.2.2) $\hat{d}(\sigma(t), \sigma_j(t)) < \varepsilon_j$ for all $t \in [t_0, t_1]$ and for all j,

(4.2.3) $\lim_{j\to\infty}\sigma_j=\sigma\,|\,[t_0, t_1]$ in the sense of C^∞ topology,

(4.2.4) any $\hat{d}$-ray emanating from a point in $\sigma([t_0, t_1])$ which is asymptotic to γ intersects every σ_j.

We denote by Γ_j (resp., Γ) the set of all $\hat{d}$-rays emanating from all points on σ_j (resp., $\sigma([t_0, t_1])$) which are asymptotic to γ and set

$$\theta_j := \min\{\measuredangle(\dot{\sigma}_j(t), \dot{\mu}(0));\ \mu\in\Gamma_j,\ t\in[t_0, t_1] \text{ and } \sigma_j(t)=\mu(0)\}\ ,$$

$$\theta := \min\{\measuredangle(\dot{\sigma}(t), \dot{\mu}(0));\ \mu\in\Gamma,\ t\in[t_0, t_1] \text{ and } \sigma(t)=\mu(0)\}\ .$$

We will show that θ_j tends to θ. Indeed, we get $\mu\in\Gamma$ such that $\theta=\measuredangle(\dot{\sigma}(t), \dot{\mu}(0))$. If we set $\theta'_j := \measuredangle(\dot{\sigma}_j(t'_j), \dot{\mu}(s_j))$, where $t'_j\in[t_0, t_1]$ and $s_j\ge 0$ are numbers satisfying $\sigma_j(t'_j)=\mu(s_j)$, then this tends to θ. Moreover, $\theta_j\le\theta'_j$ for all j. Thus θ_j tends to θ. We have

$$\hat{F}_\gamma\circ\sigma(t_1)\le\hat{F}_\gamma\circ\sigma_j(t_1)+\hat{d}(\sigma(t_1), \sigma_j(t_1))\le\int_{t_0}^{t_1}\langle\nabla\hat{F}_\gamma(\sigma_j(t)), \dot{\sigma}_j(t)\rangle dt+\hat{F}_\gamma\circ\sigma_j(t_0)+\varepsilon_j$$
$$\le(t_1-t_0)\cos\theta_j+\hat{F}_\gamma\circ\sigma_j(t_0)+\varepsilon_j$$

for all j, where $\langle\ ,\ \rangle$ is the Riemannian metric of M and ∇f is the gradient of a function f. Hence

$$\hat{F}_\gamma\circ\sigma(t_1)\le(t_1-t_0)\cos\theta+\hat{F}_\gamma\circ\sigma(t_0)\ .$$

The same argument yields the other inequality. This completes the proof.

LEMMA 4.3. *For arbitrary rays σ and γ, let c be a fixed simple closed smooth curve bounding a tube of M and having the properties* (a), (b) *and* (c) *in Section* 1. *Then we have*

$$\lim_{t\to\infty}\frac{\hat{F}_\gamma\circ\sigma(t)}{t}=\cos\min\{L(\sigma, \gamma), \pi\}\ .$$

PROOF. First we consider the case where $\{\gamma_t\}$ converges. Let $\gamma_{t,s}$ be a $\hat{d}$-segment from $\sigma(t)$ to $\gamma(s)$. Since $\{\gamma_t\}$ converges, there exists a number $r>0$ such that $\hat{d}(\gamma_t, c)<r$ for all $t\ge t_\sigma$. Hence for any $t\ge t_\sigma$ there exists a number s_t such that $\hat{d}(\gamma_{t,s}, c)<r$ for all $s\ge s_t$. If $q_{t,s}$ is a point on $\gamma_{t,s}$ such that $\hat{d}(q_{t,s}, c)=\hat{d}(\gamma_{t,s}, c)$, then for all $t\ge t_\sigma$ and for all $s\ge s_t$,

$$\hat{d}(\sigma(t), \gamma(s))=\hat{d}(\sigma(t), q_{t,s})+\hat{d}(\gamma(s), q_{t,s})\ge[\hat{d}(\sigma(t), q_{t,s})+\hat{d}(q_{t,s}, c)]+[\hat{d}(\gamma(s), q_{t,s})+\hat{d}(q_{t,s}, c)]-2r$$
$$\ge\hat{d}(\sigma(t), c)+\hat{d}(\gamma(s), c)-2r\ge t+s-L(c)-2r\ .$$

