Pacific Journal of Mathematics

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# THE IDEMPOTENTS OF A CLASS OF 0-SIMPLE INVERSE SEMIGROUPS 

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#### Abstract

An $\omega$-semigroup is a semigroup whose idempotents form an $\omega$-chain $e_{0}>e_{1}>e_{2}>\cdots$. In this paper we characterize the semilattice of idempotents of a 0 -simple inverse semigroup whose nonzero $\mathscr{D}$-classes form $\omega$-semigroups.


A semilattice $E$ is an interlaced union of $\omega$-chains $C_{\alpha}=$ $\left\{e_{\alpha, 0}>e_{\alpha, 1}>\cdots\right\}, \alpha \in A$, if $E=\bigcup_{a \in A} C_{\alpha}$ and if $\alpha, \beta \in A, i \geqq 0$, then there exists a unique $j \geqq 0$ such that

$$
e_{\beta, j}<e_{\alpha, i} \text { but } e_{\beta, j} \nless e_{\alpha, i+1} .
$$

It will be shown that $Y$ is the semilattice of a 0 -simple inverse semigroup whose nonzero $\mathscr{D}$-classes form $\omega$-semigroups if and only if $Y$ is an interlaced union of $\omega$-chains, with zero adjoined. One such 0 -simple inverse semigroup with semilattice $Y$ will be explicitly displayed.

In the semigroups under consideration, every nonzero $\mathscr{D}$-class is an $\omega$-semigroup, that is, a bisimple $\omega$-semigroup. Since bisimple $\omega$ semigroups were described completely by N. R. Reilly, [8], our semigroups are unions of well-known semigroups; it is the manner in which the idempotents of these $\omega$-semigroups relate to each other that is of interest here. This class of semigroups includes several which have already been explored, for example, simple $\omega$-semigroups, [4] and [7], and certain simple inverse semigroups whose idempotents form the ordinal product of a $\omega$-chain and a semilattice with identity, [6]. Bisimple $\omega$-semigroups occur in abundance within most regular semigroups (see [1]), so it is natural to consider, as a first step, those semigroups whose $\mathscr{D}$-classes are all $\omega$-semigroups.

1. Preliminaries. Let $S$ be an inverse semigroup. For an element $a$ of $S, a^{-1}$ denotes the unique element of $S$ for which $a a^{-1} a=$ $a$ and $a^{-1} a a^{-1}=a^{-1}$. For any subset $D$ of $S, E_{D}$ is the set of idempotents of $S$ contained in $D$. Equivalences $\mathscr{D}$ and $\mathscr{J}$ denote the usual Green's relations.

For inverse semigroups, the property of being 0 -simple is easily seen to be equivalent to the condition: if $e$ and $f$ are nonzero idempotents then there exists an idempotent $g$ such that $g \leqq f$ and $g \mathscr{D} e$, where $\leqq$ is the usual partial order on idempotents.

Let $e$ and $f$ be idempotents with $e \mathscr{D} f$. Then there exists $a$ in
$S$ such that $a a^{-1}=e$ and $a^{-1} a=f$. Furthermore, the mapping $\sigma_{a}: x \rightarrow$ $a^{-1} x a$ is an isomorphism of $E_{s} e$ onto $E_{s} f$, [3].

The following result is crucial to our development of the structure of the semigroups under consideration.

Lemma 1.1. Let $S$ be an inverse semigroup in which every nonzero $\mathscr{D}$-class is an $\omega$-semigroup. Then $S$ is 0 -simple if and only if for any two distinct nonzero $\mathscr{D}$-classes $D, D^{\prime}$, if $g, h \in E_{D}$ with $g<h$, then there exists $d \in E_{D^{\prime}}$ such that $d<h$ but $d \nless g$.

