THE IDEAL CENTER OF PARTIALLY ORDERED VECTOR SPACES

BY

WILBERT WILS

Massachusetts Institute of Technology, Cambridge, Mass., U.S.A.

Introduction and summary

This paper is concerned with a partially ordered vector space E over R such that $E=E^+-E^+$. The *ideal center* Z_E of E is the algebra of endomorphisms of E which are bounded by a multiple of the identity operator I. Z_E turns out to be a very useful tool in digging up remnants of lattice structure. It provides e.g. a missing link between the theory of simplicial spaces introduced by Effros [13] and the theory of C^*-algebras. As a result a simple proof is obtained of the general extension theorem [1, 5.2.] for certain functions on the extreme boundary of compact convex sets in locally convex spaces proved by Andersen and Alfsen. A unified treatment is given of maximal measures on simplices and central measures on state spaces of C^* algebras.

In § 1 the algebraic foundations for the subsequent theory are laid.

If the ordering on E is Archimedean then Z_E is isomorphic to a dense subalgebra of $C(\Omega)$ where Ω is a compact Hausdorff space. The relation of Z_E with relics of lattice structure becomes clear in studying the idempotents in Z_E which are precisely the extremal points of $(Z_E)_1^+$. The images SE^+ of such elements are called split-faces of E^+ because they induce a splitting of E, which is similar to the decomposition in disjoint complementary bands in the lattice setting. An important property of the set of split-faces is that it is a Boolean algebra. The concept of split-faces can be "localized" by considering spaces $F-F$, with F a face of E^+ and split-faces of F within $F-F$. This gives rise to a *disjointness* relation, $\circlearrowleft$, for faces and elements of E^+. A geometric characterization of disjointness for two faces F, G is that if $0\leqslant k\leqslant f+g$ with $f\in F$, $g\in G$ then k admits a unique decomposition $k=k_1+k_2$ with $0\leqslant k_1\leqslant f$; $0\leqslant k_2\leqslant g$. Then k_1 is the infimum in E of k and f. These notions and propositions can be generalized to more than two of course.

A map R from one partially ordered vector space into another is said to be *bipositive* if R is positive and $Rk\geqslant 0$ implies $k\geqslant 0$. If $k\in E^+$ and $J_k=\{T\in Z_E \mid Tk=0\}$ then J_k is a closed ideal in Z_E and if π: $Z_E \to Z_E/J_k$ is the canonical projection then, if finally Z_E is complete for the order-unit topology, the map $Z_E/J_k \ni \pi(T) \to Tk$ is a bipositive map onto a sublattice of E.

A similar result has been obtained simultaneously by Alfsen and Andersen [2]. After this paper was finished F. Perdrizet informed the author about work of his and F. Combes [5] in which they introduce a generalisation of split-faces and they seem to be the first in the literature to consider the operators $0 \leqslant R = R^2 \leqslant I$ in End (E). The emphasis in [5] however is more on ideal theory than on decomposition theory. The relation of the split-faces, as defined here, with the split-faces of convex sets studied by Alfsen and Andersen [1] is as follows. Suppose that K is a convex set in a linear space F such that there exists a linear functional e on F with $K \subseteq e^{-1}(1)$ then we put $E^+ = \bigcup_{\lambda \geqslant 0} \lambda K$ and $E = E^+ - E^+$. The intersections with K of split-faces of E^+ are the split-faces of K. See also [20].

In § 2 we assume that E is the dual of a regular ordered Banach space and thus E is a regular Banach space itself, with a weak* closed positive E^+ and $K = E^+ \cap E_1$ is weak*-compact. Then A can be identified with the set of all weak*-continuous affine functions on K, which vanish at 0, with the natural order and a norm which is equivalent to the uniform norm on K.

It is not difficult to see that under those circumstances Z_E is order complete and even a dual Banach space. If Δ is the union of elements on extremal rays of $E^+ \setminus \{0\}$ then as in [1] the intersections of Δ with closed split-faces of E^+ defines a topology on Δ. We denote by $C_f(\Delta)$ the algebra of bounded continuous functions on Δ and by $\bar{Z}_E$ ($\cong Z_A{}^*$) the algebra of weak*-continuous elements in Z_E. If $f \in \Delta$ and $T \in Z_E$ then $Tf = \lambda_T(f) \cdot f$ with $\lambda_T(f)$ a real-number. Thus for $T \in \bar{Z}_E$ a function λ_T on Δ is defined. It is shown that the map $\bar{Z}_E \ni T \to \lambda_T \in C_f(\Delta)$ is an isomorphism. The proof is much simpler than in [1] and [2] where Andersen and Alfsen prove the same theorem in a special case, and it rests on the simple fact that if $0 \leqslant b$ is an affine u.s.c. function on a closed split-face $\bar{G}E^+$, $\bar{G} = \bar{G}^2 \in \bar{Z}_E$ then $b \circ \bar{G}$ is an affine u.s.c. extension to E^+.

Several more results, also obtained in [1] and [19], come out as direct corollaries of the last assertion.

In § 3 we make a special assumption on A, as in § 2, to the effect that the dual norm on E is additive on E^+. The problems are different and the techniques more elaborate. If $g \in E^+$ we denote by C_g the smallest face in E^+, which contains g. Let $V_g = C_g - C_g$ and Z_g the ideal center of V_g. We note that V_g, with g as an order-unit, is an order complete order-unit space. If μ is a positive measure on K representing g then it is well known that for every $\varphi \in L^\infty(K, \mu)$ there exists a unique element $\Phi_\mu(\varphi) \in V_g$ with $a(\Phi_\mu(\varphi)) = \int \varphi a d\mu$ for all $a \in A$. There are four different results in this section.

Let $g \in W \subseteq V_g$ be an order complete linear lattice in V_g, $\|g\| = 1$. It is shown that there is a unique probability measure μ on K such that Φ_μ is a lattice isomorphism of $L^\infty(K, \mu)$ onto W.

Next we introduce the notion of central measure. The idea is to write g as a convex combination of disjoint elements. From the fact that the set of split-faces of C_g is a Boolean algebra it follows that these convex splittings of g are directed and give rise to an increasing net of representing measures for g. The supremum of these measures is the *central measure* μ_g representing g. From the first result we infer that μ_g is the unique measure such that Φ_{μ_g} maps $L^\infty(K, \mu_g)$ isomorphically onto $Z_g g$.

From the construction of μ_g we expect that μ_g gives a finest splitting in disjoint elements and in particular that μ_g is concentrated on the set of those elements in E^+ which do not admit any non-trivial splitting. Such elements are called *primary* and the union of them is $\partial_{\mathrm{pr}} E^+$. A measure μ is said to be *pseudo-concentrated* on a set D iff $\mu(O) = 0$ for every Baire set O with $O \cap D = \emptyset$. With what seems to be a new technique, and where in fact the whole concept of ideal center was born, it is proved that μ_g is pseudo-concentrated on $\partial_{\mathrm{pr}} E^+$. The idea is to consider the cone B^+ of positive maps from V_g into E. The element $\bar{g} \in B^+$ is the embedding operator from V_g into E and r: $B^+ \to E^+$ is given by $r\bar{h} = \bar{h}g$, $\bar{h} \in B^+$. It is shown that $r(\partial_{\mathrm{extr}} B^+) \subseteq \partial_{\mathrm{pr}}(E^+)$, $r\mu_{\bar{g}} = \mu_g$ and μ_g is a maximal measure. The essential point is that $V_{\bar{g}} = Z_{\bar{g}}\bar{g}$ and that r maps $Z_{\bar{g}}\bar{g}$ isomorphically onto $Z_g g$. The assertion concerning μ_g and $\partial_{\mathrm{pr}}(E^+)$ follows without difficulty using the Bishop–de Leeuw–Choquet theorem for maximal measures [21, § 4].

We remark that the measurability of $\partial_{\mathrm{pr}} E^+$, also in the metrizable case, is an open problem.

In the third part of § 3 the behavior of central measure with respect to, loosely speaking, measurable split-faces is considered. First it is shown that if μ is central and $\varphi \in L^\infty(K,\mu)^+$ then $\varphi\mu$ is central. Then let $\bar{G} = \bar{G}^2 \in Z_E$ and $p_G(f) = \|\bar{G}f\|$, $f \in E^+$. There is a large class of $\bar{G}$, including those for which $\bar{G}E^+$ is closed and their complements, so that if $\Phi_\mu(1) = g$, then $\Phi_\mu(p_G) = \bar{G}g$. A little refinement of this tells us that central-measure-theoretically closed split-faces and their complementary faces can be considered as direct summands. The relation of this with the work of Effros on "ideal center" [12] is indicated.

The last section of § 3 deals with a larger class of measures. A measure $\mu \geqslant 0$ with $\Phi_\mu(1) = g$, $\|g\| = 1$, is called sub-central if Φ_μ is an isomorphism onto an order-complete sublattice of $Z_g g$. If $\nu \geqslant 0$ is another measure with $\Phi_\nu(1) = g$, then μ, ν have a supremum $\varrho_{\mu,\nu}$ with respect to the order of Choquet. A consequence of this is that every sub-central measure and in particular μ_g is majorized by all maximal measures which represent g.

In § 4 applications to simplicial spaces and C^*-algebras are given. The very last section gives some possibilities for further study.

The disjointness relation as defined in § 1 reduces for the lattice-case to ordinary disjointness as it is commonly defined for lattices. Consequently in this case $f \in \partial_{\mathrm{pr}} E^+$, in the same set up as in § 2 again, iff f lies on an extremal ray of E^+. For the metrizable case this implies conversely that E is a lattice. Only half proved but still true is the assertion that E is a lattice iff every central measure is maximal. One way follows immediately because every maximal measure representing $g \in E^+$ majorizes μ_g and thus μ_g is unique maximal. The other side is proved in the text.

The applications to C^* algebras are less trivial and rest upon the following two facts. If A is the self-adjoint part of a C^* algebra $\mathcal{A}$ and $g \in E^+ (\cong A')$ then there can be associated to g a representation π_g of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_g$ [10, 2.4.4]. There exists an isomorphism $V_g \ni h \to \tilde{h} \in (\pi_g(\mathcal{A})')_{\mathrm{sa}}$ of the order-unit space V_g onto the self-adjoint part of the commutant $\pi_g(\mathcal{A})'$ with the operator norm [10, 2.5.1]. The second observation is that if e is a unit for $\mathcal{A}$ then the map $Z_A \ni T \to Te \in A$ is an isomorphism onto the self-adjoint part of the center of $\mathcal{A}$.

A combination of the above remarks shows that $Z_g g$ is isomorphic with the self-adjoint part of the center of $\pi_f(\mathcal{A})'$ (or $\pi_f(\mathcal{A})''$). Corollaries are that $g \in \partial_{\mathrm{pr}}(E^+)$ iff $\pi_g(\mathcal{A})''$ is a factor and $g \circlearrowleft f$ iff $\pi_g \circlearrowleft \pi_f$ [10, 5.2.2]. Also the central measure of Sakai [23] coincides with the central measure as it is given here.

We remark that the general Plâncherel theorem of Dixmier [9] can be proved using central measures. The proof is not included because it is hardly an improvement over Dixmier's elegant proof. Also the invariant of Kadison [16] for quasi-equivalence can be formulated entirely within the frame work of central measures.

For separable C^* algebras, where $\partial_{\mathrm{pr}} E^+ \cap K$ is measurable [22], $\mu_g, g \in E^+$ can be characterized as the minimal measure representing g and concentrated on $\partial_{\mathrm{pr}} E^+ \cap K$. For general C^* algebras μ_g is the greatest measure which represents g and which is majorized by all maximal measures representing g.

In the last section of § 4 a condition is considered, which is satisfied in the simplicial and C^* algebra case, and which ensures that local splittings (see § 1) extend to global splittings. Some directions for further study are indicated.

This paper had its origin in conversations with Erik Alfsen and Gert Kjaergaard Pedersen after a talk on central measures for C^* algebras at a functional-analysis gathering at Aarhus, Denmark [29]. I owe much to Erik Alfsen for a lively correspondence and personal contacts.

Parts of the results of this paper were announced in [30].

It is a pleasure to thank E. B. Davies for a careful reading of the manuscript and suggesting several improvements in the presentation.

Some preliminaries on ordered vector spaces

An *ordered space* E over the real field is defined as a real vector space E with a cone E^+ which is *proper* in the sense that $E^+ \cap -E^+ = \{0\}$ and with the partial order given by saying $x \leqslant y$ iff $y-x \in E^+$. E is said to be *positively generated* if $E = E^+ - E^+$. The order on E is *Archimedean* means, $x \leqslant 0$ whenever there is a y such that $nx \leqslant y$ for all $n \geqslant 0$. An element $e \in E$ is an *order-unit* if for all $y \in E$ there is an n such that $-ne \leqslant y \leqslant ne$. For an Archimedean space with order-unit e an *order-unit-norm* can be defined by $\|x\| = \inf \{\lambda \mid -\lambda e \leqslant x \leqslant \lambda e\}$. An ordered space E is *order-complete* if every increasing and bounded net in E has a supremum.

An *order-ideal* in an ordered space E is a linear-subspace M such that $y \leqslant x \leqslant z$ with $y, z \in M$ implies $x \in M$. If M is an order-ideal, an order can be defined on the *quotient space* E/M by $(E/M)^+ = \pi(E^+)$, where π is the canonical projection.

If E and F are ordered spaces we can define a preorder on the linear maps from E into F by saying $S \leqslant T$ iff $Sx \leqslant Tx$ for all $x \in E^+$. A map $T: E \to F$ is said to be *bipositive* if $T \geqslant 0$ and if $Tx \geqslant 0$ implies $x \geqslant 0$.

In particular a linear *functional* f for an ordered space is said to be *positive* if $f(x) \geqslant 0$ for all $x \in E^+$. The preorder so defined on the algebraic dual of E is called the *dual order*. A positive functional f on E is *order-continuous* if for every increasing net $\{x_\alpha\}_\alpha$ such that $x = \sup_\alpha \{x_\alpha\}$ we have $f(x) = \sup_\alpha f(x_\alpha)$.

A subset F of a convex set $K \subseteq E$ is said to be a *face* of K if $\varrho x + (1-\varrho) y \in F$ for $\varrho \in (0, 1)$ x, $y \in K$ implies $x, y \in F$. In particular F is a face of the positive cone E^+ iff F itself is a cone and $0 \leqslant f \leqslant h$ with $h \in F$ implies $f \in F$. A face consisting of a single point is called an *extremal point*. A ray of E^+ which is a face is called an *extremal ray*.

1. The ideal center of a partially ordered vector space

Let E be a partially ordered real vector space such that $E = E^+ - E^+$.

Definition 1.1. The set of all $T \in \mathrm{End}\,(E)$, with $-\alpha I \leqslant T \leqslant \alpha I$ for some $\alpha \geqslant 0$ and I the identity map on E is called the *ideal center* of E. It is denoted by Z_E.

Obviously Z_E^+ is the smallest face in $(\mathrm{End}\,(E))^+$, which contains I. We note that Z_E is an algebra and $Z_E^+ \cdot Z_E^+ \subseteq Z_E^+$, that is, Z_E is an ordered algebra. The algebra unit I of Z_E is also an order-unit.

THEOREM 1.2. *The ideal center Z_E of an Archimedean space E is isomorphic with a dense subalgebra of $C(\Omega)$ where Ω is the $\sigma(Z_E^*, Z_E)$-compact set of real homomorphisms of Z_E.*

Proof. If E is Archimedean, so is Z_E and Stone's algebra theorem applies. We sketch the proof.

Let $\|\cdot\|$ be the order-unit norm on Z_E and $K=\{f \in Z_E^* \mid \|f\|=f(e)\leqslant 1\}$. Then K is a weak*-compact subset of the norm-dual Z_E^* of Z_E. If $0\leqslant a\leqslant ne$ for $a\in E$ then $0\leqslant ne-a\leqslant ne$ so that $\|ne-a\|\leqslant n$ and for $f\in K$, $|f(ne)-f(a)|\leqslant n\|f\|$. There follows $f(a)\geqslant 0$ for $a\geqslant 0$.

