The identity component of the isometry group of a compact Lorentz manifold

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1 Introduction

The goal of the present article is to prove the following:

Theorem 1.1 *Let the affine group AG act isometrically on a compact Lorentz manifold $(M, <,>)$. Then some finite cover $PSL_k(2, \mathbf{R})$ of $PSL(2, \mathbf{R})$ acts isometrically on M. In fact the initial action of AG is contained in an isometric action of $PSL_k(2, \mathbf{R}) \times \mathbf{T}$, where $\mathbf{T}$ is a torus of some dimension.*

This result may be compared with a Theorem of E. Ghys [Ghy] (see also [Bel]), asserting a similar conclusion, but assuming that M has dimension 3, and that the action is just volume preserving and locally free. The statement there, is that the action of AG may be extended to an action of a finite cover of $PSL(2, \mathbf{R})$, or to an action of the solvable 3-dimensional Lie group SOL.

Here we have another motivation. We want to understand the structure of Lie groups acting isometrically on compact Lorentz manifold. The first results in the subject are due to [Zim] and [Gro]. A "final" result is due to [A-S] and [Zeg1], independently. Necessary and sufficient conditions were given in order that a Lie group acts isometrically (and locally faithfully) on a compact Lorentz manifold. Note however, that if a group acts in such a fashion, then its subgroups also act in the same way. For instance, all known examples of isometric actions of AG are obtained by viewing it as a subgroup of $SL(2, \mathbf{R})$. So a natural question is: what are the maximal (connected) Lie groups acting isometrically on a compact Lorentz manifold? Equivalently:

Question 1.2 *What is be the identity component $Isom^0(M)$ of the isometry group of a compact Lorentz manifold M ?*

In dimension 3, there is a result of [Zeg2] which describes the geometric structure of a compact Lorentz manifolds (of dimension 3) M, with $Isom^0(M)$ non compact. It has the following corollary:

Theorem 1.3 *([Zeg2]) If a compact Lorentz 3-manifold M, has $Isom^0(M)$ non compact, then $Isom^0(M)$ is isomorphic to $\mathbf{R}$ or to a finite cover of $PSL(2, \mathbf{R})$.*

Let us now recall the result of [A-S] and [Zeg1]. To simplify, we will state the following results only in the Lie algebra level:

Theorem 1.4 *([A-S] and [Zeg1]) Let G be a (connected) Lie group acting isomtrically on a compact Lorentz manifold M. Then the Lie algebra $\mathcal{G}$ has a direct decomposition (in the sense of algebras): $\mathcal{G} = \mathcal{K} + \mathcal{A} + \mathcal{S}$, where $\mathcal{K}$ is the Lie algebra of a compact semi-simple Lie group, $\mathcal{A}$ is an abelian algebra. The Lie subalgebra $\mathcal{S}$ is trivial, or isomorphic to one of the following: $sl(2, \mathbf{R})$, the Lie algebra of the affine group, a Heisenberg algebra (of some dimension), or finally to a "warped Heisenberg algebra".*

We recall that a warped Heisenberg group [Zeg1] is a semi-direct product of the circle S^1 with an Heisenberg group. The action by automorphism of S^1 must satisfy some rationality and positiveness conditions. Our next result is:

Theorem 1.5 (Structure of the Lie algebra of the isometry group) *The Lie algebra of the isometry group of a compact Lorentz manifold is isomorphic to a direct sum* $\mathcal{K} + \mathbf{R}^k + \mathcal{S}$, *where* $\mathcal{K}$ *is the Lie algebra of a compact semi-simple Lie group,* k *is an integer and* $\mathcal{S}$ *is trivial or isomorphic to: a Heisenberg algebra (of some dimension), a warped Heisenberg algebra or finally to* $sl(2,\mathbf{R})$.

Conversely, any such algebra is isomorphic to the Lie algebra of the isometry group of some compact Lorentz manifold.

This theorem is an improvement of the previous one. It means essentially that the affine group can not occur as exactly the identity component of the isometry group of a compact Lorentz manifold.

