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The identification and control of unknown linear discrete systems†

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The identification and control of unknown linear discrete systems†

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This investigation is concerned with the optimal control of unknown, time-invariant, linear discrete systems. Of particular interest is the relationship among the identification problem, the specification of the control law assuming the system is known and the overall optimalizing control. For single-stage control with noiseless observations, conditions under which the use of separate identification and control procedures results in overall optimal control are established. The complexity inherent in the control problem is illustrated with a simple single-stage example wherein optimal control calls for filtering of the observed data by a time-varying, data-dependent operator, for which no simple recursive implementation exists.

1. Introduction

There is a large body of literature dealing with the identification of deterministic linear systems from noisy output (and possibly input) records. For discrete systems, these investigations range from statistically optimum procedures (Levin 1964, Ho and Lee 1965, Aström 1967, Rogers and Steiglitz 1967) to various recursive techniques (Ho and Whalen 1963, Cox 1964, Sakrison 1967). Presumably, the purpose of the identification effort is the subsequent control of the plant in some optimum fashion. Concurrent with the work on the identification problem there has been a number of investigations dealing with the control of linear systems. The initial work by Kalman and Koepcke (1958) on the optimal control of completely deterministic systems with various performance indices has been extended to include both noisy observations (Joseph and Tou 1961), and systems with random parameters (Gunckel and Franklin 1963, Farison 1964, Farison *et al.* 1967, Bar-Shalom and Sivan 1968).

It is surprising that not more has been done with the problem of controlling a linear system with unknown, constant parameters. It would appear that this is the situation more often met in practice than the independent random parameter case. In a sense, the control of linear systems with constants ('highly dependent random parameters') has been considered before (Drenick and Shaw 1964, Zadicario and Sivan 1966, Bar-Shalom and Sivan 1968). However, the problem formulation and solution are either of such generality as to preclude the possibility of abstracting much useful information, or they are too restrictive.

The present investigation is concerned primarily with 'single-stage' control in contrast to the usual N-stage control‡. Having thus narrowed the problem we are able to investigate somewhat more deeply the relationship between

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‡ Single-stage control is the logical sub-optimum procedure to use. N-stage control is discussed in § 4.

identification and control than has been previously reported, e.g. Sworder (1966).

The investigation deals exclusively with the noiseless observation case. It is our contention that the complexity of the problem is not introduced by noisy observations, although this certainly makes the problem more difficult. Rather, the complexity is introduced by assuming the unknown parameters are constants and by the specification of an N-stage control policy, whether it be closed or open-loop optimal, or open-loop feedback†.

It is our intention here to clarify the relationship between the identification problem, the control laws which may be derived with the assumption that the parameters are known, and the overall optimization problem. We make the following assumptions, which in many cases of practical interest are not overly restrictive: the performance index depends only on a scalar output signal; only scalar input and output signals are observed. In certain cases it will turn out that an overall optimum solution can be obtained by first estimating the system parameters using conditional mean estimates, and then using these in the control law derived assuming a known plant. We call this situation *complete separation*.

2. Preliminaries

Consider the system governed by the operator equation:

$$x_n = \frac{A}{B} u_n + \frac{C}{D} e_n, \tag{2.1}$$

x_n is the scalar output and u_n the scalar control. e_n represents a sequence of independent noise disturbances with $E\{e_i\} = 0$, $E\{e_i e_j\} = \sigma^2 \delta_{ij}$. The operators A, B, C and D are polynomials in the delay operator z^{-1} and are given by‡:

$$\left.\begin{aligned} A &= 1 + \sum_{i=1}^{k_1} a_i z^{-i}, \quad & B &= 1 + \sum_{i=1}^{k_2} b_i z^{-i}, \\ C &= 1 + \sum_{i=1}^{k_3} c_i z^{-i}, \quad & D &= 1 + \sum_{i=1}^{k_4} d_i z^{-i}, \end{aligned}\right\} \tag{2.2}$$

with the roots of B, C and D inside the unit circle. The above model and operator notation has been useful in previous theoretical and practical investigations (Aström 1967, Rogers and Steiglitz 1967). The operator notation is the natural one to choose if the output of the system and not the state can be observed, or only the output is included in the performance index. Furthermore, using this notation, the operations specified by the optimum control law can be stated more succinctly and are more readily interpreted in terms of filters which can subsequently be realized in a natural way on a digital computer. We assume that the plant parameters in (2.2) are unknown constants, and we denote this parameter set by the vector $\boldsymbol{\theta}$.

