THE IDEAL STRUCTURE OF CERTAIN
NONSELFADJOINT OPERATOR ALGEBRAS

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ABSTRACT. Let (X, ϕ) be a locally compact dynamical system, and $\mathbf{Z}^+ \times_\phi C_0(X)$ the norm-closed subalgebra of the crossed product $\mathbf{Z} \times_\phi C_0(X)$ generated by the nonnegative powers of ϕ in case ϕ is a homeomorphism. If ϕ is just a continuous map, $\mathbf{Z}^+ \times_\phi C_0(X)$ can still be defined by a crossed product type construction. The ideal structure of these algebras is determined in case ϕ acts freely. A class of strictly transitive Banach modules is described, indicating that for the nonselfadjoint operator algebras considered here, not all irreducible representations are on Hilbert space. Finally in a special case, the family of all invariant maximal right ideals is given.

I. Introduction.

I.1. Nearly twenty years ago Effros and Hahn [5] investigated the primitive ideal structure of certain C^*-crossed products, and their results have been extensively generalized in recent years. However, the study of nonselfadjoint subalgebras of C^*-crossed products is much more recent, and yet the same questions regarding the interplay between topological dynamics and ideal theory can be asked in the nonselfadjoint case as well. For example, if X is a locally compact metric space, ϕ a freely acting homeomorphism of X and α the corresponding automorphism of $C_0(X)$, $\alpha(f) = f \circ \phi$, then the primitive ideals of the C^*-crossed product $\mathbf{Z} \times_\alpha C_0(X)$ correspond naturally with the space of "pseudo orbits", or orbit closures $\{\phi^n(x): n \in \mathbf{Z}\}$ $(x \in X)$. Denote by $\mathbf{Z}^+ \times_\alpha C_0(X)$ the closed subalgebra of $\mathbf{Z} \times_\alpha C_0(X)$ generated by the nonnegative powers of the automorphism. What can be said about the primitive ideals of this algebra? Unfortunately, it seems that knowing $\mathrm{Prim}(\mathbf{Z} \times_\alpha C_0(X))$ shed no light on this problem. Inherent in this problem is the fact that $\mathbf{Z}^+ \times_\alpha C_0(X)$ is not a star-algebra. And although the norm in $\mathbf{Z}^+ \times_\alpha C_0(X)$ is defined by means of Hilbert space representations, it turns out that the irreducible representations are not necessarily realizable in Hilbert space.

While for (separable) C^*-algebras, the primitive and prime ideals coincide, for a Banach algebra that need not happen. Returning to the algebra $\mathbf{Z}^+ \times_\alpha C_0(X)$, we obtain results similar to those of Effros-Hahn for C^*-crossed products, but which require some topological dynamical refinements. For $x \in X$, let $\mathcal{O}^+(x)$ be the forward orbit $\{\phi^n(x): n = 0, 1, \dots\}$. For example, the prime ideals of $\mathbf{Z}^+ \times_\alpha C_0(X)$ correspond to $\{\overline{\mathcal{O}^+(x)}: x \text{ is a recurrent point}\}$. (Theorem III.10; cf. II. 8, II. 9 for definitions.)

I.2. *Notation.* The framework of our results is actually somewhat more general than that alluded to above. Let X be a locally compact metric space; instead of requiring a homeomorphism for the dynamics, we only require that $\phi: X \to X$ be a continuous and proper surjection. That gives rise to a continuous injective (star) endomorphism α of $C_0(X)$, $\alpha(f) = f \circ \phi$. Let U be an operator defined formally by the condition $fU = Uf \circ \phi$, $f \in C_0(X)$, and denote by $K(\mathbf{Z}^+, C_0(X))$ the associative algebra over the complex field generated by U and $C_0(X)$. A typical element F of this algebra will have the form $F = f_0 + Uf_1 + \cdots + U^n f_n$ for some nonnegative integer n and functions $f_0, f_1, \ldots, f_n \in C_0(X)$. Let l_2^+ be the space of square summable sequences $\xi = (\xi_n)_{n=0}^\infty$, $\xi_n \in \mathbf{C}$, $\|\xi\|_2 = (\sum_{n=0}^\infty |\xi_n|^2)^{1/2}$, and, for each $x \in X$, let Π_x denote the representation of $K(\mathbf{Z}^+, C_0(X))$ in $\mathcal{B}(l_2^+)$ given on the generators by

$$\Pi_x(f)\xi = (f(x)\xi_0, f \circ \phi(x)\xi_1, f \circ \phi^2(x)\xi_2, \ldots),$$

and

$$\Pi_x(U)\xi = (0, \xi_0, \xi_1, \xi_2, \ldots).$$

A norm can be defined on $K(\mathbf{Z}^+, C_0(X))$ by $\|F\| = \sup\{\|\Pi_x(F)\|: x \in X\}$. The completion of $K(\mathbf{Z}^+, C_0(X))$ with respect to this norm is called the semicrossed product of $C_0(X)$ with ϕ, and is denoted by $\mathbf{Z}^+ \times_\phi C_0(X)$.

To $F \in \mathbf{Z}^+ \times_\phi C_0(X)$ there corresponds, formally, a unique series $\sum_{n=0}^\infty U^n f_n$ ($f_n \in C_0(X)$) (see III.3), called the Fourier series of F; the f_n are called the Fourier coefficients. Although the partial sums of $\sum_{n=0}^\infty U^n f_n$ may not converge, it is Cesàro summable to F [**11**, II.2, Remark]. For convenience we will often write $F = \sum_{n=0}^\infty U^n f_n$.

I.3. *Contents.* §II contains examples and notions from topological dynamics which will be used later. The ideal structure of $\mathbf{Z}^+ \times_\phi C_0(X)$ is determined in §III for a free dynamical system (X, ϕ). §IV concerns the conjugate space of $\mathbf{Z}^+ \times_\phi C_0(X)$. A partial classification of irreducible Banach modules of $\mathbf{Z}^+ \times_\phi C_0(X)$ is presented in §V, and some related questions are discussed.

II. Topological dynamics.

II.1. By a dynamical system (X, ϕ) we mean a locally compact metrizable space X together with a continuous and proper surjection $\phi: X \to X$. (Recall that a map ϕ is proper if the inverse image $\phi^{-1}(K)$ is compact for every compact $K \subset X$.) Let α be in the injective endomorphism of $C_0(X)$, $\alpha(f) = f \circ \phi$.

II.2. DEFINITIONS. If (X, ϕ), (Y, ψ) are dynamical systems, (Y, ψ) is a *factor* of (X, ϕ) if there is a continuous and proper surjection $p: X \to Y$ with $\psi \circ p = p \circ \phi$. We also say that (X, ϕ) is an *extension* of (Y, ψ). By a *covering system* for a dynamical system (X, ϕ) we mean an extension $(\tilde{X}, \tilde{\phi})$ such that $\tilde{\phi}$ is a homeomorphism and if (X_1, ϕ_1) is any other extension of (X, ϕ) with ϕ_1 a homeomorphism, then $(\tilde{X}, \tilde{\phi})$ is a factor of (X_1, ϕ_1). (This terminology is nonstandard.)

Although the existence of covering systems for dynamical systems is not new, it is perhaps worth noting that it follows from a general principle concerning extendability of endomorphisms. By Proposition I.8 of [**11**], if α is an injective endomorphism of a C^*-algebra $\mathfrak{A}$, there is a C^*-algebra $\mathcal{B}$ containing $\mathfrak{A}$ as a subalgebra, and an automorphism β of $\mathcal{B}$ such that $\alpha(x) = \beta(x)$, $x \in \mathfrak{A}$. Furthermore it follows (from the proof) that $(\mathcal{B}, \beta)$ can be chosen so that $\bigcup_{n>0} \beta^{-n}(\mathfrak{A})$ is dense in $\mathcal{B}$ and hence

$\mathcal{B}$ is commutative if $\mathfrak{A}$ is. So if $\mathfrak{A} = C_0(X)$, $\mathcal{B} = C_0(\tilde{X})$, $\beta(\tilde{f}) = \tilde{f} \circ \tilde{\phi}$ for some homeomorphism $\tilde{\phi}$ of $\tilde{X}$ ($\tilde{f} \in C_0(\tilde{X})$). If $j: \mathfrak{A} \to \mathcal{B}$ is the inclusion, take p to be the transpose restricted to the extreme points of the probability measures on $\tilde{X}$. It is not hard to verify that $(\tilde{X}, \tilde{\phi})$ is a factor of any extension (X_1, ϕ_1) of (X, ϕ) where ϕ_1 is a homeomorphism. Thus $(\tilde{X}, \tilde{\phi})$ is a covering system of (X, ϕ).

Now we recall some standard terminology from dynamical systems. Let (X, ϕ) be a dynamical system and $x_0 \in X$. The point x_0 is said to be *recurrent* if, for any neighborhood V of x_0, there is an integer $n \geq 1$ with $\phi^n(x) \in V$. (Definition 1.1 of [6].) A subset $S \subset \{0, 1, 2, \dots\} = \mathbf{Z}^+$ is called *syndetic* if there is a finite set $\mathcal{F} \subset \mathbf{Z}^+$ such that $S + \mathcal{F} = \mathbf{Z}^+$. x_0 is called *almost periodic* if for any neighborhood V of x_0, $\{n \in \mathbf{Z}^+: \phi^n(x_0) \in V\}$ is syndetic. (Some authors, e.g. [6], call this property uniform recurrence.) A subset $A \subset X$ is called a *wandering set* if the sets $A, \phi^{-1}(A), \phi^{-2}(A), \dots$ are pairwise disjoint. A point $x_0 \in X$ is a *wandering point* if x_0 has an open neighborhood which is a wandering set. If $Y \subset X$ is closed and $\phi(Y) = Y$, $(Y, \phi|_Y)$ (or simply (Y, ϕ)) will be referred to as a *subsystem* of (X, ϕ).

As these concepts will be fundamental to the next section, it is appropriate to give some examples here.

EXAMPLE II.2. Fix an orthogonal coordinate system and a unit length in the Euclidean plane $\mathbf{R}^2$. Let X_n be the circle in $\mathbf{R}^2$ centered at $(0, n + 1/n)$ with radius n, and $\mathbf{R} = \{(x, y): y = 0\}$. Set $X = \bigcup_{n=1}^{\infty} X_n \cup \mathbf{R}$, and give X the topology induced from $\mathbf{R}^2$. Define a homeomorphism $\phi: X \to X$ as follows: for each n, ϕ maps X_n onto itself by counterclockwise rotation through an arc of unit length; ϕ maps $\mathbf{R}$ to $\mathbf{R}$ by $\phi(x, 0) = (x + 1, 0)$. Since ϕ rotates X_n through an irrational angle, the forward orbit $\{\phi^k(x_0): k = 0, 1, \dots\}$ of any $x_0 \in X_n$ is dense in X_n, and x_0 is almost periodic. Since every point $(x, 0) \in \mathbf{R}$ is a limit of points of X_n ($n \uparrow \infty$), the almost periodic points of X are dense. Although the system (X, ϕ) has no wandering points, every point of the subsystem $(\mathbf{R}, \phi)$ is a wandering point of the subsystem $(\mathbf{R}, \phi)$.

II.3. Here we will construct a dynamical system (X, ϕ) with X compact, ϕ non-invertible, such that the recurrent points of X form a dense subset and such that no recurrent point, with the exception of a single fixed point, is almost periodic.