This theorem determines the Lie algebra of the isometry group, but not the group itself. Indeed, it is remarkable, as we will notice in §4, that this group is generally not simply connected.

2 Partial hyperbolic structure

2.1 Notations

We will manipulate a lot of vector fields. For this we will use the (convenient) notation U^t for the flow of a vector field U. The image by U^t of a vector field V will be simply denoted by U^tV (instead of the classical $(U^t)^*V$). We will also use the same notation for images of other quantities, such as functions, plane fields, differential forms...

For a collection of vector fields (or vectors) $X_1, ..., X_k$, we denote by $\{X_1, ..., X_k\}$ the plane field that they generate.

Let $(M, <,>)$ be a compact Lorentz manifold. For a subbundle E of TM, we will denote its orthogonal by $E^\perp$.

Suppose that M is endowed with an action of the affine group AG. This is equivalent to giving two Killing fields X and Y, with $[X,Y] = -Y$. This integrates to the identity: $X^tY^s = Y^{s\exp(-t)}X^t$.

2.2 The unstable space of X

Fact 2.1 *The action of* AG *is everywhere locally free. Furthermore* Y *is isotropic and orthogonal to* X, *which is everywhere spacelike:* $< X(x), X(x) >> 0$, $\forall x \in M$.

Proof. The fact that AG acts locally freely is due to [A-S] (see also [Zeg3]). To see that Y, is isotropic, consider its length function $f(x) =< Y(x), Y(x) >$. Taking the image by X^t, yields: $X^tf = \exp -2tf$. Thus $f = 0$ since it is bounded. By the same argument we prove that $< X, Y >= 0$, everywhere. Since the action of AG is locally free, X is nowhere isotropic: $< X, X >\neq 0$ (otherwise $\{X, Y\}$ would be a 2 dimensional isotropic plane). Necessarily X is spacelike, since if X was timelike, the metric on its orthogonal would be positive definite, but here, we know that the orthogonal contains a non trivial isotropic vector, Y. □

Fact 2.2 *The stable Lyapunov space of* X^t *is exactly* $\mathbf{R}Y$. *It is associated the the unique negative Lyapunov exponent* -1.

Proof. This is a consequence of the following lemma, which may be proved in a standard way (see for example [Wal] for definitions):

Lemma 2.3 *A Lyapunov space of an isometric flow X^t on a finite volume Lorentz manifold, associated to a non trivial Lyapunov exponent, is isotropic. In particular the stable and unstable Lyapunov spaces are isotropic and have dimension ≤ 1. Furthermore, these spaces are orthogonal to X.* □

Fact 2.4 *There is a unique measurable isotropic vector field Z such that: $X^t Z = \exp(t) Z$, and $< Z, Y >= 1$.*

Z is unique, up to multiplication by a measurable X^t-invariant function, among measurable vector fields U such that $X^t U = \exp(t) U$.

Furthermore, any measurable vector field U satisfying almost everywhere $X^{t_n} U \to 0$, for some $t_n \to +\infty$, is a multiple of Z by a measurable function.

Proof. The theory of Lyapunov exponents [Wal] implies the existence of an unstable space. From the lemma above, this space has exactly dimension 1. Thus it is associated to the (unique positive) exponent $+1$ (the sum of positive and negative exponents is 0). Of course it is not orthogonal to the stable space, since both of them are isotropic. Thus we can choose uniquely a generating vector field Z such that $< Z, Y >= 1$. Write $X^t Z = \rho(x, t) Z$. Then since X^t preserves the metric $<,>$, and $X^t Y = \exp -tY$, we have $\rho(x, t) = \exp t$. The rest of the proof is standard. □

2.3 The easiest case: dimension 3

In what follows, we give a quick proof valid in dimension 3. Observe that, everywhere, the metric on the orthogonal of X, is Lorentzian. Since this orthogonal has dimension 2, it has exactly 2 isotropic directions, which thus coincide with the direction of Y and Z. In particular the direction of Z is smooth, and hence Z itself is smooth since its size is determined by the algebraic condition $< Y, Z >= 1$.