We consider the following control problem: choose u_n to maximize $E\{x_n{}^2\}$, having observed the input sequence $u^{n-1} = (u_{n-1}, u_{n-2}, \ldots)$ and the output

† See Dreyfus (1964) and Zadicario and Sivan (1966) and the recent survey paper by Larson (1967).

‡ For simplicity, we have assumed that the leading term of A is unity; this assumption can be relaxed.

sequence $x^{n-1} = (x_{n-1}, x_{n-2}, \ldots)$. To minimize the mean-square output $E\{x_n^2\}$, clearly it is only necessary to minimize the quantity conditioned on the past input–output records, $E\{x_n^2 \mid x^{n-1}, u^{n-1}\}$. The technique used to perform this minimization is well established in the literature (Meier 1965, Aoki 1967). Here, we only outline the steps:

$$\begin{aligned} E\{x_n^2 \mid x^{n-1}, u^{n-1}\} &= \int_{x_n}\int_{\boldsymbol{\theta}} x_n^2 p(x_n \mid x^{n-1}, u^{n-1}, \boldsymbol{\theta})\, p(\boldsymbol{\theta} \mid x^{n-1}, u^{n-1})\, d\boldsymbol{\theta}\, dx_n \\ &= \int_{\boldsymbol{\theta}} p(\boldsymbol{\theta} \mid x^{n-1}, u^{n-1})\, E\{x_n^2 \mid x^{n-1}, u^{n-1}, \boldsymbol{\theta}\}\, d\boldsymbol{\theta} \\ &= \langle E\{x_n^2 \mid x^{n-1}, u^{n-1}, \boldsymbol{\theta}\}\rangle_{n-1}, \end{aligned} \tag{2.3}$$

where we have defined $\langle f(\boldsymbol{\theta})\rangle_{n-1}$ as the *a posteriori* conditional mean:

$$\langle f(\boldsymbol{\theta})\rangle_{n-1} = \int f(\boldsymbol{\theta})\, p(\boldsymbol{\theta} \mid x^{n-1}, u^{n-1})\, d\boldsymbol{\theta}. \tag{2.4}$$

The inner expression in (2.3) is now written as:

$$\begin{aligned} E\{x_n^2 \mid x^{n-1}, u^{n-1}, \boldsymbol{\theta}\} &= \int p(u_n \mid x^{n-1}, u^{n-1}, \boldsymbol{\theta})\, E\{x_n^2 \mid x^{n-1}, u^n, \boldsymbol{\theta}\}\, du_n \\ &= E\{x_n^2 \mid x^{n-1}, u_n^*, u^{n-1}, \boldsymbol{\theta}\}. \end{aligned} \tag{2.5}$$

The second line of (2.5) follows from the observation that if u_n is chosen to minimize $E\{x_n^2 \mid x^{n-1}, u^n, \boldsymbol{\theta}\}$ for every given set x^{n-1}, u^{n-1}, the left-hand side is minimized if $p(u_n \mid x^{n-1}, u^{n-1}, \boldsymbol{\theta})$ is chosen as a delta function $\delta(u_n - u_n^*)$; i.e. non-randomized control is optimum (see Aoki 1967, p. 28). In the sequel, we denote the minimizing u_n by u_n^*.

Proceeding to calculate (2.5), we rewrite (2.1) as:

$$x_n = e_n + \frac{A}{B} u_n + \left(\frac{C}{D} - 1\right) e_n. \tag{2.6}$$

Observe that the expression $(C/D - 1)\, e_n$ is a function of e^{n-1} and not e_n. Hence, if x^{n-1} and u^{n-1} are observed, $(C/D - 1)\, e_n$ can be calculated prior to choosing u_n. Substituting $e_n = (D/C)(x_n - A/Bu_n)$ in (2.6) and rearranging terms gives:

$$x_n = e_n + u_n + \left(\frac{D}{C}\frac{A}{B} - 1\right) u_n + \left(1 - \frac{D}{C}\right) x_n. \tag{2.7}$$

Then, (2.5) becomes:

$$\begin{aligned} E\{x_n^2 \mid x^{n-1}, u^{n-1}\} &= \langle E\{x_n^2 \mid x^{n-1}, u^n, \boldsymbol{\theta}\}\rangle_{n-1} \\ &= \left\langle \sigma^2 + \left[u_n + \left(\frac{D}{C}\frac{A}{B} - 1\right) u_n + \left(1 - \frac{D}{C}\right) x_n\right]^2 \right\rangle_{n-1}. \end{aligned} \tag{2.8}$$