For $a\in A$ let $|a|=\inf\{\lambda \mid a\leqslant \lambda e\}$ then $|\lambda a|\geqslant \lambda |a|$ for all scalars λ. Define f: $\alpha a+\beta e\to \alpha|a|+\beta$, then f is linear, $\|f\|=f(e)=1$. Let $\bar{f}$ be a norm-preserving extension of f to Z_E, then $\bar{f}\in K$. This shows that the evaluation map Λ: $a\in Z_E\to \Lambda a\in C(K)$ with $\Lambda a(f)=f(a)$, $f\in K$ is a bipositive, isometric map.

An element $0\neq f\in K$ is extremal iff $\|f\|=1$ and $0\leqslant g\leqslant f$ implies $g=\lambda f$ for some $\lambda\in[0,1]$. If $0\leqslant b\leqslant e$, $b\in Z_E$ and $a\in Z_E$ then $0\leqslant ba\leqslant a$ and so for an extremal $f\neq 0$ in K, $f(ab)=\lambda f(a)$ for all $a\in Z_E$ and some λ. Substitution $a=e$ shows $\lambda=f(b)$ so that f is multiplicative. Let Ω be the weak*-closed subset of K consisting of the real non-trivial homomorphisms of Z_E, then Ω contains all extremal points of K, so that by the Krein–Milman theorem and what is proved already $Z_E\ni a\to \Lambda a|_\Omega\in C(\Omega)$ is an isometric bipositive homomorphism of ordered-algebras. The image is dense in $C(\Omega)$ since $\Lambda Z_E|_\Omega$ separates points contains the constants and is a real algebra.

Our first aim is to study Z_E and some of the related concepts.

LEMMA 1.3. *$S\in(Z_E)_1^+=\{T\in Z_E \mid 0\leqslant T\leqslant I\}$ is extremal iff $S^2=S$. Any two extremal points of $(Z_E)_1^+$ commute.*

Proof. For $S\in(Z_E)_1^+$, $S=\frac{1}{2}(2S-S^2)+\frac{1}{2}(S^2)$ so that S extremal implies $S=S^2$.

Conversely if $0\leqslant S=S^2\leqslant I$ and $S=\alpha T+(1-\alpha)T'$, $\alpha\in(0,1)$, then $S=S^2=\alpha ST+(1-\alpha)ST'$ with ST, $ST'\leqslant S$ so that $ST=ST'=S$. We obtain also $0=\alpha(I-S)T+(1-\alpha)(I-S)T'$ so that $0=(I-S)T=(I-S)T'$. There follows $T=ST=S$ and S is extremal.

Next let S, $T\in(Z_E)_1^+$ be extremal. We note that $0\leqslant TS(I-T)\leqslant TI(I-T)=0$ so that $TS(I-T)=0$. Similarly $(I-T)ST=0$. Now we have $TS=ST=TST$.

COROLLARY. *The set of extreme points of $(Z_E)_1^+$ is a Boolean algebra with* $\sup(S,T)=S+T-ST$.

We note that if Z_E is Archimedean ordered the idempotents in Z_E of course correspond with open and closed subsets of Ω but in general there might be open and closed subsets of Ω, which do not correspond to idempotents in Z_E. This can be readily seen by taking for E the restriction to disjoint intervals of $\mathbf{R}$ of the polynomials on $\mathbf{R}$.

There is a simple geometric characterization of the sets SE^+ with $0 \leqslant S = S^2 \leqslant I$.

Definition 1.4. Two faces G, H of E^+ are said to be *disjoint*, notation $G \between H$, if $G+H$ is a face of E^+ and $(G-G) \cap (H-H) = \{0\}$.

A face G of E^+ is called a *split-face* if there exists another face G' of E^+ with $G' \between G$ and $G+G' = E^+$. Then G, G' are called complementary faces.

It is clear that if G, G' are complementary faces of E^+ then for $p \in E^+$ we have $p = p_1 + p_2$ with $p_1 \in G$, $p_2 \in G'$ and p_1, p_2 are unique. The map $\bar{G}$: $p \to p_1$ can be extended to a linear map of E with $0 \leqslant \bar{G} = \bar{G}^2 \leqslant I$. Thus to every split face of E^+ corresponds a unique extremal element of $(Z_E)_1^+$. A moments thought shows that the converse is true also.

PROPOSITION 1.5. *The map $\partial_{\text{extr}}(Z_E)_1^+ \ni T \to TE^+$ is one-one and onto the set of split-faces of E^+.*

If G, H are split-faces then $H+G = (\bar{H} + \bar{G} - \bar{H}\bar{G})E^+$ so that $H+G$ is also a split-face. Also if G is a split-face then $[I - \bar{G}]E^+$ is the split-face complementary to G.

COROLLARY. *The set of split-faces of E^+ is a Boolean algebra.*

We remark that $E = \bar{G}E + (I - \bar{G})E$ as a direct sum of ordered vector-spaces, that is, this is a direct sum of vector spaces and $E^+ = \bar{G}E^+ + (I-\bar{G})E^+$. Note that $\bar{G}E$ is an order-ideal in E since $x \leqslant y \leqslant z$ with $x, z \in \bar{G}E$ implies $[I-\bar{G}]x = 0 \leqslant [I-\bar{G}]y \leqslant 0$ so that $y = \bar{G}y \in \bar{G}E$. Let π: $E \to E/\bar{G}E$ be the canonical projection and u the restriction of π to $(I-\bar{G})E$, then π is an isomorphism of ordered spaces, i.e., π is bipositive and onto.

Thus we are led to the following definition.

Definition 1.6. The set of all split-faces of E^+ is called the central Boolean algebra of E. It is denoted by $B(E^+)$.

A family of mutually disjoint split-faces $\{G_i\}_{i \in I}$ is said to be a *splitting* of E^+ if $\Sigma_i G_i = E^+$, where $\Sigma_i G_i$ is the set of all finite sums $\Sigma_i g_i$, $g_i \in G_i$.

One could ask whether a finest splitting of E^+ exists or equivalently whether $B(E^+)$ is atomic. In general this question does not have a positive answer and we shall reformulate the problem. The concept of disjointness and splitting can easily be carried over to elements of E^+. We introduce some more notation.

If G is a face of E^+, we put $V_G = G - G$. For $p \in E^+$ we put

$$F_p = \{q \in E^+ \mid q \leqslant p\}, \quad C_p = \bigcup_{\lambda \geqslant 0} \lambda F_p, \quad V_p = C_p - C_p.$$

Two elements $p, p' \in E^+$ are said to be disjoint, notation $p \between p'$, if $C_p \between C_{p'}$. And $p = \Sigma_{i=1}^n p_i$ is a splitting of p if $\{C_{p_i}\}_{i=1}^n$ is a splitting of C_p.

In R_3 we consider $C = \{(x, y, z) \mid \|y\| + \|z\| \leqslant x\}$. Then C is a cone which does not admit any non-trivial splittings, but which contains many elements that can be split. The problem is among others that although splittings of bigger faces induce splittings of smaller faces, the converse is not true in general. Therefore it seems better to study not so much splittings of E^+ as of the faces C_p, $p \in E^+$. This shows that our point of view will be essentially local.

The right setting for studying splittings of $p \in E^+$ is representing measures. In order then to be able to take limits and introduce the analysis of the problem we shall have to assume that the cone E^+ has a compact base, say. We shall pursue this line of thought in § 3 and elaborate first somewhat more on disjointness.

PROPOSITION 1.7. *Suppose G, H are faces of E^+. The following conditions are equivalent.*

(i) $G \mathrel{\underset{\circ}{|}} H$.

(ii) $G \cap H = \{0\}$ *and for all* $g \in G$, $h \in H$ *we have* $F_{g+h} = F_g + F_h$.

If there are elements $g \in G$, $h \in H$ *such that* $G = C_g$, $H = C_h$. *Then* (i) *and* (ii) *are equivalent to*

(iii) $F_{g+h} = F_g + F_h$ *and* $F_g \cap F_h = \{0\}$.

Proof. (i)$\rightarrow$(ii). Let $g \in G$, $h \in H$ and $0 \leqslant k \leqslant g + h$. Then with $k' = g + h - k$ we have $k + k' \in G + H$, which is a face. Thus $k = k_1 + k_2$, $k' = k_1' + k_2'$ with $k_1, k_1' \in G$ and $k_2, k_2' \in H$. Hence because of $V_G \cap V_H = \{0\}$ we see $k_1 + k_1' = g$, $k_2 + k_2' = h$ so that $k = k_1 + k_2 \in F_{g+h}$.

(ii)$\rightarrow$(i). If $0 \leqslant k \leqslant g + h$ then $k = k_1 + k_2$ with $k_1 \leqslant g$, $k_2 \leqslant h$.

Since $G + H$ is additive too, it is a face. To show $V_g \cap V_H = \{0\}$ it suffices to prove that if $g + h = g' + h'$ with $g', g \in G$; $h, h' \in H$ then $g = g'$, $h = h'$. Well, $0 \leqslant g \leqslant g' + h'$ and so $g = g_1 + g_2$ with $g_1 \leqslant g'$ and $g_2 \leqslant h'$. Since $g_2 \in G \cap H$, $g_2 = 0$ and $g = g_1 \leqslant g'$. By symmetry $g = g'$, $h = h'$.

(ii)$\rightarrow$(iii) is obvious. So we assume that $G = C_g$, $H = C_h$ and prove (iii)$\rightarrow$(ii). We consider $g' \leqslant g$, $h' \leqslant h$ and show $F_{g'+h'} = F_{g'} + F_{h'}$. This will be enough to prove (ii). So let $k \leqslant g' + h'$. Then $k = k_1 + k_2$ with $k_1 \leqslant g$, $k_2 \leqslant h$. Let $k' = g - g'$. Then $k' \leqslant g$ and $k_1 + k' \leqslant g + h$. Therefore $k_1 + k' = l_1 + l_2$ with $l_1 \leqslant h$, $l_2 \leqslant g$. Also $l_1 \leqslant k_1 + k' \leqslant 2g$ so that $l_1 \in G \cap H = \{0\}$ and $k_1 + k' = l_2 \leqslant g$. We obtain $k_1 \leqslant g - k' = g'$. Similarly $k_2 \leqslant h'$ and we are through.

We see that for lattices, where we always have the Riesz decomposition property $F_{g+h} = F_g + F_h$, disjointness is equivalent to $G \cap H = \{0\}$. In general disjointness is much stronger. We infer from this proposition that disjointness is hereditary in the sense that $G \mathrel{\underset{\circ}{|}} H$, $G' \subseteq G$, $H' \subseteq H$ implies $G' \mathrel{\underset{\circ}{|}} H'$, where of course G, G', H, H' are faces of E^+.

It is simple to find examples of disjoint faces of cones in $\mathbf{R}^2$ or $\mathbf{R}^3$. Other non-trivial examples will be studied later. It is an amusing exercise to find a counter example to the statement $F \mathrel{\underset{\circ}{|}} G$, $G \mathrel{\underset{\circ}{|}} H$, $H \mathrel{\underset{\circ}{|}} F$ implies $F \mathrel{\underset{\circ}{|}} G + H$.

As long as we consider disjoint elements it is as if we are in a lattice.

LEMMA 1.8. *Suppose* $g \perp h$; $g, h \in E^+$ *and* $0 \leqslant k \leqslant g+h$.

(i) *Then* $k = k_1 + k_2$ *with* $k_1 \leqslant g$, $k_2 \leqslant h$. *If* $k' \leqslant k$, $k' \leqslant g$ *then* $k' \leqslant k_1$.

(ii) *If* $k \leqslant g'$ *with* $g' \perp h$, *then* $k \leqslant g$.

Proof.

(i) Since $k' \leqslant k_1 + k_2$ we have $k' = k_1' + k_2'$ with $k_1' \leqslant k_1$, $k_2' \leqslant k_2$. But then $k_2' \in F_g \cap F_{k_2} = \{0\}$ and therefore $k' = k_1' \leqslant k_1$.

(ii) If $k = k_1 + k_2$ with $k_1 \leqslant g$, $k_2 \leqslant h$ then $k_2 \leqslant k \leqslant g'$ and $k_2 \leqslant h$. Thus $k_2 = 0$ and $k = k_1 \leqslant g$.

If we want to generalize these results to more than two faces (or elements) we cannot consider merely families of two by two disjoint faces. If $C' \subseteq E^+$, let $F(C')$ be the smallest face of E^+, which contains C'. A family $\{G_i\}_{i\in I}$ of faces of E^+ is said to be *split* if $\{G_i\}_{i\in I}$ is a splitting of $F(\bigcup_{i\in I} G_i)$.

LEMMA 1.9. *Let* $\{G_i\}_{i\in I}$ *be a split-family of faces of* E^+. *If* $0 \leqslant k \leqslant \sum_i h_i$, *with* $h_i \in G_i$, *then* $k = \sum_i k_i$ *with* $k_i \in G_i$. *The* k_i *are unique and* $k_i \leqslant h_i$.

Proof. Let $\tilde{E} = F(\bigcup_i G_i) - F(\bigcup_i G_i)$ and consider the idempotents S_i in $Z_{\tilde{E}}$ corresponding to the G_i. Then $\sum_i S_i = I$ in the sense that $\sum_i S_i g = g$ for all $g \in \tilde{E}$. Let $k = \sum_i S_i k = \sum_i k_i'$ with $k_i' \in G_i$. Then $S_i k = k_i'$ since $S_i G_j = 0$ for $i \neq j$ and $S_i^2 = S_i$. Because $0 \leqslant S_i \leqslant I$ also $0 \leqslant S_i k \leqslant S_i(\sum_i h_i) = h_i$.

From this we deduce readily that splittings of bigger faces induce splittings of smaller faces.

COROLLARY. *Let* $\{G_i\}_{i\in I}$ *be a split-family of faces of* E^+ *and* $J \subseteq I$.

(i) $\{G_i\}_{i\in J}$ *is a split-family.*

(ii) *If* $H_i \subseteq G_i$ *are faces, then* $\{H_i\}_{i\in I}$ *is a split-family.*

(iii) $F(\bigcup_{i\in I} G_i)$ *is the set of all finite sums* $\sum_i g_i$ *with* $g_i \in G_i$.

(iv) *If* $\{H_i^\alpha\}_{\alpha \in J_i}$ *are split-families such that* $H_i^\alpha \in G_i$, $\alpha \in J_i$, *then* $\{H_i^\alpha\}_{i,\alpha}$ *is a split-family.*

Proof. Straightforward.

PROPOSITION 1.10. *Let* $\{G_i\}_{i\in J}$ *be a family of faces of* E^+. *The family* $\{G_i\}_{i\in J}$ *is split iff every* $k \in F(\bigcup_i G_i)$ *admits a unique decomposition* $k = \sum_i k_i$, $k_i \in G_i$.

Proof. The only if part follows from 1.9 and its corollary. For $J' \subseteq J$ we put $G_{J'} = F(\bigcup_{i\in J'} G_i)$. In order to prove the if part we have to show that each G_i is a split-face of

G_J and that $G_i \between G_{J\setminus\{i\}}$. If $k \in G_J$ then $k = k_i + \sum_{j \neq i} k_j$ with $k_j \in G_j$ so that $G_i + G_{J\setminus\{i\}} = G_J$. From the uniqueness assertion there follows readily $(G_{J\setminus\{i\}} - G_{J\setminus\{i\}}) \cap (G_i - G_i) = \{0\}$ and so $G_i \between G_{J\setminus\{i\}}$.

So much for disjointness and the extremal elements of $(Z_E)^+$. We return to Z_E, the ideal center of E.