Hence

$$\hat{F}_\gamma\circ\sigma(t)=\lim_{s\to\infty}[s-\hat{d}(\sigma(t), \gamma(s))]\le -t+L(c)+2r\ .$$

Moreover, $\hat{F}_\gamma$ satisfies

$$\hat{F}_\gamma\circ\sigma(t)\ge\hat{F}_\gamma\circ\sigma(0)-\hat{d}(\sigma(0), \sigma(t))=-t+\hat{F}_\gamma\circ\sigma(0)\ .$$

Therefore

$$\lim_{t\to\infty}\frac{\hat{F}_\gamma\circ\sigma(t)}{t}=-1\ .$$

On the other hand, we have $L(\sigma,\gamma)\geq\pi$ as in the proof of Lemma 4.1. The proof is completed in this case.

Next consider the case where $\{\gamma_t\}$ diverges. For an arbitrarily given positive ε there exists a positive t_0 such that

$$|\cos\sphericalangle(\dot{\sigma}(t),\dot{\gamma}_t(0))-\cos L(\sigma,\gamma)|<\varepsilon$$

for all $t\geq t_0$ by the proof of Lemma 4.1. Hence by Lemma 4.2,

$$(t-t_0)(\cos L(\sigma,\gamma)-\varepsilon)\leq\hat{F}_\gamma\circ\sigma(t)-\hat{F}_\gamma\circ\sigma(t_0)\leq(t-t_0)(\cos L(\sigma,\gamma)+\varepsilon)\ .$$

for all $t\geq t_0$. Therefore

$$\cos L(\sigma,\gamma)-\varepsilon\leq\liminf_{t\to\infty}\frac{\hat{F}_\gamma\circ\sigma(t)}{t}\leq\limsup_{t\to\infty}\frac{\hat{F}_\gamma\circ\sigma(t)}{t}\leq\cos L(\sigma,\gamma)+\varepsilon\ .$$

By the arbitrariness of ε this completes the proof.

THEOREM 4.4. *For any rays σ and γ,*

$$\lim_{t\to\infty}\frac{F_\gamma\circ\sigma(t)}{t}=\cos\min\{d_\infty(\sigma(\infty),\gamma(\infty)),\pi\}\ .$$

PROOF. Let c be a fixed simple closed smooth curve as in Lemma 4.3. Let $\{t_i\}$ be an arbitrary monotone and divergent sequence of positive numbers and let γ_i be a ray emanating from $\sigma(t_i)$ which is asymptotic to γ. If there exists a converging subsequence $\{\gamma_j\}$ of $\{\gamma_i\}$, then by a discussion similar to that in the proof of Lemma 4.3, we have

$$\text{(4.4.1)}\qquad \lim_{j\to\infty}\frac{F_\gamma\circ\sigma(t_j)}{t_j}=-1\qquad\text{and}\qquad d_\infty(\sigma(\infty),\gamma(\infty))\geq\pi\ .$$

Next consider the case where there exists a subsequence $\{\gamma_j\}$ of $\{\gamma_i\}$ such that for any compact set K, γ_j does not intersect K for all sufficiently large j. Then for all sufficiently large j, γ_j does not intersect c and hence it is contained in one of the two domains $D(\sigma,\gamma)$ and $D(\gamma,\sigma)$. Without loss of generality, we may assume that there exists a subsequence $\{\gamma_k\}$ of $\{\gamma_j\}$ such that $\gamma_k\subset D(\sigma,\gamma)$ for all k. Each γ_k is a $\hat{d}$-ray asymptotic to γ and some minimizing segment from $\sigma(t_k)$ to $\gamma(s)$ is contained in $D(\sigma,\gamma)$ for all sufficiently large $s\geq t_\gamma$. Thus the equality

$$F_\gamma\circ\sigma(t_k)=\hat{F}_\gamma\circ\sigma(t_k)$$

holds for all k. Hence by Lemma 4.3,

(4.4.2) $$\lim_{k\to\infty}\frac{F_\gamma\circ\sigma(t_k)}{t_k}=\lim_{k\to\infty}\frac{\hat{F}_\gamma\circ\sigma(t_k)}{t_k}=\cos\min\{L(\sigma,\gamma),\pi\}\,.$$