Let Λ be a finite alphabet and ϕ be the backward shift on $\Lambda^{\mathbf{Z}^+}$: thus, $\phi(\omega) = \omega'$, where $\omega'(n) = \omega(n + 1)$. A word $\omega \in \Lambda^{\mathbf{Z}^+}$ is recurrent iff ω has the form

$$(*) \qquad [(a\omega^{(1)}a)\omega^{(2)}(a\omega^{(1)}a)]\omega^{(3)}[(a\omega^{(1)}a)\omega^{(2)}(a\omega^{(1)}a)] \cdots$$

where $a \in \Lambda$ and $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \dots$ are words of finite length [6, Proposition 1.10]. In this example we take $\Lambda = \{0, 1\}$, $a = 1$, and $\omega^{(n)}$ to be the word consisting of a block of n zeros. Let ω have the form $(*)$, and set $X = \mathcal{O}^+(\omega)^-$; that is, X is the closure in $\Lambda^{\mathbf{Z}^+}$ of the forward orbit $\mathcal{O}^+(\omega) = \{\omega, \phi(\omega), \phi^2(\omega), \dots\}$, and (X, ϕ) is a subsystem of $(\Lambda^{\mathbf{Z}^+}, \phi)$. Set $\omega_0 = (0, 0, 0, \dots)$. Then $\omega_0 \in X$; indeed, let l_n be the number of letters which precede the first letter of (the first occurrence of) the word $\omega^{(n)}$ in ω. Clearly, $\phi^{l_n}(\omega) \to \omega_0$. Next we show that ω_0 is the only almost periodic point of X. For suppose $\omega_1 \neq \omega_0$ is another almost periodic point of X, and let η be a word (of finite length) in ω_1 such that not all the letters of η are zero. Now the occurrences of η in ω_1 form a syndetic set [6, 1.22]; let $d(\eta)$ be the maximum difference between successive terms of this set. Specifically, let $S \subset \mathbf{Z}^+$ be defined as follows: $n \in S$ iff $\omega_1(n)$ is a letter in (some occurrence) of η in ω_1.

So if $S = \{n_1, n_2, \dots\}$, set $d(\eta) = \max\{n_{k+1} - n_k, \ k = 1, 2, \dots\}$ which is finite since S is syndetic. If $l(\eta)$ is the length of η (i.e., the number of letters in η), set $p = l(\eta) + d(\eta)$. Then any subword of ω_1 of length at least p must contain η. For any $n \geq 1$, write $\phi^n(\omega) = (\omega', \omega'')$, where either ω' is the empty word or else $\omega' = (0, 0, \dots, 0)$ consisting of all zeros, and $\omega'' = (1, 0, \dots)$. We claim that $\phi^n(\omega)$ $(n \geq 1)$ cannot agree with ω_1 in the first $2p + l_p$ letters. If the length of ω' is at least p, this is obvious. So suppose ω' has length less than p. The number of letters which precede the first occurrence of a word $\omega^{(k)}$ in ω'', for some $k \geq p$, is at most l_p. This fact is an immediate consequence of the form $(*)$ of ω. Thus, the first $l_p + p$ letters of ω'' contain a block of p zeros, and hence $\phi^n(\omega)$ cannot agree with ω_1 in the first $l_p + 2p$ letters. Thus there is no sequence $\{n_k\}$ such that $\phi^{n_k}(\omega) \to \omega_1$, so $\omega_1 \notin X$.

As ω is recurrent and $\mathcal{O}^+(\omega)$ is dense in X, the set of recurrent points is dense and there are no wandering points. In fact, the set of recurrent points of (X, ϕ) is residual [6, Theorem 1.27].

II.4. EXAMPLE. This is an example of a dynamical system (X_1, ϕ_1) with X_1 compact, ϕ_1 a homeomorphism, and such that the set of almost periodic points of X_1 is not dense.

Let (X, ϕ) be the dynamical system constructed in the previous example, and let (X_1, ϕ_1) be a covering system of (X, ϕ). If $x_1 \in X_1$ is almost periodic and $p \colon X_1 \to X$ is a continuous surjection with $\phi \circ p = p \circ \phi_1$ then $p(x_1)$ is an almost periodic point of X. Since the only periodic point of X is the (fixed) point ω_0, it follows that any almost periodic point of X_1 is contained in the inverse image $p^{-1}(\omega_0)$, and in particular such points are not dense in X_1.

II.5. Let (X, ϕ) be a dynamical system. In [11] a point $x_0 \in X$ was called nearly recurrent if for every neighborhood V of x_0 there is a point $x_1 \in V$ such that $\{n \colon \phi^n(x_1) \in V\}$ has infinite cardinality. x_0 was called nearly periodic if, for every neighborhood V of x_0, there is a point $x_1 \in V$ such that $\{n \colon \phi^n(x_1) \in V\}$ is syndetic.

PROPOSITION. *Let (X, ϕ) be a dynamical system. Then*

(a) *If the set of wandering points of X is empty, then the set of nearly recurrent points of X is the closure of the set of recurrent points;*

(b) *The set of nearly periodic points of X is the closure of the set of almost periodic points.*

PROOF. (a) Since the set of nearly recurrent points is a closed invariant set whose complement is the set of wandering points, by hypothesis all points of X are nearly recurrent. If X is not compact, replace (X, ϕ) by (X_1, ϕ_1), where $X_1 = X \cup \{\infty\}$ is the one point compactification of X, and $\phi_1(x) = \phi(x)$, $x \in X$, $\phi_1(\infty) = \infty$. Since ϕ is onto, so is ϕ_1. To see ϕ_1 is continuous, let $x_n \to \infty$. If $\phi(x_n)$ did not approach ∞, there would be a compact set K, an integer N and a subsequence (x_{n_j}) such that $\phi(x_{n_j}) \in K$, $n_j \geq N$. But since ϕ is proper, $\phi^{-1}(K)$ is compact, and $x_{n_j} \in \phi^{-1}(K)$, $n_j \geq N$, a contradiction. Now by [6, Theorem 1.27], the set of nonrecurrent points is residual.

(b) Let $x_0 \in X$ be nearly periodic and V be a compact neighborhood of x_0. By definition, there is a point $x_1 \in V$ such that the set $S = \{n \colon \phi^n(x_1) \in V\}$ is

syndetic. If $S = \{n_k\}_{k=1}^{\infty}$, let $d = \max\{n_{k+1} - n_k : k \geq 1\} < \infty$. Then $\mathcal{O}^+(X_1) \subset \bigcup_{j=0}^{d} \phi^{-j}(V) \equiv W$. Indeed, if $m > 0$, let $n > m$ such that $n - m \leq d$ and $n \in S$. Then

$$\phi^m(x_1) \in \phi^{-(n-m)}(\phi^n(x_1)) \subset \phi^{-(n-m)}(V) \subset W.$$

Since W is compact, $\mathcal{O}^+\overline{(x_1)} \subset W$. By a theorem of Auslander and Ellis applied to the compact dynamical system (Y, ϕ), $Y = \mathcal{O}^+\overline{(x_1)}$, Y contains an almost periodic point, say y_0 [6, Theorem 8.7]. Thus $y_0 \in \phi^{-k}(V)$ for some k, $0 \leq k \leq d$. Since y_0 is almost periodic, so is every point of $\mathcal{O}^+(y_0)$, in particular $\phi^k(y_0)$. As $\phi^k(y_0) \in V$, and V was an arbitrary compact neighborhood of x_0, it follows that the almost periodic points are dense in the nearly periodic points.

In the context of group actions, Gottschalk and Hedlund have used the term regionally recurrent for what we have called nearly recurrent [7, 7.17]. Thus, if we were considering group actions instead of the semigroup $\mathbf{Z}^+$, part (a) of the proposition would follow from [7, Theorem 7.20].

II.6. There is a mistake in the proof of part (ii) of Theorem IV.1 of [11], and the "iff" in that statement should be replaced by "if". The question of necessary and sufficient conditions on the dynamical system for the semicrossed product to be semisimple is still open.

THEOREM. *Let (X, ϕ) be a dynamical system, and $\mathfrak{A} = \mathbf{Z}^+ \times_\phi C_0(X)$ the associated semicrossed product. Then*
   (i) *$\mathfrak{A}$ is semiprime iff the recurrent points of (X, ϕ) are dense;*
   (ii) *$\mathfrak{A}$ is semisimple if the almost periodic points of (X, ϕ) are dense;*
   (iii) *$\mathfrak{A}$ is strongly semisimple iff the periodic points of (X, ϕ) are dense;*
   (iv) *The normed algebra $K(\mathbf{Z}^+, C_0(X))$ is semisimple iff the almost periodic points of (X, ϕ) are dense.*

PROOF OF (iv). Let K be a compact neighborhood contained in the complement of the closure of the almost periodic points (equivalently, the complement of the nearly periodic points). The set J of all $F \in K(\mathbf{Z}^+, C_0(X))$ whose Fourier coefficients are all supported on K is a left ideal in $K(\mathbf{Z}^+, C_0(X))$. Since the radical contains every right or left nil ideal [12, 2.3.5], it follows that J belongs to the radical, since the calculation on the top of p. 528 of [11] shows precisely that every $F \in J$ is nilpotent. Thus the radical of $K(\mathbf{Z}^+, C_0(X))$ is nontrivial.

On the other hand, suppose every point of X is nearly periodic, and let $\mathcal{R}$ be the radical of $K(\mathbf{Z}^+, C_0(X))$. If $K(X)$ denotes the continuous functions with compact support, then $\mathcal{R} \neq (0)$ implies $\mathcal{R} \cap K(\mathbf{Z}^+, K(X)) \neq (0)$, since if $F \in \mathcal{R}$ then $Fu \in \mathcal{R}$ and the Fourier coefficients of Fu have compact support if $u \in K(X)$; also if $F \neq 0$, Fu will be nonzero for some $u \in K(X)$. Lemma III.2 shows that if $F \in \mathcal{R} \cap K(\mathbf{Z}^+, K(X))$, $F = \sum_{k=0}^{n} U^k f_k$, then $U^k f_k \in \mathcal{R} \cap K(\mathbf{Z}^+, K(X))$ for all $k = 0, 1, \ldots, n$. But then the calculation at the bottom of p. 528 of [11] shows that if $F \neq 0$, F cannot belong to any topologically nilpotent ideal. It follows from [12, Theorem 2.3.4] that the radical is trivial.

II.7. Next we give an example of a dynamical system (X, ϕ) such that the algebra $K(\mathbf{Z}^+, C_0(X))$ is not semisimple. Let (X, ϕ), (X_1, ϕ_1) be the dynamical systems constructed in II.3, II.4 resp. In both cases, we saw that the almost periodic points were not dense. Thus, by Theorem II.6(iv), it follows that $K(X, C(X))$ and $K(X_1, C(X_1))$ are not semisimple algebras. Whether or not $\mathbf{Z}^+ \times_\phi C(X)$ and $\mathbf{Z}^+ \times_{\phi_1} C(X_1)$ are semisimple is not clear.

II.8. A dynamical system (X, ϕ) is said to be *topologically transitive* if there exists a point $x_0 \in X$ such that $\mathcal{O}^+\overline{(x_0)} = X$. The following proposition reduces to [**14**, Theorem 5.4] when X is compact and ϕ a homeomorphism. The proof is omitted.

PROPOSITION. *The following are equivalent:*
(i) (X, ϕ) *is topologically transitive;*
(ii) *If $E \subset X$ is closed, $\phi(E) \subset E$, $E \neq X$, then E is nowhere dense;*
(iii) *If $U \subset X$ is open, nonempty, $\phi^{-1}(U) \subset U$, then U is dense.*
(iv) *If U, V are nonempty open sets in X, there exists $n \in \mathbf{Z}^+$ such that $\phi^{-n}(U) \cap V \neq \varnothing$;*
(v) $\{x \colon \mathcal{O}^+\overline{(x)} \neq X\}$ *is a set of first category.*

II.9. A dynamical system (X, ϕ) is *minimal* if no proper closed subset is invariant under ϕ; equivalently, every orbit $\mathcal{O}^+(x)$ is dense in X ($x \in X$). We do not know any examples where X is noncompact. If (X, ϕ) is a minimal dynamical system with X compact, we will say (X, ϕ) is compact minimal, and in that case every $x \in X$ is almost periodic. For any dynamical system (X, ϕ) with X compact, the set of almost periodic points is nonempty. If x is an almost periodic point of a dynamical system (X, ϕ), its forward orbit closure $\mathcal{O}^+\overline{(x)}$ is a minimal invariant closed subset of X [**6**, 1.4]. The following simple fact will be used later.