Proof. Let $S$ be 0 -simple and $D, D^{\prime}$ be two distinct nonzero $\mathscr{D}$-classes with $g<h, g, h$ in $E_{D}$. By 0 -simplicity, there exists $e \in E_{D^{\prime}}$ such that $e<g$. Since $E_{D^{\prime}}$ is inversely well-ordered, $e$ can be picked to be the maximal idempotent of $D^{\prime}$ beneath $g$. Moreover, since there is an idempotent of $D$ below $e$, there are only a finite number above $e$, so we let $g^{\prime}$ be the minimal such one. That is,

$$
e<g^{\prime} \leqq g<h .
$$

Since $g^{\prime} \mathscr{D} h$, there exists $a$ in $S$ with $a a^{-1}=h, a^{-1} a=g^{\prime}$. Now $a^{-1} e a \mathscr{D} e$ and $a^{-1} e a<g^{\prime}<g$. By maximality of $e$, it follows that $a^{-1} e a \leqq e<g^{\prime}$. If $a^{-1} e a=e$, then $\sigma_{a}$, as defined above, acts in the following manner: $\sigma_{a}(h)=g^{\prime}, \sigma_{a}(e)=e$ and $\sigma_{a}(g)=g^{\prime \prime}$ for some $g^{\prime \prime} \mathscr{D} g$. Since $e<g<h$, then $e<g^{\prime \prime}<g^{\prime}$. But by minimality of $g^{\prime}$, this is impossible. Thus $a^{-1} e a<e<g^{\prime}$.

Since $\sigma_{a^{-1}}$ is also an isomorphism, $a^{-1} e a<e<g^{\prime}$ implies

$$
a\left(a^{-1} e a\right) a^{-1}<a e a^{-1}<a g^{\prime} a^{-1} .
$$

That is, $e<a e a^{-1}<h$. Consequently $d=a e a^{-1}$ is $\mathscr{D}$-related to $e$ and $d$ satisfies the condition that $d<h$. Furthermore, since $e$ is the maximal idempotent of $D^{\prime}$ below $g, d \nless g$.

The converse follows directly from the remark preceding Lemma 1.1.

An ideal $I$ is called prime if $a b \in I$ implies $a \in I$ or $b \in I$.
Lemma 1.2. If $S$ is a 0 -simple inverse semigroup whose nonzero $\mathscr{D}$-classes are $\omega$-semigroups then 0 is a prime ideal, and $S \backslash 0$ is a simple inverse semigroup whose $\mathscr{D}$-classes are $\omega$-semigroups.

Proof. Let $e$ and $f$ be nonzero idempotents of $S$ with $e f=0$. Then $e$ and $f$ must be in distinct $\mathscr{D}$-classes, since each $\mathscr{D}$-class is closed. By 0 -simplicity, there exists an idempotent $g$ such that $g \leqq e$ and $g \mathscr{O} f$. Since $f$ and $g$ are in an $\omega$-semigroup, either $g \leqq f$ or
$f<g$. But if $f<g$, then $f \leqq e$ and $e f \neq 0$. Hence $g \leqq f$ and $g \leqq e$. But this implies that $g \leqq e f=0$. But $g \neq 0$, and thus $e f \neq 0$. Therefore, 0 is a prime ideal of $E_{s}$, and thus of $S$.
2. The idempotent structure. In light of Lemma 1.2, we now restrict ourselves to simple inverse semigroups whose $\mathscr{D}$-classes are $\omega$-semigroups. In such a semigroup, we now show that the semilattice of idempotents is an interlaced union of $\omega$-chains.

Lemma 2.1. Let $S$ be a simple inverse semigroup whose $\mathscr{D}$ classes are $\omega$-semigroups $D_{\alpha}, \alpha \in A$, and $E_{D_{\alpha}}=\left\{e_{\alpha, 0}>e_{\alpha, 1}>\cdots\right\}$. The following properties hold in $E_{s}$.
(i) If $e_{\alpha, i} \leqq e_{\beta, j}$ then $i \geqq j$.
(ii) For $\alpha, \beta \in A, i, j \geqq 0$, and for all $n$ such that $-j \leqq n<+\infty$,

$$
e_{\alpha, i}<e_{\beta, j} \longleftrightarrow e_{\alpha, i+n}<e_{\beta, j+n} .
$$