The various properties, which E can have, carry over directly to Z_E. For example if E is order complete so is Z_E; if E is Archimedean so is Z_E; if all the spaces V_p with $p \in E^+$ as their order unit and equipped with the corresponding norm-topology are complete then Z_E is complete. We note that every $T \in Z_E$ leaves the spaces V_p invariant. The proofs of the above facts are elementary and are omitted.

In the sequel we shall consider *Archimedean spaces only.*

We are interested in the maps $Z_E \ni T \to Tk$, $k \in E^+$.

PROPOSITION 1.11. *Let k be an order-unit for E. Then the map $Z_E \ni T \to Tk \in E$ is bipositive.*

Proof. It is clear that the map is linear. Let $T \in Z_E$ be such that $Tk \geq 0$. We infer from 1.2 that for all $\varepsilon > 0$ there exists a $S \in (Z_E)_1^+$ with $ST \geq -\varepsilon I$ and $(I-S)T \leq \varepsilon I$. Suppose $0 \leq f \leq k$.

$$Tf = STf + (I-S)Tf \geq [-\varepsilon I + (I-S)T]f \geq -\varepsilon k + (I-S)Tk \geq -\varepsilon k$$

so that $Tf \geq 0$. Therefore $T \geq 0$.

If Z_E is complete, we can do better than 1.11.

LEMMA 1.12. *Suppose S, $T \in Z_E$ are such that* $\sup(S, T) \in Z_E$. *If $k, f \in E$ are such that Sk, $Tk \leq f$ then* $\sup(S, T)k \leq f$.

Proof. We identify $T, S \in Z_E$ with their image in $C(\Omega)$ with Ω as in 1.2. Let $\Omega_1 = \{w \in \Omega \mid T(w) \geq S(w)\}$ and for $\varepsilon > 0$ let $\Omega_2 = \{w \in \Omega \mid T(w) \leq S(w) - \varepsilon\}$. Let $\lambda \geq 2 \max(\|T\|, \|S\|)$. There exists $U \in (Z_E)_1^+$ such that $1 \geq U(w) \geq 1 - \varepsilon/\lambda$ for $w \in \Omega_1$ and $0 \leq U(w) \leq \varepsilon/\lambda$ for $w \in \Omega_2$. We find

$$\sup(T, S) + \varepsilon I \geq UT + (I-U)S \geq \sup(T, S) - \varepsilon I$$

and we obtain $f \geq UTk + (I-U)Sk$. If $k = k_1 - k_2$ there follows

$$f \geq \sup(T, S)k - \varepsilon(k_1 + k_2).$$

Since this holds for all $\varepsilon > 0$, by Archimedicity $f \geq \sup(T, S)k$.

COROLLARY 1.13. *Suppose that Z_E is complete with respect to its order-unit norm and $k \in E^+$. Then $Z_E \ni T \to Tk \in E$ is a lattice homomorphism of Z_E onto a sublattice of E.*

Proof. It follows from 1.12 that for $T, S \in Z_E$, $\sup(T, S)k$ is the smallest element in E, which majorizes Tk and Sk.

COROLLARY 1.14. *Let Z_E be complete and $k \in E^+$. Then $T_k = \{T \in Z_E \mid Tk = 0\}$ is a closed ideal in Z_E and if $\pi: Z_E \to Z_E/T_k$ is the canonical projection then $Z_E/T_k \ni \pi(T) \to Tk \in Z_E k$ is a lattice isomorphism.*

Proof. Obvious.

2. Ordered Banach spaces

In this paragraph we introduce the analysis. The ideas in this section are strongly inspired by Alfsen and Andersen [1]. Independently Alfsen and Andersen [2] established Theorem 2.8. Also independently Perdrizet [20] found 2.9–2.12. The idea to use regular rather than GM or GL-spaces (see further on for definitions) was incorporated in a second version of this paper after seeing a preprint of [5]. We feel nevertheless that publication of this material is justified since our proofs are different and serve as a nice illustration of the use of the algebra developed in § 1.

It will be convenient to have some terminology ready. An *ordered Banach* space X over the real field is defined as a Banach space X with a partial order defined by a closed cone X^+. X is said to be *regular* [E. B. Davies [6]] if it has the properties

R_1: If $x, y \in X$ and $-y \leqslant x \leqslant y$, then $\|y\| \geqslant \|x\|$.

R_2: If $x \in X$ and $\varepsilon > 0$, then there is some y with $-y \leqslant x \leqslant y$ and $\|y\| \leqslant \|x\| + \varepsilon$.

We shall denote by K the set $K = \{x \in X \mid x \geqslant 0, \ \|x\| \leqslant 1\}$. The open unit ball in a Banach space X will be written as X_1' and the closed unit ball as X_1. If X is a regular Banach space, T is the set $T = \{k_1 - k_2 \mid k_1, k_2, k_1 + k_2 \in K\}$ and it follows from R_1 and R_2 that

$$X_1' \subseteq T \subseteq X_1. \tag{P}$$

Let X be a positively generated ordered Banach space, i.e., $X = X^+ - X^+$ with Banach dual X^*. The set $(X^*)^+$ of continuous positive functionals is a weak*-closed proper cone in X^*. By the Hahn–Banach theorem if $x \in X$, then $x \in X^+$ iff $f(x) \geqslant 0$ for all $f \in (X^*)^+$.

We firstly record some results from the duality theory for regular Banach spaces. For the proofs we refer to Ng [17].

THEOREM 2.01. *Let X be a regular Banach space.*

(1) [*E. B. Davies* [6]] *Then* X^* *is also regular.*

(2) [*Ng* [17]] *Suppose there is given on* X *a locally convex topology* τ *such that* K *is* τ*-compact. Let* A *be the space of linear functionals on* X*, whose restriction to* K *is* τ*-continuous. Then* A *is a regular Banach subspace of* X^* *such that* X *is naturally isomorphic as an ordered Banach space to the normed dual of* A.

COROLLARY. *If* A *and* E *are ordered Banach spaces such that* $E = A^*$*, then* A *is regular iff* E *is regular. In that case* A *is isomorphic, under the evaluation map on* K*, with* $A_0(K)$*, where* $K = E_1 \cap E^+$ *and* $A_0(K)$ *is the space of affine weak* continuous functions on* K*, which vanish at* 0*, with the natural order and norm* $\|a\| = \sup \{a(f) - a(g) \mid f, g, f+g \in K\}$.

Examples of regular Banach spaces are the usual L^p spaces, $1 \leqslant p \leqslant \infty$.

There are some interesting special cases. A regular Banach space X is said to be a GM (GL) space if GM: for $x, y \in X_1'$, there exists $z \in X_1'$ with $x, y \leqslant z$ (GL: for $x, y \in X^+$, we have $\|x+y\| = \|x\| + \|y\|$).

Examples of GM-spaces are the self-adjoint part of a C^*-algebra, the simplex-spaces introduced by Effros, and M-spaces. Examples of GL-spaces are the preduals of von Neumann-algebras (and duals of C^*-algebras) and L-spaces.

THEOREM 2.02. (*Asimow, Ng, Perdrizet*). *Suppose that* A *and* E *are regular Banach spaces such that* $E = A^*$*, then*

(1) A *is a* GL*-space iff* E *is a* GM*-space.*

(2) A *is a* GM*-space iff* E *is a* GL*-space.*

In the rest of this paragraph A shall denote a fixed regular Banach space, which is represented as $A_0(K)$ as in 2.01 with $K = E_1 \cap E^+$ where $E = A^*$. We shall consider E always with the $\sigma(E, A)$ topology. So far the introduction.

LEMMA 2.1. *If* X *is an order-complete space and* F *is a face of* X *then* Z_{F-F} *is order-complete.*

Proof. It suffices to show that $F - F$ is order-complete. Let $\{g_\alpha\}_\alpha$ be an increasing net in $F - F$, which is bounded above by $g \in F - F$. Let $h = \sup_\alpha g_\alpha \in X$ then $0 \leqslant h - g_\alpha \leqslant g - g_\alpha \leqslant k - l$ with $k, l \in F$. Hence $0 \leqslant h - g_\alpha \leqslant k$ and because F is a face of X^+, $h - g_\alpha \in F$ and $h \in F - F$.

COROLLARY. *For* F *as in the lemma, the central Boolean algebra* $B(F)$ *is complete.*

If we let $X=E$, then the conditions of Lemma 2.1 are certainly satisfied.

LEMMA 2.2. *The set of closed split-faces of E^+ is closed under arbitrary intersections and finite sums.*

Proof. The intersection of an arbitrary family of closed (split-) faces of E^+ is again a closed (split-) face according to 2.1.

If F and G are closed split-faces of E^+, then $F+G$ is a split-face of E^+ and since $(F+G)\cap K=3\mathrm{co}(F\cap K, G\cap K)\cap K$ it is also a closed face.

Let Δ be the set consisting of the elements in extremal rays of $E^+/\{0\}$. Then the closed convex hull of Δ is E^+. Following [1] we define the facial topology on Δ by taking as closed sets the intersections of Δ with closed split-faces of E^+. According to Lemma 2.2, this defines a (non-Hausdorff) topology on Δ.

Also let $\bar{Z}_E$ be the algebra of weak* continuous elements in Z_E. Then $T\in\bar{Z}_E$ iff there exists $S\in Z_A$ with $S^*=T$ and for the order-unit norms $\|\cdot\|_0$ clearly $\|S^*\|_0=\|S\|_0$. In order to show that $\bar{Z}_E$ is a complete sub-algebra of Z_E, it suffices to prove that Z_A is complete.

LEMMA 2.3. *Let X be a regular Banach space. Then for $T\in Z_X$, $\|T\|_0=\|T\|$, where $\|T\|$ is the operator norm and $\|T\|_0$ the order-unit norm.*

Proof. (a) $\|T\|\leqslant\|T\|_0$. Let $x\in X$, $\varepsilon>0$. There exists $y\in X$, $\|y\|\leqslant\|x\|+\varepsilon$, $-y\leqslant x\leqslant y$ according to R_2. Suppose $T\in Z_X$ satisfies $-\lambda I\leqslant T\leqslant\lambda I$. We obtain $(\lambda I+T)x\leqslant(\lambda I+T)y$ and $-(\lambda I-T)x\leqslant(\lambda I-T)y$. Adding these inequalities gives $Tx\leqslant\lambda y$. The same holds for $-x$ so that by R_1, $\|Tx\|\leqslant\lambda\|y\|\leqslant\lambda\|x\|+\lambda\varepsilon$ there follows $\|T\|\leqslant\|T\|_0$.

(b) $\|T\|\geqslant\|T\|_0$. In the proof of (b) we need Lemma 2.4. In the proof of 2.4 we only use $\|T\|\leqslant\|T\|_0$. Thus we assume $(Z_X,\|\cdot\|_0)$ is complete. Let $T\in Z_E$ and $\alpha=\|T\|_0=\inf\{\lambda\geqslant 0\,|\,\lambda I\geqslant T\}$ (otherwise consider $-T$). We identify $T\in Z_E$ with its image in $C(\Omega)$ with Ω as in 1.2. Standard arguments show that there exists $U\in(Z_E)_1^+$ such that $TU+\varepsilon U\geqslant\alpha U$, $U\neq 0$. Let $x\in X$, such that $y=Ux\neq 0$. By R_2 we may suppose $x>0$ so that $y\geqslant 0$. Then we have $Ty+\varepsilon y\geqslant\alpha y\geqslant 0$ so that $\|Ty\|\geqslant(\alpha-\varepsilon)\|y\|$ and $\|T\|\geqslant\alpha-\varepsilon$. The conclusion follows.

LEMMA 2.4. *Let X be a regular Banach space, then $(Z_X,\|\cdot\|_0)$ is complete.*

Proof. Suppose $\{T_n\}_n$ is a Cauchy sequence in $(Z_X,\|\cdot\|_0)$. Then there is a $\lambda>0$ such that $-\lambda I\leqslant T_n\leqslant+\lambda I$. By 2.3(a) the sequence $\{T_n\}_n$ converges to a $T\in\mathrm{End}\,(X)$ such that $\|T\|\leqslant\lambda$ and by Archimedicity $-\lambda I\leqslant T\leqslant\lambda I$ so that $T\in Z_X$. Clearly also $T=\lim_n T_n$ for $\|\cdot\|_0$.

PROPOSITION 2.5. *If $f \in \Delta$ and $T \in \bar{Z}_E$, then there exists a (unique) constant $\lambda_T(f)$ such that $\lambda_T(f)f = Tf$. Let $\lambda_T: f \to \lambda_T(f)$. The map $\bar{Z}_E \ni T \to \lambda_T$ is a bipositive algebra homomorphism of the ordered algebra $\bar{Z}_E$ into a sub ordered algebra of $C_f(\Delta)$, the set of bounded facially continuous functions on Δ.*

Proof. If $0 \leqslant T \leqslant I$, $T \in \bar{Z}_E$ we have $0 \leqslant Tf \leqslant f$ and so $Tf = \lambda_T(f)f$. Obviously $T \to \lambda_T$ is linear, positive and multiplicative. If $\lambda_T \geqslant 0$, then $Tf \geqslant 0$ for $f \in \Delta$. Since $E^+ = \overline{\mathrm{co}(\Delta)}$, $T \geqslant 0$.

It is left to show that $\lambda_T \in C_f(\Delta)$. Because $\lambda_I = 1$ and the λ_T form an algebra it suffices to prove that for $0 \leqslant T \leqslant I$, λ_T is lower semi-continuous. We consider $F' = \{f \in \Delta \mid \lambda_T(f) \leqslant \beta\}$ with $0 \leqslant T \leqslant I$ and $\beta \geqslant 0$. Let $T' = \inf\{T, \beta I\}$ and $T'' = T - T'$. Then $\lambda_T = \lambda_{T'} + \lambda_{T''}$ and $\lambda_T(f) > \beta$ iff $\lambda_{T''}(f) \neq 0$, so that $F' = \{f \in \Delta \mid T''f = 0\}$. We may suppose $0 \leqslant T'' \leqslant I$ and if $\bar{T} = I - T''$ we have $F' = \{f \in \Delta \mid \bar{T}f = f\}$. If $F = \{f \in E^+ \mid \bar{T}f = f\}$, then F is a closed face of E^+ with $F \cap \Delta = F'$. We put $S = \inf_n \{\bar{T}^n\}$, then $0 \leqslant S = S^2 \leqslant I$ and since $\bar{T}S = S\bar{T} = S$ we have $SE^+ \subseteq F$. Because also $SE^+ \supseteq F$ we obtain $SE^+ = F$ so that F is split and we are done.

We note that the map $T \to \lambda_T$ can be extended to Z_E in a straightforward way. It is possible however that the ideal $\{T \in Z_E \mid \lambda_T = 0\}$ is non-empty. Indeed we only have to produce a split-face without extremal rays. An example is the set of normal positive functionals on a von Neumann algebra of type II or III, which is a face in the cone of positive functionals and which does not contain any extremal rays.

The question is, of course, whether $Z_E \ni T \to \lambda_T \in C_f(\Delta)$ is onto.

THEOREM 2.6. *Let $G \subseteq E^+$ be a closed split face and b a non-negative upper-semi-continuous (u.s.c.) affine function on G. We define $\bar{b}(k) = b(\bar{G}k)$, $k \in E^+$. Then b is u.s.c. and affine on E^+.*

Proof. We show that $\{k \in E^+ \mid \bar{b}(k) \geqslant \alpha\} \cap nK$ is closed for all n. Thus let $\{g_i\}_i$ be a net in nK with $\bar{b}(g_i) \geqslant \alpha$ and $\lim_i g_i = g$. The net $\{\bar{G}g_i\}_i$ is contained in $nK \cap G$, has a cluster-point $g_1 \in nK \cap G$ and $g_1 \leqslant g$, $b(g_1) \geqslant \alpha$. We find $g_1 = \bar{G}g_1 \leqslant \bar{G}g$ and therefore $\bar{b}(g) = b(\bar{G}g) \geqslant b(g_1) \geqslant \alpha$.