LEMMA. *The dynamical system (X, ϕ) is compact minimal iff for every open set $\mathcal{O} \subset X$, there exists a positive integer N such that $\bigcup_{k=0}^{N} \phi^{-k}(\mathcal{O}) = X$.*

Recall that the support of a nonnegative Borel measure on a separable metric space is the complement of the union of all open sets of measure zero. A dynamical system (X, ϕ) is *uniquely ergodic* if there is a unique invariant probability measure on X. The support of an invariant measure is an invariant subset, and if (X, ϕ) is uniquely ergodic and $Y \subset X$ is the support of the invariant measure, then (Y, ϕ) is minimal. In particular, if (X, ϕ) is minimal and uniquely ergodic with invariant measure μ, then $\mu(\mathcal{O}) > 0$ for every open set $\mathcal{O} \subset X$.

If (X, ϕ) is a dynamical system such that the family $\{\phi^n \colon n = 1, 2, \dots\}$ is equicontinuous, then it is uniquely ergodic [**4**, Theorem 3.1]. But the converse is false, even if (X, ϕ) is compact minimal (e.g., let $X = T^2$, the two-torus, and let ϕ be the "skew product", $\phi(z, w) = (e^{2\pi i \alpha} z, zw)$, α a fixed irrational [**3**, p. 52]). In general, a compact minimal dynamical system need not be uniquely ergodic.

III. Ideal structure.
III.1. Let (X, ϕ) be a dynamical system, and let $B_b(X)$ denote the bounded Borel functions on X. If F is an element of the algebra $K(\mathbf{Z}^+, B_b(X))$ generated by the operator U and $B_b(X)$, say $F = \sum_{n=0}^{N} U^n f_n$ ($f_n \in B_b(X)$), the representations Π_x of §I can be defined on $K(\mathbf{Z}^+, B_b(X))$ by the same formulas given earlier, and we set $\|F\| = \sup_{x \in X} \|\Pi_x(F)\|$. The completion $\mathbf{Z}^+ \times_\phi B_b(X)$ of $K(\mathbf{Z}^+, B_b(S))$ with respect to this norm is an operator algebra containing $\mathbf{Z}^+ \times_\phi C_0(X)$ as a (norm closed) subalgebra.

LEMMA. *Let $h_i, k_i \in B_b(X)$, $1 \leq i \leq m$, and assume*

$$\sum_{i=1}^{m} |h_i(x)|^2 \leq 1, \qquad \sum_{i=1}^{m} |k_i(x)|^2 \leq 1 \quad \textit{for all } x \in X.$$

If $F_i \in \mathbf{Z}^+ \times_\phi B_b(X)$, $1 \leq i \leq m$, then $\| \sum_{i=1}^m h_i F_i k_i \| \leq \sup_i \{ \|F_i\| \}$.

PROOF. Let $G = \sum_{i=1}^m h_i F_i k_i$, and let $\xi, \eta \in l_2^+$, $\xi = (\xi_n)_{n=0}^\infty$, $\eta = (\eta_n)_{n=0}^\infty$, $\|\xi\| = \|\eta\| = 1$. Then

$$
\begin{aligned}
(\Pi_x(G)\xi, \eta) &= \sum_{i=1}^m (\Pi_x(h_i F_i k_i)\xi, \eta) \\
&= \sum_{i=1}^m (\Pi_x(F_i)\Pi_x(k_i)\xi, \Pi_x(\overline{h_i})\eta) \\
&= \left(\begin{bmatrix} \Pi_x(F_i) & & \\ & \ddots & \\ & & \Pi_x(F_m) \end{bmatrix} \begin{bmatrix} \xi^{(1)} \\ \vdots \\ \xi^{(m)} \end{bmatrix}, \begin{bmatrix} \eta^{(1)} \\ \vdots \\ \eta^{(m)} \end{bmatrix} \right)
\end{aligned}
$$

where $\xi^{(i)} = \Pi_x(k_i)\xi$, $\eta^{(i)} = \Pi_x(\overline{h_i})\eta$. Now $\xi^{(i)} = (k_i(x)\xi_0, k_i \circ \phi(x)\xi_1, k_i \circ \phi^2(x)\xi_2, \dots)$ and so $\|\xi^{(i)}\|^2 = \sum_{n=0}^\infty |k_i \circ \phi^n(x)|^2 |\xi_n|^2$; hence,

$$
\begin{aligned}
\sum_{i=1}^m \|\xi^{(i)}\|^2 &= \sum_{i=1}^m \sum_{n=0}^\infty |k_i \circ \phi^n(x)|^2 |\xi_n|^2 \\
&= \sum_{n=0}^\infty \left(\sum_{i=1}^m |k_i \circ \phi^n(x)|^2 \right) |\xi_n|^2 \\
&\leq \sum_{n=0}^\infty |\xi_n|^2 \leq 1.
\end{aligned}
$$

Similarly, $\sum_{i=1}^m \|\eta^{(i)}\|^2 \leq 1$. Thus the vectors $[\xi^{(1)}, \dots, \xi^{(m)}]$, $[\eta^{(1)}, \dots, \eta^{(m)}]$ both have norm at most one. It follows that

$$
\|\Pi_x(G)\| \leq \left\| \begin{bmatrix} \Pi_x(F_1) & & \\ & \ddots & \\ & & \Pi_x(F_m) \end{bmatrix} \right\| \leq \sup_i \{ \|\Pi_x(F_i)\| \}.
$$

III.2. Let (X, ϕ) be a dynamical system which has no periodic points; i.e., $\phi^n(x) \neq x$, $n = 1, 2, \dots$, all $x \in X$. This is also expressed by saying that ϕ acts freely. Let $K(X)$ denote the continuous functions with compact support on X.

LEMMA. *Given $F \in K(\mathbf{Z}^+, K(X))$ and $N \in \mathbf{Z}^+$ there exist $m \in \mathbf{Z}^+$ and $h_i \in K(X)$, $1 \leq i \leq m$, such that, if $F = \sum_{k=0}^n U^k f_k$, then $\sum_{i=1}^m h_i F h_i \circ \phi^N = U^N f_N$. ($f_N$ is understood to be zero if $N > n$.)*

PROOF. Let $K = \bigcup_{k=0}^n \operatorname{supp}(f_k)$. Since ϕ acts freely, given $x \in \phi^N(K)$ there exists a compact neighborhood V_x of x such that, if $n' = \max\{n, N\}$, $V_x, \phi^{-1}(V_x), \dots, \phi^{-n'}(V_x)$ are pairwise disjoint compact neighborhoods. As $\{\operatorname{int}(V_x): x \in \phi^N(K)\}$ is an open cover of $\phi^N(K)$, we can extract a finite subcover, $V_1, \dots, V_m$. Let $\{h_1, \dots, h_m\} \subset C_0(X)$ satisfy $0 \leq h_i \leq 1$, $\operatorname{supp} h_i \subset V_i$, $1 \leq i \leq m$, and

$\sum_{i=1}^{m} h_i(x)^2 = 1$, $x \in \phi^N(K)$. (Choose a partition of unity $\{h_i'\}$ for $\phi^N(K)$ subordinate to $\{V_i\}$, and set $h_i = \sqrt{h_i'}$.) Compute

$$\sum_{i=1}^{m} h_i F h_i \circ \phi^N = \sum_{i=1}^{m} h_i \left(\sum_{k=0}^{n} U^k f_k \right) h_i \circ \phi^N$$

$$= \sum_{k=0}^{n} \sum_{i=1}^{m} h_i U^k f_k h_i \circ \phi^N$$

$$= \sum_{k=0}^{n} \sum_{i=1}^{m} U^k h_i \circ \phi^k h_i \circ \phi^N f_k.$$

Now $h_i \circ \phi^k$ is supported on $\phi^{-k}(V_i)$, which is disjoint from $\phi^{-N}(V_i)$ if $k \neq N$. Thus

$$\sum_{i=1}^{m} h_i F h_i \circ \phi^N = \sum_{i=1}^{m} U^N (h_i \circ \phi^N)^2 f_N = U^N f_N.$$

III.3. LEMMA. *Let (X, ϕ) be a dynamical system. For each $n \geq 0$ there is a linear projection $P_n \colon \mathbf{Z}^+ \times_\phi C_0(X) \to C_0(X)$ with the following properties:*

(i) $P_n(fF) = f \circ \phi^n P_n(F)$, *and* $P_n(Ff) = P_n(F)f$, $F \in \mathbf{Z}^+ \times_\phi C_0(X)$, $f \in C_0(X)$;

(ii) $\|P_n(F)\| \leq \|F\|$, $F \in \mathbf{Z}^+ \times_\phi C_0(X)$;

(iii) *If* $F = \sum_{n=0}^{N} U^n f_n$, *then* $P_n(F) = f_n$; *in general, for* $F \in \mathbf{Z}^+ \times_\phi C_0(X)$,

$$F - \sum_{n=0}^{N} U^n P_n(F) \in (U^{N+1}(\mathbf{Z}^+ \times_\phi C_0(X)).$$

Also, the statements are valid if "$B_b(X)$" is substituted for "$C_0(X)$".

PROOF. The automorphism τ_t, defined on $K(\mathbf{Z}^+, C_0(X))$ by $\tau_t(\sum_{n=0}^{N} U^n f_n) = \sum_{n=0}^{N} U^n e^{int} f_n$, is isometric and hence extends to an automorphism of $\mathbf{Z}^+ \times_\phi C_0(X)$. Define $P_n(F)$ by $U^n P_n(f) = (1/2\pi) \int_0^{2\pi} \tau_t(F) e^{-int} dt$. It is shown in [10 and 11] that P_n has the required properties.

III.4. THEOREM. *Let (X, ϕ) be a dynamical system. Then ϕ acts freely if and only if every closed, two sided ideal $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ has the property*

$$F \in J \quad implies \quad U^n P_n(F) \in J, \qquad n = 0, 1, 2, \dots.$$

PROOF. Assume ϕ acts freely, and let $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ be a closed, two sided ideal. Let $F \in J$ and $u \in C_0(X)$ with compact support, $0 \leq u \leq 1$. Now $F \in J$ implies $Fu \in J$, and if the conclusion holds for Fu, viz., $P_n(Fu) = P_n(F)u \in J$, $n = 0, 1, \dots$, then $P_n(F) \in J$. For let u run through an approximate identity for $C_0(X)$, and use the fact that J is closed.

Thus we assume $F \in J$ and there is a compact set K such that $f_n = P_n(F)$ is supported on K, $n = 0, 1, \dots$. Write $F = F' + F''$, $F' = \sum_{k=0}^{N} U^k f_k$, $F'' = F - F'$ ($N \in \mathbf{Z}^+$ is arbitrary but fixed). By Lemma III.2 there exist functions $h_i \in K(X)$, $1 \leq i \leq m$, such that $\sum_{i=1}^{m} h_i F' h_i \circ \phi^N = U^N f_N$. Thus,

$$\sum_{i=1}^{m} h_i F h_i \circ \phi^N = U^N f_N + F_N, \quad F_N = \sum_{i=1}^{m} h_i F'' h_i \circ \phi^N \in U^{N+1}(\mathbf{Z}^+ \times_\phi C_0(X)),$$

and $P_n(F_N)$ is supported on K, for all n. Given $\varepsilon > 0$ there exists $G \in K(\mathbf{Z}^+, C_0(X))$, $G = \sum_{k=N+1}^{M} U^k g_k$, supp $g_k \subset K$ all k, such that $\|F_N - G\| < \varepsilon$. Apply Lemma III.2 again, this time to $U^N f_N + G$. Thus there exist functions $k_i \in C_0(X)$, $0 \le k_1 \le 1$, $1 \le i \le p$, $\sum_{i=1}^{p} k_i(x)^2 = 1$ for all $x \in K$. Now $U^N f_N + F_N \in J$, as is

$$\sum_{i=1}^{p} k_i(U^N f_N + F_N) k_i \circ \phi^N = U^N f_N + \sum_{i=1}^{p} k_i F_N k_i \circ \phi^N.$$

Estimate

$$\left\| \sum_{i=1}^{p} k_i F_N k_i \circ \phi^N \right\| = \left\| \sum_{i=1}^{p} k_i (F_N - G) k_i \circ \phi^N + \sum_{i=1}^{p} k_i G k_i \circ \phi^N \right\|$$

$$\le \left\| \sum_{i=1}^{p} k_i (F_N - G) k_i \circ \phi^N \right\| + \left\| \sum_{i=1}^{p} k_i G k_i \circ \phi^N \right\|$$

$$\le \|F_N - G\| < \varepsilon$$

by Lemma III.1 and the fact $\sum_{i=1}^{p} k_i G k_i \circ \phi^N = 0$. Since J is closed, we conclude $U^N f_N \in J$.