Let $\hat{d}'$ be the inner distance of $D(\gamma, \sigma)$ and set

$$\hat{F}'_\gamma(x):=\lim_{t\to\infty}[t-\hat{d}'(x,\gamma(t))]\qquad\text{for}\quad x\in D(\gamma,\sigma)\,.$$

Under our assumption $\gamma_k\subset D(\sigma,\gamma)$, we observe that $\hat{d}'(\sigma(t_k),\gamma(s))\geq\hat{d}(\sigma(t_k),\gamma(s))=d(\sigma(t_k),\gamma(s))$ for all k and for all sufficiently large $s\geq t_\gamma$. This implies that $\hat{F}'_\gamma\circ\sigma(t_k)\leq F_\gamma\circ\sigma(t_k)$ for all k. Hence

(4.4.3) $$\lim_{k\to\infty}\frac{F_\gamma\circ\sigma(t_k)}{t_k}\geq\lim_{k\to\infty}\frac{\hat{F}'_\gamma\circ\sigma(t_k)}{t_k}=\cos\min\{L(\gamma,\sigma),\pi\}\,.$$

By (4.4.2) and (4.4.3), we have $L(\sigma,\gamma)\leq L(\gamma,\sigma)$ and hence $d_\infty(\sigma(\infty),\gamma(\infty))=L(\sigma,\gamma)$. Therefore

(4.4.4) $$\lim_{k\to\infty}\frac{F_\gamma\circ\sigma(t_k)}{t_k}=\cos\min\{d_\infty(\sigma(\infty),\gamma(\infty)),\pi\}\,.$$

By (4.4.1), (4.4.4) and the arbitrariness of $\{t_i\}$, this completes the proof of Theorem 4.4.

THEOREM 4.5. *Assume that $c(M)>-\infty$. Let c be a fixed simple closed smooth curve and γ a ray from c. For any divergent sequence $\{p_j\}$, let σ_j be a minimizing segment from c to p_j. If σ_j tends to some ray σ from c, then*

$$\lim_{j\to\infty}\frac{F_\gamma(p_j)}{d(p_j,c)}=\cos\min\{d_\infty(\sigma(\infty),\gamma(\infty)),\pi\}\,.$$

PROOF. First consider the case where $2\pi\chi(M)-c(M)>0$. Now, we will prove that for an arbitrarily given $\varepsilon\in[0,(2\pi\chi(M)-c(M))/2)$, there exist rays τ_1 and τ_2 from c such that

(4.5.1) $\sigma\subset D(\tau_1,\tau_2)$,
(4.5.2) $d_\infty(\tau_1(\infty),\tau_2(\infty))=L(\tau_1,\tau_2)<\varepsilon$,
(4.5.3) $\tau_1\neq\sigma$, $\tau_2\neq\sigma$.

Let μ be a ray from c such that $d_\infty(\sigma(\infty),\mu(\infty))=(2\pi\chi(M)-c(M))/2$ and $\mu^-=\mu$, where μ^- is as in the proof of Theorem 2.4, (1). Denote by $f_\mu\colon M(\infty)\to[0,2\pi\chi(M)-c(M))$ the bijection as in the proof of Theorem 2.4, (2). Since the restriction $f_\mu\colon M(\infty)-\{\mu(\infty)\}\to(0,2\pi\chi(M)-c(M))$ is a local isometry, there exist rays τ_1 and τ_2 from c such that $0<f_\mu(\sigma(\infty))-f_\mu(\tau_1(\infty))<\varepsilon/2$ and $0<f_\mu(\tau_2(\infty))-f_\mu(\sigma(\infty))<\varepsilon/2$. The rays τ_1 and τ_2 satisfy (4.5.1), (4.5.2) and (4.5.3).