Since the first term in the numerator of the operator $[(D/C)(A/B) - 1]$ is (const.) z^{-1}, only the first term in (2.8) depends on u_n. Hence, differentiating with respect to u_n and solving for u_n^*, we obtain:

$$\frac{\partial}{\partial u_n} E\{x_n^2 \mid x^{n-1}, u^{n-1}\} = 2\left\langle u_n + \left(\frac{D}{C}\frac{A}{B} - 1\right) u_n + \left(1 - \frac{D}{C}\right) x_n \right\rangle_{n-1} \tag{2.9}$$

and

$$u_n^* = \left\langle\left(1-\frac{DA}{BC}\right)\right\rangle_{n-1} u_n^* + \left\langle\left(\frac{D}{C}-1\right)\right\rangle_{n-1} x_n$$

$$= \langle Q_1\rangle_{n-1} u_n^* + \langle Q_2\rangle_{n-1} x_n, \tag{2.10}$$

where†

$$\left.\begin{aligned} Q_1 &= 1-\frac{DA}{CB}, \\ Q_2 &= \frac{D}{C}-1. \end{aligned}\right\} \tag{2.11}$$

Recall that $\langle\ \rangle_{n-1}$ is a conditional mean estimate and as such, is a minimum mean-square (MMS) estimate. Equation (2.10) has the following interpretation: the overall optimum control law results from the use of the MMS criterion in estimating the operators Q_1 and Q_2. This result is not surprising—a quadratic cost has been used as the performance index for a single-stage control problem.

3. The relationship between system identification and control

In most practical situations, we are usually restricted to estimating the set of parameters $\boldsymbol{\theta}$, rather than the operators Q_1 and Q_2. In such cases we are interested in the possible overall optimality of a system which uses the estimated parameters $\boldsymbol{\theta}$ in the control law determined by Q_1 and Q_2. This consideration motivates the following definition.

Definition

Complete separation of identification and control is said to take place if:

$$\left.\begin{aligned} &\langle Q_1(\boldsymbol{\theta})\rangle_{n-1} = Q_1(\langle\boldsymbol{\theta}\rangle_{n-1}) \\ \text{and}\quad &\langle Q_2(\boldsymbol{\theta})\rangle_{n-1} = Q_2(\langle\boldsymbol{\theta}\rangle_{n-1}). \end{aligned}\right\} \tag{3.1}$$

We can then state the following theorem, which follows directly from the linearity of the operation $\langle\ \rangle_{n-1}$.

Theorem 1

Complete separation takes place if Q_1 and Q_2 are both linear in $\boldsymbol{\theta}$.

The following cases are of interest.

Corollary

Complete separation takes place if any of the following situations occur:

(1) A, C known; $B = D$ unknown,

(2) A, B, C known; D unknown,

(3) B, C, D known; A unknown.

To illustrate complete separation, consider the following example.

† Note that Q_1 and Q_2 are operators on the past. If the plant parameters were known, u_n^* is given by $u_n^* = Q_1 u_n^* + Q_2 x_n$, i.e. u_n^* is a function only of the sequences u^{n-1}, x^{n-1}. See also Aoki (1964) and Dreyfus (1964).

Example 1

Take $B = D = 1+\sum_{i=1}^{k_2} b_i z^{-i}$, A and C as known operators. For simplicity, we set $A = C = 1$, and (2.1) becomes:

$$x_n = (u_n + e_n)/B, \tag{3.2}$$

or, writing out the operation explicitly:

$$x_n = -\sum_{i=1}^{k_2} b_i x_{m-i} + u_n + e_n.$$

The desired control is given by (2.10):

$$\begin{aligned} u_n^* &= \left(1-\frac{A}{C}\right) u_n^* + \left\langle\left(\frac{B}{C}-1\right)\right\rangle_{n-1} x_n \\ &= \left\langle \sum_{i=1}^{k_2} b_i z^{-i} \right\rangle_{n-1} x_n \\ &= \sum_{i=1}^{k_2} \left\{ \int b_i p(b_i \,|\, x^{n-1}, u^{n-1})\, db_i \right\} x_{n-i}. \end{aligned} \tag{3.3}$$

Since the parameters b_i appear linearly, the expectation with respect to $p(\boldsymbol{\theta} \,|\, x^{n-1}, u^{n-1})$ reduces to that shown in the last line of (3.3). Hence:

$$u_n^* = \sum_{i=1}^{k_2} \langle b_i \rangle_{n-1} x_{n-i}. \tag{3.4}$$

The identification and control problems separate completely if the MMS estimate is used as the criterion for the identification of systems parameters. With more general, but known operators A and C, u_n^* becomes:

$$u_n^* = -\sum_{i=1}^{k_1} a_i u_{n-i}^* - \sum_{i=1}^{k_3} c_i x_{n-i} + \sum_{i=1}^{k_2} \langle b_i \rangle_{n-1} x_{n-i}. \tag{3.5}$$

It should be noted that if separation does not occur, then even the single-stage control law, viewed as a sub-optimum procedure, is generally quite complicated. This is amply illustrated by the following 'simple' system.