For the converse, suppose ϕ does not act freely. In other words, there is a point $x \in X$ and a positive integer p such that $\phi^p(x) = x$. Then, as discussed in [11, III], there are representations of $\mathbf{Z}^+ \times_\phi C_0(X)$ as matrix algebras of holomorphic functions. The kernels of these representations do not have the property that $F \in J$ implies $U^n P_n(F) \in J$ for all n.

III.5. Suppose (X, ϕ) is a dynamical system where ϕ acts freely, and let $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ be a closed, two sided ideal. For $f \in C_0(X)$, let $Z(f)$ be the zero set of f, $\{x \colon f(x) = 0\}$. Set

$$X_n = \bigcap \{Z(P_n(F)) \colon F \in J\}$$

and

$$J' = \{F \in \mathbf{Z}^+ \times_\phi C_0(X) \colon Z(P_n(F)) \supseteq X_n, \ n = 0, 1, \ldots \}.$$

COROLLARY. $J = J'$.

PROOF. Clearly, $J \subseteq J'$. On the other hand, observe that $F \in J$ iff $U^n P_n(F) \in J$, $n = 0, 1, \ldots$. One direction is Theorem III.4. Conversely, if $U^n P_n(F) \in J$, then so are the nth arithmetic means of the sequence $\{U^n P_n(F)\}_{n=0}^{\infty}$, hence so is F [11, IV.2, Remark]. Let $J_n = \{f \in C_0(X) \colon U^n f \in J\}$. J_n is a closed ideal in $C_0(X)$, hence there is a closed set $X_n' \subseteq X$ such that $J_n = \{f \in C_0(X) \colon f | X_n' = 0\}$. Furthermore,

$$X_n' = \bigcap \{\mathbf{Z}(f) \colon U^n f \in J\}$$

$$= \bigcap \{Z(P_n(F)) \colon F \in J\} = X_n.$$

Let $F \in J'$, so $P_n(F) | X_n = 0$, whence $U^n P_n(F) \in J$, $n = 0, 1, \ldots$, and finally $F \in J$. Thus $J = J'$.

Notation. Let $(X_n)_{n=0}^{\infty}$ be the sequence of closed subsets of X associated with the ideal J as above; write this as $J \sim (X_n)_{n=0}^{\infty}$. Of course, not every sequence of closed subsets of X is associated with an ideal $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$. Let $J \sim (X_n)_{n=0}^{\infty}$, and $U^n f \in J$; then both $U^{n+1} f$, $U^n f U = U^{n+1} f \circ \phi \in J$, so both f, $f \circ \phi$ must vanish

on X_{n+1}. If f is chosen so that $Z(f) = X_n$ for any $x \in X_{n+1}$, $f(x) = f \circ \phi(x) = 0$ implies that both $x, \phi(x) \in X_n$. Thus

$$(*) \qquad\qquad\qquad X_{n+1} \cup \phi(X_{n+1}) \subset X_n.$$

Conversely, if $(X_n)_{n=0}^\infty$ is any sequence of closed subsets of X satisfying $(*)$, then the subspace $J \sim (X_n)_{n=0}^\infty$, $J = \{F \in \mathbf{Z}^+ \times_\phi C_0(X): P_n(F)|X_n = 0\}$ is closed under left and right multiplication by U and by functions $f \in C_0(X)$, hence is a closed, two sided ideal. This proves

THEOREM. *Let (X, ϕ) be a dynamical system with ϕ acting freely. Then there is a one-to-one correspondence between closed, two sided ideals $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ and sequences $(X_n)_{n=0}^\infty$ of closed subsets of X satisfying $(*)$.*

III.6. Even if ϕ does not act freely on X, still, corresponding to every sequence $(X_n)_{n=0}^\infty$ of closed subsets of X satisfying $(*)$ of III.5 there is a closed ideal J of $\mathbf{Z}^+ \times_\phi C_0(X)$. Recall that a (two sided) ideal J is *prime* if, whenever J_1, J_2 are two sided ideals such that $J_1 J_2 \subset J$, either $J_1 \subset J$ or $J_2 \subset J$.

LEMMA. *Let $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ be a prime ideal, $J \sim (X_n)_{n=0}^\infty$. Then either $X_n = Z$ for some closed, invariant subset $Z \subset X$, $n = 0, 1, \ldots$, or X_0 consists of a single point and $X_n = \varnothing$, $n \geq 1$.*

PROOF. Set $Y_n = X_{n+1}$, $n = 0, 1, \ldots$. As $(X_n)_{n=0}^\infty$ satisfies $(*)$ of III.5, so does $(Y_n)_{n=0}^\infty$; let J_1 be the corresponding ideal. If $U^n f \in J_1$, that is, $f|Y_n = f|X_{n+1} = 0$, then $U(U^n f) = U^{n+1} f \in J$. Consequently, $J_2 J_1 \subset J$, where J_2 is the ideal generated by U; viz., $J_2 = U(\mathbf{Z}^+ \times_\phi C_0(X))$. As J is prime, either $J_1 \subset J$ or $J_2 \subset J$. Suppose now that the sequence $(X_n)_{n=0}^\infty$ is not constant; so for some N, $X_{N+1} \subset X_N$ but $X_{N+1} \neq X_N$. Let $f \in C_0(X)$ be such that $f|X_{N+1} = 0$, $f|X_N$ is not identically zero. In that case, $U^N f \in J_1$, but $U^N f \notin J_1$ is not contained in J. Hence $J_2 \subset J$. The quotient map $\mathbf{Z}^+ \times_\phi C_0(X) \to \mathbf{Z}^+ \times_\phi C_0(X)/J_2 \cong C_0(X)$ maps $\mathbf{Z}^+ \times_\phi C_0(X)$ onto $C_0(X)$, and as J is a prime ideal in $\mathbf{Z}^+ \times_\phi C_0(X)$ which contains J_2, its image is a prime ideal in $C_0(X)$. But the prime ideals in $C_0(X)$ are exactly the maximal ideals (of codimension one), so X_0 consists of a single point, and $X_n = \varnothing$, $n \geq 1$.

If the sequence $(X_n)_{n=0}^\infty$ is constant, so $X_n = Z$ for all n, then Z satisfies $Z \cup \phi(Z) \subset Z$, so $\phi(Z) \subset Z$ and Z is a closed, invariant subset.

Notation. If an ideal $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ is associated with a sequence $(X_n)_{n=0}^\infty$, and $X_n = Z$ for all n, we will write $J \sim (Z)$.

III.7. PROPOSITION. *Let (X, ϕ) be a dynamical system, $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ a prime ideal such that $J \sim (Z)$, $Z \subset X$ a closed, invariant subset. Then (Z, ϕ) is transitive; that is, there exists $x_0 \in Z$ such that $Z = \mathcal{O}^+\overline{(x_0)}$.*

PROOF. First we claim that the recurrent points are dense in Z, or equivalently (by Proposition II.5) that every point of Z is nearly recurrent. Let X_{nr} be the set of nearly recurrent points of X. If $X \backslash X_{nr}$ intersects Z, let $f \in C_0(X)$ be such that $f|X_{nr} = 0$, $f|Z$ not identically zero. By [11, IV.4] Uf belongs to the prime radical of $\mathbf{Z}^+ \times_\phi C_0(X)$, but $Uf \notin J$. This contradiction establishes the claim.

Let $Z^{(1)} \subset Z$ be a closed, invariant subset with nonempty interior (with respect to Z). Then $Z^{(2)} = (Z \backslash Z^{(1)})^-$ is also invariant: for the recurrent points are dense

in the open set $Z\backslash Z^{(1)}$, hence are dense in $Z^{(2)}$ if $Z^{(2)}$ is nonempty. Suppose $x \in Z\backslash Z^{(1)}$ is recurrent; if $\phi(x) \in Z^{(1)}$ then by invariance of $Z^{(1)}$, $\phi^n(x) \in Z^{(1)}$ for all $n \geq 1$. By recurrence there is a sequence $\{n_k\}$ such that $\phi^{n_k}(x) \to X$, so $x \in Z^{(1)}$. This contradiction shows that the recurrent points form an invariant subset of $Z^{(2)}$, and since they are dense, $Z^{(2)}$ is invariant.

Let J_1, J_2 be ideals in $\mathbf{Z}^+ \times_\phi C_0(X)$ associated with the constant sequences $(Z^{(1)})$, $(Z^{(2)})$, respectively. Now $J_1 J_2 \subset J_1 \cap J_2 \sim (Z^{(1)} \cup Z^{(2)}) = (Z)$, and since J is prime either $J_1 \subset J$ or $J_2 \subset J$. That is, either $Z \subset Z^{(1)}$ or $Z \subset Z^{(2)}$. But $Z^{(2)} = (Z\backslash Z^{(1)})^- \subset Z\backslash(\text{int } Z^{(1)})$, so $Z^{(2)}$ is properly contained in Z as int $Z^{(1)} \neq \varnothing$. Thus it must be that $Z^{(1)} = Z$, and by Proposition II.8 (Z, φ) is topologically transitive, and any proper closed invariant subset must be nowhere dense.

III.8. PROPOSITION. *Let (X, ϕ) be a dynamical system such that ϕ is free. Let $Z \subset X$ be closed, invariant and transitive (so $Z = \mathcal{O}^+\overline{(x_0)}$, some $x_0 \in X$). then the ideal $J \sim (Z)$ is prime in $\mathbf{Z}^+ \times_\phi C_0(X)$.*

PROOF. Let J_1, J_2 be closed, two sided ideals in $\mathbf{Z}^+ \times_\phi C_0(X)$ such that $J_1 J_2 \subset J$. By Theorem III.5 there are sequences $(Y_n^{(1)})_{n=0}^\infty \sim J_1$ and $(Y_n^{(2)})_{n=0}^\infty \sim J_2$. Fix $n_1, n_2 \in \mathbf{Z}^+$ and let $f, g \in C_0(X)$ be such that $\{x\colon f(x) = 0\} = Y_{n_1}^{(1)}$, $\{x\colon g(x) = 0\} = Y_{n_2}^{(2)}$. Then $U^{n_1} f \in J_1$, $U^{n_2} g \in J_2$, and so for any $k \in \mathbf{Z}^+$, $U^{n_1} f U^k U^{n_2} g = U^{n_1+n_2+k} f \circ \phi^{k+n_2} g \in J$. This means $f \circ \phi^{k+n_2} g$ vanishes on Z. Suppose there is a transitive point x_1 (i.e., $\mathcal{O}^+\overline{(x_1)} = Z$) such that $g(x_1) \neq 0$. Then $f \circ \phi^{k+n_2}(x_1) = 0$, so $\phi^{k+n_2}(x_1) \in Y_{n_1}^{(1)}$ for all $k \geq 0$. Since x_1 is recurrent, $\mathcal{O}^+\overline{(x_1)} = \mathcal{O}^+(\phi^{n_2}\overline{(x_1)}) = Z$, and so $Z \subset Y_{n_1}^{(1)}$. On the other hand, if $g(x) = 0$ for every transitive point $x \in Z$, since by Proposition II.8 the transitive points are dense, it follows g vanishes on Z, and so $Z \subset Y_{n_2}^{(2)}$. Thus, for every n, either $Z \subset Y_n^{(1)}$ or $Z \subset Y_n^{(2)}$, and hence, either $Z \subset Y_n^{(1)}$ for infinitely many n or else $Z \subset Y_n^{(2)}$ for finitely many n. Say $Z \subset Y_n^{(1)}$ for infinitely many n. As $Y_{n+1}^{(1)} \subset Y_{n+1}^{(1)} \cup \phi(Y_{n+1}^{(1)}) \subset Y_n^{(1)}$, it follows $Z \subset Y_n^{(1)}$ for all n, and hence $J_1 \subset J$. In the other case we conclude $J_2 \subset J$. Thus J is prime.