By (4.5.1) and (4.5.3), $\sigma_j\subset D(\tau_1,\tau_2)$ for all sufficiently large j. If $\{t_j\}$ is a sequence of non-exceptional t-values satisfying $|t_j-d(p_j,c)|<\varepsilon$, then since $d(p_j,S(t_j)\cap D(\tau_1,\tau_2))<\varepsilon$, we have

$$|F_\gamma(p_j)-F_\gamma\circ\sigma(t_j)|\leq d(p_j,\sigma(t_j))<L(S(t_j)\cap D(\tau_1,\tau_2))+\varepsilon$$

and hence

$$\left|\frac{F_\gamma(p_j)}{t_j}-\frac{F_\gamma\circ\sigma(t_j)}{t_j}\right|<\frac{L(S(t_j)\cap D(\tau_1,\tau_2))+\varepsilon}{t_j}$$

for all sufficiently large j. Moreover, by Theorem 4.4 and Lemma 3.1,

$$\lim_{j\to\infty}\frac{F_\gamma\circ\sigma(t_j)}{t_j}=\cos\min\{d_\infty(\sigma(\infty),\gamma(\infty)),\pi\}$$

and

$$\lim_{j\to\infty}\frac{L(S(t_j)\cap D(\tau_1,\tau_2))}{t_j}=L(\tau_1,\tau_2)<\varepsilon\,.$$

Therefore the proof in this case is completed.

Next consider the case where $2\pi\chi(M)-c(M)=0$. Similarly, for any $\varepsilon>0$ if $\{t_j\}$ is as above, then

$$|F_\gamma(p_j)-F_\gamma\circ\sigma(t_j)|\leq d(p_j,\sigma(t_j))<L(S(t_j))+\varepsilon\,.$$

In this case we have

$$\lim_{j\to\infty}\frac{L(S(t_j))}{t_j}=0\,.$$

Therefore this completes the proof of Theorem 4.5.

THEOREM 4.6. *Assume that $c(M)>-\infty$. Let c be a fixed simple closed smooth curve bounding a tube of M and let σ_1, σ_2, γ be rays from c such that $\gamma\subset D(\sigma_1,\sigma_2)$. Then the following* (1) *and* (2) *hold:*

(1) *If $L(\sigma_1,\gamma)$, $L(\gamma,\sigma_2)<\pi/2$ and if $\{p_j\}\subset D(\sigma_1,\sigma_2)$ is a sequence such that $\{d(p_j,c)\}_j$ is a monotone and divergent sequence, then*

$$\lim_{j\to\infty}F_\gamma(p_j)=+\infty\,.$$

(2) *If $L(\sigma_1,\gamma)$, $L(\gamma,\sigma_2)>\pi/2$ and if $\{p_j\}\subset D(\sigma_2,\sigma_1)$ is a sequence such that $\{d(p_j,c)\}_j$ is a monotone and divergent sequence, then*

$$\lim_{j\to\infty}F_\gamma(p_j)=-\infty\,.$$

PROOF. We will show (1). Suppose that there exists a sequence $\{p_j\}$ satisfying the assumption in (1) as well as $\lim_{j\to\infty}F_\gamma(p_j)<+\infty$. Let σ_j be a minimizing segment from c to p_j. There exists a subsequence $\{\sigma_k\}$ of $\{\sigma_j\}$ which tends to some ray σ from

c. Then σ is contained in $D(\sigma_1, \sigma_2)$ and $d_\infty(\sigma(\infty), \gamma(\infty)) < \pi/2$. Thus by Theorem 4.5, $\lim_{k\to\infty} F_\gamma(p_k) = +\infty$. This is a contradiction. (2) is derived from Theorem 4.5 in the same manner.