All the following results are more or less simple consequences of 2.6.

PROPOSITION 2.7. *Let $0 \leqslant \varphi \leqslant 1$ be a facially u.s.c. function on Δ. If $G'_\alpha = \{f \in \Delta \mid \varphi(f) \geqslant \alpha\}$ and G_α is the unique closed split-face with $G_\alpha \cap \Delta = G'_\alpha$, then for $a \in A^+$ we put*

$$[\Phi a](k) = \int_0^1 a(\bar{G}_\alpha k)\, d\alpha \quad k \in E.$$

Then Φa is u.s.c. on E^+ and linear on E. On Δ we have $\Phi a = \varphi \cdot a$.

Proof. The $\bar{G}_\alpha$ are decreasing and so for $k\in E^+$, $\alpha\to a(\bar{G}_\alpha k)$ is decreasing. This shows that the above integral exists in the Riemann sense. Plainly Φa is linear. If $f\in\Delta$ and $\alpha\leqslant\varphi(f)$ we have $\bar{G}_\alpha f=f$ and for $\alpha>\varphi(f)$, $\bar{G}_\alpha f=0$ so that $[\Phi a](f)=\varphi(f)a(f)$.

From 2.6 we infer that the upper sums of the integral, considered as functions of k, are u.s.c. on E^+. But then Φa itself is u.s.c. as well.

THEOREM 2.8. *The map $\bar{Z}_E\ni T\to\lambda_T\in C_f(\Delta)$ defined in 2.4 is an isomorphism.*

Proof. If φ in 2.7 is continuous, then so is $1-\varphi$. We apply 2.7 to $1-\varphi$ and a and find that $a-\Phi a$ is u.s.c. also so that Φa is continuous. The map $a\to\Phi a$ defines an element $T\in Z_A$ and obviously $\lambda_{T*}=\varphi$. This ends the proof.

We see that we have obtained a natural representation of $Z_A=\bar{Z}_E$.

Let G be a fixed closed split-face of E^+. We put $V=G-G=\bar{G}E$, $W=(I-\bar{G})E$ and

$$G^0=V^0=\{a\in A\,|\,\langle a,g\rangle=0 \text{ for } f\in G\}.$$

We want to study G^0, A/G^0, V, $A|_V$ and $G^0|_W$.

PROPOSITION 2.9. *Let X be a regular Banach space and H a split-face of X^+.*

(1) *$\bar{H}X=H-H$ is a closed subspace of X and a regular Banach space.*

(2) *The restriction $\pi=\pi|_{(I-\bar{H})X}$ of the canonical projection $\pi\colon X\to X/\bar{H}X$ to $(I-\bar{H})X$ is an isomorphism of ordered Banach spaces.*

Proof. (1) It follows from Lemma 2.3 that the projection $\bar{H}$ is continuous on the Banach space X and so has closed range. R_1 is trivially satisfied by $\bar{H}X$ and R_2 follows since $0\leqslant\bar{H}\leqslant\bar{I}$.

(2) It follows from the remark preceding definition 1.6 that π is an isomorphism of ordered spaces and since by Lemma 2.5 $\|I-\bar{H}\|\leqslant 1$, π is isometric.

We take up G, V, W and G^0 as indicated. It follows from 2.9 that V and W are regular Banach spaces and that W and E/V are isomorphic, via π, as ordered Banach spaces.

LEMMA 2.10. *V is closed.*

Proof. It suffices to show that $V\cap E_1=V_1$ is closed. Well,

$$V\cap E_1=3\mathrm{co}\,(G\cap K,\ -G\cap K)\cap E_1,$$

where all sets on the right are compact and so are closed.

Let $[E/V]$ denote the linear space E/V with quotient-order, quotient-norm and the weak*-quotient topology. Similarly $[W]$ denotes W with the induced order and norm and the $\sigma(W, G^0)$ topology. Finally $[G^{0*}]$ will be the Banach dual of G^0 with the weak* topology and the dual order. Let $\pi: E \to E/V$ be the canonical projection; $\dot{\pi} = \pi|_W$; $\varrho: E/V \to G^{0*}$ the natural isomorphism and $i: W \to G^{0*}$ the restriction map.

PROPOSITION 2.11. *Let the notation be as above.*

(1) *Then the diagram*

$$\begin{tikzcd} & {[W]} \arrow[dl, "i"'] \arrow[dr, "\dot{\pi}"] & \\ {[G^{0*}]} & & {[E/V]} \arrow[ll, "\varrho"'] \end{tikzcd}$$

is commutative and the maps $\dot{\pi}$, ϱ and i are isomorphisms.

(2) *The set $K \cap W$ in $[W]$ is compact. The cone W^+ in $[W]$ is closed.*

(3) *The map $r: G^0 \ni a \to a|_{K \cap W} \in A_0(K \cap W)$ is an isomorphism of ordered spaces.*

Proof. (1) It is well known that ϱ is an isomorphism of Banach spaces and it follows from 2.9 that so is $\dot{\pi}$. Since for $k \in W$ and $a \in G^0$ we have

$$\langle \varrho \circ \dot{\pi}(k), a \rangle = \langle k, a \rangle = \langle ik, a \rangle$$

there follows $\varrho \circ \dot{\pi} = i$. Thus also i is an isomorphism of Banach spaces. We see from the definition that i is a linear homeomorphism for the weak*-topologies and so is ϱ. Therefore the same holds for $\dot{\pi}$. There follows that a functional a on $[W]$ is continuous iff $a|_{W_1}$ is continuous. Since $W \cap K = \dot{\pi}^{-1} \circ \pi(K)$, where K is compact and $\dot{\pi}^{-1} \circ \pi$ is continuous, $W \cap K$ is compact. As in 2.01 we find $W_1 = 2\mathrm{co}(W \cap K, -W \cap K) \cap W_1$ so that a is continuous iff $a|_{W \cap K}$ is continuous, i.e., $a|_{W \cap K} \in A_0(K \cap W)$. Therefore we have $G^0|_{K \cap W} = A_0(K \cap W)$.

Now 2.01 shows that i is bipositive. Since also $\dot{\pi}$ is bipositive $\varrho = i \circ \dot{\pi}^{-1}$ is bipositive.

(2) This follows from 2.01 since $G_0|_{K \cap W} = A_0(K \cap W)$.

(3) r is onto and obviously bipositive, hence one-one.

COROLLARY. *G^0 is a regular Banach space.*

Proof. This follows from 2.10 and 2.01.

There is a similar proposition for V. Let $\sigma: A \to A/G^0$ be the canonical projection. Remark that G^0 is an order-ideal in A and that consequently $\sigma(A^+)$ is a proper cone in A/G^0 and defines an order. Let $[(A/G^0)^*]$ be $(A/G^0)^*$ with the dual norm and order and weak*-topology. Similarly let $[V]$ be V with the induced order, norm and topology. We consider $A|_V$ with the norm $\|a|_V\| = \sup\{|\langle a, f \rangle| \mid f \in V_1\}$ and A/G^0 with the quotient norm.

PROPOSITION 2.11. (1) *Let* $\tau\colon A/G^0 \to A|_V$ *be defined by* $\tau(\sigma(a)) = a|_V$, $a \in A$. *Then* τ *is an isomorphism of ordered Banach spaces.*

(2) $\sigma^*\colon [(A/G^0)^*] \to [V]$ *is an isomorphism.*

Proof. (1) It is well known that τ is a Banach space isomorphism. Also τ is positive. It follows from 2.12 that τ^{-1} is positive too.

(2) Again it is well known that ϱ^* maps $(A/G^0)^*$ into $V = G^{00} \subseteq E$ and is an isomorphism of topological vector spaces. If $a \in A^+$ and $f \in (A/G^0)^*$ we have $\langle \sigma^*(f), a\rangle = \langle f, \sigma(a)\rangle$ which shows that σ^* is bipositive.

COROLLARY. A/G^0 *is a regular Banach space.*

Proof. This is a consequence of 2.11 and 2.01.

PROPOSITION 2.12. *Let a be a non-negative continuous linear function on G. Suppose that* φ *is a l.s.c. function on K such that* $\varphi \geqslant \bar{a}$, *with* $\bar{a}$ *as in 2.6. Then there exists for every* $\varepsilon > 0$ *a* $c \in A^+$ *with*

$$a = c|_G \quad \text{and} \quad c|_K \leqslant \varphi + \varepsilon$$

Proof. It follows as in [17, Lemma 9.7] that the set $\{b \in A(K) \mid b > a\}$, where $A(K)$ is the set of continuous affine functions on K, is directed downwards with infimum a. Using the compactness of K, that φ is l.s.c. and that a is continuous we find that for $\varepsilon' = \varepsilon/8$ there exists $c_1 \in A(K)$ such that

$$a \leqslant c_1|_{G\cap K} \leqslant a + \varepsilon' \quad \text{and} \quad \bar{a} \leqslant c_1 \leqslant \varphi + \varepsilon'.$$

Let $c_2 = c_1 - c_1(0)$ and extend c_2 to a linear functional on E. Then $c_2 \in A$ and

$$-\varepsilon' \leqslant a - c_2|_{G\cap K} \leqslant \varepsilon'.$$

Because V is regular $V_1 = (2+\delta)\operatorname{co}(G \cap K, -G \cap K) \cap E_1$ for all $\delta > 0$ and hence $\|a - c_2|_V\| \leqslant 2\varepsilon'$. With τ as in 2.11 being isometric, there must exist $c_3 \in A$ such that $\|c_3\| \leqslant 3\varepsilon'$ and $c_3|_V = a - c_2|_V$. Let $c_4 = c_2 + c_3$. We obtain $c_4 \in A$ and

$$c_4|_V = a \quad \text{and} \quad -4\varepsilon' \leqslant c_4|_K \leqslant \varphi + 4\varepsilon'$$

since $0 \leqslant c_1(0) \leqslant \varepsilon'$.

The function $c_4 - \bar{a}$ is l.s.c. on E^+ and $c_4 - \bar{a}|_G = 0$. We note that $c_4 - \bar{a} = c_4$ on W. There follows that $c_4|_{W^+}$ is a l.s.c. affine function on W^+ in $[W]$ (which is isomorphic with $\pi(E^+) \subseteq E/V$ with the quotient topology).

We apply the Hahn-Banach theorem to the cone $\{(f, \alpha) \mid \alpha \geqslant c_4(f), f \in W^+\}$ and $\{(f, -5\varepsilon') \mid f \in K \cap W\}$ in $[W] \times \mathbf{R}$. In the usual way we obtain a $c_5 \in [W]^* \cong G^0$ such that

$$-5\varepsilon' \leqslant c_5|_{K\cap W} \leqslant c_4|_{K\cap W} \quad \text{and} \quad 0 = c_5|_G \leqslant c_4|_G.$$

Since c_5 and c_4 are linear $c_5|_{W^+} \leqslant c_4|_{W^+}$ and we obtain $c_5|_{E^+} \leqslant c_4|_{E^+}$. We put $c = c_4 - c_5$. Then c satisfies the requirements.

With an extra assumption on φ we can extend a theorem of T. B. Andersen [3].

PROPOSITION 2.13. *Suppose a is as in 2.11 and φ is a l.s.c. concave, positively homogeneous function on E^+ such that there exists $\delta > 0$ with*

$$\varphi|_G \geqslant a \quad \text{and} \quad \varphi|_{W^+} \geqslant \delta \| \cdot \| |_{W^+}.$$

Then there exists a $c \in A^+$ such that $0 \leqslant c|_{E^+} \leqslant \varphi$ and $c|_G = a$.

Proof. By induction we construct a sequence $\{c_n\}_n \subseteq A^+$ such that (1) $c_n|_G = a/2^n$, (2) $\|c_n\| \leqslant 2\|a|_V\|/2^n$ and (3) $\varphi - \Sigma_{i=1}^n c_i \geqslant \delta \| \cdot \| /4^n$ on W^+. It is a consequence of (1) and $a \leqslant \varphi|_G$ that also $\varphi - \Sigma_{i=1}^n c_i|_G \geqslant a/2^n|_G \geqslant 0$.

The series $\{c_n\}_n$ converges uniformly on E_1 so that $c = \Sigma_{i=1}^\infty c_i$ is contained in A. It satisfies

$$c|_G = \sum_{i=1}^\infty a/2^i = a.$$

Therefore $\varphi \geqslant c$ on G. But also $\varphi \geqslant c$ on W^+ by (3), so that $\varphi \geqslant c$ on E^+.

It suffices to treat $n = 1$. Because with $a_k = a/2^k$, $\varphi_k = \varphi - \Sigma_{i=1}^k c_i$ and $\delta_k = \delta/4^k$ we reduce $n = k + 1$ to $n = 1$.

If $\|a\| = 0$, let $c_1 = 0$. If $\|a\| \neq 0$, let $\varphi' = \min(\varphi, \|a\|)$ and apply 2.12 to φ', a and $\varepsilon = \min(\delta/2, \|a\|)$. Indeed since $\varphi|_{W^+} \geqslant 0 = \bar{a}|_{W^+}$ and φ is concave, $\varphi \geqslant \bar{a}$. We obtain $2c_1 \in A^+$ such that $2c_1|_G = a$ and $0 \leqslant 2c_1 \leqslant \varphi' + \varepsilon$ on K. There follows $0 \leqslant 2c_1 \leqslant \|a\| + \varepsilon \leqslant 2\|a\|$ on K and $\|c_1\| \leqslant \|a\|$. We also have $0 \leqslant c_1 \leqslant \varphi/2 + \delta/4 \leqslant \varphi - \delta/4$ on $W \cap K$. Hence for $f \in W$, $c_1(f/\|f\|) \leqslant \varphi(f/\|f\|) - \delta/4$ and so $c_1(f) \leqslant \varphi(f) - \delta\|f\|/4$.

We remark that the concavity of φ on W^+ and on G is not used in the proof.

Let us now quickly consider the case where A is a GM-space and hence E a GL-space. The following refinements can be given. The notation will always be as in the un-primed propositions.

PROPOSITION 2.9′. (a) *Let X be a* GM-*space and a H split-face of X^+.*

(a_1) *$\bar{H}X$ is a* GM-*space.*

(a_2) *If $0 \leqslant T \leqslant I$ for $T \in Z_E$ and $x \in X$, then* $\|x\| = \max(\|Tx\|, \|(I-T)x\|)$.

(a_3) $K = \bar{H}K + (I - \bar{H})K$.

(b) *Let X be a* GL-*space and H a split-face of X^+.*

(b_1) *$\bar{H}X$ is a* GL-*space.*

(b_2) *If $0 \leqslant T \leqslant I$ for $T \in Z_E$ and $x \in X$, then* $\|x\| = \|Tx\| + \|(I-T)x\|$.

(b_3) $K = \mathrm{co}(\bar{H}K, (I - \bar{H})K)$.

Proof. (a_1) and (b_1) follow as in 2.9 (1).

(a_2) Suppose $\|Tx\|, \|(I-T)x\| < \alpha$, then there exists $z \in X$, $\|z\| < \alpha$ such that Tx, $(I-T)x \leqslant z$. From 1.12 we see $x \leqslant z$. The same applies to $-x$ so that $+z' \leqslant x \leqslant z$ with $\|z\|, \|z'\| < \alpha$. Then there is a z'' with $-z', z \leqslant z''$, $\|z''\| \leqslant \alpha$. Therefore $-z'' \leqslant x \leqslant z''$, so that $\|x\| \leqslant \alpha$.

(a_3) This follows from (a_2).

(b_2) For $x \in X$, there exist $x_1, x_2 \in X^+$ such that $\|x\| = \|x_1\| + \|x_2\|$, $x = x_1 - x_2$.

$$\|x\| \leqslant \|Tx\| + \|(I-T)x\| \leqslant \|Tx_1\| + \|(I-T)x_1\| + \|Tx_2\| + \|(I-T)x_2\| \leqslant \|x_1\| + \|x_2\|.$$

(b_3) This follows from (b_2).

