THE IDEAL STRUCTURE OF IDEMPOTENT-GENERATED TRANSFORMATION SEMIGROUPS

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1. Introduction

Let X be a set and $\mathcal{T}_X$ the semigroup (under composition) of all total transformations from X into itself. In ([6], Theorem 3) Howie characterised those elements of $\mathcal{T}_X$ that can be written as a product of idempotents in $\mathcal{T}_X$ different from the identity. We gather from review articles that his work was later extended by Evseev and Podran [3, 4] (and independently for finite X by Sullivan [15]) to the semigroup $\mathcal{P}_X$ of all partial transformations of X into itself. Howie's result was generalized in a different direction by Kim [8], and it has also been considered in both a topological and a totally ordered setting (see [11] and [14] for brief summaries of this latter work). In addition, Magill [10] investigated the corresponding idea for endomorphisms of a Boolean ring, while J. A. Erdos [2] resolved the analogous problem for linear transformations of a finite-dimensional vector space.

In this paper we return to Howie's original article and first determine the ideals of the semigroup $\mathscr{E}_X$ generated by the idempotents in $\mathcal{T}_X$ different from the identity. Next we characterise Green's relations on $\mathscr{E}_X$ and use our result to produce a new class of 0-bisimple regular semigroups. Finally we consider the extension of our work to the partial case.

2. Ideal structure

Throughout this paper we shall in the main use the notation of [1] but occasionally abbreviate it for the purpose of convenience. In particular, if $\alpha \in \mathcal{T}_X$ we write $r(\alpha) = |X\alpha|$ and can call this the *rank* of α.

Howie showed in ([6], Theorem 1) that if X is finite then $\mathscr{E}_X = \{\alpha \in \mathcal{T}_X : r(\alpha) < |X|\}$. Since the ideals of this semigroup are well-known (cf. [1], Vol. 2, Theorem 10.59) we assume for the remainder of this section that $|X| = \kappa \geq \aleph_0$. In [6] Howie described the elements of $\mathscr{E}_X$ in this case via three concepts: if $\alpha \in \mathcal{T}_X$ we put

$$D(\alpha) = X \setminus X\alpha \qquad \text{and } d(\alpha) = |D(\alpha)|$$
$$S(\alpha) = \{x \in X : x\alpha \neq x\} \qquad \text{and } s(\alpha) = |S(\alpha)|$$
$$C(\alpha) = \cup\{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\} \quad \text{and } c(\alpha) = |C(\alpha)|$$

319

and we refer to the cardinals $d(\alpha)$, $s(\alpha)$ and $c(\alpha)$ as the *defect*, *shift* and *collapse* of α, respectively. Howie proved in ([6], Theorem 3) that $\mathscr{E}_X$ is the disjoint union of two semigroups:

$$V = \{\alpha \in \mathscr{T}_X : s(\alpha) < \aleph_0 \text{ and } d(\alpha) \neq 0\}$$

$$H = \{\alpha \in \mathscr{T}_X : s(\alpha) = d(\alpha) = c(\alpha) \geqq \aleph_0\}.$$

That V is in fact a semigroup follows from ([6], Lemmas 2 and 5), and that H is a semigroup follows from ([6], Lemmas 6 and 7). Since we will need to refer to the last of these Lemmas quite often, we re-state it here for convenience (and note in passing that the original proof contained a significant error that was neatly corrected in [7]).

Lemma 1.   *If* $\alpha \in H$, $\beta \in \mathscr{T}_X$ *and* $s(\beta) < s(\alpha)$ *then both* $\alpha\beta$ *and* $\beta\alpha$ *have shift, defect and collapse equal to that of* α.

In this section we aim to describe the ideals I of $\mathscr{E}_X$: note that $I = (I \cap V) \cup (I \cap H)$ and if $I \cap V$ and $I \cap H$ are non-empty they are ideals of V and H respectively; hence our first task will be to determine the ideals of V and of H. It seems that Vorobev [18] has described the ideals of a semigroup closely allied to V: namely, the set of all $\alpha \in \mathscr{T}_X$ with $s(\alpha) < \aleph_0$ ([17] may also be relevant: it is listed in [9] but has not been reviewed and was unavailable to us). For completeness we provide a proof of the following.

Theorem 1.   *Let* $n \in \mathbb{Z}^+$ *and* $V_n = \{\alpha \in V : d(\alpha) \geqq n\}$. *Then* V_n *is an ideal of* V *and every ideal of* V *equals some* V_n. *Moreover, each* V_n *is principal and generated by an element with defect* n.

Proof.   Let $\alpha \in V_n$ and $\beta \in V$. Since $D(\alpha) \subseteq D(\beta\alpha)$, we have $\beta\alpha \in V_n$. Although after some reflection it is intuitively clear that also $\alpha\beta \in V_n$, a convincing argument is somewhat longer. Firstly we assert that

$$[S(\beta) \cup D(\alpha)] \cap X\alpha\beta \subseteq [S(\beta) \backslash D(\alpha)]\beta.$$

For, if $x \in S(\beta) \cup D(\alpha)$ and $x = y\alpha\beta$ for some $y \in X$ then $y\alpha \in S(\beta)$: otherwise, $x = (y\alpha)\beta = y\alpha \notin S(\beta)$ implies $x = y\alpha \in D(\alpha)$, a contradiction. Hence $y\alpha \in S(\beta) \backslash D(\alpha)$, and our assertion follows. Now we put $Y = S(\beta) \cup D(\alpha)$ and note that

$$|Y \cap X\alpha\beta| \leqq |S(\beta) \backslash D(\alpha)|$$

where

$$|Y| = |Y \cap X\alpha\beta| + |Y \cap D(\alpha\beta)|$$

Hence we have

$$d(\alpha\beta) \geqq |Y \cap D(\alpha\beta)| = |Y| - |Y \cap X\alpha\beta| \geqq |Y| - |S(\beta) \backslash D(\alpha)| = d(\alpha) \geqq n.$$