III.9. Let (X, ϕ) be a dynamical system, and set $\mathfrak{A} = \mathbf{Z}^+ \times_\phi C_0(X)$. A closed, two sided ideal $J \subset \mathfrak{A}$ is called right primitive if $\mathfrak{A}/J$ is a (right) primitive algebra; in other words, $\mathfrak{A}/J$ can be faithfully represented as a strictly irreducible (or strictly transitive) right $\mathcal{M}$-module, for some vector space $\mathcal{M}$. $\mathcal{M}$ is isomorphic to $\mathfrak{A}/\mathcal{R}$ for some maximal right ideal $\mathcal{R} \subset \mathfrak{A}$, and $\mathfrak{A}/\mathcal{R}$ is a Banach space under a suitable norm, and so $\mathfrak{A}/J$ is represented as an algebra of continuous operators on a Banach space (acting on the right) (see [**12**, 2.4.7]).

III.10. The proof of the next proposition is deferred until V.5.

PROPOSITION. *Let (X, ϕ) be a dynamical system, and $Z \subset X$ a compact minimal invariant subset. If $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ is the closed two sided ideal associated with the constant sequence (Z), then J is right primitive.*

III.11. Let (X, ϕ) be a dynamical system with ϕ freely acting. We summarize the results from this section regarding the maximal, primitive and prime ideals of $\mathbf{Z}^+ \times_\phi C_0(X)$.

THEOREM. (i) *There is a one-to-one correspondence between the maximal two sided ideals J of $\mathbf{Z}^+ \times_\phi C_0(X)$ and the points of X as follows: J corresponds to $x \in X$ iff $J \sim (Y_n)_{n=0}^\infty$, with $Y_0 = \{x\}$, $Y_n = \varnothing$, $n \geq 1$.*

(ii) *There is a one-to-one mapping from the compact minimal invariant subsets* $Z \subset X$ *into the* (*right*) *primitive ideals* J *of* $\mathbf{Z}^+ \times_\phi C_0(X)$ *as follows:* $Z \mapsto J$ *if*

$$J = \{F \in \mathbf{Z}^+ \times_\phi C_0(X) \colon P_n(F)| Z = 0, \ n = 0, 1, 2, \dots \}.$$

(iii) *There is a one-to-one correspondence between prime ideals* $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ *which are not maximal, and closed, invariant transitive sets* $Z \subset X$ *as follows:* $J \sim (Z)$ *iff*

$$J = \{F \in \mathbf{Z}^+ \times_\phi C_0(X) \colon P_n(F)|Z = 0, \ n = 0, 1, \dots \}.$$

III.12. We conclude this section with some remarks concerning the hull-kernel topology on the various spaces of ideals. Effros and Hahn [5] showed that if ϕ is a freely acting homeomorphism of a locally compact space X, there is a one-to-one correspondence between the primitive ideals of the C^*-crossed product $\mathbf{Z} \times_\phi C_0(X)$ and the set of orbit closures in X. In their case, the orbit was two sided: $\mathcal{O}(x) = \{\phi^n(x) \colon n \in \mathbf{Z}\}$. Define a map p from X to the collection of closed orbits by $p(x) = \overline{O(x)}$. Then $p(X)$, provided with the quotient topology, is homeomorphic with $\mathrm{Prim}(\mathbf{Z} \times_\phi C_0(X))$.

The results for $\mathbf{Z}^+ \times_\phi C_0(X)$ are similar, through somewhat more complicated to state. Let ϕ be freely acting. The correspondence described in III.11(i) is a homeomorphism of X with the space of maximal, two sided ideals (i.e., the strong structure space) of $\mathbf{Z}^+ \times_\phi C_0(X)$. This was proved in [11, IV.8] and also follows easily from the results of this section. Next consider the hull-kernel topology on the collection of prime ideals which are not maximal. Let $\mathcal{S} = \{J_i\}$, $J_i \sim (Z^{(i)})$, be a collection of prime ideals which are not maximal. A nonmaximal prime ideal J, $J \sim (Z)$, belongs to the closure of $\mathcal{S}$ iff $J \supset \bigcap J_i$, iff $Z \subset (\bigcup Z^{(i)})^-$.

IV. The conjugate space.

IV.1. Let (X, ϕ) be a dynamical system and let $\xi, \eta \in l_1^+$; the map $F \in \mathbf{Z}^+ \times_\phi C_0(X) \to \Lambda_x(F) = (\Pi_x(F)\xi, \eta)$ defines a continuous linear functional on $\mathbf{Z}^+ \times_\phi C_0(X)$.

LEMMA. *Let* $J \subset \mathbf{Z}^+ \times_\phi C_0(X)$ *be a closed two sided ideal,* $J \sim (Z)$, *where* $Z \subset X$ *is closed, invariant and* $\phi^{-1}(Z) = Z$. *Then* $\Lambda_x \in J^\perp$ *iff* $x \in Z$.

PROOF. Suppose $x \in Z$ and $F = \sum_{n=0}^\infty U^n f_n \in J$. Then $f_n|Z = 0$ for all n, so $\Pi_x(F) = 0$. Thus $\Lambda_x \in J^\perp$.

Conversely, let $\Lambda_x \in J^\perp$; if $\Lambda_x \neq 0$, then $\Lambda_x(U^n g) \neq 0$ for some $n \in \mathbf{Z}^+$ and $g \in C_0(X)$. If $\xi = (\xi_n)_{n=0}^\infty$, $\eta = (\eta_n)_{n=0}^\infty$, then $\Lambda_x(U^n g) = \sum_{i=0}^\infty g \circ \phi^i(x)\xi_i \bar{\eta}_{n+i}$. Adjusting g by a scalar, we may assume $\Lambda_x(U^n g) = 1$. Choose N large enough so that both

$$\left| \sum_{i=0}^N g \circ \phi^i(x)\xi_i \bar{\eta}_{n+i} \right| > \frac{2}{3} \quad \text{and} \quad \sum_{i=N+1}^\infty |\xi_i \bar{\eta}_{n+i}| < \frac{1}{3\|g\|}.$$

Now if $x \notin Z$, $\phi^i(x) \notin Z$ for all $i = 0, 1, \dots$, and so there exists $f \in C_0(X)$ such that $f|Z = 0$, $f(\phi^i(x)) = g(\phi^i(x))$, $i = 0, 1, \dots, N$, and $\|f\| \leq \|g\|$. On the one

hand, since $\Lambda_x \in J^\perp$ and $U^n f \in J$, $\Lambda_x(U^n f) = 0$. However,

$$
\begin{aligned}
|\Lambda_x(U^n f)| &= \left| \sum_{i=0}^\infty f \circ \phi^i(x) \xi_i \overline{\eta}_{n+i} \right| \\
&\geq \left| \sum_{i=0}^N f \circ \phi^i(x) \xi_i \overline{\eta}_{n+i} \right| - \left| \sum_{i=N+1}^\infty f \circ \phi^i(x) \xi_i \overline{\eta}_{n+i} \right| \\
&\geq \left| \sum_{i=0}^N g \circ \phi^i(X) \xi_i \overline{\eta}_{n+i} \right| - \|f\| \sum_{i=N+1}^\infty |\xi_i \eta_{n+i}| \\
&> \frac{2}{3} - \|g\| \frac{1}{3\|g\|} = \frac{1}{3}.
\end{aligned}
$$

This contradiction shows that $x \in Z$, as asserted.

Let $F \in K(\mathbf{Z}^+, C_0(X))$, $F = \sum_{k=0}^N U^k f_k$. The element $F' = \sum_{k=0}^N U^k f_k'$, where $f_k' = f_k | Z$, belongs to $K(\mathbf{Z}^+, C_0(Z))$, whose norm (in $\mathbf{Z}^+ \times_\phi C_0(Z)$) is given by $\sup_{x \in Z} \|\Pi_x(F')\|$, which is no greater than $\sup_{x \in X} \|\Pi_x(F)\| = \|F\|$. This mapping extends to a map $\Pi: \mathbf{Z}^+ \times_\phi C_0(X) \to \mathbf{Z}^+ \times_\phi C_0(Z)$, and $\|\Pi(F)\| \leq \|F\|$, $F \in \mathbf{Z}^+ \times_\phi C_0(X)$. As $J = \ker \Pi$, $\Pi(\mathbf{Z}^+ \times_\phi C_0(X))$ and $\mathbf{Z}^+ \times_\phi C_0(X)/J$ are algebraically isomorphic. If Z is just invariant ($\phi(Z) \subset Z$ or even $\phi(Z) = Z$), it is an open question whether Π is surjective.

COROLLARY. *Under the hypotheses of the Lemma, the map Π is surjective, and $\mathbf{Z}^+ \times_\phi C_0(X)/J$ and $\mathbf{Z}^+ \times_\phi C_0(Z)$ are isomorphic as Banach algebras.*

PROOF. First we claim $J^\perp$ is the closed linear span of $\{\Lambda_x = \Lambda_x(\xi, \eta): x \in Z, \xi, \eta \in l_2^+\}$. Let K be this closed subspace, and let $\Lambda_0 \in J^\perp$ and suppose $\Lambda_0 \notin K$. By the Hahn-Banach Theorem, there is an $F \in \mathbf{Z}^+ \times_\phi C_0(X)$ such that $\mathrm{Re}(\Lambda_0(F)) > 0$, $\mathrm{Re}(\Lambda(F)) \leq 0$ for all $\Lambda \in K$. But as K is a subspace, it must be that $\Lambda(F) = 0$ for all $\Lambda \in K$. So $\Pi_x(F) = 0$, $x \in Z$, and hence, by the Lemma, $F \in J$. But then $\Lambda_0(F) = 0$.

The quotient norm is given by

$$
\|F + J\| = \inf_{G \in J} \|F + G\| = \sup_{\substack{\Lambda \in J^\perp \\ \|\Lambda\| \leq 1}} |\Lambda(F)|.
$$

By the claim, the linear functionals Λ_x $(x \in Z)$ are total in $J^\perp$, so that

$$
\|F + J\| = \sup_{\|\Lambda_x\| \leq 1, \ x \in Z} |\Lambda_x(F)|.
$$

But the right side is precisely the norm of $\Pi(F)$ in $\mathbf{Z}^+ \times_\phi C_0(Z)$. Thus Π maps onto $\mathbf{Z}^+ \times_\phi C_0(Z)$, and $\mathbf{Z}^+ \times_\phi C_0(X)/J$ and $\mathbf{Z}^+ \times_\phi C_0(Z)$ are isomorphic as Banach algebras.

IV.2. Let (X, ϕ) be a dynamical system, and set $\mathfrak{A} = \mathbf{Z}^+ \times_\phi C_0(X)$. Let $\mathfrak{A}^*$ denote the conjugate space of $\mathfrak{A}$ (the continuous linear functionals on $\mathfrak{A}$).

A. LEMMA. *Let $F \in \mathfrak{A}$, $F = \sum_{n=0}^\infty U^n f_n$. Then $\lim_{n \to \infty} \|f_n\| = 0$.*

PROOF. Since $K(\mathbf{Z}^+, C_0(X))$ is dense in $\mathfrak{A}$, given $F \in \mathfrak{A}$ and $\varepsilon > 0$ there exists $G \in K(\mathbf{Z}^+, C_0(X))$ such that $\|F - G\| < \varepsilon$. By [11, II.7] the norm of an element

of $\mathfrak{A}$ is not less than the norms of any of its Fourier coefficients. If $G = \sum_{j=0}^{N} U^i g_i$, then for $n > N$ the nth Fourier coefficient of $F - G$ is f_n, and $\|f_n\| \leq \|F - G\| < \varepsilon$.