COROLLARY 2.10′. *G^0 is a* GM-*space.*

Proof. $(W, \|\cdot\|)$ is a GL-space and so the proof follows from 2.02.

COROLLARY 2.11′. *A/G^0 is a* GM-*space.*

Proof. $(V, \|\cdot\|)$ is a GL-space and 2.02 applies again.

We remark that if A is a GM-space, then in 2.12 we can take for φ a concave l.s.c. function, which is non-negative and $\varphi \geqslant a$ on $V \cap K$. Since $\bar{a}$ is affine and $\varphi \geqslant \bar{a} = 0$ on $W \cap K$, $\varphi \geqslant \bar{a}$ on $\text{co}(V \cap K, W \cap K) = K$ according to Corollary 2.9′.

Similarly one can in 2.13 use a l.s.c. concave φ on K such that $\varphi \geqslant a$ on $V \cap K$ and $\varphi \geqslant \delta\|\cdot\|$ on $W \cap K$, and obtain an extension c of a to E such that $0 \leqslant c|_K \leqslant \varphi$. The same proof applies.

In [1] Alfsen and Andersen study the case where A is a GM-space with order-unit and they use an equivalent form of 2.12 as point of departure. In [2] they introduce also the center of A and prove Theorem 2.8 using the results of [1]. For the same case Andersen [3] proves 2.13 under slightly stronger conditions on φ.

In a first version of this paper the results of this paragraph were only stated for the case where A is a GM-space. The same proofs however applied to the more general case of regular spaces. The advantage of regular spaces is that a duality theory can be set up. This was the point of view, which prevailed in Combes and Perdrizet [5] and Perdrizet [20]. An extensive study is undertaken, on how various properties of $M \subseteq A$ carry over to properties of A/M, $M^0 \subseteq E$ and E/M^0. In [20] e.g., subspaces M are considered such that $M^0 \cap E^+$ is a closed split-face of E^+. In the course of that study results similar to 2.9–2.12 are proved.

3. Central decomposition

We return to the ideas on decomposition, which were developed in § 1. The situation will be the same as in § 2 and the notation will be consistent with the one already introduced. We assume A *is a* GM-*space*, and hence E will be GL by 2.02.

The first theorem will state the existence and uniqueness of certain representing measures on K and will be applied several times in the sequel. The measure which corresponds to point-evaluation at $f \in K$, will be denoted by δ_f.

Definition 3.1. If μ is a positive measure on K, we let Φ_μ: $L^\infty(K, \mu) \to E$ be the map

$$a(\Phi_\mu(\varphi)) = \int \varphi a d\mu \quad a \in A, \ \varphi \in L^\infty(K, \mu)$$

THEOREM 3.2. *Let $g \in K$, $\|g\| = 1$ and $g \in W \subseteq V_g$, where W is a complete linear lattice in the induced ordering.*

The set of discrete probability measures $\Sigma \alpha_i \delta_{f_i}$ with $f_i \in W^+ \cap K$, $\alpha_i \geqslant 0$ and $\Sigma \alpha_i f_i = g$ is directed in the ordering of Choquet–Meyer. Let μ be the supremum of this net of measures then μ is the unique probability measure such that Φ_μ is a lattice isomorphism from $L^\infty(K, \mu)$ onto W.

Proof. The first part of the theorem is an elementary consequence of the fact that W is a lattice.

Since W is a complete linear lattice with order unit it is lattice isomorphic to $C(S)$ for some completely disconnected Hausdorff space S [24, 8.4 and 8.5]. Let us denote this isomorphism by $W \ni f \to \hat{f} \in C(S)$. Since $\|\cdot\|$ is additive on W^+ we can define a linear functional e on W by $e(f) = \|f_1\| - \|f_2\|$, where $f_1 - f_2 = f$; $f_1, f_2 \in W^+$. Then e is order-continuous on W. For $a \in A$ and e a measure on S is determined by

$$\mu_a(\hat{f}) = a(f); \quad \mu_e(\hat{f}) = e(f) \quad f \in W.$$

The support of μ_e is S and so we can embed $C(S)$ isometrically in $L^\infty(S, \mu_e)$. Since both e and μ_e are order-continuous this embedding is order-continuous too. Hence the image of $C(S)$ in $L^\infty(S, \mu_e)$ is an order-complete subspace and thus coincides with $L^\infty(S, \mu_e)$. The measures μ_a are absolutely continuous with respect to μ_e and so $\mu_a = \varphi_a \mu_e$. In fact $|\varphi_a(s)| \leqslant \|a\|$ for all $s \in S$ and $\varphi_a \geqslant 0$ if $a \geqslant 0$, so that $\varphi_a \in L^\infty(S, \mu)$ and we may even take $\varphi_a \in C(S)$.

For $s \in S$ we define $\varrho_s \in K$ by $a(\varrho_s) = \varphi_a(s)$. The map ϱ: $S \ni s \to \varrho_s \in K$ is continuous, due to the continuity of the φ_a and therefore $\varrho \mu_e$ is a well-defined measure on K. The map R: $L^1(K, \varrho\mu_e) \ni \varphi \to \varphi \circ \varrho \in L^1(S, \mu_e)$ maps $L^1(K, \varrho\mu_e)$ isometrically onto a closed subspace of $L^1(S, \mu_e)$, which contains all φ_a, $a \in A$. If $\hat{f} \in L^\infty(S, \mu_e)$ is such that

$$0 = \int \hat{f} \cdot \varphi_a d\mu_e = a(f) \quad a \in A$$

then $f=0$. Consequently $\hat{f}=0$ also and R is isomorphic. The dual map $R^*: L^\infty(S, \mu_e) \to L^\infty(K, \varrho\mu_e)$ is then also a lattice isomorphism. In total we have for $f \in W$

$$a(\Phi_{\varrho\mu_e}(R^*\hat{f})) = \int (R^*\hat{f}) a \, d\varrho\mu_e = \int \hat{f}\varphi_a d\mu_e = a(f), \quad a \in A$$

so that $\Phi_{\varrho\mu_e}(R^*\hat{f}) = f$ for $f \in W$. This shows that $\Phi_{\varrho\mu_e}$ maps $L^\infty(K, \varrho\mu_e)$ isomorphically onto W. Let ν be a probability measure which represents f and such that $\Phi_\nu: L^\infty(K, \nu) \to W$ is an isomorphism and let μ be as in the statement of the theorem. We show $\nu = \mu$.

Thus let $\Sigma_i \alpha_i \delta_{f_i}$ be a discrete measure as mentioned, then

$$\Sigma_i \alpha_i \delta_{f_i} \prec \Sigma_i \alpha_i [\Phi_\nu^{-1}(f_i)]\nu = \nu$$

so that $\mu \prec \nu$. But conversely ν is the supremum of all discrete measures $\Sigma_i \alpha_i \delta_{g_i}$, where $g_i = \Phi_\nu(\varphi_i)$ with $\varphi_i \in L^\infty(K, \nu)^+$ and $\Sigma_i \alpha_i \varphi_i = 1$. [Cf. 20, Lemma 9.6.] By assumption we have $g_i \in W^+$ and $\Sigma_i \alpha_i g_i = g$ so that also $\nu \prec \mu$ and we are done.

Next we give the "central" definition of this section.

Definition 3.3. For $h \in E^+$, we denote Z_{V_h} by Z_h. A (positive) measure μ on K, which represents $h \in K$ is said to be *central* iff Φ_μ maps $L^\infty(K, \mu)$ isomorphically onto the lattice $Z_h h \subseteq V_h$ and $\|\mu\| = \|h\|$. We note that $E \cong A^*$ has a norm, and that the intersection of E^+ with the unit ball of E is K.

THEOREM 3.4. *For $g \in K$, there exists a unique central measure μ which represents g.*

Proof. This follows directly from 3.2 and 3.3.

Let us keep $g \in K$ fixed and μ as in 3.4. In order to study the properties of μ we introduce the cone B^+ of positive linear maps of V_g into E. Let $B = B^+ - B^+$. Then B^+ defines an ordering on B. We provide B with the topology of pointwise weak* convergence. A universal cap for B^+ is

$$\bar{K} = \{\bar{h} \in B^+ \mid \bar{h}g \in K\}.$$

The compactness of $\bar{K}$ follows since $\bar{K}$ is homeomorphic to the set of all affine maps $h: F_g \to K$, with $h(0)=0$ and F_g as in 1.7, with the topology of point-wise convergence. According to Tychonoff's theorem the latter set is compact.

The embedding of V_g into E is denoted by $\bar{g}$. There exists a natural map $\tau: B \to E$, given by

$$\tau\bar{h} = \bar{h}g.$$

Obviously τ is continuous, $\tau \geqslant 0$ and $\tau\bar{K} \subseteq K$.

Definition 3.5. An element $h \in E^+$ is said to be primary iff Z_h consists of scalar multiples of the identity map in V_h.

The set of primary points in E^+ is $\partial_{\mathrm{pr}} E^+$ and $\partial_{\mathrm{pr}} K = \partial_{\mathrm{pr}} E^+ \cap K$.

There follows that $h \in E^+$ is primary iff C_h and h do not admit any non-trivial splittings. Indeed $(Z_h)^+$ is compact (cf. 2.1) to the effect that Z_h is trivial iff the only extremal points of $(Z_h)_1^+$ are 0 and I. On the other hand, the set of extremal points of $(Z_h)_1^+$ is in a one-to-one correspondence with the split-faces of C_h (cf. 1.5). Plainly also $\partial_{\mathrm{extr}} K \subseteq \partial_{\mathrm{pr}} K$.

We want to split $g \in E^+$ into primary elements, that is, we like to show that the central measures are carried by $\partial_{\mathrm{pr}} E^+$ in some sense.

PROPOSITION 3.6. *τ maps $\partial_{\mathrm{pr}}\bar{K}$ into $\partial_{\mathrm{pr}} K$.*

Proof. Let $\bar{h} \in \bar{K}$, $\bar{f} \in B$ and $T \in Z_{\tau\bar{h}}$. If $-\alpha\bar{h} \leqslant \bar{f} \leqslant \alpha\bar{h}$ with $\alpha \geqslant 0$ and $0 \leqslant k \leqslant g$ we have

$$0 \leqslant (\alpha\bar{h} \pm \bar{f})k = \alpha\bar{h}k \pm \bar{f}k \leqslant \alpha(\tau\bar{h}) \pm \bar{f}k$$

so that $\bar{f}$ maps V_g into $V_{\tau\bar{h}}$. Thus we put (by abuse of notation)

$$\bar{T}\bar{f} = T \circ \bar{f}$$

If $T \geqslant 0$, we see $\bar{T} \geqslant 0$. Conversely if $\bar{T} \geqslant 0$, then $0 \leqslant (\bar{T}\bar{h})g = T(\tau\bar{h})$ and we infer from 1.11, $T \geqslant 0$. There follows that $\bar{T} \in Z_{\bar{h}}$ and $Z_{\tau\bar{h}} \ni \bar{T} \to \bar{T} \in Z_{\bar{h}}$ is bipositive. In particular $\bar{h} \notin \partial_{\mathrm{pr}} \bar{K}$ if $\tau\bar{h} \notin \partial_{\mathrm{pr}} K$.

PROPOSITION 3.7. *Let $T \in Z_g$ and $\bar{T} \in Z_{\bar{g}}$ be as in* 3.6 (*note $\tau\bar{g} = g$*). *The map $Z_g \ni T \to \bar{T}\bar{g} \in V_g$ is an order-isomorphism.*

Proof. Plainly the map is bipositive. If $\bar{h} \in V_{\bar{g}}$ then $\bar{h}$ maps V_g into itself and because $\bar{h}$ is bounded by multiples of $\bar{g}$, which "acts as the identity operator" on V_g, we see that $\bar{h}$ defines an element $T \in Z_g$ by $Tk = \bar{h}k$ for $k \in V_g$. Obviously $\bar{h} = \bar{T}_{\bar{g}}$ and we are done.

COROLLARY. *$V_{\bar{g}}$ is a complete linear lattice and there exists a unique maximal representing measure $\bar{\mu}$ on $\bar{K}$ for $\bar{g}$. Moreover, $\bar{\mu}$ is central.*

Proof. We apply 3.2 to $\bar{g}$ and $W = V_{\bar{g}}$ and infer that $\bar{\mu}$ is the supremum of all discrete measures on $\bar{K}$, which represent $\bar{g}$, so that $\bar{\mu}$ is unique maximal. Since $\bar{T} \in Z_{\bar{g}}$ we have $V_{\bar{g}} = Z_{\bar{g}}g$ and therefore $\bar{\mu}$ is central as well.

PROPOSITION 3.8. *Let $\mu = \tau\bar{\mu}$. Then μ is the* (*unique*) *central representing measure on K for g.*

Proof. Suppose $T_i \in (Z_g)_1^+$ and $\alpha_i > 0$, $\Sigma_{i=1}^n \alpha_i = 1$ are such that $\Sigma_i \alpha_i T_i = I$. Because τ is linear and $\tau \overline{T}\overline{g} = Tg$ for $T \in Z_g$ we get

$$\tau\left(\sum_{i=1}^n \alpha_i \delta_{\overline{T}_i \overline{g}}\right) = \sum_{i=1}^n \alpha_i \delta_{T_i g}.$$

If $\bar{\mu} \succ \bar{\nu}$ on $\overline{K}$, then $\tau\bar{\mu} \succ \tau\bar{\nu}$ on K and so we find that $\tau\bar{\mu} = \mu$ since $\tau\bar{\mu}$ is the supremum of all measures $\Sigma_{i=1}^n \alpha_i \delta_{T_i g}$ by the above equality and the continuity of τ.

The observation that the use of B^+ as above, rather than of the set of positive bilinear functionals on $A \times V_g$, which featured in an earlier draft of this paper, would clarify the proofs considerably, is due to E. B. Davies.

THEOREM 3.9. *Every $g \in K$, $\|g\| = 1$ can be represented by a unique central measure μ and $\mu(O) = 0$ for every Baire set $O \subseteq K$ with $O \cap \partial_{\mathrm{pr}} K = \varnothing$.*

Proof. Existence and uniqueness were established in 3.4. Let $O \subseteq K$ be a Baire set with $O \cap \partial_{\mathrm{pr}} K = \varnothing$. Then $\tau^{-1}(O) \cap \partial_{\mathrm{pr}} \overline{K} = \varnothing$, because of 3.6 and since $\tau^{-1}(O)$ is also a Baire set we have $0 = \bar{\mu}(\tau^{-1}(O)) = \tau\bar{\mu}(O) = \mu(O)$ by the maximality of $\bar{\mu}$ and 3.8 [21, ch. IV].

COROLLARY. *If μ is a central measure on K, then $\mu(K \setminus \overline{\partial_{\mathrm{pr}} K}) = 0$.*

Proof. If $f \in C(K)$ and $f|_{\partial_{\mathrm{pr}} K} = 0$ then $\{k \in K \mid f(k) \neq 0\}$ is a Baire set disjoint from $\partial_{\mathrm{pr}} K$ and so $\int |f| d\mu = 0$. If we take the supremum over all $f \in C(K)_1^+$ with $f|_{\partial_{\mathrm{pr}} K} = 0$ we obtain $\mu(\mathrm{K} \setminus \overline{\partial_{\mathrm{pr}} K}) = 0$.

Remarks. 1. If $g \neq 0$, we let its central measure be $\|g\|\mu$, where μ is the central measure associated to $g/\|g\|$.

2. A problem, of course, is the measurability of $\partial_{\mathrm{pr}} K$ and for e.g. separable A it would be desirable to have that supp $\mu \cap \partial_{\mathrm{pr}} K \neq \varnothing$. A possible way to prove that this is the case could be to show that supp $\bar{\mu} \cap \partial_{\mathrm{pr}} \overline{K} \neq \varnothing$ or even supp $\bar{\mu} \cap \partial_{\mathrm{extr}} \overline{K} \neq \varnothing$. In general $\overline{K}$ is not metrizable, even if K is, so that we do not know much about the measurability of $\partial_{\mathrm{extr}} \overline{K}$.