Example 2

The operators A, C and D are known and taken equal to unity. B is given by $B = 1-bz^{-1}$, with the single parameter b unknown. From (2.10), the optimum control is given by:

$$u_n^* = -\left(\left\langle \frac{1}{B} \right\rangle_{n-1} - 1\right) u_n^*. \tag{3.6}$$

Note that $1/B$ can be written as:

$$\frac{1}{B} = \frac{1}{1-bz^{-1}} = \sum_{i=0}^{\infty} b^i z^{-i}. \tag{3.7}$$

Then:

$$u_n^* = -\sum_{i=1}^{\infty} \langle b^i \rangle_{n-1} u_{n-i}^*. \tag{3.8}$$

Therefore, at each stage, optimality requires the estimation of all moments of b:

$$\langle b^i \rangle_{n-1} = \int b^i p(b \mid x^{n-1}, u^{n-1})\, db.$$

The control law (3.8) can be interpreted as follows: $\langle b^i \rangle_{n-1}$ represents the (time-varying) pulse response of the system which generates the optimum control u_n^* by filtering the previous controls u_{n-i}^*.

Interestingly, the pulse response for fixed n is not, in general, a rational function of z, and there is no simple recursive implementation for the filter.

Having encountered these difficulties for the single-stage, noiseless observation case, it is clear that the N-stage control problem even with perfect observations is considerably more complex. It is not surprising, then that many investigations have dealt with sub-optimum procedures (Aoki 1964, Dreyfus 1964, Sworder 1966, Zadicario and Sivan 1966, Farison *et al.* 1967, Bar-Shalom and Sivan 1968).

4. Closed-loop N-stage control

The difficulty in studying complete separation for the N-stage control problem is related to the fact that, in general, the form of the control sequence, even for the known parameter problem, is unknown. Nevertheless, a partial characterization can be given.

By a closed-loop control policy we mean that the optimal control sequence for N-stages, associated with each possible state of the system, is specified (Dreyfus 1964). This implies, for example, that at a particular state, future outputs are anticipated through their correlations with the data on hand, and are incorporated into the present control. Consider, first, the last two stages, where it is desired to minimize $E\{(x_n^2 + x_{n-1}^2)\}$. Again, this quantity is minimized if u_n and u_{n-1} are chosen to minimize:

$$E\{(x_n^2 + x_{n-1}^2) \mid x^{n-2}, u^{n-2}\},$$

for every set $\{x^{n-2}, u^{n-2}\}$. This conditional expectation can be written as:

$$\begin{aligned} &E\{(x_n^2 + x_{n-1}^2) \mid x^{n-2}, u^{n-2}\} \\ &\quad = E\{(x_n^2 \mid x^{n-2}, u^{n-2})\} + E\{(x_{n-1}^2 \mid x^{n-2}, u^{n-2})\} \\ &\quad = E\{(x_n^2 \mid x^{n-2}, u^{n-2})\} + \langle [\sigma^2 + (u_{n-1} - Q_1 u_{n-1} - Q_2 x_{n-1})^2] \rangle_{n-2}, \end{aligned} \tag{4.1}$$

where the last line follows from (2.10) and the definitions introduced in (2.11). The first expression on the right-hand side of (4.1) is minimized by choosing u_n. Since u_n is applied after x_{n-1} has been observed, and since a *closed-loop policy* has been specified, it is required to incorporate the pair x_{n-1}, u_{n-1}, into the expectation. This is accomplished through the application of Bayes' rule (Meier 1965, Aoki 1967). The result is:

$$\begin{aligned} E\{(x_n^2 \mid x^{n-2}, u^{n-2})\} = \Big\langle \int & p(x_{n-1} \mid x^{n-2}, u^{n-2}, \boldsymbol{\theta}) \\ & \times \langle \sigma^2 + (u_n - Q_1 u_n - Q_2 x_n)^2 \rangle_{n-1}\, dx_{n-1} \Big\rangle_{n-2}. \end{aligned} \tag{4.2}$$

Equation (4.2) is substituted for the first expression on the right-hand side of (4.1), and the resulting expression is minimized with respect to the controls u_{n-1}, u_n.