(iii) If $e_{\alpha, i} e_{\beta, j}=e_{r, k}$ then $e_{\alpha, i+n} e_{\beta, j+n}=e_{r, k+n}$, for all $n \geqq-\min \{i, j\}$.
(iv) For $\alpha \in A$, if $a a^{-1}=e_{\alpha, i}, a^{-1} a=e_{\alpha, j}$ then $\sigma_{a}: E e_{\alpha, i} \rightarrow E e_{\alpha, j}$ defined by $x \sigma_{a}=a^{-1} x a$, is an isomorphism such that if $e_{\beta, k} \leqq e_{\alpha, i}$, then

$$
\begin{equation*}
e_{\beta, k} \sigma_{a}=e_{\beta, k+(j-i)} \tag{1}
\end{equation*}
$$

Proof. (i) Let $e_{\alpha, i}<e_{\beta, j}$. Consider the set

$$
M=\left\{k \mid e_{\alpha, k}<e_{\beta, 0}, e_{\alpha, k} \nless e_{\beta, j}\right\} .
$$

Then if $k$ is in $M, k<i$ since $e_{\alpha, i}<e_{\beta, j}$. On the other hand, by Lemma 1.1, for all $p<j$, there exists $p^{\prime}$ such that $e_{\alpha, p^{\prime}}<e_{\beta, p}, e_{\alpha, p^{\prime}} \nless$ $e_{\beta, p+1}$; each $p^{\prime}$ is in $M$ and they are all distinct. Consequently $j-1 \leqq$ $|M|<i$, so $i>j$.

We know from [3] that $\sigma_{a}$ is an isomorphism and thus preserves $\mathscr{D}$-classes. Therefore, if $e_{\beta, k} \leqq e_{\alpha, i}$, then $e_{\beta, k} \sigma_{a}=e_{\beta, m}$ for some $m$. In addition it is clear that for $e_{\alpha, k} \leqq e_{\alpha, i}, e_{\alpha, k} \sigma_{a}=e_{\alpha, k+(j-i)}$, since there must be a one-to-one correspondence between the sets $\left\{e_{\alpha, k}<\cdots<e_{\alpha, i}\right\}$ and $\left\{e_{\alpha, k} \sigma_{a}<\cdots<e_{\alpha, j}\right\}$. The proof of (1) for arbitrary $\beta$ will be made after (ii) and (iii) are proved.
(ii) Let $e_{\alpha, i}<e_{\beta, j}$. It will first be shown that $e_{\alpha, i+1}<e_{\beta, j+1}$. Either $e_{\alpha, i}<e_{\beta, j+1}$ and thus $e_{\alpha, i+1}<e_{\alpha, i}<e_{\beta, j+1}$, or $e_{\alpha, i} \nless e_{\beta, j+1}$. We may assume the latter. By simplicity, there exists $e_{\beta, k}<e_{\alpha, i}$, so let $r=\min \left\{k \mid e_{\beta, k}<e_{\alpha, i}\right\}$. That is, using 1.1

$$
e_{\beta, r}<e_{\alpha, i}<e_{\beta, i} \text { and } e_{\beta, r} \nless e_{\alpha, i+1} .
$$

Let $a a^{-1}=e_{\beta, j}$ and $a^{-1} a=e_{\beta, j+1}$. Then

$$
a^{-1} e_{\beta, r} a<a^{-1} e_{\alpha, i} a<a^{-1} e_{\beta, j} a,
$$

where the strict inequalities hold since $\sigma_{a}$ is an isomorphism. That is,

$$
\begin{equation*}
e_{\beta, r+1}<a^{-1} e_{\alpha, i} a<e_{\beta, j+1} \tag{2}
\end{equation*}
$$