If we consider a subspace $g \in W \subseteq V_g$ and the positive maps from W into E, we can proceed as before. Proposition 3.6 actually still holds. The map $Z_g \in T \to \overline{T}\overline{g} \in V_g$ maps Z_g onto a complete sublattice, say $\overline{W}$, of $Z_{\bar{g}}\,\bar{g}$. The map q: $\overline{W} \to Z_g g$ is an isomorphism. We can "diagonalize" $\overline{W}$ by means of a measure $\bar{\mu}$, as can be seen from 3.2 and again we have 3.9. However we do not know whether $\bar{\mu}$ is maximal. By taking for W a separable subspace of V_g, with respect to the order-unit norm, we construct indeed a $\overline{K}$ which is metrizable. It is unclear whether one can chose W at the same time such that $\bar{\mu}$ is maximal. A sufficient condition would be that V_g is the smallest order closed subspace of V_g which contains $\overline{W}$.

For separable C^* algebras we shall see that W can be chosen to be separable and so that $\bar{\mu}$ is maximal [cf. § 4].

We want to investigate the behavior of central measures with respect to split-faces of E^+ and we need the following result on extension of central elements.

LEMMA 3.10. *Let $g \in E^+$ and $T \in Z_g^+$. Then every $S \in Z_{Tg}$ is the restriction to V_{Tg} of an element $\bar{S} \in Z_g$.*

Proof. Let $S \in (Z_{Tg})_1^+$. We suppose $T \in (Z_g)_1^+$. For $k \in V^+$ we put

$$\bar{S}k = \sup_n \{S[k-(I-T)^n k]\}.$$

We note that $\bar{S}k$ is well-defined, since $k-(I-T)^n k \in V_{Tg}$ and the sequence is bounded in V_g by k. We have $0 \leqslant \bar{S}k \leqslant k$, $\bar{S}$ is additive and positive homogeneous on V_g^+, so that it can be extended to a linear operator on V_g, which is central. If $0 \leqslant k \leqslant Tg$ we find

$$0 \leqslant S(I-T)^n k \leqslant (I-T)^n Tg$$

The sequence of maps $(I-T)^n T$ converges pointwise to 0 and thus the right side of the inequality tends to $0 \in V_g$. There follows $\bar{S}k = Sk$, so that $\bar{S}$ is an extension of S.

PROPOSITION 3.11. *If μ is a central measure on K and $\varphi \in L^\infty(K,\mu)^+$ then $\varphi\mu$ is central.*

Proof. Let g be the resultant of μ and $T \in Z_g^+$ such that $\Phi_\mu(\varphi) = Tg$. If χ is the support function of φ in $L^\infty(K,\mu)$, then $L^\infty(K,\varphi\mu)$ can be identified with $\chi L^\infty(K,\mu)$. If $\psi \in (L^\infty(K,\varphi\mu))_1^+$ and $\Phi_\mu(\psi) = Sg$, $S \in (Z_g)_1^+$, then we obtain

$$\Phi_{\varphi\mu}(\psi) = \Phi_\mu(\varphi\psi) = STg.$$

Therefore $\Phi_{\varphi\mu}$ maps into $Z_{Tg}Tg$. If $\Phi_{\varphi\mu}(\psi) \geqslant 0$, then $\Phi_\mu(\varphi\psi) \geqslant 0$ and thus $\psi\varphi \geqslant 0$, $\psi \geqslant 0$ for $\psi \in L^\infty(K,\varphi\mu)$. This shows that $\Phi_{\varphi\mu}$ is bipositive. If $S \in Z_{Tg}$, there is a $\psi \in L^\infty(K,\mu)$ such that $\Phi_\mu(\psi) = \bar{S}g$. We have

$$\Phi_{\varphi\mu}(\chi\psi) = \Phi_\mu(\chi\psi\varphi) = \Phi_\mu(\psi\varphi) = STg$$

so that the image of $\Phi_{\varphi\mu}$ is all of $Z_{Tg}Tg$ and $\Phi_{\varphi\mu}$ is an isomorphism. We infer from 3.4 that $\varphi\mu$ is the central measure which represents Tg.

We consider the set of split-faces of E^+, $B(E^+)$. If $G \in B(E^+)$, we define p_G by $p_G(f) = \|\bar{G}f\|$, $f \in E^+$. Then p_G is additive and positive homogeneous on E^+. A measurable affine function a on K is said to satisfy the central barycentric calculus if

$$\int a\,d\mu = a(g)$$

for all $g \in E^+$ and where μ is the central measure associated to g. We call $G \in B(E^+)$ *admissible* if p_G is measurable and satisfies the central barycentric calculus.

PROPOSITION 3.12. *Let $G \in B(E^+)$ be admissible. For every $g \in E^+$, with associated central measure μ, we have $\Phi_\mu(p_G) = \bar{G}g$.*

Proof. There exists an idempotent $\varphi \in L^\infty(K, \mu)$ with $\Phi_\mu(\varphi) = \bar{G}g$. We get

$$\int p_G \, d\mu = \|\bar{G}g\| = \int \varphi \, d\mu = p_G(\bar{G}g) = \int \varphi p_G \, d\mu.$$

Since $0 \leqslant p_G \leqslant 1$ on K we have $p_G = 1$ modulo $\varphi\mu$ and because φ is an idempotent $p_G \geqslant \varphi$ in $L^\infty(K, \mu)$. Then we readily obtain $p_G = \varphi$ modulo μ and that ends the proof.

COROLLARY. *If G is admissible and $g \in K \cap G$ then we have for the central measure μ associated to g, $\mu(\{k \in K \,|\, p_G(k) = 1\}) = \mu(K)$.*

Proof. Indeed we have $\|g\| = p_G(g) = \int p_G \, d\mu$ and because $\mu \geqslant 0$, $\int 1 \, d\mu = \|g\|$ and $p_G \leqslant 1$ on K, the conclusion obtains.

PROPOSITION 3.13. *The set of all admissible $G \in B(E)$ is closed for relative complementation, that is, G, H admissible and $G \subseteq H$, then $G' \cap H$, with G' the complement of G, is admissible. The set is closed for monotone sequential limits and contains the closed split-faces, their complements and intersections of those.*

Proof. If G, H are admissible faces and $G \subseteq H$, then $p_{G' \cap H} = p_H - p_G$ so that $G' \cap H$ is admissible. Since E^+ is admissible there follows G' is admissible if G is. For an increasing sequence of split-faces $\{G_n\}_n$ and $G = \sup_n \{G_n\}$ we have $p_G = \sup_n \{p_{G_n}\}$ so that G is admissible.

If G is a closed split face $\{a \circ \bar{G}' \,|\, a \in A^+, \|a\| < 1\}$ is an increasing family of lower semi-continuous functions (2.6) and

$$p_{G'} = \sup \{a \circ \bar{G}' \,|\, a \in A, \|a\| < 1\} \quad \text{on } E^+.$$

There follows that $p_{G'}$ is of the right kind so that G' and G are admissible. If G, H are closed split-faces then $G' \cap H' = (G + H)'$ is admissible and also $G' \cap (G + H) = G' \cap H$.

PROPOSITION 3.14. *Let G be a closed split-face and H' the complement of a closed split-face. If T is a Baire subset of K such that $(T \cap G \cap H') \cap \partial_{\mathrm{pr}} K = \emptyset$ and μ a central measure, then $\mu(T \cap G \cap H') = 0$.*

Proof. We have

$$\mu = \|\cdot\|\mu = p_G\mu + p_{G'}\mu$$

since according to the corollary of 3.12, μ is concentrated on $\{k\in K \mid \|k\|=1\}$. Because $(p_{G'}\mu)(G)=0$ and $p_G\mu$ is central on $G\cap K$ in the induced topology, and because $T\cap G$ is a Baire subset of $G\cap K$, we can restrict our attention to the case where $G\cap K=K$, that is $G=E^+$ and $p_G\mu=\mu$.

For $a\in H^0$ we let $\Omega_a=\{k\in K \mid a(k)\neq 0\}$ then $\Omega_a\cap T$ is a Baire subset of K and $\Omega_a\cap T\cap \partial_{\mathrm{pr}}K=\emptyset$. Indeed if $h\in\partial_{\mathrm{pr}}K\cap H'$, then $h\notin T$ since $T\cap H'\cap\partial_{\mathrm{pr}}K=\emptyset$ and if $h\in\partial_{\mathrm{pr}}K\cap H$ then $h\notin\Omega_a$ so that

$$\int_T a\,d\mu = 0, \quad a\in H^0$$

If we take the supremum over all $\min(a, 1)$, $a\in H^0$ we get $\mu(T\cap(K\setminus H))=0$, so that $\mu(T\cap H')=0$ and we are done.

An earlier version of this proof for the C^*-algebra case originated in a discussion with F. Combes.

For separable A, that is, metrizable K, this does not give anything new since then closed split-faces and their complements are Baire sets so that 3.9 can be applied directly.

It follows from 3.12, 3.13 and 3.14 that measure theoretically closed split-faces and their complements can be considered as direct summands. One can conclude from 3.13 and general facts from § 1 that the smallest σ-algebra in $B(E^+)$, which contains the closed split-faces, consists entirely of admissible split-faces. This σ-algebra corresponds with a monotone-sequentially closed subalgebra Z'_E of Z_E. The restriction of Z'_E to V_g, $g\in E^+$ is a weakly closed subalgebra of Z_g, with respect to which one can decompose g. This is the approach which Effros takes in [12] for the case of C^*-algebras. Effros relates the obtained measures with measures on the prime ideal of A. Using the ideas of Andersen and Alfsen [1] one can see that this can be done also in this general setting.

In proving the "Extension theorem" in [16] Kadison introduces the monotone sequential closure, that is, A^m is the smallest set of bounded affine functions on K, which contains A and is closed for the taking of monotone sequential limits.

It was remarked by Pedersen [19] that every $a\in A^m$ is in the monotone sequential closure of some separable subspace of A. Using this it is not hard to see that also if A does not have a unit, still as in [16], A^m is a linear space closed with respect to the uniform norm of K and $A^m=(A^m)^+-(A^m)^+$.

PROPOSITION 3.15. *For every $T\in Z_{A^m}$ there is a unique $\overline{T}\in Z_E$ such that for all $a\in A^m$, $Ta=a\circ\overline{T}$.*

Proof. Let $T \in (Z_{A^m})_1^+$ and $h \in E$. We define $\overline{T}h$ by $a(\overline{T}h) = Ta(h)$, $a \in A$. Then the map $E \ni h \to \overline{T}h$ is linear and satisfies $0 \leqslant \overline{T} \leqslant I$ so that $\overline{T} \in Z_E$. The subspace of those $a \in A^m$ for which the defining equality for $\overline{T}$ holds contains A of course and is closed for monotone sequential limits. Indeed if $\{a_n\}_n \subseteq A^m$ is an increasing sequence with $a_n(\overline{T}h) = Ta_n(h)$ and $\sup_n \{a_n\} = a \in A^m$. Then we have

$$0 \leqslant Ta - Ta_n \leqslant a - a_n$$

so that $Ta = \sup_n \{Ta_n\}$ and there follows

$$a(\overline{T}h) = \sup_n \{a_n(\overline{T}h)\} = \sup_n \{Ta_n(h)\} = Ta(h).$$

The same holds for decreasing sequences.

We shall denote $\overline{Z_{A^m}} \subseteq Z_E$ by Z^m and use it later [4.8]. In [4] Combes studies the space spanned by the semi-continuous affine function on K and one can prove the same proposition. Davies [7] considers the sequential closures of A and it is not clear whether in general then the above proposition holds.

It would be worthwhile to find out how far the theory of the centers of these larger spaces can be pushed in this general setting. The principal aim then should be to show that these centers are large in some sense (cf. Davies [7] and [8]). It seems probable that some extra conditions on E or A are needed in order to make things work. An additional axiom like Ext [4.C] seems to be the least one has to require.

Another closely related problem is to try to obtain, by using a suitable set of split-faces, a global decomposition of E and realize E either as a measurable or a continuous direct sum.

We have seen that there might be some interest in decomposing with respect to sub-algebras of Z_g, $g \in E^+$ and so we consider a somewhat wider class of measures than the central ones.

Definition 3.16. Let $g \in E^+$, $\|g\| = 1$. A positive probability measure on K, which represents g, is said to be *subcentral* if for every Borel-set $B \subseteq K$ with $0 < \mu(B) < 1$, the resultants of the restricted measures μ_B and $\mu_{K \setminus B}$ are disjoint.

In other words μ is sub-central if

$$\Phi_\mu(\chi) \between \Phi_\mu(1-\chi) \quad \text{for all} \quad \chi = \chi^2 \in L^\infty(K, \mu).$$

PROPOSITION 3.17. *Let μ be a positive measure on K, which represents $g \in K$, $\|g\| = 1$. Then μ is subcentral iff Φ_μ maps $L^\infty(K, \mu)$ isomorphically onto a complete linear sublattice W_μ of $Z_g g$.*

Proof. We suppose that Φ_μ is an isomorphism onto a complete linear sub-lattice $W_\mu \subseteq Z_g g$ and let $\chi = \chi^2 \in L^\infty(K, \mu)$. Then $\inf(\Phi_\mu(\chi), \Phi_\mu(1-\chi)) = 0$ in $Z_g g$ and thus $\Phi_\mu(\chi) = Tg$ with $T = T^2 \in Z_g$. But then $\Phi_\mu(\chi) = Tg \perp (I-T)g = g - \Phi_\mu(\chi)$ so that μ is sub-central.

Conversely let μ be sub-central and $\chi \in L^\infty(K, \mu)$ an idempotent. Then again $\Phi_\mu(\chi) = Tg$, with T an idempotent in Z_g (we used 1.11). There follows that Φ_μ maps $L^\infty(K, \mu)$ into $Z_g g$. If $\varphi \in L^\infty(K, \mu)$ and $\Phi_\mu(\varphi) \geqslant 0$ then $\Phi_\mu(\varphi^+) - \Phi_\mu(\varphi^-) \geqslant 0$ and since $\Phi_\mu(\varphi^+) \perp \Phi_\mu(\varphi^-)$, $\Phi_\mu(\varphi^-) = 0$, so that Φ_μ is bipositive. Since, as we have seen, Φ_μ maps idempotents in $L^\infty(K, \mu)$ onto idempotents of $Z_g g$ the conclusion readily obtains.

COROLLARY. *The set of sub-central probability measures, which represent gK, $\|g\| = 1$, is a complete lattice for Choquet–Meyer ordering. The map $\mu \to W_\mu$ is a lattice isomorphism of the lattice of sub-central measures onto the lattice of complete linear sub-lattices of $Z_g g$ and the central measure is the unique maximal sub-central measure.*

Proof. This is an immediate consequence of 3.18 and 3.2.

In order to obtain our next results we remind the reader of the following result of Cartier–Fell–Meyer ([21, pg. 112]).

THEOREM 3.18. (*Cartier–Fell–Meyer*). *If μ and ν are positive measures on K then $\mu \succ \nu$, iff for each subdivision $\Sigma_{j=1}^n \nu_j = \nu$ of ν there exists a subdivision $\Sigma_{j=1}^n \mu_j$ of μ such that μ_j and ν_j have the same resultant.*

We are interested in the comparison of measures with sub-central measures.