By the Lemma, $\mathfrak{A}$ is a subspace of the Banach space $c_0(C_0(X))$ of sequences $(f_n)_{n=0}^{\infty}$ of functions in $C_0(X)$, with $\lim \|f_n\| = 0$. Thus,

$$l_1(C_0(X)) \subset \mathfrak{A} \subset c_0(C_0(X)).$$

So, $l_{\infty}(\mathcal{M}(X)) \supset \mathfrak{A}^* \supset l_1(\mathcal{M}(X))$, where $\mathcal{M}(X)$ is the space of complex regular Borel measures on X with finite total variation. In other words, to every $\Gamma \in \mathfrak{A}^*$ there corresponds a unique sequence $(\mu_n)_{n=0}^{\infty}$, $\mu_n \in \mathcal{M}(X)$, such that for every $F = \sum_{n=0}^{N} U_n f_n \in K(\mathbf{Z}^+, C_0(X))$

$$\langle \Gamma, F \rangle = \sum_{n=0}^{N} \int f_n \, d\mu_n.$$

Also, $\|\Gamma\| \geq \sup_n \|\mu_n\|$. It will be convenient to write $\Gamma = (\mu_n)_{n=0}^{\infty}$. (Note: from now on we will use the bracket $\langle \cdot, \cdot \rangle$ notation for linear functionals in place of the notation of (IV.1).)

B. REMARKS. (i) If $F \in \mathfrak{A}$ with Fourier series $F = \sum_{n=0}^{\infty} U^n f_n$, and if $\Gamma_x \in \mathfrak{A}^*$ is identified with the sequence $(\xi_n \delta_x)_{n=0}^{\infty}$ where δ_x is the point mass at X, the series $\sum_{n=0}^{\infty} \xi_n f_n(x)$ may not necessarily converge, just as the partial sum of the Fourier series $\sum_{n=0}^{\infty} U^n f_n$ need not converge to F. (However, both are Cesàro summable.)

(ii) As an example, let $(\xi_n)_{n=0}^{\infty} \in l_2^+$. Then $\Gamma_x = (\xi_n \delta_x)_{n=0}^{\infty} \in \mathfrak{A}^*$, for $\langle \Gamma_x, F \rangle = (\Pi_x(F)\eta, \xi)$, where $\eta = (1, 0, 0, \dots) \in l_2^+$, and $\xi = (\bar{\xi}_n)_{n=0}^{\infty}$.

V. Irreducible Banach modules.

V.1. Let (X, ϕ) be a dynamical system, and $\mathcal{R} \subset \mathfrak{A} = \mathbf{Z}^+ \times_\phi C_0(X)$ a closed right ideal. $\mathcal{R}$ is thus a closed linear subspace of $\mathfrak{A}$ which is a right $C_0(X)$-module and invariant under right multiplication by U; conversely, any closed linear subspace $\mathcal{R} \subset \mathfrak{A}$ with these properties is a closed right ideal.

If $\mathcal{R}$ is a subset of $\mathfrak{A}$, denote by $\mathcal{R}^\perp = \{\Gamma \in \mathfrak{A}^* : \langle \Gamma, F \rangle = 0, F \in \mathcal{R}\}$. If $\mathcal{R}$ is a right ideal of $\mathfrak{A}$, then $\mathcal{R}^\perp$ is a (left) $C_0(X)$-module under $\langle f\Gamma, F \rangle = \langle \Gamma, Ff \rangle$ $(F \in \mathfrak{A})$. Furthermore, $\mathcal{R}^\perp$ is invariant under the operator T defined to be the adjoint of right multiplication by U: $\langle T\Gamma, F \rangle = \langle \Gamma, FU \rangle$.

Conversely, if $S \subset \mathfrak{A}^*$ is a closed linear subspace which is (i) a left $C_0(X)$ module, and (ii) invariant under T, then $S_\perp = \{F \in \mathfrak{A} : \langle \Gamma, F \rangle = 0, \Gamma \in S\}$ is a closed right ideal of $\mathfrak{A}$.

As observed in [11, IV.3], the endomorphism α of $C_0(X)$, $\alpha(f) = f \circ \phi$, admits an extension to an endomorphism of the Banach algebra $\mathfrak{A}$; this extension will also be denoted by α. For $F = \sum_{n=0}^{N} U^n f_n \in K(\mathbf{Z}^+, C_0(X))$, $\alpha(F) = \sum_{n=0}^{N} U^n f_n \circ \phi$. One computes that for $F \in K(\mathbf{Z}^+, C_0(X))$, $\alpha(UF) = U(\alpha(F)) = FU$, and so these equalities hold for $F \in \mathfrak{A}$ as well. Hence if S denotes the adjoint of left multiplication by U, and $^+\alpha$ the adjoint of α,

$$T = {}^+\alpha S = S^+\alpha.$$

If $\mathcal{R} \subset \mathfrak{A}$ is a right ideal, $\mathcal{R}^\perp$ will not, in general, be invariant under either $^+\alpha$ or S.

V.2. Let λ be a complex number, $0 \leq |\lambda| < 1$. Since $(\lambda^n)_{n=0}^{\infty} \in l_2^+$, it follows from Remark (ii) of IV.2 that for each $x \in X$ the linear functional corresponding to

the sequence $(\lambda^n \delta_x)_{n=0}^\infty$ is bounded, and hence there is a unique positive constant $c(\lambda, x)$ such that $c(\lambda, x)((\lambda^n \delta_x)_{n=0}^\infty)$ has norm one. Now as $^+\alpha$ has norm one,

$$^+\alpha(c(\lambda, x)(\lambda^n \delta_x)_{n=0}^\infty) = c(\lambda, x)(\lambda^n \delta_{\phi(x)})_{n=0}^\infty$$

has norm at most one. The same can be said about $c(\lambda, x)(\lambda^n \delta_{\phi^N(x)})_{n=0}^\infty$ for any N, and hence $\|c(\lambda, x)(\lambda^n \delta_y)_{n=0}^\infty\| \le 1$ for any $y \in \mathcal{O}^+\overline{(x)}$. Thus $c(\lambda, y) \ge c(\lambda, x)$. Suppose now that (X, ϕ) is a compact, minimal dynamical system; then $x \in \mathcal{O}^+\overline{(y)}$, so $c(\lambda, x) \ge c(\lambda, y)$. It follows that $c(\lambda, x) = c_\lambda$ depends only on λ. Let $\Gamma_x^{(\lambda)} = c_\lambda(\lambda^n \delta_x)_{n=0}^\infty$.

If S is the operator defined in V.1, $\Gamma \in \mathfrak{A}^*$, $\Gamma = (\mu_n)_{n=0}^\infty$, then $S\Gamma = (\nu_n)_{n=0}^\infty$, where $\nu_n = \mu_{n+1}$, $n \ge 0$. In particular, $S\Gamma_x^{(\lambda)} = \lambda \Gamma_x^{(\lambda)}$, $x \in X$. Let S be the closed linear subspace of $\mathfrak{A}^*$ generated by the linear functionals $\{\Gamma_x^{(\lambda)} : x \in X\}$. As every complex measure in $\mathcal{M}(X)$ is a weak-$*$ limit of linear combinations of point masses, $S = \{\Gamma \in \mathfrak{A}^* : \langle \Gamma, F \rangle = \int_X \langle \Gamma_x^{(\lambda)}, F \rangle \, d\mu(x)$, for some $\mu \in \mathcal{M}(X)\}$. Because the set $\{\Gamma_x^{(\lambda)} : x \in X\}$ is invariant under $^+\alpha$ and S, it follows that S is invariant under $^+\alpha$ and S, hence invariant under T.

LEMMA. *Let* $\mathcal{R}^{(\lambda)} = S_\perp = \{F \in \mathfrak{A} : \langle \Gamma_x^{(\lambda)}, F \rangle = 0 \text{ for all } x \in X\}$. *Then* $\mathcal{R}^{(\lambda)}$ *is a closed, α-invariant right ideal in* $\mathfrak{A}$. *Furthermore, the unit ball of S is the closed, balanced convex hull of* $\{\Gamma_x^{(\lambda)} : x \in X\}$.

PROOF. It remains only to be noted that S is a (left) $C(X)$-module, which is clear.

COROLLARY. *For* $G \in \mathfrak{A}$, *the quotient norm in* $\mathfrak{A}/\mathcal{R}^{(\lambda)}$,

$$\|G + \mathcal{R}^{(\lambda)}\| = \inf\{\|G + H\| : H \in \mathcal{R}^{(\lambda)}\} = \sup\{|\langle \Gamma, G \rangle| : \Gamma \in S\}$$
$$= \sup\{|\langle \Gamma_x^{(\lambda)}, G \rangle| : x \in X\}.$$

V.3. If $\mathcal{R}$ is a closed right ideal of a Banach algebra $\mathfrak{A}$, then $\mathfrak{A}/\mathcal{R}$ is a right $\mathfrak{A}$-module under right multiplication, and by a standard calculation (e.g. [12, p. 50]),

$$\|GF + \mathcal{R}\| \le \|G + \mathcal{R}\| \, \|F\|.$$

Let $\mathfrak{A}$, $\mathcal{R}^{(\lambda)}$ be as in V.2 and define a map $\Phi_\lambda : \mathfrak{A}/\mathcal{R}^{(\lambda)} \to C(X)$ by the formula

$$\Phi_\lambda(F + \mathcal{R}^{(\lambda)})(x) = \langle \Gamma_x^{(\lambda)}, F \rangle, \qquad F \in \mathfrak{A}.$$

The map is well defined, and by Corollary V.2 it is an isometry. As $C(X) \subset \mathfrak{A}$ and $\Phi_\lambda(f + \mathcal{R}^{(\lambda)}) = c_\lambda f$, Φ_λ is onto. Also, observe

(i) $\qquad \Phi_\lambda(Ff + \mathcal{R}^{(\lambda)})(x) = \langle \Gamma_x^{(\lambda)}, Ff \rangle = \langle \Gamma_x^{(\lambda)}, F \rangle f(x)$
$$= \Phi_\lambda(F + \mathcal{R}^{(\lambda)})(x) f(x);$$

(ii) $\qquad \Phi_\lambda(FU + \mathcal{R}^{(\lambda)})(x) = \langle T\Gamma_x^{(\lambda)}, F \rangle = \langle S^+\alpha(\Gamma_x^{(\lambda)}), F \rangle = \langle S\Gamma_{\phi(x)}^{(\lambda)}, F \rangle$
$$= \langle \lambda \Gamma_{\phi(x)}^{(\lambda)}, F \rangle = \lambda \Phi_\lambda(F + \mathcal{R}^{(\lambda)}) \circ \phi(x).$$

Thus, Φ implements an $\mathfrak{A}$-module isomorphism of $\mathfrak{A}/\mathcal{R}^{(\lambda)}$ and $C(X)$; under this isomorphism, an element $F = \sum_{n=0}^N U^n f_n$ acts on $C(X)$ on the right by

$$[g]F = \left[\sum_{n=0}^N \lambda^n g \circ \phi^n f_n \right], \qquad g \in C(X).$$

Since $\|[g]F\| \leq \|g\| \|F\|$, we obtain, taking $g \equiv 1$,

$$(*) \qquad \left\| \sum_{n=0}^{N} \lambda^n f_n \right\| \leq \|F\|.$$

COROLLARY. $c_\lambda = 1$; in other words, the linear functional $\Gamma_x^{(\lambda)} = (\lambda^n \delta_x)_{n=0}^{\infty}$ has norm one, $0 \leq |\lambda| < 1$.

PROOF. Let $F = \sum_{n=0}^{N} U^n f_n$; $|\langle (\lambda^n \delta_x)_{n=0}^{\infty}, F \rangle| = |\sum_{n=0}^{N} \lambda^n f_n(x)| \leq \|F\|$ by $(*)$, so $\|(\lambda^n \delta_x)_{n=0}^{\infty}\| \leq 1$. But $|\langle (\lambda^n \delta_x)_{n=0}^{\infty}, f \rangle| = |f(x)| = \|f\|$ if $f \in C_0(X)$ achieves its supremum at the point x. Thus $\|(\lambda^n \delta_x)_{n=0}^{\infty}\| = 1$.