COROLLARY 4.7 ([Sh2]).
(1) *If* $2\pi\chi(M) - c(M) < \pi$, *then all Busemann functions are exhaustive.*
(2) *If* $2\pi\chi(M) - c(M) > \pi$, *then all Busemann functions are non-exhaustive.*

PROOF. When $2\pi\chi(M) - c(M) < +\infty$, Corollary 4.7 is an immediate consequence of Theorem 4.6. We claim that if $2\pi\chi(M) - c(M) > 2\pi$, then for any ray γ there exists a straight line σ such that $t \mapsto \sigma(-t)$ is asymptotic to γ. If this is the case, then for any ray γ, such a straight line σ satisfies $d_\infty(\sigma(\infty), \gamma(\infty)) \geq \pi$, and hence by Theorem 4.4 $F_\gamma \circ \sigma(t)$ tends to $-\infty$ as $t \to \infty$, which means that F_γ is non-exhaustive.

We will show this claim. Suppose that $2\pi\chi(M) - c(M) > 2\pi$ and that for a ray γ, M does not contain any straight line σ such that $t \mapsto \sigma(-t)$ is asymptotic to γ. Take $\varepsilon \in [0, 2\pi\chi(M) - c(M) - 2\pi)$. There exists a compact domain K such that $M - K$ is a tube and

$$\int_{M-K} G^+ dM < \varepsilon \qquad \text{and} \qquad c(K) < 2\pi(\chi(M) - 1) - \varepsilon, \tag{4.7.1}$$

because $c(M) < 2\pi(\chi(M) - 1) - \varepsilon$. By the non-existence of straight lines as above, for all sufficiently large $t > 0$ there are no rays emanating from $\gamma(t)$ which intersect K. We get an unbounded open domain D bounded by two rays α and β emanating from $\gamma(t)$ such that D contains K and contains no rays emanating from $\gamma(t)$. Note that $\partial D = \alpha([0, +\infty)) = \beta([0, +\infty))$ may happen. If θ denotes the inner angle of D at $\gamma(t)$, then

$$c(D) = 2\pi(\chi(M) - 1) + \theta,$$

which is due to [Sg] (see also [Sh5] and [Sy1]). On the other hand, (4.7.1) implies

$$c(D) = c(K) + c(D - K) < 2\pi(\chi(M) - 1),$$

which contradicts $\theta \geq 0$. This completes the proof.

LEMMA 4.8. *Assume that there exists a compact subset K of M such that the Gaussian curvature G of M is nonnegative outside K. If $L(\sigma, \gamma) = \pi/2$ holds for rays σ and γ, then there exists a positive number t_0 such that $\hat{F}_\gamma \circ \sigma$ is monotone nonincreasing on $[t_0, +\infty)$.*

PROOF. We use the same notation as in the proof of Lemma 4.1. Since $L(\sigma, \gamma) = \pi/2$, $\{\gamma_t\}$ diverges. $\{c(D_t)\}_{t \geq t_0}$ is a monotone nondecreasing sequence if $D(\sigma, \gamma) - D_{t_0}$ does not intersect K for some number t_0. Indeed, such a number t_0 exists. Since

$$c(D_t) = -\measuredangle(\dot\sigma(t), \dot\gamma_t(0)) - \int_{I(\sigma,\gamma)} \kappa\, ds,$$

the sequence $\{\sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0))\}_{t\geq t_0}$ is monotone nonincreasing. Hence if $t\geq t_1\geq t_0$, then

$$\hat{F}_\gamma\circ\sigma(t)-\hat{F}_\gamma\circ\sigma(t_1)\leq(t-t_1)\cos\sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0))$$

by Lemma 4.2. Moreover since Lemma 4.1 implies that $\sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0))$ tends to $\pi/2$ as $t\to\infty$, we have

$$\sphericalangle(\dot{\sigma}(t), \dot{\gamma}_t(0))\geq\frac{\pi}{2} \qquad \text{and hence} \quad \hat{F}_\gamma\circ\sigma(t)\leq\hat{F}_\gamma\circ\sigma(t_1)\,.$$

This completes the proof.