since $e_{\beta, r} \sigma_{a}=e_{\beta, r+1}$ as we have seen earlier. Now $a^{-1} e_{\alpha, i} a \mathscr{D} e_{\alpha, i}$ and thus $a^{-1} e_{\alpha, i} a<e_{\alpha, i}$ since $e_{\alpha, i} \nless e_{\beta, j+1}$. If $a^{-1} e_{\alpha, i} a<e_{\alpha, i+1}$ then by 1.1, there exists $p$ such that $e_{\beta, p}<e_{\alpha, i+1}, e_{\beta, p} \nless a^{-1} e_{\alpha, i} a$. By definition of $r, p \geqq r$ and in fact $p>r$ since $e_{\beta, r} \nless e_{\alpha, i+1}$. But then by (2) $e_{\beta, p} \leqq e_{\beta, r+1}<\alpha^{-1} e_{\alpha, i} a$, contrary to the assumption. Hence $a^{-1} e_{\alpha, i} a=$ $e_{\alpha, i+1}$ and thus $e_{\alpha, i+1}<e_{\beta, j+1}$.

That $e_{\alpha, i+n}<e_{\beta, j_{+n}}$ for all $n \geqq 0$ follows by induction.
Now consider the case $n=-1$. Let $j>0$. Then $i>j>0$ by (i). Either $e_{\alpha, i}$ is the maximal idempotent of $D_{\alpha}$ less than $e_{\beta, j}$, or $e_{\alpha, i}<e_{\alpha, i-1}<e_{\beta, j}<e_{\beta, j-1}$. Thus we may assume that the former holds. By 1.1, there exists $m$ such that $e_{\alpha, m}<e_{\beta, j-1}, e_{\alpha, m} \nless e_{\beta, j}$. Since $e_{\alpha, i}<$ $e_{\beta, j}$, it follows that $m \leqq i-1$. Hence $e_{\alpha, i-1} \leqq e_{\alpha, m}<e_{\beta, j-1}$. The proof for $n$ such that $-j \leqq n \leqq-1$ is by induction.
(iii) The proof of (iii) is made using repeated applications of (ii).

To see that (1) holds for arbitrary $\beta$, let $\sigma_{a}$ be defined as in (iv). Then, as we have stated, for $e_{\beta, k}<e_{\alpha, i}, a^{-1} e_{\beta, k} a=e_{\beta, p}$ for some $p$. By (ii), $e_{\beta, k}<e_{\alpha, i}$ if and only if $e_{\beta, k+(j-i)} \leqq e_{\alpha, i+(j-i)}=e_{\alpha, j}$. Since $\sigma_{a}$ is one-to-one and preserves $\mathscr{D}$-classes, $e_{\beta, k} \sigma_{a}=e_{\beta, k+(j-i)}$.

THEOREM 2.2. If $S$ is a simple inverse semigroup whose $\mathscr{D}$ classes are $\omega$-semigroups, then $E_{S}$ is an interlaced union of $\omega$-chains.

Proof. We know that $E_{s}$ is a union of $\omega$-chains $E_{D_{\alpha}}=\left\{e_{\alpha, 0}\right\rangle$ $\left.e_{\alpha, 1}>\cdots\right\}, \alpha \in A$, where $D_{\alpha}$ is a $\mathscr{D}$-class. Let $\alpha, \beta \in A, i \geqq 0$. We must find a unique $j \geqq 0$ such that $e_{\beta, j}<e_{\alpha, i}, e_{\beta, j} \nless e_{\alpha, i+1}$. Consider the set

$$
K=\left\{j \mid e_{\beta, j}<e_{\alpha, i}\right\} .
$$

By Lemma 1.1, $K$ is nonempty, and thus $K$ must have a least element, call it $m$. Then $e_{\beta, m}<e_{\alpha, i}$. If $e_{\beta, m}<e_{\alpha, i+1}$, then by Lemma 2.1 (ii), $e_{\beta, m-1}<e_{\alpha,(i+1)-1}$. That is, $e_{\beta, m-1}<e_{\alpha, i}$. By minimality of $m$, this is impossible. Thus $e_{\alpha, m} \nless e_{\alpha, i+1}$.