For the converse we assume I is an ideal of V, choose $\alpha \in I$ with minimal defect, and put $d(\alpha) = n$. Then $I \subseteq V_n$. Let $\beta \in V_n$ and put $Z = E(\alpha) \cup E(\beta)$, where $E(\gamma)$ denotes $S(\gamma) \cup S(\gamma)\gamma$ for each $\gamma \in V$. Note that both α and β map Z into itself, and fix $X \backslash Z$ pointwise. Hence $D(\alpha) \cup D(\beta) \subseteq Z$. Put $\alpha_1 = \alpha | Z$ and $\beta_1 = \beta | Z$, and note that since α_1 fixes $E(\beta) \backslash E(\alpha)$, we have $d(\alpha_1) = d(\alpha)$. Likewise $d(\beta_1) = d(\beta)$ and, since

$$r(\alpha_1) + d(\alpha_1) = r(\beta_1) + d(\beta_1) = |Z| < \aleph_0$$

where $d(\beta_1) \geqq d(\alpha_1) \neq 0$, we conclude that $r(\beta_1) \leqq r(\alpha_1) < |Z|$. At this point we can invoke the well-known characterisation of Green's $\mathcal{J}$-relation on $\mathcal{T}_Z$ (cf. [1], Vol. 1, pp. 52–53) to assert that $\beta_1 = \lambda_1 \alpha_1 \mu_1$ for some $\lambda_1, \mu_1 \in \mathcal{T}_Z$. In fact, since $r(\alpha_1) < |Z|$, we can ensure that $d(\lambda_1)$ and $d(\mu_1)$ are both non-zero. So, by extending λ_1 and μ_1 to the whole of X in an obvious way, we obtain $\beta = \lambda \alpha \mu$ for some $\lambda, \mu \in V$; that is, $\beta \in I$ and we have shown $I = V_n$. That V_n is a principal ideal is clear from the foregoing discussion.

According to ([1], Vol. 2, Theorem 10.59) the ideals of $\mathcal{T}_X$ take the form I_ξ where for $1 < \xi \leqq \kappa'$,

$$I_\xi = \{\alpha \in \mathcal{T}_X : r(\alpha) < \xi\}.$$

If ξ is finite then the Rees quotient semigroup $I_{\xi+1}/I_\xi$ is completely 0-simple ([1], Vol. 2, Lemma 10.54). We assert that this is also true for the semigroups V_n/V_{n+1} where $1 \leqq n < \aleph_0 \leqq \kappa$. To show this we again use the set $E(\alpha) = S(\alpha) \cup S(\alpha)\alpha$ (called by Symons [16] the *essential domain* of $\alpha \in \mathcal{T}_X$); namely, if $\alpha, \beta \in V$ with $d(\alpha) = d(\beta) = n$, we put $Y = E(\alpha) \cup E(\beta)$ and observe that $\alpha_1 = \alpha | Y$ and $\beta_1 = \beta | Y$ are elements of $\mathcal{T}_Y$ with $r(\alpha_1) = r(\beta_1) < |Y|$. We can now follow the proof of ([1], Vol. 2, Lemma 10.54) to eventually conclude that V_n/V_{n+1} is 0-simple. Clearly, $V_n \backslash V_{n+1}$ contains idempotents. To show each of these is primitive, we again put $Y = E(\alpha) \cup E(\beta)$ where α, β are idempotents in V with $\alpha\beta = \beta\alpha = \alpha$ and $d(\alpha) = d(\beta)$. Then $\alpha_1\beta_1 = \beta_1\alpha_1 = \alpha_1$ for idempotents $\alpha_1, \beta_1 \in \mathcal{T}_Y$ with $r(\alpha_1) = r(\beta_1) < \aleph_0$. An argument similar to that in the reference already cited eventually leads us to $\alpha = \beta$. We have therefore shown

Theorem 2. *If* $1 \leqq n < \aleph_0$ *then* V_n/V_{n+1} *is a completely* 0-*simple semigroup.*

We assert that V_n/V_{n+1} is not isomorphic to any I_{m+1}/I_m with m finite ... simply because the cardinal of the first is κ while that of the second is 2^κ. To see this, recall that the set $\mathcal{F}$ of all finite subsets of X has cardinal κ ([12], Theorem 22.17). If $\mathcal{F}$ is any finite subset of X such that $|F| = n + 1 \geqq 2$ then V_n/V_{n+1} contains an idempotent that is constant on F and fixes $X \backslash F$; hence, if $|V_n/V_{n+1}| = \varepsilon$ then $\varepsilon \geqq \kappa$. Now, to each $\alpha \in V_n \backslash V_{n+1}$ we can associate in a one-to-one fashion the element $\alpha | E(\alpha)$ of $\mathcal{T}_{E(\alpha)}$. Hence if m_F denotes the (finite) cardinal of $\mathcal{T}_F$ for each $F \in \mathcal{F}$ then $\varepsilon \leqq \Sigma m_F = \kappa$. On the other hand, if $F \in \mathcal{F}$ and $|F| = m$ then I_{m+1}/I_m contains all maps from X onto F and there are 2^κ such maps (since there are $(2^\kappa)^m = 2^\kappa$ ways of partitioning X into a family of m subsets of X: cf. [12], Exercise 22.20). However the cardinal of $\mathcal{T}_X$ is 2^κ and so we have $|I_{m+1}/I_m| = 2^\kappa$.

It will become apparent after we have determined Green's $\mathcal{H}$-relation on V in Section 3 that V_n/V_{n+1} and I_{m+1}/I_m are non-isomorphic for a less trivial reason: namely, the

non-zero group $\mathcal{H}$-classes of I_{m+1}/I_m are all isomorphic to the symmetric group $\mathcal{G}_m$ on m letters (as observed in [1], Vol. 2, p. 226) whereas those of V_n/V_{n+1} are all isomorphic to the group $\mathcal{G}(\kappa, \aleph_0)$ of all permutations of κ letters with finite shift (as can be readily checked by mimicking the proof of Theorem 2.10(ii) in [1], Vol. 1).

We now turn to the problem of describing the ideal of H: it happens that, just as in V, they form a chain, even though two cardinals are required for their description. To show this, we let

$$H(\delta, \xi) = \{\alpha \in H : d(\alpha) \geq \delta \quad \text{and} \quad r(\alpha) < \xi\}$$

where $\aleph_0 \leq \delta \leq \kappa$ and $2 \leq \xi \leq \kappa'$.