Consider $T = [Y, Z]$. The jacobi's identity implies that T commutes with X. However, from the continuity of Z, one conclude that X^t is an Anosov flow. Therefore, T is collinear to X. In fact by ergodicity of X (as a volume preserving Anosov flow), T is a (constant) multiple of X. Therefore, the three vector fields X, Y and Z determine a Lie algebra $\mathcal{G}$. One easily see that $\mathcal{G}$ is isomorphic to $sl(2, \mathbf{R})$ unless $T = 0$, in which case $\mathcal{G}$ would be isomorphic to the solvable algebra sol. But in this case, Z commutes with Y, and in particular Y^t would preserve the orthogonal of Z. This last orthogonal is a 2 -plane field containing X, but transverse to Y. However, the identity: $[X, Y] = -Y$, integrates to $Y^t X = X - tY$. In particular, under Y^t, the direction of X tends to the direction of Y, when $t \to \infty$. This contradicts the fact that Y^t preserves a transverse plane field which contains X. Therefore $\mathcal{G}$ is isomorphic to $sl(2, \mathbf{R})$.

It is clear that the group G determined by $\mathcal{G}$ acts transitively and locally freely. Thus M can be identified with a quotient G/Γ, where Γ is a discrete subgroup of G.

The metric on M, lifts to a right invariant metric on G, since the scalar products of the fundamental fields X, Y and Z are constant. Such a metric is determined by giving a Lorentzian scalar product on $\mathcal{G}$. In order that the left action of G to be isometric, it is sufficient and necessary that this scalar product is bi-invariant. But in the case of $sl(2, \mathbf{R})$, this means that it is a multiple of the Killing form, or equivalently, that our scalar product vanishes on the light cone of the Killing form. This last cone is exactly the set of nilpotent matrices of $sl(2, \mathbf{R})$.

In our case, we know that Y and Z which (by construction) correspond to nilpotent matrices, are isotropic. On the other hand, since Y^t acts isometrically, all the elements : $(\exp(tY))Z$ are isotropic. But any nilpotent matrix in $sl(2, \mathbf{R})$ is a multiple of Y or some $(\exp(tY))Z$. Therefore our scalar product is a multiple of the Killing form. □

2.4 Asymptotic foliations

Here we present a special result concerning asymptotic foliations associated to non equicontinuous Killing fields (see [D-G] and [Zeg3]). A fundamental fact about these foliations is the following proposition. It describes a uniformity property of isometry of Lorentz metrics, due to their "linearisability":

Proposition 2.5 *Let M be a compact Lorentz manifold, $x_0 \in M$, and f_i a sequence of isometries. Let $V_{x_0} \subset T_{x_0}M$, be such that $\exp_{x_0}(V_{x_0})$ is a convex neighborhood of x_0. Let $P \subset T_{x_0}M$ be a vector subspace, and suppose that the derivatives $D_{x_0}f_i/P$ are uniformly bounded (in the sense of an auxiliary norm). Then $D_x f_i$ are uniformly bounded when x runs over $\exp_{x_0}(P \cap V_{x_0})$*

Proof. If all the f_i fix x_0: $f_i(x_0) = x_0$, then after conjugacy by $\exp_{x_0}$, the problem becomes linear. Observe that by our hypothesis, some fixed neighborhood of 0 in P is contained in the domain of linearization of all the f_i. The conclusion of the proposition is hence clear in this case.

When x_0 is not a fixed point of the sequence f_i, we just compost with linear isometries identifying $T_{f_i(x_0)}M$ with $T_{x_0}M$ (observe that we consider just a sequence of isometries, that has not necessarily a group-like structure. So making the previous identification does not cause a loss of structure). □

Proposition 2.6 *The subbundles of codimension 1, $\mathbf{R}Y^\perp$ and $\mathbf{R}Z^\perp$ are integrable and have geodesic leaves.*