REMARK. The linear functionals $\Gamma_x^{(\lambda)}$ are defined for all $|\lambda| < 1$. For $|\lambda| = 1$, define $\Gamma_x^{(\lambda)}$ on $K(\mathbf{Z}^+, C_0(X))$ by $\langle \Gamma_x^{(\lambda)}, F \rangle = \sum_{n=0}^{N} \lambda^n f_n(x)$, $F = \sum_{n=0}^{N} U^n f_n$. Now $(*)$ implies, for every $0 < r < 1$,

$$\left| \sum_{n=0}^{N} \lambda^n r^n f_n(x) \right| \leq \|F\|.$$

Taking the limit as $r \uparrow 1$, we obtain $|\langle \Gamma_x^{(\lambda)}, F \rangle| \leq \|F\|$. As $K(\mathbf{Z}^+, C(X)) \subset \mathfrak{A}$ is dense, $\Gamma_x^{(\lambda)}$ has a unique extension to a (norm one) linear functional on $\mathfrak{A}$, again denoted $\Gamma_x^{(\lambda)}$. Of course this does not imply that for $F \in \mathfrak{A}$ arbitrary, with Fourier series $\sum_{n=0}^{\infty} U^n f_n$, that $\sum_{n=0}^{\infty} \lambda^n f_n(x)$ converges; but it is summable by a summability kernel.

V.4. Let (X, ϕ) be a compact minimal dynamical system, and $\mathcal{R}^{(\lambda)}$ the closed, α-invariant right ideal $\mathcal{R}^{(\lambda)} = \{\Gamma_x^{(\lambda)} : x \in X\} \perp$.

THEOREM. Let $0 < |\lambda| \leq 1$. The action of $\mathfrak{A} = \mathbf{Z}^+ \times_\phi C(X)$ on $C(X)$, which is determined by $[g]f = [gf]$, $[g]U = \lambda[g \circ \phi]$, $g, f \in C(X)$, is strictly transitive. In fact, the action of the subalgebra $K(\mathbf{Z}^+, C(X))$ is strictly transitive on $C(X)$.

PROOF. In V.3 it was shown that $\mathfrak{A}$ acts on $C(X)$ as described above, and $\|[g]F\| \leq \|g\| \|F\|$, $g \in C(X)$, $F \in \mathfrak{A}$. To show strict transitivity, let $0 \neq g$, $h \in C(X)$ be given. By the minimality of (X, ϕ) there exists $N \in \mathbf{Z}^+$ such that $\bigcup_{j=0}^{N} \phi^{-j}(\mathcal{O}) = X$, where $\mathcal{O} = \{x : g(x) \neq 0\}$ (Lemma II.9). Thus,

$$|g|^2 + |\lambda|^2 |g \circ \phi|^2 + \cdots + |\lambda|^{2N} |g \circ \phi^N|^2$$

is strictly positive on X. Note that

$$[g]U^k(\overline{\lambda}^k g \circ \phi^k) = [|\lambda|^{2k} |g \circ \phi^k|^2].$$

Let

$$f = \left(\sum_{k=0}^{N} |\lambda|^{2k} |g \circ \phi^k|^2 \right)^{-1}.$$

Then $f \in C(X)$, and $[h] = [g](\sum_{k=0}^{N} U^k \overline{\lambda}^k g \circ \phi^k) fh$.

COROLLARY. The ideals $\mathcal{R}^{(\lambda)}$ $(0 < |\lambda| \leq 1)$ are maximal right ideals.

PROOF. Since $\mathfrak{A}/\mathcal{R}^{(\lambda)}$ and $C(X)$ with the action $[g]f = [gf]$, $[g]U = \lambda[g \circ \phi]$ $(f, g \in C(X))$ are isomorphic as right $\mathfrak{A}$-modules, the maximality of $\mathcal{R}^{(\lambda)}$ follows from the Theorem and [8, 1.1.3].

It would be interesting to know if the ideals $\mathcal{R}^{(\lambda)}$ $(0 < |\lambda| \leq 1)$ are the only maximal, α-invariant right ideals in $\mathfrak{A}$. In conclusion we will show, under the additional assumption of unique ergodicity, that this is in fact the case.

V.5. The proof of Proposition III.10 can now be given. Let (X, ϕ) be a dynamical system and $Z \subset X$ be a compact minimal invariant subset, and let J be the closed two sided ideal in $\mathbf{Z}^+ \times_\phi C_0(X)$, $J \sim (Z)$ (Notation III.6). Let $j: \mathbf{Z}^+ \times_\phi C_0(X) \to \mathbf{Z}^+ \times_\phi C(Z)$ be the canonical map (see [11, II.12]), which is defined on the subalgebra $K(\mathbf{Z}^+, C_0(X))$ by $j(\sum_{n=0}^N U^n f_n) = \sum_{n=0}^N U^n f_n'$, where $f_n' = f_n|Z$. The image of j contains $K(\mathbf{Z}^+, C(Z))$, $\ker j = J$, and $\|j(F)\| \leq \|F\|$ for $F \in \mathbf{Z}^+ \times_\phi C_0(X)$. Since by Theorem V.4 there exist faithful right $\mathbf{Z}^+ \times_\phi C(Z)$-modules such that the subalgebra $K(\mathbf{Z}^+, C(Z))$ acts strictly transitively, it follows that J is a (right) primitive ideal in $\mathbf{Z}^+ \times_\phi (X)$.

V.6. Assume that the dynamical system (X, ϕ) is transitive and X is compact. Let $\mathfrak{A} = \mathbf{Z}^+ \times_\phi C(X)$ and $\mathfrak{A}_0$ be the subalgebra generated by $\{1, U\}$. $\mathfrak{A}_0$ is a commutative Banach algebra; it is the subalgebra of all $F \in \mathfrak{A}$ whose Fourier coefficients are all constant functions.

PROPOSITION. $\mathfrak{A}_0 \subset \mathfrak{A}$ *is a maximal abelian subalgebra.*

PROOF. Let $F \in \mathfrak{A}$ be such that $FU = UF$. If $F = \sum_{n=0}^\infty U^n f_n$, then $f_n \circ \phi = f_n$. This means that f_n is constant on each orbit $\mathcal{O}^+(x)$. Since (X, ϕ) is transitive, there exists $x_0 \in X$ with $\mathcal{O}^+(\overline{x_0}) = X$. Thus f_n is constant, $n = 0, 1, \ldots$, and $F \in \mathfrak{A}_0$.

V.7. The disk algebra is the commutative Banach algebra of continuous functions on the closed unit disk $\{z \in \mathbf{C}: |z| \leq 1\}$ which are holomorphic in the interior. For the next result, we need only assume X is compact. As before, $\mathfrak{A} = \mathbf{Z}^+ \times_\phi C(X)$ and $\mathfrak{A}_0$ is the closed subalgebra generated by $\{1, U\}$.

PROPOSITION. *The mapping which takes an element $F \in \mathfrak{A}_0$ with Fourier series $\sum_{n=0}^\infty U^n c_n$ to the analytic function with power series $\sum_{n=0}^\infty c_n z^n$ is an isometric Banach algebra isomorphism of $\mathfrak{A}_0$ with the disk algebra.*

PROOF. Since for any $F \in \mathfrak{A}_0$, the norms $\|\Pi_x(F)\|$ $(x \in X)$ all coincide, $\mathfrak{A}_0$ is naturally isomorphic with $\mathbf{Z}^+ \times_\phi C(X_0)$ where X_0 consists of a single point and ϕ_0 is the identity map. Now $\mathbf{Z}^+ \times_{\phi_0} C(X_0)$ is the disk algebra [11, Theorem III.2 (case $k_0 = 1$)], and apply [11, Corollary II.10].

Note. It would not be hard to give a direct proof of the proposition.

V.8. Assume that (X, ϕ) is a compact minimal dynamical system which is uniquely ergodic. Let ω be the unique invariant probability measure on X. For $f \in C(X)$, the averages $[f + f \circ \phi + \cdots + f \circ \phi^n]/(n+1)$ converge uniformly to a constant, which is $\int_X f \, d\omega$ [6, Theorem 2.3.5]. Define $Q_n: K(\mathbf{Z}^+, C(X)) \to K(\mathbf{Z}^+, C(X))$ by $Q_n(F) = \sum_{k=0}^n \alpha^k(F)/(n+1)$. Since α is a contraction, so is Q_n. If $F = \sum_{j=0}^N U^j f_j$, $Q_n(F) = \sum_{j=0}^N U^j Q_n(f_j)$, and since $\{Q_n(f_j)\}_{n=0}^\infty$ converges uniformly, $0 \leq j \leq N$, $\{Q_n(F)\}_{n=0}^\infty$ converges in the norm of $\mathfrak{A} = \mathbf{Z}^+ \times_\phi C(X)$. Denote the limit of the sequence $\{Q_n(F)\}_{n=0}^\infty$ by $F^\sharp$. Now let $F \in \mathfrak{A}$ and $\varepsilon > 0$ be given. If $G \in K(\mathbf{Z}^+, C(X))$ is such that $\|F - G\| < \varepsilon$, and $M \in \mathbf{Z}^+$ such that $n \geq M$ implies $\|Q_n(G) - G^\sharp\| < \varepsilon$, then

$$\|Q_n(F) - G^\sharp\| \leq \|Q_n(F - G)\| + \|Q_n(G) - G^\sharp\| < 2\varepsilon,$$

and so $\|Q_n(F) - Q_m(F)\| < 4\varepsilon$, $n, m \geq M$. As $\alpha(G^\sharp) = G^\sharp$, it follows

$$\|\alpha^k(Q_n)(F) - Q_n(F)\| < 4\varepsilon, \qquad k = 0, 1, \dots .$$

Thus the sequence $\{Q_n(F)\}_{n=0}^\infty$ is Cauchy in $\mathfrak{A}$, and its limit is α-invariant, and consequently belongs to the subalgebra $\mathfrak{A}_0$ generated by $\{1, U\}$. Denote the limit by $F^\sharp$.

PROPOSITION. *The operator* $\sharp\colon \mathfrak{A} \to \mathfrak{A}_0$ *has the following properties*:
(i) $\|F^\sharp\| \leq \|F\|$;
(ii) $\sharp$ *is a linear projection*;
(iii) $F^\sharp$ *belongs to the closed convex hull of* $\{\alpha^n(F)\}_{n=0}^\infty$;
(iv) *If* $F \in \mathfrak{A}$ *and* $G \in \mathfrak{A}_0$, *then* $(FG)^\sharp = F^\sharp G$, *and* $(GF)^\sharp = GF^\sharp$.

PROOF. (i) and (iii) have already been verified, while (ii) and (iv) are straightforward.

V.9. Let (X, ϕ) be as in V.6. If $\mathcal{R} \subset \mathfrak{A}$ is an α-invariant right ideal, set $\mathcal{R}^\sharp = \{F^\sharp \colon F \in \mathcal{R}\}$. If $I \subset \mathfrak{A}_0$ is an ideal, set $\tilde{I} = \{F \in \mathfrak{A} \colon (\alpha^n(F)G)^\sharp \in I, \text{ for all } G \in \mathfrak{A}, n = 0, 1, \dots \}$.

A. PROPOSITION. (i) *If* $\mathcal{R}$ *is an* α-*invariant closed right ideal in* $\mathfrak{A}$, $\mathcal{R}^\sharp$ *is a closed ideal in* $\mathfrak{A}_0$, *and* $\mathcal{R}^\sharp = \mathcal{R} \cap \mathfrak{A}_0$.

(ii) *If* I *is a closed ideal in* $\mathfrak{A}_0$, $\tilde{I}$ *is an* α-*invariant right ideal in* $\mathfrak{A}$, *and* $\tilde{I}$ *is the unique maximal element of the set* $\{J \colon J \text{ an } \alpha\text{-invariant right ideal of } \mathfrak{A}, J^\sharp \subset I\}$.