LEMMA 3.19. *Let μ and ν be positive measures on K, which represent $g \in K$, $\|g\| = 1$. If either μ or ν is sub-central, then $\mu \succ \nu$ iff $\Phi_\mu(L^\infty(K, \mu)_1^+) \supseteq \Phi_\nu(L^\infty(K, \nu)_1^+)$.*

Proof. In general it follows from 3.18 that if $\mu \succ \nu$ then $\Phi_\mu(L^\infty(K, \mu)_1^+) \supseteq \Phi_\nu(L^\infty(K, \nu)_1^+)$. So now let μ be sub-central and $\Phi_\mu(L^\infty(K, \mu)_1^+) \supseteq \Phi_\nu(L^\infty(K, \nu)_1^+)$. If $\varphi_j \in L^\infty(K, \nu)_1^+$ and $\Sigma_{j=1}^m \varphi_j = 1$, then there exist unique $\psi_j \in L^\infty(K, \mu)_1^+$ with $\Phi_\mu(\psi_j) = \Phi_\nu(\varphi_j)$. Moreover $\Sigma_{j=1}^m \Phi_\mu(\psi_j) = g$ so that $\Sigma_{j=1}^n \psi_j = 1$ and we conclude from 3.18, $\mu \succ \nu$.

Next we assume that ν is central. If $\chi \in L^\infty(K, \nu)$ is an idempotent we have $\Phi_\nu(\chi) \perp g - \Phi_\nu(\chi)$. Thus there is a $\varphi \in L^\infty(K, \mu)_1^+$ with $\Phi_\mu(\varphi) = \Phi_\nu(\chi)$. Then $\Phi_\mu(1-\varphi) \perp \Phi_\mu(\varphi)$ and thus φ must be an idempotent in $L^\infty(K, \mu)$. If $\varphi' \in L^\infty(K, \mu)_1^+$ also satisfies $\Phi_\mu(\varphi') = \Phi_\nu(\chi)$ then $\Phi_\mu(\varphi') \perp \Phi_\mu(1-\varphi)$ and so φ' has support disjoint from $1-\varphi$. Because φ' also has to be a characteristic function, there follows first $\varphi' \leqslant \varphi$ and then $\varphi' = \varphi$ since $\Phi_\mu(\varphi) = \Phi_\mu(\varphi')$. In this way we see that there exists a map Ψ: $\Phi_\nu(L^\infty(K, \nu)) \to \Phi_\mu(L^\infty(K, \mu))$, which carries 1 into 1 and is bipositive. Moreover, we have $\Phi_\mu(\Psi(\varphi)) = \Phi_\nu(\varphi)$ for all $\varphi \in L^\infty(K, \nu)$. We conclude easily using 3.18 again that $\mu \succ \nu$.

COROLLARY. *If μ_α is a net of sub-central measures increasing in the sense of Choquet and $\mu = \sup_\alpha \mu_\alpha$, then μ is sub-central.*

Proof. We have $\mu_\alpha \leftrightarrow W_{\mu_\alpha}$ as in the corollary to Prop. 3.17 and if $W_{\mu'} = \overline{\bigcup_\alpha W_{\mu_\alpha}}$, (the σ-closure), then $W_{\mu'} \leftrightarrow \mu'$. We have $\mu' \succ \mu_\alpha$ for all α because of 3.20. Since $\mu \succ \mu_\alpha$, we have

$$\Phi_\mu(L^\infty(K,\mu)_1^+) \supseteq \Phi_{\mu_\alpha}(L^\infty(K,\mu_\alpha)_1^+)$$

for all α and therefore $\Phi_\mu(L^\infty(K,\mu)_1^+) \supseteq \Phi_{\mu'}(L^\infty(K,\mu')_1^+)$ so that $\mu \succ \mu'$. There follows $\mu' = \mu$.

PROPOSITION 3.20. *Suppose μ and ν represent $g \in K$, $\|g\| = 1$, and ν is sub-central. There exists a smallest measure $\varrho_{\mu,\nu}$ such that $\varrho_{\mu,\nu} \succ \mu, \nu$. Moreover, if also μ is sub-central, $\varrho_{\mu,\nu}$ is sub-central.*

Proof. We start by supposing that both μ and ν are discrete, $\nu = \sum_{i=1}^n \alpha_i \delta_{f_i}$ and $\mu = \sum_{j=1}^m \beta_j \delta_{h_j}$ with f_i, $h_j \in K$, $\sum_i \alpha_i = \sum_j \beta_j = 1$, $\alpha_i, \beta_j \geqslant 0$. We have $\alpha_i f_i = T_i g$, with $T_i \in (Z_g)_1^+$, $T_i^2 = T_i$ and $g = \sum_{i,j} \beta_j T_i h_j$. Let $T_i h_j / \|T_i h_j\| = k_{ij}$ for those i, j with $T_i h_j \neq 0$ and

$$\varrho_{\mu,\nu} = \sum\nolimits'_{i,j} \beta_j \|T_i h_j\| \, \delta_{k_{ij}},$$

where Σ' indicates that we only add over i, j with $T_i h_j \neq 0$. Obviously if also μ is sub-central then $\varrho_{\mu,\nu}$ is sub-central. It is clear too that $\varrho_{\mu,\nu} \succ \mu, \nu$. Next we assume that $\varrho \succ \mu, \nu$. There are φ_i, $\psi_j \in L^\infty(K,\varrho)_1^+$ with $\varphi_i \varrho \succ \alpha_i \delta_{f_i}$, $\psi_j \varrho \succ \beta_j \delta_{h_j}$ as we see from 3.18 again. We put $k'_{ij} = \Phi_\varrho(\varphi_i \psi_j)$. Then $\sum_{i,j} k'_{ij} = g$ and $\sum_i k'_{ij} = \beta_j h_j$. We obtain $k'_{ij} \leqslant \alpha_i f_i$ for $j = 1, \dots m$. Since $\{\alpha_i f_i\}_i$ is a splitting of g we have by the uniqueness assertion in 1.10 $k'_{ij} = T_i h_j$. But then

$$\varphi_i \psi_j \varrho \succ \beta_j \|T_i h_j\| \delta_{k_{ij}}$$

if $\varphi_i \psi_j \neq 0$ and so $T_i h_j \neq 0$. This proves $\varrho \succ \varrho_{\mu,\nu}$ and thus the minimality of $\varrho_{\mu,\nu}$.

There follows immediately for discrete μ', ν' with ν' sub-central and $\mu' \succ \mu$; $\nu' \succ \nu$ that $\varrho_{\mu',\nu'} \succ \varrho_{\mu,\nu}$.

If μ is arbitrary then it is well known that μ is the supremum of the net of discrete measures $\mu_d = \sum_j \gamma_j \delta_{h_j}$ with $\gamma_j h_j = \Phi_\mu(\chi_j)$ with $\sum_j \chi_j = 1$ and $\chi_j = \chi_j^2 \in L^\infty(K,\mu)$. Moreover, if ν sub-central, the ν_d are sub-central. We put for μ arbitrary and ν sub-central

$$\varrho_{\mu,\nu} = \sup_{d,d'} \varrho_{\mu_d, \nu_{d'}}.$$

Then $\varrho_{\mu,\nu} \succ \mu_d, \nu_{d'}$ and so $\varrho_{\mu,\nu} \succ \mu, \nu$. If μ is sub-central, then all μ_d are sub-central, so that $\varrho_{\mu_d, \nu_{d'}}$ is sub-central. We infer from the corollary to 3.19 that then $\varrho_{\mu,\nu}$ is sub-central. If $\varrho \succ \mu, \nu$ we have $\varrho \succ \mu_d, \nu_{d'}$ and so $\varrho \succ \varrho_{\mu_d, \nu_{d'}}$. There follows $\varrho \succ \varrho_{\mu,\nu}$ and we are done.

THEOREM 3.21. *Let $g \in K$, $\|g\| = 1$ and μ a maximal measure on K, which represents g. If ν is the central measure associated to g then $\mu \succ \nu$.*

Proof. We obtain from 3.20 that $\varrho_{\mu,\nu} \succ \mu$ and so $\varrho_{\mu,\nu} = \mu$ because μ is maximal. But then we see $\mu = \varrho_{\mu,\nu} \succ \nu$.

The idea that the central measure ν is the maximal measure with the above property 3.20 is readily proved to be wrong by considering a rectangular cone in $\mathbf{R}^3$. For simplices of course the central measures are maximal and for the case of the positive cone in the dual of a C^*-algebra we shall see that indeed the central measure is the maximal measure with the property that it is majorized by all maximal measures representing a point $g \in K$, $\|g\| = 1$. [4.10.]

4. Examples

A. Let E be an order complete vector lattice. Since E has the Riesz decomposition property, $f \between g$ iff $C_f \cap C_g = \{0\}$ for $f, g \in E^+$.

4.1. *An element $f \in \partial_{\mathrm{pr}} E^+$ iff f lies on an extremal ray of E^+.*

Plainly if f lies on an extremal ray of E^+ then $f \in \partial_{\mathrm{pr}} E^+$. Conversely let $f \in \partial_{\mathrm{pr}} E^+$ and $0 \leqslant h \leqslant f$. If $\alpha = \sup \{\alpha' \geqslant 0 \mid \alpha' h \leqslant f\}$, we take $0 < \alpha < \beta \leqslant 2\alpha$ and $k = \beta h - f = k^+ - k^-$ with $0 \leqslant k^+ \leqslant \beta h$, $0 \leqslant k^- \leqslant f$ and $k^+ \between k^-$. Then $k^+ + k^- \leqslant 3f$ so that $f = f_1 + f_2$ with $0 \leqslant k^+ \leqslant 3f_1$, $0 \leqslant k^- \leqslant 3f_2$ and $f_1 \between f_2$ [20, V.1.3]. There follows either $f_2 = k^- = 0$ and $\beta h \geqslant f$ or $f_1 = k^+ = 0$, that is, $\beta h \leqslant f$, which contradicts the choice of β. Thus $f \leqslant \beta h$ for all $\beta > \alpha$ and by Archimedicity $f = \alpha h$ so that f lies on an extremal ray.

4.2. *If every $f \in \partial_{\mathrm{pr}} E^+$ lies on an extremal ray and E^+ has a metrizable universal cap K then K is a simplex and E a vector lattice.*

For $g \in K$, $\|g\| = 1$ its associated central measure is concentrated on the extremal points of K and since K is metrizable there follows that every central measure is maximal. We infer from 3.20 that there exists for $g \in K$ a unique maximal measure and so K is a simplex (21, § 9).

4.3. *If E is a vector lattice with a universal cap K for E^+, then every central measure μ is maximal.*

Let $g \in K$, $\|g\| = 1$. Then V_g is a complete linear lattice and it follows e.g. from 3.2 that $Z_g g = V_g$ and the conclusion is obvious from 3.2 again.

We see that for the vector lattice case the notions which we introduced reduce to familiar concepts. For the closed split-faces and examples we refer to [1].

Let us next look at the C^*-algebra case.

B. A will denote the self-adjoint part of a C^*-algebra $\mathcal{A}$ and E is the real dual of A. The set $\{f \in E^+ \mid \|f\| \leqslant 1\}$ is a universal cap for E^+ and the setting is as in § 2.

The Gelfand–Neumark–Segal construction [10, 2.4.4] associates to every $f \in E^+$ a Hilbert space H_f, a vector $w \in H_f$, $\|w\|^2 = \|f\|$, and a representation $\pi_f \colon A \to L(H_f)$ such that $\pi_f(A)w$ is dense in H_f and $f(a) = (\pi_f(a)w, w)$ for $a \in A$. It is then well known [10, 2.5.1] that for $g \in V_f$ there exists a unique operator $\tilde{g}$ in the commutant $\pi_f(A)'$ of $\pi_f(A)$ such that $g(a) = (\tilde{g}\pi_f(a)w, w)$ for $a \in A$. The map $g \to \tilde{g}$ is an isomorphism of ordered Banach spaces from V_f onto the self-adjoint part of $\pi_f(A)'$.

We need the following probably well-known lemma.

4.4. *If $\mathcal{A}$ is a C^*-algebra with unit e then the map $Z_A \ni T \to Te \in A$ is an isomorphism onto the self-adjoint part of the center of A.*

Obviously every self-adjoint element in the center of $\mathcal{A}$, operating as a multiplicator on $\mathcal{A}$, induces an element of Z_A.

By using the second dual [10, § 12] we see that to prove the converse it suffices to consider von Neumann algebras. Then Z_A is order-complete and so Z_A is the norm closed linear span of the idempotents in Z_A. Thus let $P = P^2 \in Z_A$. We shall show that Pe is a central projection in A. First since $\inf(Pe, (I-P)e) = 0$ [1.13] we see that $[Pe]^2 = Pe$ so that Pe is a projection. For $a \in (A)_1^+$ we have $a = Pa + (I-P)a$. But since $0 \leqslant Pa \leqslant Pe$ with Pe a projection we see that Pa commutes with Pe and similarly Pe commutes with $(I-P)a$ and so with a. There follows that Pe is central, and we are done.

Then every element $T \in Z_f$ induces an element $\bar{T} \in Z_{\pi_f(A)'}$ by $\bar{T}\tilde{g} = Tg$, $g \in V_f$ and conversely. We infer from 4.4 that $\widetilde{Z_f f} = \bar{Z}_f \tilde{f} = \bar{Z}_f e$, with e the identity in $L(H_f)$, is the self-adjoint part of the center of $\pi_f(A)'$. We draw an immediate conclusion.

4.5. *$f \in \partial_{\mathrm{pr}} E^+$ iff π_f is primary, that is, the center of $\pi_f(\mathcal{A})'$ consists of multiples of the identity.*

Two representations π_f and π_g are said to be disjoint, notation $\pi_f \mathrel{\overset{|}{\circ}} \pi_g$, if there does not exist a non-trivial partial isometry $U \colon H_f \to H_g$ such that $U\pi_f(a) = \pi_g(a)U$, $a \in A$. It can be proven then that $\pi_f \mathrel{\overset{|}{\circ}} \pi_g$ iff there exists a central projection $p \in \pi_{f+g}(A)'$ such that π_f is unitarily equivalent to the restriction of π_{f+g} to pH_{f+g} [10, 2.5.1].

4.6. *Two elements $f, g \in E^+$ are disjoint iff $\pi_f \mathrel{\overset{|}{\circ}} \pi_g$.*

If $f \mathrel{\overset{|}{\circ}} g$, then there exists $S = S^2 \in Z_{f+g}$ with $S(f+g) = f$, so that $\tilde{f}$ is a central projection in $\pi_{f+g}(A)'$ and $\tilde{g}$ is the complementary projection. Because for $a \in A$

$$(\tilde{f}\pi_{f+g}(a)w, w) = f(a) = (\pi_{f+g}(a)\tilde{f}w, \tilde{f}w)$$

and similarly for g, we see that π_f and π_g correspond to the restrictions of π_{f+g} to respectively $\tilde{f}H_{f+g}$ and $\tilde{g}H_{f+g}$ so that $\pi_f \circlearrowright \pi_g$.

Conversely if $\pi_f \circlearrowright \pi_g$ there is a central projection $p \in \pi_{f+g}(A)'$ such that π_f is the restriction of π_{f+g} to pH_{f+g} and similarly for π_g. Then there is a $P = P^2 \in Z_{f+g}$ such that $P(f+g) = f$ and $\tilde{f} = p$. There follows $f \circlearrowright g$.

The second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ can be identified with the space of all (complex valued) bounded affine functions on K, which vanish at 0 and A^{**} with the subspace of real elements, since $E_1 = \text{co}\,(K, -K)$. Then $\mathcal{A}^{**}$ is again a C^*-algebra and A^{**} is the self-adjoint part of $\mathcal{A}^{**}$. We denote the center of $\mathcal{A}^{**}$ by Z. Every $f \in E$ has a natural extension to $\mathcal{A}^{**}$ as evaluation at f.