Proof. Observe that the 2-plane field $\{Y, Z\}$ is Lorentzian since the restriction of the metric to it has exactly two isotropic directions (those of Y and Z). Hence the metric on the orthogonal $\{Y, Z\}^\perp$ is Riemannian. Therefore X^t acts on $\{Y, Z\}^\perp$ equicontinuously, that is, if $u \in \{Y, Z\}^\perp$, $\{X^t u / t \in \mathbf{R}\}$ is bounded (in the sense of an auxiliary norm). Observe that $\mathbf{R}Y^\perp = \mathbf{R}Y \bigoplus \{Y, Z\}^\perp$ and also $\mathbf{R}Z^\perp = \mathbf{R}Z \bigoplus \{Y, Z\}^\perp$. Note the following straightforward dynamical interpretation of $\mathbf{R}Y^\perp$ and $\mathbf{R}Z^\perp$:

Fact 2.7 *Let $u \in TM$. Then $u \in \mathbf{R}Y^\perp$ (resp. $\mathbf{R}Z^\perp$), if and only if $X^t u$, is bounded for $t > 0$ (resp. $t < 0$).*

From the proposition above (2.5), one easily deduce that for $x \in M$, $\exp_x \mathbf{R}Y^\perp$ (resp. $\exp_x \mathbf{R}Z^\perp$) is everywhere (not only at x) tangent to $\mathbf{R}Y^\perp$ (resp. $\mathbf{R}Z^\perp$). This finishes the proof of the Proposition 2.6. □

Corollary 2.8 *The vector field Z is Lipschitz. Moreover, both Y and Z have isotropic geodesic orbits.*

Proof. A codimension 1 geodesic foliation in a pseudo-reimannian manifold is locally Lipschitz. This is well known in the riemannian case, but the proof extends straightforwardly to the pseudo-riemannian case (see for instance [Zeg4] and [Sol]). Thus $\mathbf{R}Z^\perp$ is Lipschitz, and hence the direction of Z which is nothing but $(\mathbf{R}Z^\perp)^\perp$, is also Lipschitz. Thus Z itself is Lipschitz since it is defined by just adding the algebraic equation: $< Z, Y >= 1$.

Let us now show that Y and Z have geodesic orbits. From the proposition above, a geodesic which is somewhere tangent to $\mathbf{R}Y$ (resp. $\mathbf{R}Z$) is everywhere isotropic and tangent to $\mathbf{R}Y^\perp$ (resp. $\mathbf{R}Z^\perp$).

Observe now that the orbits of Y (resp. Z) are characterized by being the unique (geometric) isotropic curves tangent to the leaves of $\mathbf{R}Y^\perp$ (resp. $\mathbf{R}Z^\perp$). □

In fact, there is another method (valid only for Killing fields) to prove that Y has geodesic orbits:

Lemma 2.9 *A Killing field Y on a pseudo-riemannian manifold with constant length $< Y, Y >$ (for example Y isotropic) has geodesic orbits: $\nabla_Y Y = 0$.*

Proof. Recall the fact that Y is a Killing field is equivalent to that: the linear map $DY : u \to \nabla_u Y$ is antisymmetric with respect to $<,>$, i.e. $< \nabla_u Y, v > + < u, \nabla_v Y >= 0$. In particular: $< \nabla_Y Y, u > + < \nabla_u Y, Y >= 0$. Thus $< \nabla_Y Y, u >= -(1/2)u < Y, Y >= 0$. □

2.5 Partial ergodicity

Proposition 2.10 *If a measurable bounded function is invariant by X^t, then it is invariant by Y^t and Z^t.*

Proof. Let f be a X^t-invariant bounded measurable function. Consider for fixed s, the function $g = Z^s f - f$. Then: $X^t g = X^t Z^s f - f = Z^{s \exp t} f - f$. Thus $\int |g| = \int |X^t g| = \int |Z^u f - f|$, for any u with the same sign as s. But since Z^u is Lipschitz, the last integral tends to 0, when $u \to 0$. Therefore $g = 0$, that is f is Z^s-invariant. The same argument works for Y^s. □

3 Proof of Theorem 1.1

3.1 The bracket $T = [Y, Z]$ and the subbundle $\mathcal{E}$

The Lie bracket is well defined for two Lipschitz vector fields. So the relation $X^t Z = \exp(t)Z$, translates to the meaningful identity: $[X, Z] = Z$.