Lemma 2. *Each $H(\delta, \xi)$ is an ideal of H and the set of all such distinct ideals forms a chain:*

$$H(\kappa, 2) \subseteq \cdots \subseteq H(\kappa, \xi) \subseteq \cdots \subseteq H(\kappa, \kappa') \subseteq \cdots \subseteq H(\aleph_1, \kappa') \subseteq H(\aleph_0, \kappa') \qquad (*)$$

Proof. Let $\alpha \in H(\delta, \xi)$ and $\beta \in H$, and suppose $s(\alpha) = d(\alpha) = c(\alpha) = a$ and $s(\beta) = d(\beta) = c(\beta) = b$. If $b \leq a$ then ([6], Lemmas 6 and 7) imply that both $\alpha\beta$ and $\beta\alpha$ have defect equal to $a \ (\geq \delta)$. If $a > b$ then Lemma 1 above implies that both $\alpha\beta$ and $\beta\alpha$ have defect equal to $b > a \geq \delta$. Since $r(\alpha\beta) \leq \min\{r(\alpha), r(\beta)\}$ it therefore follows that $\alpha\beta, \beta\alpha \in H(\delta, \xi)$ and $H(\delta, \xi)$ is an ideal of H.

Now consider an arbitrary $H(\delta, \xi)$. If $\delta = \kappa$ (and $2 \leq \xi \leq \kappa'$) we have an ideal in the first portion of the above chain, and if $\xi = \kappa'$ (and $\aleph_0 \leq \delta \leq \kappa$) we are in the second portion of the chain. On the other hand, since $|X| = \kappa \geq \aleph_0$ and $X = X\alpha \cup (X \setminus X\alpha)$ for each $\alpha \in H$, we must have $d(\alpha) = \kappa$ if $r(\alpha) < \xi \leq \kappa$; that is if $\delta < \kappa$, $\xi \leq \kappa$ and $\alpha \in H(\delta, \xi)$ then $\alpha \in H(\kappa, \xi)$. Since $H(\kappa, \xi) \subseteq H(\delta, \xi)$, we deduce that $H(\delta, \xi) = H(\kappa, \xi)$ when $\delta < \kappa$ and $\xi \leq \kappa$.

Following ([1], Vol. 2, p. 241), for each $\alpha \in \mathcal{T}_X$, we write

$$\alpha = \begin{pmatrix} C_m \\ x_m \end{pmatrix}$$

where $X\alpha = \{x_m : m \in M\}$ for some index set M and $C_m = x_m\alpha^{-1}$ for each $m \in M$. To abbreviate notation, we adopt the convention (as in the reference just cited) of writing $\{x_m\}$ for $\{x_m : m \in M\}$, taking the subscript m to signify the index set M within a specific context.

Theorem 3. *Every ideal of H has the form $H(\delta, \xi)$ for some δ, ξ. In particular, the principal ideals of H are $H(\kappa, \eta')$ and $H(\varepsilon, \kappa')$ for some η, ε satisfying $1 \leq \eta \leq \kappa$ and $\aleph_0 \leq \varepsilon \leq \kappa$.*

Proof. Suppose I is an ideal of H. Let δ be the defect of an element of I with minimal defect and let ξ be the least cardinal greater than the ranks of all the elements of I. We assert that $I = H(\delta, \xi)$. Since $I \subseteq H(\delta, \xi)$, we therefore proceed to show that if $\beta \in H(\delta, \xi)$ then there exist $\alpha \in I$ and $\lambda, \mu \in H$ such that $\beta = \lambda\alpha\mu$. So, let $\beta \in H(\delta, \xi)$ and note

that $r(\alpha) < r(\beta) < \xi$ *for all* $\alpha \in I$ contradicts the choice of ξ. Hence there exists $\alpha \in I$ with $r(\beta) \leqq r(\alpha)$. Put

$$\beta = \begin{pmatrix} B_m \\ x_m \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} A_n \\ y_n \end{pmatrix}$$

and choose a cross-section $\{a_n\}$ of $\{A_n\}$. Write $\{a_n\} = \{a_m\} \;\dot\cup\; \{a_s\}$, which is possible since $r(\alpha) \geqq r(\beta)$, and put

$$\lambda = \begin{pmatrix} B_m \\ a_m \end{pmatrix}.$$

Now suppose $d(\alpha) = d(\beta) = \kappa$. Since $\beta \in H$, we have $c(\beta) = \kappa$ and this means $c(\lambda) = s(\lambda) = \kappa$ (using [6], Lemma 3). If $r(\beta) < \kappa$ then $|X \backslash \{a_m\}| = \kappa$ and so $d(\lambda) = \kappa$; that is, $\lambda \in H$. If on the other hand $r(\beta) = \kappa$ then $r(\alpha) = \kappa$ and we can ensure that $|S| = \kappa$; that is, $\{a_s\} \subseteq D(\lambda)$ and again $\lambda \in H$. To define μ, put $C = X \backslash \{y_m\}$ and note that $D(\alpha) \subseteq C$. Hence if we choose $z \in C$ and define

$$\mu = \begin{pmatrix} y_m & C \\ x_m & z \end{pmatrix}$$

then $c(\mu) = s(\mu) = \kappa$. In addition, $D(\beta) \backslash z \subseteq D(\mu)$ and so $d(\mu) = \kappa$. That is, $\mu \in H$ and we have $\beta = \lambda \alpha \mu$ as required.

Before considering the next case, note that if $r(\beta) \leqq r(\alpha) < \kappa$ then $d(\alpha) = d(\beta) = \kappa$ as above. Hence we may suppose $r(\alpha) = \kappa$. Suppose further that $d(\beta) = \kappa$. In this case, with the same notation as before, we immediately have $c(\lambda) = s(\lambda) = \kappa$. Moreover, since $r(\alpha) = \kappa$ we can ensure that $|S| = \kappa$. Then $d(\lambda) = \kappa$ and, since $\{y_s\} \subseteq C$, we also have $c(\mu) = s(\mu) = \kappa$ together with $d(\mu) = \kappa$ (as before).

Hence we may now assume $r(\alpha) = \kappa$ and $\delta \leqq d(\beta) < \kappa$. This implies $r(\beta) = \kappa$. In addition, by choice of δ, there exists $\gamma \in I$ with $d(\gamma) = \delta < \kappa$ (in this case) and so $r(\gamma) = \kappa$; that is, we can assume without loss of generality that $d(\alpha) = \delta \leqq d(\beta) = \varepsilon$, say. Given all this, we now restrict α, β (as in the proof of Theorem 1) to $Y = E(\alpha) \cup E(\beta)$ and obtain $\alpha_1, \beta_1 \in \mathscr{T}_Y$ with the same shift, defect and collapse as α, β respectively. However, $|Y| = \varepsilon$ and so, from our very first case, $\beta_1 = \lambda_1 \alpha_1 \mu_1$ for some $\lambda_1, \mu_1 \in \mathscr{T}_Y$ where both λ_1 and μ_1 have equal infinite shift, defect and collapse. By extending this equation to the whole of X in an obvious way, we have $\beta = \lambda \alpha \mu$ for some $\lambda, \mu \in H$ and so $\beta \in I$.