4.7. *The central measure of* $f \in K$, $\|f\| = 1$, *introduced by Sakai* [23] *coincides with the central measure defined here.*

Sakai calls a probability measure μ on K, representing f, central iff there exists a weak*-continuous homomorphism $\Psi: Z \to L^\infty(K, \mu)$ such that

$$f(za) = \int_K \Psi(z)\, a\, d\mu, \quad z \in Z, \quad a \in A.$$

The representation π_f has a weak*-continuous extension to $\mathcal{A}^{**}$ such that π_f maps Z homomorphically onto the center of $\pi_f(A)'$. Since we have $f(za) = (\pi_f(z)\pi_f(a)w, w)$, $z \in Z$, $a \in A$ we find that it is sufficient if there exists a weak*-continuous homomorphism Ψ' of the center of $\pi_f(A)'$ onto $L^\infty(K, \mu)$ such that

$$f(za) = (\pi_f(z)\pi_f(a)\, w, w) = \int_K \Psi'(\pi_f(z))\, a\, d\mu \quad a \in A,\ z \in Z.$$

For the measure μ' introduced in 3.3 there exists an isomorphism $\Phi_{\mu'}: L^\infty(K, \mu') \to Z_f f$ such that

$$a(\Phi_{\mu'}(\varphi)) = \int \varphi a\, d\mu' \quad a \in A,\ \varphi \in L^\infty(K, \mu')$$

and we have
$$(\widetilde{\Phi_{\mu'}(\varphi)}\, \pi_f(a)\, w, w) = a(\Phi_{\mu'}(\varphi)) = \int \varphi a\, d\mu'.$$

We put $\Psi'(\widetilde{\Phi_{\mu'}(\varphi)}) = \varphi$. Then Ψ' has the required properties and since the central measure of Sakai is unique we see that μ' is central in the sense of Sakai.

Feldman and Effros [23] have proved that for separable C^*-algebras $\partial_{\text{pr}} E^+ \cap K$ is a Baire subset of K so that according to 3.9 for every central measure μ, $\mu(\partial_{\text{pr}} E^+ \cap K) = \mu(K)$.

This can also be shown by taking for $W \subseteq V_g$ the set which corresponds with the self-adjoint part of a weak* dense separable subalgebra $D \subseteq \pi_f(\mathcal{A})'$ and proceeding as indicated in the remarks following 3.9. The fact that the measure $\bar{\mu}$, then constructed, is maximal follows essentially from [10, 2.5.1]. We note that the positive maps from V_g into E correspond with the positive linear functionals on the algebraic tensor product $A \otimes W$. If $D \subseteq \pi_f(\mathcal{A})'$ is as above, then $\pi_f(\mathcal{A}) \otimes_{Z_f} D$ can be identified with a weak* dense subalgebra of the commutant of the center of $\pi_f(\mathcal{A})'$ [16, § 3.1] and this indicates the relation between the approach taken here and the one of [28] in which the algebra generated by D and $\pi_f(\mathcal{A})$ is considered.

The central measure for C^*-algebras was introduced by S. Sakai [23] and he proved existence and uniqueness as well as the fact that it is concentrated on $\partial_{\mathrm{pr}} E^+ \cap K$, using the von Neumann decomposition theory for separable C^*-algebras. The extension to general C^*-algebras was made in [28], also [29]. Independently Ruelle [22] and Guichardet, Kastler [15] established existence and uniqueness; all using the same method. Similar ideas are already present in much of the older theory on decomposition theory for von Neumann algebras, see e.g. Tomita [27].

The relevance of the central measure in statistical mechanics is that if $\{\alpha_t\}_{t \in \mathbf{R}}$ is a one-parameter group of automorphisms of $\mathcal{A}$ and K_0 is the set of states $f \in E^+$, $e(f)=1$, where e is the identity of $\mathcal{A}$, which satisfy the K.M.S. boundary condition, then K_0 is a simplex and, by abuse of language, the central measure of $f \in K_0$ on K coincides with the maximal measure on K_0 representing f. This was established by Takesaki [26] and Emch, Knops and Verboven [14].

There are several more conditions than separability of $\mathcal{A}$, which ensure $\mu(\partial_{\mathrm{pr}} E^+ \cap K) = \mu(K)$ for central measures. There is e.g. condition "S" of Ruelle [22] which is pertinent for applications to locally normal states. A more classical condition is that $\mathcal{A}$ is generated by its center B and a separable C^*-algebra $\mathcal{A}'$. Then it is readily verified that $\partial_{\mathrm{pr}} E^+ = \{f \in E^+ \mid f|_B \text{ is multiplicative}\} \cap \{f \in E^+ \mid f|_{\mathcal{A}'} \text{ is primary for } \mathcal{A}'\}$. The complements of both sets have measure 0 for all central measures and thus $\mu(\partial_{\mathrm{pr}} E^+ \cap K) = \mu(K)$ for all those measures. A condition of a different type again is that $\mathcal{A}$ is of type I. This follows from [3.9] and [10, 4.5.5 and 4.5.3 and 4.7.6].

A more detailed problem is the measurability of the sets of primary states of type I, II, II_∞ or III. It is possible to prove that if $f \in K$ is of type I, that is, $\pi_f(\mathcal{A})''$ is of type I, then the central measure μ of f is pseudo-concentrated on the set of primary states of type I and if f is of type II_1 then μ is concentrated on the set of primary states of type II_1. Just as in the separable case it is not easy to do something for type III or II_∞.

As applications of the theory of central measures we point out that e.g. the general Plâncherel theorem of Dixmier [10] can be proved using central measures. Also the invariants which Kadison gives for quasi-equivalence [16] can be formulated entirely within the frame work of central measures on K, instead of with the set of all extensions of states on A to measures on K. We shall not go into these matters and rather derive another property and characterization of central measures for the case of separable C^*-algebras.

We remind the reader of the monotone sequential closure A^m of A and $\bar{Z}^m \subseteq Z_E$ [3.15]. If A is separable then A^m contains the function $\|\cdot\|$ on K. Let e denote the linear extension of $\|\cdot\|$ to E, then e acts as a unit for A^m. It is proved by Kadison [16, pages 317, 318] that A^m is a J-algebra and Pedersen [18, th. 1] showed that A^m actually is the self-adjoint part of a C^*-algebra. Every $x \in A^m$ has a central support. Indeed if $x \in A^m$, then the separable algebra generated by xA, has a countable approximate identity. The supremum of this monotone sequence is the central support of x in A^m.

4.8. *Let $\mathcal{A}$ be a separable C^*-algebra and $f \in E_1^+$. Then $(Z_f)_1^+ f = (\bar{Z}^m)_1^+ f$.*

Note that Z^m is defined globally, i.e. Z^m operates on all of E and Z_f operates only on V_f. Obviously $(Z^m)_1^+ f \subseteq (Z_f)_1^+ f$. Let $T = T^2 \in Z_f$. Then $\widetilde{Tf}$ is a central projection in $\pi_f(\mathcal{A})'$. Because $\mathcal{A}$ is separable, $\mathcal{H}_f$ is separable and so the identity operator on $\mathcal{H}_f$ is countably decomposable for $\pi_f(\mathcal{A})''$. We infer from [16, pp. 322–323] that there is a $a \in A^m$ with $\pi_f(a) = \widetilde{Tf}$. Let p be the central support of a, then $\pi_f(p) = \pi_f(a)$. It follows from 4.4 that there exists $T' = T'^2 \in Z_{A^m}$ with $T'e = p$. In total we have for $b \in A$ with w as before and using 4.4 again

$$\begin{aligned} b(Tf) &= (\widetilde{Tf}\pi_f(b)w, w) = (\pi_f(p)\pi_f(b)w, w) = (\pi_f(pb)w, w) \\ &= (\pi_f(T'b)w, w) = (T'b)(f) = b(T'^*f). \end{aligned}$$

Consequently $Tf = \bar{T}'f$ and $(\bar{Z}^m)_1^+ f$ contains the idempotents of $(Z)_1^+ f$. Since Z^m is closed for monotone sequential convergence the statement of 4.8 follows:

4.9. *Let $\mathcal{A}$ be a separable C^*-algebra and $f \in K$, $\|f\| = 1$. Then the central measure μ on K representing f is the unique minimal measure representing f and concentrated on $\partial_{\mathrm{pr}} E^+ \cap K$.*

Suppose $\nu \geqslant 0$ represents f and $\nu(\partial_{\mathrm{pr}} E^+ \cap K) = 1$. We see from 3.19 that we have to show that $\Phi_\nu(L^\infty(K, \nu)_1^+) \supseteq (Z_f f)_1^+ = (\bar{Z}^m)_1^+ f$. Take $T \in (Z^m)_1^+$.

If $g \in \partial_{\mathrm{pr}} E^+$ then $Z_g g$ is trivial and thus $\bar{T}g = \lambda_{\bar{T}}(g)g$ for some constant $\lambda_{\bar{T}}(g)$. We have $e(\bar{T}g) = Te(g) = \lambda_{\bar{T}}(g)e(g)$ with $e(g) \neq 0$ for $g \neq 0$. There follows $\lambda_{\bar{T}}$ is the restriction to $\partial_{\mathrm{pr}} E^+ \cap K$ of a Baire function on K. We also have $0 \leqslant \lambda_{\bar{T}} \leqslant 1$. Armed with this knowledge we note that for $a \in A$.

$$(Ta)(f) = a(\bar{T}f) = \int_K Ta\, d\nu.$$

Now ν is concentrated on $\partial_{\mathrm{pr}} E^+ \cap K$ and there $(Ta)(g) = \lambda_{\bar{T}}(g)a(g)$ so that

$$a(\bar{T}f) = \int_K \lambda_{\bar{T}}\, a\, d\nu.$$

We found $\Phi_\nu(\lambda_{\bar{T}}) = \bar{T}f$ and we can get all of $(Z_f f)_1^+$ in this way. There follows $\nu \succ \mu$ and we are done.

In order to balance this characterization of central measures as minimal measures we give now a last description of central measures for C^*-algebras as maximal measures.

If $\mathcal{B} \subseteq \pi_f(\mathcal{A})'$ is an abelian von Neumann algebra then the self-adjoint part B of $\mathcal{B}$ corresponds with a complete lattice in V_f so that 3.1 applies. We denote the corresponding measure with μ_B. Exactly as in 3.9 it is possible to prove that a measure μ representing f satisfies $\mu \prec \mu_B$ iff $\Phi_\mu(L^\infty(K, \mu)_1^+) \subseteq B_1^+$. Also Tomita [26] showed already long ago that μ_B is maximal if B is maximal abelian. A proof of this fact follows by using again the techniques of 3.19 and 4.4.

4.10. *If ν is a measure on K representing $f \in K$, $\|f\| = 1$ and $\nu \prec \mu$ for all maximal measures μ representing f then $\nu \prec \bar{\mu}$, where $\bar{\mu}$ is f's central measure.*

Proof. Since $\nu \prec \mu_B$ for all B, B maximal abelian, $\Phi_\nu(L^\infty(K, \mu)_1^+) \subseteq B_1^+$ for all such B and so $\Phi_\nu(L^\infty(K, \nu)_1^+) \subseteq Z_1^+$ because Z is the intersection of all maximal abelian $B \subseteq \pi_f(\mathcal{A})'$. We conclude with the help of 3.19, $\nu \prec \bar{\mu}$.

The first one to have been considering measures on the state space of a C^*-algebra in connection with decomposition theory is I. Segal [25]. The idea of using a construction like in the first part of 3.1 to decompose states can be found in practically every article on desintegration of von Neumann algebras since 1949.

A last remark concerning this example is that the name "ideal center" is taken from Effros [12] and Dixmier [11]. The algebra A' in [11, th. 8] can be constructed by using the regular representation of A and $Z_A \subseteq \mathrm{End}\,(A)$, just as is done in adjoining an identity [10, 1.3.8]. For the connection with the ideal center of Effros we refer to [12] and the remarks following 3.14.

C. Another example can be found in the realm of positive definite functions on groups. Also one could try to find motivation for studying central measures in the theory of function algebras. Another direction would be applications to weakly complete cones. On none of these subjects very much is known in this connection and I should rather like to indicate the relevance for a classification theory of a condition, which is satisfied by both examples discussed so far.

Let E be a partially ordered vector space with as usual $E = E^+ - E^+$. We suppose that the following axiom is satisfied.

Ext: *If $h, f, g \in E^+$ and $g \geqslant h + f$ with $h \,\delta\, f$, then $g = g_1 + g_2$ with $g_1 \,\delta\, g_2$ and $g_1 \geqslant h$, $g_2 \geqslant f$.*

Plainly Ext is satisfied in the lattice case. For the C^*-algebra case Ext follows from [9, A15] and some standard reasoning as in B. The disjointness relation behaves very nicely in the presence of Ext.

4.11. *If $f, f', h \in E^+$ and $f \mathrel{\overset{\shortmid}{\circ}} h$, $f' \mathrel{\overset{\shortmid}{\circ}} h$, then $f+f' \mathrel{\overset{\shortmid}{\circ}} h$.*

The proof is simple algebra as in 1.7. A consequence is

4.12. *$\{h \in E^+ \mid h \mathrel{\overset{\shortmid}{\circ}} g\} = K_g$ is a face of E^+, $g \in E^+$.*

We introduce the following definition.

4.13. *Definition.* If h, $g \in E^+$, we write $h << g$ iff $h \mathrel{\overset{\shortmid}{\circ}} k$ for all $k \in E^+$ with $g \mathrel{\overset{\shortmid}{\circ}} k$. Also $H_g = \{h \in E^+ \mid h << g\}$.

4.14. *H_g and K_g are complementary split-faces of E^+.*

Suppose $f \in E^+$ and let $T = T^2 \in Z_{f+g}$ be minimal so that $Tg = g$. Then $(I-T)f \mathrel{\overset{\shortmid}{\circ}} g$. We want to show $Tf << g$. Suppose $E^+ \ni k \mathrel{\overset{\shortmid}{\circ}} g$ and let T' be the smallest idempotent in Z_{f+g+k} with $T'g = g$. It follows from Ext and 1.5 that $T'k = 0$. The restriction of T' to V_{f+g} satisfies $T'g = g$ so that $T' \geqslant T$ and $T'f \geqslant Tf$. Consequently $Tf \mathrel{\overset{\shortmid}{\circ}} k$ and $H_g + K_g = E^+$. If h_1, $h_1' \in K_g$ and h_2, $h_2' \in H_g$ satisfy $h_1 + h_2 = h_1' + h_2'$ then it follows from the definition h_1, $h_1' \mathrel{\overset{\shortmid}{\circ}} h_2'$, h_2' and from 1.8, $h_1 = h_1'$, $h_2 = h_2'$. This ends the proof.

Thus Ext has a consequence that local splitting extend to global ones. The effect of Ext can be seen by considering examples in finite dimensional spaces. An interesting problem is e.g. whether a proposition like 4.10 holds in general in the presence of Ext.

A statement like $f << g$ iff $H_f \subseteq H_g$ or $f \mathrel{\overset{\shortmid}{\circ}} g$ iff $H_f \mathrel{\overset{\shortmid}{\circ}} H_g$ is readily verified and shows that $<<$ defines a preorder on E^+. The resulting equivalence relation should be called *quasi-equivalence*.

For the lattice case H_g is just the band generated by g in E^+ and K_g the complementary band. For the C^*-algebra case H_g must be interpreted as the pull-back to $E(\cong A')$ of the restriction to $\pi_g(\mathcal{A})$ of the normal linear functionals on $\mathcal{L}(\mathcal{H}_g)$ and then H_g serves as invariant for quasi-equivalence [10, 5.3]. This is e.g. the point of view in [16, th. 2.2.5]. The representation π_g roughly corresponds with the restriction of A to H_g and the sandwiched algebras $p\pi_f(\mathcal{A})''p$ with p a countably decomposable projection in $\pi_f(\mathcal{A})'$ are the "restrictions" of the set of all bounded affine functions on K to C_f, $f \in H_g$. It seems certain that quite a bit more than Ext is needed to obtain the results of [16] in a general setting.

Ext however is sufficient to give a good definition of the equivalent of the quasi-spectrum as for C^*-algebras [10, 7.2]. A decent Borel structure on this object as e.g. defined by Davies [8, § 4] by means of the center of A^m is again a big problem. The same applies to the measurability of $\partial_{\mathrm{pr}} E^+$, which is a connected question.

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