THEOREM 4.9. *Assume that* $2\pi\chi(M)-c(M)=\pi$ *and that there exists a compact subset* K *of* M *such that the Gaussian curvature* G *of* M *is nonnegative outside* K. *If* $d_\infty(\sigma(\infty), \gamma(\infty))=\pi/2$ *holds for rays* σ *and* γ *of* M, *then there exists a positive number* t_0 *such that* $F_\gamma\circ\sigma$ *is monotone nonincreasing on* $[t_0, +\infty)$.

PROOF. By assumption, we have $L(\sigma, \gamma)=L(\gamma, \sigma)=\pi/2$. Let c be a simple closed smooth curve bounding a tube U of M such that $K\subset M-U$ and c has the properties (a), (b) and (c). For large $t>0$ let τ be a ray emanating from $\sigma(t)$ which is asymptotic to γ. Now, since (1.11) implies that M contains no straight lines, τ is contained in U if t is sufficiently large. Then τ is a $\hat{d}$-ray asymptotic to γ, or is a $\hat{d}'$-ray asymptotic to γ, where $\hat{d}$, $\hat{d}'$ are the inner distances of $D(\sigma, \gamma)$, $D(\gamma, \sigma)$, respectively. Hence by the definition of Busemann functions,

$$F_\gamma\circ\sigma(t)=\min\{\hat{F}_\gamma\circ\sigma(t), \hat{F}'_\gamma\circ\sigma(t)\}$$

for all sufficiently large t. Therefore by Lemma 4.8, this completes the proof.

Corollary 4.7 and Theorem 4.9 imply:

COROLLARY 4.10. *If the Gaussian curvature* G *is nonnegative outside some compact subset of* M, *then the following hold*:

(1) $2\pi\chi(M)-c(M)<\pi$ *if and only if all Busemann functions are exhaustive.*

(2) $2\pi\chi(M)-c(M)\geq\pi$ *if and only if all Busemann functions are non-exhaustive.*

5. M having more than one end.

In this section we assume that M is finitely connected with k ends and admits the total curvature. Let K be a compact domain in M such that $M-\mathrm{int}(K)$ is a union of disjoint closed tubes $U_1, \cdots, U_k$ and ∂K consists of k simple closed smooth curves. If we set

$$s_i(M):=-c(U_i)-\kappa(U_i) \qquad \text{for} \quad i=1, \cdots, k\,,$$

then

$$\sum_{1\leq i\leq k} s_i(M)=2\pi\chi(M)-c(M)\,.$$

The value $s_i(M)$ does not depend on the choice of U_i by the Gauss-Bonnet theorem. For each $i=1, \cdots, k$, let M_i be a complete open Riemannian 2-manifold with one end such that there exists an isometric embedding $\iota_i\colon U_i \cup K \to M_i$ and $M_i - \iota_i(U_i \cup K)$ consists of $k-1$ open disk domains. Then the Gauss-Bonnet theorem implies

$$s_i(M) = 2\pi\chi(M_i) - c(M_i) .$$

For any ray γ let $n(\gamma) = 1, \cdots, k$ be such that some subray of γ is contained in a tube $U_{n(\gamma)}$. Rays σ and γ are said to be *equivalent* and denoted by $\sigma \sim \gamma$ if $n(\sigma) = n(\gamma) =: i$ and if the two rays $\iota_i \circ \sigma_1$ and $\iota_i \circ \gamma_1$ are equivalent in the sense of Section 1, where σ_1, γ_1 are subrays of σ, γ, respectively. Here we remark that there exist subrays σ_1, γ_1 of σ, γ such that $\iota_i \circ \sigma_1$ and $\iota_i \circ \gamma_1$ are rays in M_i. We denote the equivalence class of a ray γ by $\gamma(\infty)$ and the set of all equivalence classes by $M(\infty)$. We define the distance function $d_\infty\colon M(\infty) \times M(\infty) \to \boldsymbol{R} \cup \{+\infty\}$ by

$$d_\infty(\sigma(\infty), \gamma(\infty)) := \begin{cases} d^i_\infty(\iota_i \circ \sigma_1(\infty), \iota_i \circ \gamma_1(\infty)) & \text{if } \ i := n(\sigma) = n(\gamma) \\ +\infty & \text{if } \ n(\sigma) \neq n(\gamma) , \end{cases}$$

where σ_1, γ_1 are subrays of the rays σ, γ and d^i_∞ is the distance of $M_i(\infty)$.