Finally, observe that we have indirectly proved

$$H(\delta_1, \xi_1) \subseteq H(\delta_2, \xi_2) \text{ if and only if } \delta_1 \geqq \delta_2 \text{ and } \xi_1 \leqq \xi_2,$$

and $H(d(\alpha), r(\alpha)') = H^1 \alpha H^1$ for each $\alpha \in H$.

As noted in ([1], Vol. 2, p. 227, Exercise 3), each $I_{\xi'}/I_\xi$ is a 0-bisimple semigroup for $\aleph_0 \leqq \xi \leqq \kappa$. Hence, since $I_{\xi'}/I_\xi$ contains non-zero idempotents, it is also regular (by [1], Vol. 1, Theorem 2.11). We shall consider the Rees factor semigroups corresponding to the ideals in (*) after we have determined Green's $\mathscr{D}$ and $\mathscr{J}$ relations on H in Section 3.

At this point we simply remark that $H(\kappa, \xi) = I_\xi$ for each ξ satisfying $1 < \xi \leq \kappa$. For. if $\alpha \in I_\xi$ and $1 < \xi \leq \kappa$ then $d(\alpha) = \kappa$ and, since $D(\alpha) \subseteq S(\alpha)$, we also have $s(\alpha) = \kappa$. But α can be written as

$$\alpha = \begin{pmatrix} A_m & a_n & a_p \\ x_m & x_n & a_p \end{pmatrix}$$

where $C(\alpha) = \cup\{A_m : m \in M\}$, $a_n \neq x_n$ for all $n \in N$ and $\{a_p\} = X \setminus [C(\alpha) \cup S(\alpha)]$. Since $|M \cup N| < \kappa$ and $s(\alpha) = \kappa$, we must have $|C(\alpha) \setminus \{x_n\}| = \kappa$ and so $\alpha \in H(\kappa, \xi)$.

On the other hand, $H(\kappa, \kappa')$ is a proper subset of $I_{\kappa'}$. For, if $\xi < \kappa$ we can partition X into sets A and $B_1, B_2, B_3, \ldots$ where $|A| = \xi$ and $|B_i| = \kappa$ for each $i \geq 1$, choose bijections $\theta_i : B_i \rightarrow B_{i+1}$ and $a \in A$, and then define $\alpha \in \mathscr{T}_X$ by

$$x\alpha = a \quad \text{if} \quad x \in A,$$
$$= x\theta_i \quad \text{if} \quad x \in B_i.$$

Then $D(\alpha) = B_1$, $S(\alpha) = \cup\{B_i : i \geq 1\}$ and $C(\alpha) = A$; that is $\alpha \notin H(\kappa, \kappa')$.

Theorem 4. *The ideals of $\mathscr{E}_X$ are precisely the ideals of H together with the sets $V_n \cup H$ for $n \geq 1$.*

Proof. By Lemma 1 and Theorem 3 the ideals of H, as well as the sets $V_n \cup H$, are all ideals of $\mathscr{E}_X$. Conversely, suppose I is an ideal of $\mathscr{E}_X$; the desired result follows immediately from Theorem 1 since if $I \cap H$ is a proper subset of H we can use Lemma 1 to obtain a contradiction.

3. Green's Relations

For convenience we start this section by re-stating certain information from ([1], Vol. 1, pp. 52–53).

Lemma 3. *If $\alpha, \beta \in \mathscr{T}_X$ then*

(a) $\beta = \lambda\alpha$ for some $\lambda \in \mathscr{T}_X$ if and only if $X\beta \subseteq X\alpha$,

(b) $\beta = \alpha\mu$ for some $\mu \in \mathscr{T}_X$ if and only if $\alpha \circ \alpha^{-1} \subseteq \beta \circ \beta^{-1}$,

(c) $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in \mathscr{T}_X$ if and only if $r(\alpha) \leq r(\beta)$,

(d) $\mathscr{D} = \mathscr{J}$.

Comparable statements can be made for $\mathscr{P}_X$ and the symmetric inverse semigroup $\mathscr{I}_X$ on X (see [5] for a brief summary of this idea and its extension to a categorical setting). Our task in this section is to show that statements analogous to (a), (b) and (d) hold for both V and H, but that something different occurs for (c).

Theorem 5. *If $\alpha, \beta \in V$ then*

(a) $\beta = \lambda\alpha$ for some $\lambda \in V$ if and only if $X\beta \subseteq X\alpha$,

(b) $\beta = \alpha\mu$ for some $\mu \in V$ if and only if $\alpha \circ \alpha^{-1} \subseteq \beta \circ \beta^{-1}$,

(c) $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in V$ if and only if $d(\beta) \geqq d(\alpha)$,

(d) $\mathscr{D} = \mathscr{J}$.

Proof. Suppose $X\alpha \subseteq X\beta$ where $X\beta$ is a proper subset of X. Put $Y = E(\alpha) \cup E(\beta)$ and restrict α, β to Y to produce $\alpha_1, \beta_1 \in \mathscr{T}_Y$ where $Y\beta_1$ is a proper subset of Y. Now Y is finite and $Y\beta_1 \subseteq Y\alpha_1$: since if $y \in Y$ and $y\beta = x\alpha$ for $x \in X$ then $x\alpha \neq x$ (and so $x \in Y$) or $x\alpha = x$ (and so $x = y\beta \in Y$). By Lemma 3(a), $\beta_1 = \lambda_1\alpha_1$ for some $\lambda_1 \in \mathscr{T}_Y$ which moreover can be chosen with $d(\lambda_1) \neq 0$ (since $d(\beta_1) \neq 0$). Then $\beta = \lambda\alpha$ where $\lambda \in V$ and part (a) follows; a similar argument establishes part (b).