Defining the operator:

$$L(u_n, x_n) \equiv \sigma^2 + (u_n - Q_1 u_n - Q_2 x_n)^2 \tag{4.3}$$

and the conditional expectation:

$$E_j\{f(x_j)\} = \int f(x_j)\, p(x_j \,|\, x^{j-1}, u^{j-1}, \boldsymbol{\theta})\, dx_j, \tag{4.4}$$

the quantity to be minimized is:

$$\begin{aligned} &E\{(x_n{}^2 + x_{n-1}{}^2 \,|\, x^{n-2}, u^{n-2})\} \\ &\quad = \langle E_{n-1}\{\langle \sigma^2 + (u_n - Q_1 u_n - Q_2 x_n)^2\rangle_{n-1}\}\rangle_{n-2} \\ &\qquad\qquad + \langle \sigma^2 + (u_{n-1} - Q_1 u_{n-1} - Q_2 x_{n-1})^2\rangle_{n-2} \\ &\quad = \langle L(u_{n-1}, x_{n-1}) + E_{n-1}\{\langle L(u_n, x_n)\rangle_{n-1}\}\rangle_{n-2}. \end{aligned} \tag{4.5}$$

By way of comparison, the single-stage control policy requires the minimization of [see (2.8)]:

$$E\{x_j{}^2 \,|\, x^{j-1}, u^{j-1}\} = \langle L(u_j, x_j)\rangle_{j-1}, \quad j = n-1, n. \tag{4.6}$$

It is clear that complete separation will occur if the conditions of the above corollary are met and if the N-stage control problem factors into N single-stage controls. Though it is not possible to give a complete characterization of the conditions for factorization, special cases can be investigated. Consider a specific case of Example 1.

Example 3

Take $A = C = 1$, $B = D = 1 + bz^{-1}$, with the parameter b unknown. From (3.3), the last control is:

$$u_n{}^* = \langle b\rangle_{n-1} x_{n-1}.$$

Let us now assume that the *a posteriori* variance of the estimate is independent of the past input–output record, but does depend on the number of observations†:

$$\langle (b - \langle b\rangle_{m-i})^2\rangle_{n-i} = \int (b - \langle b\rangle_{n-i})^2 p(b \,|\, x^{n-i}, u^{n-i})\, db = K_{n-i}, \quad i = 1, 2, \ldots. \tag{4.7}$$

We then have:

$$\langle L(u_n{}^*, x_n)\rangle_{n-1} = \sigma^2 + K_{n-1} x_{n-1}{}^2 \tag{4.8}$$

and from (4.4):

$$E_{n-1}\{\langle L(u_n{}^*, x_n)\rangle_{n-1}\} = \sigma^2 + K_{n-1} L(u_{n-1}, x_{n-1}).$$

Hence, for this example, (4.5) and (4.6) are equivalent:

$$u_{n-i} = \langle b\rangle_{n-i} x_{n-i}, \quad i = 1, 2.$$

This example can be generalized to N stages by noting that to minimize:

$$R_{k+1} = E\{(x_N{}^2 + x_{N-1}{}^2 + \ldots + x_{N-k}{}^2 \,|\, x^{N-k-1}, u^{N-k-1}\},$$

† Note that (4.7) does not necessarily require a Gaussian assumption.

one can obtain the recurrence relation†:

$$R_{k+1} = \min_{u_{N-k}}\{\langle L(u_{N-k}, x_{N-k}) + E_{N-k}\{R_k\}\rangle_{N-(k+1)}.$$

Then, assuming that

$$\left.\begin{aligned} u_{N-(k+1)}{}^* &= \langle b\rangle_{N-k}\, x_{N-k}, \\ R_k &= \alpha_k + \beta_k x_{n-k}{}^2, \end{aligned}\right\} \tag{4.9}$$

it can be established by induction that $u_{N-k}{}^*$ and R_{k+1} are of this form with α and β satisfying:

$$\left.\begin{aligned} \alpha_{k+1} &= \alpha_k + \sigma^2(1+\beta_k), \\ \beta_{k+1} &= (1+\beta_k)\, K_{N-(k+1)}. \end{aligned}\right\} \tag{4.10}$$

Consequently, $u_{N-j}{}^* = \langle b\rangle_{N-j-1}\, x_{N-j-1}$, $j = 0, 1, \ldots, N-1$, i.e. both factorization and complete separation occur. Processes for which the *a posteriori* variance satisfies (4.7), and conditions under which complete separation occurs without factorization are presently under investigation.

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† See Aoki (1967) for the details.