Since $e_{\alpha, i} \mathscr{D} e_{\alpha, i+1}$, there exists $a \in S$ such that $a a^{-1}=e_{\alpha, i}, a^{-1} a=e_{\alpha, i+1}$ and $\sigma_{a}$ defined by $e_{r, k} \sigma_{a}=e_{r, k+1}$ is an isomorphism of $E e_{\alpha, i}$ onto $E_{\alpha, i+1}$, by Lemma 2.1(iv). Now $e_{\beta, m}<e_{\alpha, i}$ so $e_{\beta, m} \sigma_{a}=e_{\beta, m+1}<e_{\alpha, i+1}$. Hence $e_{\beta, k}<e_{\alpha, i+1}$ for all $k>m$. From this and minimality of $m$, it follows that $e_{\beta, m}$ is the unique idempotent in $D_{\beta}$ such that $e_{\beta, m}<e_{\alpha, i}$ and $e_{\beta, m} \nless e_{\alpha, i+1}$. Therefore, $E_{s}$ is an interlaced union of $\omega$-chains $E_{D_{\alpha}}, \alpha \in A$.
3. An interlaced union of $\omega$-chains. Given an interlaced union
of $\omega$-chains, we now construct a simple inverse semigroup associated with it.

Let $E$ be an interlaced union of $\omega$-chains $e_{\alpha, 0}>e_{\alpha, 1}>\cdots, \alpha \in A$. Recall that this means that for all $\alpha, \beta \in A, i \geqq 0$, there exists a unique $j \geqq 0$ such that $e_{\beta, j}<e_{\alpha, i}, e_{\beta, j} \nless e_{\alpha, i+1}$.

Lemma 3.1. For $E$ as described, the following hold.
(i) If $e_{\alpha, i} \leqq e_{\beta, i}$ then $i \geqq j$.
(ii) If $e_{\alpha, i} \leqq e_{\beta, j}$ then $e_{\alpha, i+n} \leqq e_{\beta, j+n}$ for all $n \geqq-j$.
(iii) If $e_{\alpha, i} e_{\beta, j}=e_{r, k}$ then $e_{\alpha, i+n} e_{\beta, j+n}=e_{\gamma, k+n}$ for all $n \geqq 0$.

Proof. First we prove (ii) for all $n \geqq-\min \{i, j\}$. Assume that $e_{\alpha, i} \leqq e_{\beta, j}$. Let $n \geqq 0$ and assume $e_{\alpha, i+n} \leqq e_{\beta, j+n}$. If $e_{\alpha, i+n} \leqq e_{\beta, j+n+1}$, then $e_{\alpha, i+n+1}<e_{\alpha, i+n} \leqq e_{\beta, j+n+1}$ and the result holds. If $e_{\alpha, i+n} \nless e_{\beta, j+n+1}$ then $e_{\alpha, i+n}$ is the unique element below $e_{\beta, j+n}$ which is not below $e_{\beta, j+n+1}$. Consider $e_{\alpha, i+n+1}$. We know $e_{\alpha, i+n+1}<e_{\beta, j+n}$ since $e_{\alpha, i+n+1}<$ $e_{\alpha, i+n}$; therefore, by uniqueness of $i+n$, we have $e_{\alpha, i+n+1} \leqq e_{\beta, j+n+1}$. By induction, (ii) holds for all $n \geqq 0$.