Suppose $\beta = \lambda\alpha\mu$ where $d(\alpha) = n$. By Theorem 1, $V^1\alpha V^1 = V_n$ and so $d(\beta) \geqq n$. Conversely, suppose $d(\beta) \geqq d(\alpha)$. Then, using our customary notation, $r(\beta_1) \leqq r(\alpha_1)$ and Lemma 3(c) implies $\beta_1 = \lambda_1\alpha_1\mu_1$ for some $\lambda_1, \mu_1 \in \mathscr{T}_Y$. In fact, since Y is finite and $d(\beta_1) \neq 0$, both λ_1 and μ_1 can be chosen with non-zero defect; hence we have $\beta = \lambda\alpha\mu$ with $\lambda, \mu \in V$, as required.

Finally, if $V^1\alpha V^1 = V^1\beta V^1$ then $d(\alpha) = d(\beta) \neq 0$ and so $r(\alpha_1) = r(\beta_1) \neq |Y|$. By Lemma 3(d), this implies $\alpha_1\mathscr{L}\gamma_1\mathscr{R}\beta_1$ for some $\gamma_1 \in \mathscr{T}_Y$ which can in fact be chosen with non-zero defect. Hence, $\alpha\mathscr{L}\gamma\mathscr{R}\beta$ for some $\gamma \in V$ and the proof is complete.

The proof of the corresponding result for H is much longer since our technique of restricting $\alpha, \beta \in \mathscr{T}_X$ to $Y = E(\alpha) \cup E(\beta)$ does not seem to help matters.

Theorem 6. *If $\alpha, \beta \in H$ then*

(a) $\beta = \lambda\alpha$ for some $\lambda \in H$ if and only if $X\beta \subseteq X\alpha$,

(b) $\beta = \alpha\mu$ for some $\mu \in H$ if and only if $\alpha \circ \alpha^{-1} \subseteq \beta \circ \beta^{-1}$,

(c) $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in H$ if and only if $r(\beta) \leqq r(\alpha)$ and $d(\beta) \geqq d(\alpha)$,

(d) $\mathscr{D} = \mathscr{J}$.

Proof. Suppose $X\beta \subseteq X\alpha$ and put $Z = X \setminus [C(\beta) \cup S(\beta)]$. Then $d(\beta) \geqq d(\alpha)$ and $|Z \cap S(\alpha)| \leqq s(\alpha) \leqq s(\beta)$. Now write

$$\beta = \begin{pmatrix} B_p & b_q & b_m & b_n \\ x_p & x_q & b_m & b_n \end{pmatrix}$$

where $C(\beta) = \cup \{B_p : p \in P\}$, $b_q \neq x_q$ for all $q \in Q$, $\{b_m\} = Z \cap S(\alpha)$, and $Z = \{b_m\} \cup \{b_n\}$. Then $b_n\alpha = b_n$ for all $n \in N$ and we can write

$$\alpha = \begin{pmatrix} A_p & A_q & A_m & A_n & A_s \\ x_p & x_q & b_m & b_n & x_s \end{pmatrix}$$

where $b_n \in A_n$ for each $n \in N$ and $\{x_s\} = X\alpha \setminus X\beta$ (if non-empty). We now choose a partial cross-section $\{a_p\} \cup \{a_q\} \cup \{a_m\} \cup \{b_n\}$ of $X/\alpha \circ \alpha^{-1}$ and put

$$\lambda = \begin{pmatrix} B_p & b_q & b_m & b_n \\ a_p & a_q & a_m & b_n \end{pmatrix}.$$

Note that since $|M| \leq s(\beta)$, we have $c(\lambda) = s(\lambda) = s(\beta)$. In addition, we have

$$D(\lambda) \subseteq [C(\beta) \cup S(\beta) \cup \{b_m\}] \setminus [\{a_p\} \cup \{a_q\} \cup \{a_m\}]$$

and so $d(\lambda) \leq s(\beta)$. If $d(\alpha) < d(\beta)$ then $|S| = d(\beta)$ and so, since $\cup\{A_s : s \in S\} \subseteq D(\lambda)$, we have $d(\lambda) = s(\beta)$. Hence, we may suppose $d(\alpha) = d(\beta) = \varepsilon$ say. Now write $C(\alpha) = \cup\{A_t : t \in T\}$ and note that, when selecting the partial cross-section of $X/\alpha \circ \alpha^{-1}$ to form λ, we choose at most one element from each A_t. Consider the worst case and suppose we have in fact chosen some $a_t \in A_t$ for each $t \in T$. Since each A_t contains at least 2 elements, we have $|C(\alpha) \setminus \{a_t\}| = \varepsilon$. However, $C(\alpha) \setminus \{a_t\} \subseteq D(\lambda)$ and so $d(\lambda) = \varepsilon$. That is, $\lambda \in H$ and $\beta = \lambda\alpha$.

For part (b), we now suppose $\alpha \circ \alpha^{-1} \subseteq \beta \circ \beta^{-1}$ and write

$$\beta = \begin{pmatrix} B_m & B_p & x_i & w_r & w_s \\ b_m & b_p & y_i & w_r & w_s \end{pmatrix}$$

$$\alpha = \begin{pmatrix} B_{mn} & B_p & x_i & w_r & w_s \\ c_{mn} & c_p & z_i & v_r & w_s \end{pmatrix}$$

where the sets B_m and B_p contain at least 2 elements, $B_m = \cup\{B_{mn} : n \in N_m\}$ for some index set N_m, and $x_i \neq y_i$, $w_r \neq v_r$ (note that possibly $x_i = z_i$ for some i, and also some B_{mn} may consist of a single element). The above display is possible since each $\beta \circ \beta^{-1}$-class is the union of one or more $\alpha \circ \alpha^{-1}$-classes; the sets N_m are therefore chosen to satisfy $2 \leq |N_m| \leq c(\beta)$. Put $C_m = \{c_{mn} : n \in N_m\}$ and $D = D(\alpha)$, choose $d \in D$ and let

$$\mu = \begin{pmatrix} C_m & c_p & z_i & v_r & w_s & D \\ b_m & b_p & y_i & w_r & w_s & d \end{pmatrix}$$