In this notation we extend results in Sections 1, 2, 3 and 4 as follows:

THEOREM 5.1. *Assume that M with k ends admits the total curvature. If a ray σ of M is asymptotic to a ray γ, then σ and γ are equivalent.*

PROOF. If a ray σ is asymptotic to a ray γ, then σ, γ have subrays σ_1, γ_1 in a common tube U_i and $\iota_i \circ \sigma_1$ is asymptotic to $\iota_i \circ \gamma_1$. Thus this completes the proof by Theorem 2.1.

Theorem 5.2 is an immediate consequence of the definition of $M(\infty)$.

THEOREM 5.2. *Assume that M with k ends admits the total curvature. Let M_i for $i=1, \cdots, k$ be as above. Then*

$$M(\infty) = M_1(\infty) \cup \cdots \cup M_k(\infty) \quad \text{(disjoint union)} .$$

Let c be a simple closed smooth curve in M and set $S(t) := \{p \in M\,;\, d(p, c) = t\}$. Then there exists a number $R > 0$ such that for any non-exceptional $t > R$, $S(t)$ consists of disjoint k simple closed piecewise smooth curves (cf. [Sh4]).

THEOREM 5.3. *Assume that M with k ends admits the total curvature. Let c be an arbitrarily fixed simple closed smooth curve. For any rays σ and γ from c,*

$$\lim_{t\to\infty} \frac{d_t(\sigma(t), \gamma(t))}{t} = d_\infty(\sigma(\infty), \gamma(\infty)) ,$$

where d_t is the inner distance of $S(t)$.

PROOF. If $n(\sigma) \neq n(\gamma)$, then $d_t(\sigma(t), \gamma(t)) = +\infty$ for all sufficiently large $t > 0$ and

$d_\infty(\sigma(\infty), \gamma(\infty)) = +\infty$. If $i := n(\sigma) = n(\gamma)$, then in the same way as in the proof of Lemma 3.2, there exist a number $R > 0$ and a simple closed smooth curve $c_1 \subset M_i$ such that $\iota_i(S(t+R) \cap U_i) = S_1(t) := \{x \in M_i;\ d(x, c_1) = t\}$ for all sufficiently large $t > 0$ and thus this completes the proof by Theorem 3.3.

THEOREM 5.4. *Assume that M with k ends admits the total curvature $c(M) > -\infty$. Let c be an arbitrarily fixed simple closed smooth curve. Then for any rays σ and γ,*

$$\lim_{t \to \infty} \frac{d_t(\sigma \cap S(t), \gamma \cap S(t))}{t} = d_\infty(\sigma(\infty), \gamma(\infty)) .$$

The proof of Theorem 5.4 is similar to that of Theorem 5.3.

THEOREM 5.5. *Assume that M with k ends admits the total curvature. Then for any rays σ and γ,*

$$\lim_{t \to \infty} \frac{F_\gamma \circ \sigma(t)}{t} = \cos \min\{d_\infty(\sigma(\infty), \gamma(\infty)), \pi\} .$$

PROOF. First consider the case where $n(\sigma) \neq n(\gamma)$. Then $d_\infty(\sigma(\infty), \gamma(\infty)) = +\infty$. Moreover, for each sufficiently large number $t > 0$ and for each ray τ emanating from $\sigma(t)$, if τ is asymptotic to γ, then it intersects K, where K is as above, and hence

$$\lim_{t \to \infty} \frac{F_\gamma \circ \sigma(t)}{t} = -1 ,$$

as in the proof of Lemma 4.3.

If $i := n(\sigma) = n(\gamma)$, then there exist subrays σ_1, γ_1 of σ, γ such that $\iota_i \circ \sigma_1$, $\iota_i \circ \gamma_1$ are rays of M_i, which satisfy the equality of Theorem 4.4. Thus this completes the proof.

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