Now let $n>-\min \{i, j\}$ and let $e_{\alpha, i-n} \leqq e_{\beta, j-n}$. Either $e_{\alpha, i-n-1} \leqq$ $e_{\beta, j-n}<e_{\beta, j-n-1}$, or else $e_{\alpha, i-n-1} \nless e_{\beta, j-n}$. There exists a unique $k \geqq 0$ such that $e_{\alpha, k}<e_{\beta, j-n-1}$ and $e_{\alpha, k} \nless e_{\beta, j-n}$. If $e_{\alpha, i-n-1} \nless e_{\beta, j-n}$ then it must be that $k \leqq i-n-1$ and $e_{\alpha, i-n-1} \leqq e_{\alpha, k}<e_{\beta, j-n-1}$. Consequently, for all $n$ such that $-\min \{i, j\} \leqq n<+\infty$, (ii) holds.
(i) Let $e_{\alpha, i} \leqq e_{\beta, j}$ and assume $i<j$. Then by the above paragraph, $e_{\alpha, i-i} \leqq e_{\beta, j-i}$. That is, $e_{\alpha, 0} \leqq e_{\beta, j-i}<e_{\beta, 0}$. Since $E$ is an interlaced union of $\omega$-chains, there exists $k \geqq 0$ such that $e_{\alpha, k}<e_{\beta, 0}$ and $e_{\alpha, k} \nless e_{\beta, 1}$. But $j-i \geqq 1$ and $e_{\alpha, k} \leqq e_{\beta, 0} \leqq e_{\beta, j-i} \leqq e_{\beta, 1}$. This is impossible. Therefore $i \geqq j$. This also shows that (ii) is true for all $n \geqq-j=-\min \{i, j\}$.
(iii) Let $e_{\alpha, i} e_{\beta, j}=e_{r, k, k}$. Then $e_{\gamma, k} \leqq e_{\alpha, i}$ and $e_{r, k} \leqq e_{\beta, j}$, so that by (ii), $e_{r, k+1} \leqq e_{\alpha, i+1}, e_{T, k+1} \leqq e_{\beta, j+1}$. That is,

$$
e_{r, k+1} \leqq e_{\alpha, i+1} e_{\beta, j+1}<e_{\alpha, i} e_{\beta, j}=e_{i, k}
$$

Let $e_{\alpha, i+1} e_{\beta, j+1}=e_{\delta, p}$. Then $e_{\delta, p} \leqq e_{\alpha, i+1}, e_{\sigma, p} \leqq e_{\beta, j+1}$, so by (ii), $e_{\dot{\delta}, p-1} \leqq$ $e_{\alpha, i}, e_{\partial, p-1} \leqq e_{\beta, j}$. That is, $e_{i, p-1} \leqq e_{\alpha, i} e_{\beta, j}=e_{\gamma, k, k}$. Consequently, $e_{r, k+1} \leqq$ $e_{\partial, p}<e_{\partial, p-1} \leqq e_{r, k}$. But then by uniqueness in the definition of $E$, both $e_{i, p}$ and $e_{i, p-1}$ can not be strictly between $e_{r, k+1}$ and $e_{i, k}$. Thus $e_{i, p}=$ $e_{r, k+1}$ and $e_{r, k+1}=e_{\alpha, i+1} e_{\beta, j+1}$. By induction, (iii) holds for all $n \geqq 0$.

Theorem 3.2 Let $E$ be an interlaced union of $\omega$-chains $\left\{e_{\alpha, 0}\right\rangle$ $\left.e_{\alpha, 1}>\cdots\right\}, \alpha \in A$. For $\alpha \in A, m, n \geqq 0$, let $\tau_{(m, \alpha, n)}$ be the mapping from Ee $\alpha_{\alpha, m}$ onto $E e_{\alpha, n}$ defined by

$$
e_{\beta, j} \tau_{(m, \alpha, n)}=e_{\beta, j+(n-m)}
$$

Then $W=\left\{\tau_{(m, \alpha, n)} \mid \alpha \in A, m, n \geqq 0\right\}$, under composition, is a simple inverse semigroup whose $\mathscr{D}$-classes are $\omega$-semigroups, and $E_{W} \cong E$.

Proof. By Theorem 3.2 of [5], to see that $W$ is a simple inverse semigroup, it suffices to show that $W$ is a subtransitive inverse subsemigroup of $T_{E}$, the set of isomorphisms of principal ideals of $E$. Using (ii) and (iii) of 3.1, it is not difficult to show that $\tau_{(m, \alpha, n)}$ is an isomorphism of $E e_{\alpha, m}$ onto $E e_{\alpha, n}$, and thus $W$ is contained in $T_{E}$.