Then $\beta = \alpha\mu$ and $D(\mu) = D(\beta) \setminus d$: that is, $d(\mu) = d(\beta)$. In addition, $C(\mu) = (\cup C_m) \cup D$ and

$$S(\mu) \subseteq (\cup C_m) \cup \{c_p\} \cup \{z_i\} \cup \{v_r\} \cup D.$$

However, $C(\alpha) \subseteq C(\beta)$ and so $d(\alpha) \leq d(\beta)$; also, $|M \cup P| \leq c(\beta)$, $|I| \leq s(\beta)$, $|R| \leq s(\alpha)$ and $|C_m| \leq c(\beta)$ for each m. Hence both $c(\mu)$ and $s(\mu)$ are at most $d(\beta)$. In fact, it is clear from the very definition of μ that $c(\mu) = s(\mu) = d(\beta)$ when $d(\alpha) = d(\beta)$. So, suppose $d(\alpha) < d(\beta)$. This implies $|\cup B_m| = c(\beta)$ since $|\cup B_p| \leq c(\alpha)$; also $|(\cup B_m) \cap S(\alpha)| < c(\beta)$. Hence, $|(\cup B_m) \cap F(\alpha)| = c(\beta)$ where $F(\alpha) = X \setminus S(\alpha)$. But $\cup B_m = \bigcup_m \bigcup_n B_{mn}$ and so in this case there are $c(\beta)$ elements in $\cup C_m$ that are fixed by α. Consequently, $|\cup C_m| = c(\beta)$ and so $c(\mu) = c(\beta)$. Moreover, each C_m contains at least 2 elements and $|M| \leq c(\beta)$. So, $|\cup(C_m \setminus b_m)| = c(\beta)$ and therefore $s(\mu) \geq c(\beta)$. That is, $\mu \in H$ as required.

To prove parts (c) and (d), we first show that $\alpha \mathcal{D} \beta$ in H if and only if $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\beta)$. Suppose $\alpha \mathcal{L} \gamma \mathcal{R} \beta$ for some $\gamma \in H$. By parts (a) and (b), $X\alpha = X\gamma$ and $\gamma \circ \gamma^{-1} = \beta \circ \beta^{-1}$. Hence, $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\gamma) = c(\gamma) = c(\beta) = d(\beta)$. For the converse we assume $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\beta)$, and consider two cases. If $d(\alpha) = \kappa$ we choose any $\gamma \in \mathcal{T}_X$ with $X\alpha = X\gamma$ and $\gamma \circ \gamma^{-1} = \beta \circ \beta^{-1}$ (such γ's exist since $r(\alpha) = r(\beta)$). Then $d(\gamma) = d(\alpha) = \kappa$ (and so $s(\gamma) = \kappa$ since $D(\gamma) \subseteq S(\gamma)$) and $c(\gamma) = c(\beta) = \kappa$; that is, $\gamma \in H$ and we are finished.

If on the other hand $d(\alpha) = \delta < \kappa$ then $r(\alpha) = \kappa$ and our task of finding a suitable γ is much harder. However, before accomplishing this we note in passing that this case cannot be reduced to the one already considered by restricting α, β to $Y = E(\alpha) \cup E(\beta)$. For, we might now have

$$\alpha = \begin{pmatrix} U \cup V & c_n \\ x & c_n \end{pmatrix} \qquad \beta = \begin{pmatrix} U & V & c_n \\ y & z & c_n \end{pmatrix} \tag{**}$$

where U, V and $\{c_n\}$ partition X, $|U| = |V| = \delta < \kappa$, and x, y, z are distinct elements of $U \cup V$; if this were so then $Y = U \cup V$ and $d(\alpha_1) = d(\beta_1) = \delta$ but $r(\alpha_1) \neq r(\beta_1)$.

Now let $A = X\alpha \setminus [C(\beta) \cup S(\beta)]$ and note that $|A| = \kappa$ and $a\beta = a$ for all $a \in A$. Put

$$B = X\alpha \cap [C(\beta) \cup S(\beta)]$$

$$C = D(\alpha) \cup C(\beta) \cup S(\beta)$$

$$\varepsilon = \max(|B|, |C\beta|, \aleph_0)$$

and note that $\aleph_0 \leq \varepsilon \leq \delta < \kappa$ since $|C| = \delta$. Choose a subset D of A with $|D| = \varepsilon$ and let $E = C \cup D$, $F = B \cup D$. Now $|E\beta| = |C\beta| + |D\beta| = \varepsilon$ (since $D\beta = D$) and $|F| = \varepsilon$. Let $\theta: E\beta \to F$ be any bijection and define $\gamma \in \mathcal{T}_X$ by

$$x\gamma = x \quad \text{if } x \in A \setminus D$$
$$= x\beta\theta \text{ if } x \in E.$$

The domain of γ is X since $E \cup (A \setminus D)$ contains $C \cup A$ which equals $X\alpha \cup D(\alpha)$. Moreover, if $x\gamma = y\gamma$ then either (1) $x = y \in A \setminus D$, or (2) $x \in A \setminus D$, $y \in E$ and $x = y\beta\theta$, or (3) $x, y \in E$ and $x\beta\theta = y\beta\theta$ (we omit the dual of (2)). If (1) occurs then $x\beta = y\beta$; if (2) occurs then $x \in F$ and so $x \in B$ (since $x \notin D$), contradicting the assumption that $x \in A$; and if (3) occurs then $x\beta = y\beta$ since θ is one-to-one. That is, $\gamma \circ \gamma^{-1} \subseteq \beta \circ \beta^{-1}$. On the other hand, if $x\beta = y\beta$ then either $x = y$ (and so $x\gamma = y\gamma$) or $x \neq y$ (in which case $x, y \in C(\beta) \subseteq E$ and so $x\gamma = x\beta\theta = y\beta\theta = y\gamma$). Hence, $\gamma \circ \gamma^{-1} = \beta \circ \beta^{-1}$. In addition,

$$X\gamma = [E \cup (A \setminus D)]\gamma = F \cup (A \setminus D) = A \cup B = X\alpha.$$

Thus, $c(\gamma) = c(\beta) = d(\alpha) = d(\gamma)$. Clearly, $S(\gamma) \subseteq E$ and $|E| = \delta$. But $D(\gamma) \subseteq S(\gamma)$ and $d(\gamma) = \delta$; thus, $s(\gamma) = \delta$ and we have found some $\gamma \in H$ such that $\alpha \mathcal{L} \gamma \mathcal{R} \beta$.