Let $T = [Y, Z]$, it is a priori merely measurable (and bounded). If T were sufficiently smooth, Jacobi's identity yields: $[X, T] = [X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] = 0$, that is T is X^t-invariant.

One may give the following justification to this last claim, that is, without any regularity assumption, T is invariant by X^t. Indeed let τ be a small transversal to X, and $\bar{Z}_k$ is a sequence of smooth vector fields on τ converging in the Lipschitz topology to the restriction Z/τ. Extend $\bar{Z}_k$ along the X-orbits, in a neighborhood of τ such that $X^t \bar{Z}_k = \exp(t)\bar{Z}_k$, that is $[X, \bar{Z}_k] = \bar{Z}_k$. Here the Jacobi identity implies that $T_k = [Y, Z_k]$ is X^t-invariant. The same fact is valid for T since it is a simple limit of T_k. So we have proved:

Fact 3.1 *X^t preserves T.*

Remark 3.2 *The previous argument is in fact essentially equivalent to taking Lie derivative in the sense of distributions. That is for X smooth and V a distribution valued vector field, one may consider the derivative $L_X V = [X, V]$ as a distribution valued vector field. If X and Y are smooth, then we have the meaningful Jacobi's identity: $L_{[X,Y]} V = L_X L_Y V - L_Y L_X V$.*

Fact 3.3 *For almost every $x \in M$, $T(x) \neq 0$.*

Proof. The proof is as in 2.3. If $T = 0$, Z will be preserved by Y^t, and hence the orthogonal $\mathbf{R}Z^\perp$ is also preserved by Y^t. This codimension 1 subbundle contains X but does not contain Y. However $Y^t X = X + tY$, and hence this converges in direction to Y. Contradiction. □

Notation: Let $\mathcal{E} = \{Y, Z, T\}$. It has almost everywhere dimension 3.

Fact 3.4 *(i) The flow Y^s preserves $\mathcal{E}$, and hence also $\mathcal{E}^\perp$.*

(ii) $\mathcal{E}$ is Lipschitz, and hence so $\mathcal{E}^\perp$ is too.

Proof. Consider the difference $V = Y^s T - T$. Then $X^t V = X^t Y^s T - X^t T = Y^{s \exp -t} X^t T - X^t T = Y^{s \exp -t} T - T$, since $X^t T = T$. This tends to 0, when $t \to +\infty$. Thus V is collinear to Y. In particular $Y^s T \in \mathcal{E}$.

Consider now $V = Y^s Z - Z - sT$. Then $X^t V = X^t Y^s Z - X^t Z - s X^t T = Y^{s \exp -t} X^t Z - X^t Z - s X^t T$. But $X^t Z = \exp tZ$, and $X^t T = T$. Therefore: $X^t V = \exp t (Y^{s \exp -t} Z - Z) - sT$. Since $[Y, Z] = T$, we have $Y^{s \exp -t} Z - Z = s \exp -tT + (s \exp -t)\epsilon$, where ϵ is vector field converging to 0, when $t \to +\infty$. Therefore $X^t V \to 0$, when $t \to +\infty$. As above this implies that V is collinear to Y. In particular $Y^s Z \in \mathcal{E}$.

To see that $\mathcal{E}$ is Lipschitz, observe that it is generated by Y, Z and $Y^s Z$, for any $s \neq 0$. □

Fact 3.5 *The restriction of the metric on $\mathcal{E}$ is Lorentzian, and its restriction on $\mathcal{E}^\perp$ is Riemannian. In particular, AG acts equicontinuously on $\mathcal{E}^\perp$, that is the trajectories under the AG action of the elements of $\mathcal{E}^\perp$ are bounded. In contrast the action of AG on $\mathcal{E}$ is quasi-Anosov, that is for $u \in \mathcal{E}$, $u \neq 0$, the trajectory $\{gu, g \in AG\}$ is unbounded.*

Proof. $\mathcal{E}$ is lorentzian since $\{Y, Z\}$ itself is lorentzian (this last fact is true, as in the proof of 2.6, since both Y and Z are isotropic).