To see that $W$ is closed, let $\tau_{(m, \alpha, n)}, \tau_{(i, \beta, j)}$ be in $W$. Certainly $\tau_{(m, \alpha, n)} \tau_{(i, \beta, j)}$ is an isomorphism from one subset of $E$ to another. We need to show its domain is $E e_{i, p}$ and its range is $E e_{\partial, q}$ for some $\delta \in$ $A, p, q \geqq 0$.

Now, $e_{\gamma, k} \in$ domain of $\tau_{(m, \alpha, n)} \tau_{(i, \beta, j)}$ if and only if

$$
e_{\gamma, k} \leqq e_{\alpha, m} \quad \text { and } \quad e_{\gamma, k+(n-m)} \leqq e_{\beta, i},
$$

which by Lemma 3.1 (ii) is equivalent to

$$
e_{\gamma, k} \leqq e_{\alpha, m} \quad \text { and } \quad e_{\gamma, k} \leqq e_{\beta, i-(n-m)} .
$$

This is equivalent to

$$
e_{r, k} \leqq e_{\alpha, m} e_{\beta, i-(n-m)} .
$$

Thus the domain of $\tau_{(m, \alpha, n)} \tau_{(i, \beta, j)}$ is $E e_{\alpha, m} e_{\beta, i-(n-m)}$.
Now, $e_{\tilde{j}, s}$ is in the range of $\tau_{(m, \alpha, n)} \tau_{(i, \beta, j)}$ if and only if

$$
e_{\partial, s} \leqq e_{\beta, j} \quad \text { and } \quad e_{0, s-(j-i)} \leqq e_{\alpha, n},
$$

which is equivalent to

$$
e_{i, s} \leqq e_{\beta, j} \quad \text { and } \quad e_{i, s} \leqq e_{\alpha, n+(j-i)} .
$$

This in turn is equivalent to

$$
e_{i, s} \leqq e_{\alpha, n+(i-j)} e_{\beta, j} .
$$

Therefore, the range of $\tau_{(m, \alpha, n)} \tau_{(i, \beta, j)}$ is $E e_{\alpha, n+(j-i)} e_{\beta, j}$.
If $(n-m)+(j-i) \geqq 0$, and $e_{\alpha, m} e_{\beta, i-(n-m)}=e_{o, p}$ for some $\delta \in A$, $p \geqq 0$, then by Lemma 3.1 (iii),

$$
e_{\alpha, m+(n-m)+(j-i)} e_{\beta, i-(n-m)+(n-m)+(j-i)}=e_{\delta, p+(n-m)+(j-i)} .
$$

That is,

$$
e_{\alpha, n+(j-i)} e_{\beta, j}=e_{\delta, p+(n-m)+(j-i)}=e_{i, q},
$$

for some $q \geqq 0$, and $\tau_{(m, \alpha, n)} \tau_{(i, \beta, j)}=\tau_{(p, 0, q)}$. If $(n-m)+(j-i) \leqq 0$, a similar argument works for $e_{\alpha, n+(i-j)} e_{\beta, j}$. Thus $W$ is closed and is a subsemigroup of $T_{E}$. It is clearly an inverse semigroup since $\tau_{(n, \alpha, m)}=\tau_{(m, \alpha, n)}^{-1}$.

In order that $W$ be subtransitive, it must satisfy the condition: for $e, f$ in $E$, there exists $\theta \in W$ such that domain of $\theta=E e$, range of $\theta \subseteq E f$. For $e_{\alpha, i}, e_{\beta, j}$ in $E$, there exists $k \geqq 0$ such that $e_{\alpha, k} \leqq e_{\beta, j}$, since $E$ is interlaced. Thus $\theta=\tau_{(i, \alpha, k)}$ satisfies the necessary condition.

Since idempotents of $W$ are of the form $\tau_{(i, \alpha, i)}, E_{W}$ is an interlaced union of $\omega$-chains, isomorphic to $E$ under the map: $e_{\alpha, i} \rightarrow \tau_{(i, \alpha, i)}$. By Lemma 1.2 of [5], it is clear the $\tau_{(i, \alpha, i)} \mathscr{D} \tau_{(j, \beta, j)}$ if and only if $\alpha=\beta$, so the $\mathscr{D}$-classes of $W$ are $\omega$-semigroups.