PROOF. (i) By Proposition V.6(iv), $\mathcal{R}^\sharp$ is an ideal in $\mathfrak{A}_0$. If $F \in \mathcal{R}$, $F^\sharp$ is in the closed convex hull of $\{\alpha^n(F)\}_{n=0}^\infty \subset \mathcal{R}$, so $F^\sharp \in \mathcal{R} \cap \mathfrak{A}_0$. Thus $\mathcal{R}^\sharp \subset \mathcal{R} \cap \mathfrak{A}_0$. Since $\sharp$ is a projection, $\mathcal{R} \cap \mathfrak{A}_0 = (\mathcal{R} \cap \mathfrak{A}_0)^\sharp \subset \mathcal{R}^\sharp$.

(ii) By its definition, $\tilde{I}$ is an α-invariant right ideal. Let $\mathcal{R}$ be any α-invariant right ideal such that $\mathcal{R}^\sharp \subset J$. If $F \in \mathcal{R}$, $\alpha^n(F)G \in \mathcal{R}$ for all $n \geq 0$, $G \in \mathfrak{A}$. So $(\alpha^n(F)G)^\sharp \in \mathcal{R}^\sharp \subset I$. It follows that $\mathcal{R} \subset \tilde{I}$.

B. PROPOSITION. *If* $\mathcal{M}_0 \subset \mathfrak{A}_0$ *is a maximal ideal, then* $\tilde{\mathcal{M}}_0 \subset \mathfrak{A}$ *is a maximal* α-*invariant right ideal*.

PROOF. Observe that $(\tilde{\mathcal{M}}_0)^\sharp = \mathcal{M}_0$, and hence $\tilde{\mathcal{M}}_0$ is proper. Let $\mathcal{R} \subset \tilde{\mathcal{M}}_0$ be an α-invariant right ideal. Now $\mathcal{R}^\sharp$ is proper, for if $\mathcal{R}^\sharp = \mathfrak{A}_0$, then $\mathcal{R} \cap \mathfrak{A}_0 = \mathfrak{A}_0$, so $1 \in \mathcal{R}$, a contradiction. By the maximality of $\mathcal{M}_0$, $\mathcal{R}^\sharp = (\tilde{\mathcal{M}}_0)^\sharp = \mathcal{M}_0$. By Proposition A(ii), $\mathcal{M}_0 = \mathcal{R}$, and hence $\mathcal{M}_0$ is a maximal α-invariant right ideal.

C. COROLLARY. *If* $\mathcal{R} \subset \mathfrak{A}$ *is any* α-*invariant right ideal*, $\mathcal{R} \subset (\mathcal{R}^\sharp)^\sim$. *Hence if* $\mathcal{R}$ *is a maximal* α-*invariant right ideal*, $\mathcal{R} = (\mathcal{R}^\sharp)^\sim$.

D. COROLLARY. *If* $\mathcal{R} \subset \mathfrak{A}$ *is a maximal* α-*invariant right ideal, then* $\mathcal{R}^\sharp \subset \mathfrak{A}_0$ *is a maximal ideal*.

PROOF. Let $\mathcal{M}_0 \supset \mathcal{R}^\sharp$ be a maximal ideal in $\mathfrak{A}_0$. By Proposition B, $\tilde{\mathcal{M}}_0$ is a maximal α-invariant right ideal, and since $\tilde{\mathcal{M}}_0 \supset (\mathcal{R}^\sharp)^\sim = \mathcal{R}$ (Corollary C), it follows that $\tilde{\mathcal{M}}_0 = \mathcal{R}$. Therefore $\mathcal{M}_0 = (\tilde{\mathcal{M}}_0)^\sharp = \mathcal{R}^\sharp$ is maximal.

E. SUMMARY. There is a one-to-one correspondence between the set $\underline{R_\alpha}$ of maximal α-invariant right ideals of $\mathfrak{A}$, and the set $\underline{M}$ of maximal ideals of $\mathfrak{A}_0$, given by

$$\mathcal{R} \in \underline{R_\alpha} \mapsto \mathcal{R}^\sharp \in \underline{M} \quad \text{and} \quad \mathcal{M} \in \underline{M} \mapsto \tilde{\mathcal{M}} \in \underline{R_\alpha}.$$

V.10. Assume (X, ϕ) is compact, minimal, free and uniquely ergodic.

LEMMA. *Each $\gamma \in \mathfrak{A}_0^*$ extends to a unique $+\alpha$-invariant element Γ of $\mathfrak{A}^*$.*

PROOF. Let $\Gamma, \Gamma' \in \mathfrak{A}^*$ be $+\alpha$-invariant, and suppose $\langle \Gamma, G \rangle = \langle \Gamma', G \rangle$ for all $G \in \mathfrak{A}_0$. If $F \in \mathfrak{A}$, $\langle \Gamma, F \rangle = \langle \Gamma, Q_n(F) \rangle$ by invariance, hence $\langle \Gamma, F \rangle = \langle \Gamma, F^\sharp \rangle$ (cf. V.6). Since the same holds for Γ', we have $\langle \Gamma, F \rangle = \langle \Gamma, F^\sharp \rangle = \langle \Gamma', F^\sharp \rangle = \langle \Gamma, F^\sharp \rangle = \langle \Gamma', F \rangle$. Thus, the restriction map from the subspace of $+\alpha$-invariant elements of $\mathfrak{A}^*$ to $\mathfrak{A}_0^*$ is injective. To see that it is surjective, let $\gamma \in \mathfrak{A}_0^*$; by the Hahn-Banach Theorem there is a $\Gamma_1 \in \mathfrak{A}^*$ such that $\Gamma_1|\mathfrak{A}_0 = \gamma$. Define $\Gamma \in \mathfrak{A}^*$ by $\langle \Gamma, F \rangle = \langle \Gamma_1, F^\sharp \rangle$. As $\Gamma = \Gamma_1 = \gamma$ on $\mathfrak{A}_0$, and Γ is $+\alpha$-invariant, we are done.

By V.7, $\mathfrak{A}_0$ can be identified with the disk algebra, and so if $G \in \mathfrak{A}_0$ we will let $G(\lambda)$ denote the value of the analytic function corresponding to G at the point λ in the closed unit disk. By the Lemma, there is a unique $+\alpha$-invariant linear functional $\Gamma^{(\lambda)} \in \mathfrak{A}^*$ such that

$$\langle \Gamma^{(\lambda)}, F \rangle = F^\sharp(\lambda), \qquad 0 \le |\lambda| \le 1,$$

$F \in \mathfrak{A}$. Let $\Gamma_x^{(\lambda)}$ be as in V.2, V.3, and $\mathcal{R}^{(\lambda)} = \{\Gamma_x^{(\lambda)} : x \in X\}_\perp$. On the dense subalgebra $K(\mathbf{Z}^+, C_0(X)) \subset \mathfrak{A}_0$, $\langle \Gamma_x^{(\lambda)}, F \rangle = F^\sharp(\lambda)$, $x \in X$. As observed in V.2, $\mathcal{R}^{(\lambda)\perp} = \{\Gamma \in \mathfrak{A}^* : \Gamma = \int_X \Gamma_x^{(\lambda)} d\mu(x)$ for some $\mu \in \mathcal{M}(X)\}$; since $\mathcal{M}(X)$ contains a unique invariant probability measure ω and $\int_X \langle \Gamma_x^{(\lambda)}, F \rangle d\omega(x) = F^\sharp(\lambda)$, it follows that $\Gamma^{(\lambda)} = \int_X \Gamma_x^{(\lambda)} d\omega(x)$. Observe that the smallest closed $C(X)$-submodule $\mathcal{S}$ of $\mathfrak{A}^*$ which contains $\Gamma^{(\lambda)}$ is exactly $\mathcal{R}^{(\lambda)\perp}$: let $\{f_i\}_{i=1}^\infty \subset C(X)$ satisfy $f_i \ge 0$, $\int f_i d\omega = 1$, all i, and $\mathrm{supp}(f_i)$ runs through a neighborhood base at a given $x_0 \in X$. Since the map $x \mapsto \Gamma_x^{(\lambda)}$ is continuous from X to $\mathfrak{A}^*$ in the $\sigma(\mathfrak{A}^*, \mathfrak{A})$ topology, $f_i \Gamma^{(\lambda)} \to \Gamma_{x_0}^{(\lambda)}$. Thus $\Gamma_{x_0}^{(\lambda)} \in \mathcal{S}$ for each $x_0 \in X$, and so $\mathcal{S} = \mathcal{R}^{(\lambda)\perp}$.

THEOREM. *Every maximal α-invariant right ideal of $\mathfrak{A}$ is of the form $\mathcal{R}^{(\lambda)}$ for some complex number λ, $|\lambda| \le 1$. If $\lambda \ne 0$, the ideal $\mathcal{R}^{(\lambda)}$ is maximal among all right ideals of $\mathfrak{A}$.*

PROOF. The second statement is Corollary V.4. Let $\mathcal{R}$ be a maximal α-invariant right ideal. By V.8, $\mathcal{R}^\sharp$ is a maximal ideal in the disk algebra, so $\mathcal{R}^\sharp$ is the kernel of the evaluation map at some λ in the closed unit disk [9, p. 87, Corollary]. As the functional $\gamma_\lambda \colon G \in \mathfrak{A}_0 \to G(\lambda)$ belongs to $\mathcal{R}^{\sharp\perp}$, it follows that $\Gamma^{(\lambda)} \subset \mathcal{R}^\perp$. Since $\mathcal{R}^{(\lambda)\perp}$ is the smallest closed $C(X)$-submodule of $\mathfrak{A}^*$ containing $\Gamma^{(\lambda)}$, and $\mathcal{R}^{(\lambda)\perp}$ is $+\alpha$-invariant, the minimality $\mathcal{R}^\perp$ forces $\mathcal{R}^\perp = \mathcal{R}^{(\lambda)\perp}$, or $\mathcal{R} = \mathcal{R}^{(\lambda)}$.

It remains only to show that $\mathcal{R}^{(0)}$ is not maximal among all right ideals (although it is a maximal element in the collection of all α-invariant right ideals). Let $F \in \mathfrak{A}$, $F = \sum_{n=0}^\infty U^n f_n$; now $\langle \Gamma_x^{(0)}, F \rangle = f_0(x)$, and $F \in \mathcal{R}^{(0)}$ iff $F \in \ker \Gamma_x^{(0)}$ for all $x \in X$, iff $f_0 = 0$. Thus $\mathcal{R}^{(0)} = U\mathfrak{A}$. Since, for each $x \in X$, $\ker \Gamma_x^{(0)}$ is a maximal two sided ideal (of codimension one) which properly contains $\mathcal{R}^{(0)}$, $\mathcal{R}^{(0)}$ is not a maximal right ideal.

V.11. Suppose $\mathfrak{A}$ is a Banach algebra with endomorphism α; let us say that α *acts* on a right $\mathfrak{A}$-module M if M is isomorphic with $\mathfrak{A}/\mathcal{R}$, where $\mathcal{R}$ is an α-invariant right ideal.

Suppose (X, ϕ) is a dynamical system and $Z \subset X$ is a compact minimal invariant subset. If $\phi^{-1}(Z) = Z$ and $J \sim (Z)$, then by Corollary IV.1, $\mathbf{Z}^+ \times_\phi C_0(X)/J$ is isomorphic with $\mathbf{Z}^+ \times_\phi C(Z)$, so there is a bijective correspondence between

the maximal invariant right ideals of $\mathbf{Z}^+ \times_\phi C_0(X)$ which contain J and those of $\mathbf{Z}^+ \times_\phi C(Z)$. If, in addition, (X, ϕ) is free and (Z, ϕ) is uniquely ergodic, Theorem V.9 gives a complete classification of the irreducible right $\mathbf{Z}^+ \times_\phi C_0(X)$-modules on which α acts, with kernel J. Unfortunately, these methods do not yield a classification in the case of an arbitrary right primitive ideal J.

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