Having characterised Green's $\mathcal{D}$ relation on H, we now consider part (c) and suppose $\beta = \lambda\alpha\mu$. Then $r(\beta) \leq r(\alpha)$ and $d(\beta) \geq d(\mu)$. Hence, if $d(\beta) < d(\alpha)$ then $d(\alpha\mu) = d(\alpha)$ (by Lemma 1) as well as $d(\beta) \geq d(\alpha\mu)$, a contradiction. Therefore $d(\beta) \geq d(\alpha)$. For the converse suppose $\alpha, \beta \in H$, $r(\beta) \leq r(\alpha)$ and $d(\beta) \geq d(\alpha)$. This means $\beta \in H(d(\alpha), r(\alpha)')$ which by Theorem 3 equals $H^1 \alpha H^1$, and so part (c) is proved. Finally, $\alpha \mathcal{J} \beta$ implies $H^1 \alpha H^1 = H^1 \beta H^1$ and this in turn implies $d(\alpha) = d(\beta)$ and $r(\alpha) = r(\beta)$; from the foregoing, we deduce $\alpha \mathcal{D} \beta$, and of course $\mathcal{D} \subseteq \mathcal{J}$ always.

It may be worthwhile illustrating the choice of γ for the α, β displayed in (**) above. Using the notation introduced in the second last paragraph of the proof, we have

$A = \{c_n\}$, $B = \{x\}$, $C = U \cup V$ and $\varepsilon = \aleph_0$. Then we in effect "blow-up" $r(\alpha_1)$ and $r(\beta_1)$ until they are equal by suitably enlarging the domain of α_1 and β_1. That is, we choose $D = \{c_m\}$ in A with $|M| = \varepsilon = \aleph_0$ and note that $E\beta = \{y, x\} \cup \{c_m\}$ has the same cardinal as $F = \{x\} \cup \{c_m\}$. If $\{c_p\} = A \setminus D$ then γ is the map

$$\begin{pmatrix} U & V & c_m & c_p \\ y\theta & z\theta & c_m\theta & c_p \end{pmatrix}$$

where θ is any bijection between $E\beta$ and F.

Before proceeding we note that the significance of Theorem 6 (especially part (d)) lies in the fact that it gives some hope of determining the congruences on H in a manner akin to that developed by Clifford and Preston for $\mathscr{T}_X$ in ([1], Vol. 2, Section 10.8); we shall explore this possibility in a subsequent paper.

We now consider the Rees quotient semigroup $H(\delta, \kappa')/H(\delta', \kappa')$ for $\aleph_0 \leq \delta < \kappa$. Clearly, each non-zero element of this semigroup has defect δ and rank κ. Moreover, a close perusal of the proof of Theorem 6 shows that if $d(\alpha) = d(\beta)$ and $r(\alpha) = r(\beta)$ for $\alpha, \beta \in H$ then there exists $\gamma \in H$ such that $\alpha \mathscr{L} \gamma \mathscr{R} \beta$ and $d(\gamma) = d(\alpha)$. In addition, if $\alpha \mathscr{L} \gamma$ in H then there exist $\lambda_1, \lambda_2 \in H$ with $\alpha = \lambda_1 \gamma$, $\gamma = \lambda_2 \alpha$ and $d(\lambda_1) = d(\alpha)$, $d(\lambda_2) = d(\gamma)$; likewise, if $\gamma \mathscr{R} \beta$ in H then there exist $\mu_1, \mu_2 \in H$ with $\gamma = \beta\mu_1$, $\beta = \gamma\mu_2$ and $d(\mu_1) = d(\gamma)$, $d(\mu_2) = d(\beta)$. In other words, each $H(\delta, \kappa')/H(\delta', \kappa')$ is 0-bisimple when $\delta < \kappa$. Each such semigroup is also regular since it contains non-zero idempotents and ([1], Vol. 1, Theorem 2.11) can be applied. However, none of them is completely 0-simple since they always contain non-zero non-primitive idempotents; for example, if

$$\alpha = \begin{pmatrix} A & b_m & b_n \\ a & b_m & b_n \end{pmatrix} \qquad \beta = \begin{pmatrix} A \cup \{b_m\} & b_n \\ a & b_n \end{pmatrix}$$

where $|A| = |\{b_m\}| = \delta$ and $a \in A$, then α, β are distinct idempotents satisfying $\alpha\beta = \beta\alpha = \beta$.

Unfortunately we cannot decide whether these 0-bisimple quotients in the "top half" of (*) are isomorphic to any of the quotients in the "bottom half" of (*). For, an argument similar to that applied to V_n/V_{n+1} in Section 2 can be used to show that for $\delta < \kappa$, $H(\delta, \kappa')/H(\delta', \kappa')$ has cardinal κ^δ (this is because the set of all subsets of X with cardinal δ has cardinal κ^δ: see [12], Exercise 22.25). It can also be readily shown that the non-zero group $\mathscr{H}$-classes of $H(\delta, \kappa')/H(\delta', \kappa')$ are all isomorphic to the group $\mathscr{G}(\kappa, \delta')$ of all permutations of κ letters with shift at most δ. On the other hand, for $\xi \leq \kappa$, $I_{\xi'}/I_\xi$ has cardinal 2^κ and its non-zero group $\mathscr{H}$-classes are all isomorphic to $\mathscr{G}_\xi$. However, without GCH, we may have $2^\delta = 2^\xi$ even though $\delta \neq \xi$ ([13], pp. 119 and 130).

Our final result in this section in effect determines Green's relations on $\mathscr{E}_X$.

Theorem 7. *If* $\alpha, \beta \in \mathscr{E}_X$ *and are related under one of Green's relations on* $\mathscr{E}_X$ *then* $\alpha, \beta \in V$ *or* $\alpha, \beta \in H$.

Proof. Suppose $\alpha \in V$, $\beta \in H$ and $\alpha = \lambda\beta$ for some $\lambda \in \mathscr{E}_X$. Then, by Lemma 1, $s(\lambda) \geq s(\beta)$ and this means $\lambda \in H$ which in turn implies $\alpha \in H$, a contradiction. A similar argument can be applied if $\alpha = \beta\mu$ or if $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in \mathscr{E}_X$.

4. Partial Transformations

In this section we consider the way in which the results of Sections 2 and 3 can be extended to the semigroup $\mathscr{E}_X^*$ that is generated by all the idempotent partial transformations of X different from the identity (it is clear that the identity cannot be written as a product of idempotents in $\mathscr{P}_X$ different from the identity).

For finite X, the elements of $\mathscr{E}_X^*$ were characterised by Evseev and Podran [3] (and independently by Sullivan [15]). As one might expect (by analogy with $\mathscr{T}_X$), if $|X| = n$ then $\mathscr{E}_X^* = \{\alpha \in \mathscr{P}_X : r(\alpha) < n\}$ and moreover each $\alpha \in \mathscr{E}_X^*$ can in fact be written as a product of idempotents in $\mathscr{P}_X$ with defect 1 (note that such idempotents can equal ι_Y for some $Y \subseteq X$ with $|Y| = n - 1$). Given this, it is easy to see what the ideals and Green's relations on $\mathscr{E}_X^*$ must be when X is finite.