Theorem 3.3. A semilattice $E$ is the semilattice of idempotents of a 0-simple inverse semigroup whose nonzero $\mathscr{D}$-classes are $\omega$ semigroups if and only if $E$ is an interlaced union of $\omega$-chains with 0 adjoined.

Proof. This follows immediately from Corollary 1.2, Theorem 2.2 and Theorem 3.2.
4. An application. The simplest example of an interlaced union of $\omega$-chains is that of an $\omega$-chain itself. The inverse semigroups corresponding are simple $\omega$-semigroups, the structure of which was determined by Kochin [4] and Munn [7]. The following result demonstrates the strength of the condition imposed on an interlaced union of $\omega$-chains.

Theorem 4.1. If $S$ is a simple inverse semigroup with exactly two $\mathscr{D}$-classes, each of which is an $\omega$-semigroup, then $S$ is itself an $\omega$-semigroup.

Proof. Let $\left\{e_{0}>e_{1}>\cdots\right\}$ and $\left\{f_{0}>f_{1}>\cdots\right\}$ be the idempotents of the two $\mathscr{D}$-classes. Since $E_{s}$ must be an interlaced union of $\omega$ chains by Theorem 2.2 , there exists unique $i \geqq 0, j \geqq 0$ such that

$$
e_{i}<f_{0}, e_{i} \nless f_{1}, \quad \text { and } f_{j}<e_{0}, f_{j} \nless e_{1} .
$$

Now $e_{0} f_{0} \in E_{S}$ so $e_{0} f_{0}=e_{k}$ or $f_{k}$ for some $k$. Without loss of generality we may assume $e_{0} f_{0}=e_{k}$. Then $e_{k}<f_{0}$. But $e_{i}<f_{0}$ implies that $e_{i}=$ $e_{i} e_{0} \leqq e_{0} f_{0}=e_{k}$, so $i \geqq k$. But if $e_{i}<e_{k}$, then $e_{k} \nless f_{1}$ since $e_{i} \nless f_{1}$. Thus by uniqueness, $k=i$ and $e_{0} f_{0}=e_{i}$. Now $f_{j}<e_{0}$ so $f_{j}<e_{0} f_{0}=e_{i}$. Since $f_{j} \nless e_{1}$, it follows that $i=0$. Hence $e_{0} f_{0}=e_{0}$, i.e., $e_{0} \leqq f_{0}$. By Lemma 3.1(ii), $e_{n} \leqq f_{n}$ for all $n$.

We need to show that $f_{1}<e_{0}$. Since $e_{1}<f_{1}$ and $e_{0}<f_{0}$, then

$$
e_{1} \leqq e_{0} f_{1}<e_{0} f_{0}=e_{0} .
$$

If $e_{1}=e_{0} f_{1}$ then $f_{j}<e_{0}$ and $f_{j} \nless e_{1}$ implies that $f_{j}=f_{j} f_{0}<e_{0} f_{1}=e_{1}$. But
this is impossible, so $e_{1}<e_{0} f_{1}<e_{0}$. Thus $e_{0} f_{1}=f_{j}$, by uniqueness, and $e_{1}<f_{j}<e_{0}$. By property (i) of Lemma 3.1, $j \leqq 1$, so $j=1$, and $e_{1}<f_{1}<e_{0}<f_{0}$. By property (ii), this means that $E_{S}$ is an $\omega$-chain.

To see that Theorem 4.1 does not hold for more than two $\mathscr{D}$ classes, consider the following semilattice $E$.


This semilattice $E$ is the interlaced union of three $\omega$-chains, each chain being a column, but $E$ is not an $\omega$-chain itself. For more than three $\mathscr{D}$-classes, one may add to $E \omega$-chains each of whose elements is put between two elements of one of the columns in the semilattice $E$.

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Received February 3, 1978.
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