Hence we again assume throughout this section that $|X| = \kappa \geq \aleph_0$. It seems from a review that Evseev and Podran [4] have also investigated $\mathscr{E}_X^*$ in this case. Since we need a straight forward characterisation of the elements of $\mathscr{E}_X^*$ in order to describe the ideals of $\mathscr{E}_X^*$, we now present such a characterisation and for completeness we include a short proof based on Howie's Theorem.

However, before proceeding to do this we recall Lyapin's method of representing $\mathscr{P}_X$ as a semigroup of total transformations: namely, let $0 \notin X$, put $Y = X \cup 0$ and

$$F_0 = \{\beta \in \mathscr{T}_Y : 0\beta = 0\},$$

and define $\theta : \mathscr{P}_X \to F_0$, $\alpha \to \alpha\theta$, where $x(\alpha\theta) = x\alpha$ if $x \in \mathrm{dom}\,\alpha$ and $x(\alpha\theta) = 0$ otherwise. Clearly θ is an isomorphism. We extend the notions of defect, collapse and shift of $\alpha \in \mathscr{T}_X$ to elements of $\mathscr{P}_X$ as follows: for each $\alpha \in \mathscr{P}_X$, let

$$D^*(\alpha) = X \backslash X\alpha \qquad\qquad \text{and} \quad d^*(\alpha) = |D^*(\alpha)|$$

$$C^*(\alpha) = C(\alpha) \cup (X \backslash \mathrm{dom}\,\alpha) \qquad \text{and} \quad c^*(\alpha) = |C^*(\alpha)|$$

$$S^*(\alpha) = \{x \in \mathrm{dom}\,\alpha : x\alpha \neq x\} \cup (X \backslash \mathrm{dom}\,\alpha) \quad \text{and} \quad s^*(\alpha) = |S^*(\alpha)|$$

Theorem 8. *An element α of $\mathscr{P}_X$ can be written as a product of idempotents in $\mathscr{P}_X$ different from the identity if and only if either $s^*(\alpha) < \aleph_0$ and $d^*(\alpha) \neq 0$ or $s^*(\alpha) = c^*(\alpha) = d^*(\alpha) \geq \aleph_0$.*

Proof. Suppose $\alpha \in \mathscr{E}_X^*$ and $\beta = \alpha\theta$. Then $\beta \in \mathscr{E}_Y$ and so, by Howie's Theorem, either $s(\beta) < \aleph_0$ and $d(\beta) \neq 0$ or $s(\beta) = c(\beta) = d(\beta) \geq \aleph_0$. Since $Y \backslash Y\beta = X \backslash X\alpha$, $C(\beta) = C^*(\alpha)$ and $S(\beta) = S^*(\alpha)$, this produces the desired result. Now suppose $\alpha \in \mathscr{P}_X$ and α satisfies $s^*(\alpha) = c^*(\alpha) = d^*(\alpha) \geq \aleph_0$. Then $\beta = \alpha\theta \in F_0$ and β satisfies the corresponding condition in Howie's Theorem. Suppose

$$\beta = \begin{pmatrix} B_i & C & y_j & a_n \\ x_i & 0 & z_j & a_n \end{pmatrix}$$

where each B_i contains at least 2 elements and $y_j \neq z_j$ for each j. Choose $b_i \in B_i$ and let

$$\lambda = \begin{pmatrix} B_i & C & y_j & a_n \\ b_i & 0 & y_j & a_n \end{pmatrix} \qquad \mu = \begin{pmatrix} b_i & 0 & y_j & a_n & D \\ x_i & 0 & z_j & a_n & d \end{pmatrix}$$

where $D = (C\backslash 0) \cup (C(\beta)\backslash\{b_i\})$ and $d \in D$. Then $\beta = \lambda\mu$ where $\lambda^2 = \lambda \in F_0$. Now since $c(\beta)$ equals either $|\cup B_i|$ or $|C|$, we have $|D| = c(\beta)$. Thus, since $|I| \leq c(\beta)$ and $|J| \leq s(\beta)$, we have $c(\mu) = s(\mu) = c(\beta)$. Moreover, $D(\mu) = D(\beta)\backslash d$ and so $d(\mu) = s(\beta)$. Therefore, by Howie's Theorem, $\mu | X$ is a product of idempotents in $\mathcal{T}_X$, each of which can be extended in an obvious way to an idempotent in F_0. By applying the isomorphism θ^{-1}, we obtain a product of idempotents in $\mathscr{P}_X$ that equals α.

Since the proof for the case when $s^*(\alpha) < \aleph_0$ and $d^*(\alpha) \neq 0$ can be carried through in an entirely similar manner, we omit the details. However we note that in this case β cannot look like

$$\begin{pmatrix} 0 & y_j & a_n \\ 0 & z_j & a_n \end{pmatrix}$$

where $|J| < \aleph_0$ for this would mean β is a permutation, contradicting $d(\beta) \neq 0$. In other words, either $|C| \geq 2$ or $I \neq \square$: this fact can be used to ensure that defects are non-zero.

We can now write $\mathscr{E}_X^* = V^* \cup H^*$ where

$$V^* = \{\alpha \in \mathscr{P}_X : s^*(\alpha) < \aleph_0 \quad \text{and} \quad d^*(\alpha) \neq 0\}$$

$$H^* = \{\alpha \in \mathscr{P}_X : s^*(\alpha) = d^*(\alpha) = c^*(\alpha) \geq \aleph_0\}$$

and $\square \in H^*$. Using the results of Section 2 and the isomorphism θ, it is a simple matter to check that the ideals of V^* take the form

$$V_n^* = \{\alpha \in V^* : d^*(\alpha) \geq n\}$$

where $n \geq 1$, and the ideals of H^* equal

$$H^*(\delta, \xi) = \{\alpha \in H^* : d^*(\alpha) \geq \delta \quad \text{and} \quad r(\alpha) < \xi\}$$

for some δ, ξ satisfying $\aleph_0 \leq \delta \leq \kappa$ and $1 \leq \xi \leq \kappa'$. In addition, Green's relations on V^* and on H^* are precisely what one would expect given the results of Section 3.

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