To see that the action of AG on $\mathcal{E}$ is quasi-Anosov, observe firstly that X^t acts in an Anosov way on $\{Y, Z\}$, and so in particular, if $u \in \{Y, Z\}$, then $\{gu/g \in AG\}$ is unbounded. If u is (somewhere) collinear to T, then by the proof above, for fixed $s \neq 0$, $Y^s u$ is not collinear to T: $Y^s u = u_1 + u_2 \in \{Y, Z\} \bigoplus \mathbf{R}T$, where $u_1 \neq 0$. The trajectory under X^t of u_1 (resp. u_2) is unbounded (resp. bounded, since X^t commutes with T). Thus the trajectory of $u_1 + u_2$ by X^t is unbounded. □

Fact 3.6 *$\mathcal{E}^\perp$ is integrable and has geodesic leaves.*

Proof. Consider the tensor $S : \mathcal{E}^\perp \times \mathcal{E}^\perp \to \mathcal{E}$, $S(A, B) =$ the orthogonal projection of $\nabla_A B$ on $\mathcal{E}$. Here ∇ is the Levi-Civita connection, and A and B are sections of $\mathcal{E}^\perp$.

This tensor is AG-equivariant: $S(gA, gB) = gS(A, B)$, for $g \in AG$. It then follows from the Fact above, that AG acts equicontinuously on the image of S. But, again by the above Fact, the action of AG on $\mathcal{E}$ is quasi-Anosov. Thus $S = 0$. This means precisely that $\mathcal{E}^\perp$ is integrable and has geodesic leaves. □

3.2 A change of T

Instead of T, we will make use of a collinear vector field H, that is the orthogonal projection of X on $\mathcal{E}$.

Fact 3.7 *H is Lipschitz, and satisfies: $[H, X] = 0$ and $[H, Y] = -Y$.*

Proof. Write $X = H + H'$, where $H' \in \mathcal{E}^\perp$. Then: $0 = [X, X] = [X, H] + [X, H'] = 0 + 0$, since X^t preserves both $\mathcal{E}$ and $\mathcal{E}^\perp$.

We have: $-Y = [X, Y] = [H, Y] + [H', Y]$. Therefore, by Y^t-invariance of both of the subbundles $\mathcal{E}$ and $\mathcal{E}^\perp$: $[H, Y] = -Y$ (and $[H', Y] = 0$). □

Fact 3.8 *H^t preserves $\mathcal{E}^\perp$ and acts isometrically on it.*

Proof. H commutes with X and satisfies the relation: $H^u Y^s = Y^{s \exp -u} H^u$. Thus, for fixed u, the subbundle $H^u(\mathcal{E}^\perp)$ is AG-invariant. In fact also the image $H^u(g)$ of the restriction g of the Lorentz metric on $\mathcal{E}^\perp$ (which is in fact a Riemannian metric) is AG-invariant. Hence AG acts equicontinuously on the projection of $H^u(\mathcal{E}^\perp)$ on $\mathcal{E}$, which contradicts 3.5, unless this projection is trivial.

Now to see that H^t preserves the metric on $\mathcal{E}^\perp$, we recall the following classical property of the geodesic foliations on pseudo-riemannian manifolds, which characterizes them. This is usually expressed for Riemannian metrics, but extends directly to pseudo-riemannian metrics:

Lemma 3.9 *(see for example [Mol)) A flow which is orthogonal to a geodesic foliation and preserves it, sends leaves to leaves isometrically.*

Fact 3.10 *H satisfies: $[H, Z] = Z$.*

Proof. H commutes with X, and hence $H^u Z$ will be a stable vector field for X. This implies that it is collinear to Z, that is $[H, Z] = fZ$, where, one easily see that f is X-invariant. Therefore by 2.10, f is invariant by the flows of Y and Z. Thus f is constant on the leaves of $\mathcal{E}$, and in particular is H-invariant. Now, we calculate the Jacobian of H^t. Since H^t preserves the metric on $\mathcal{E}^\perp$, and satisfies: $[H, Y] = -Y$, we have: $Jac(H^t) = \exp(t(f-1))$. The subset $F = \{x/f(x) > 1+\epsilon\}$ where ϵ is a positive real number, is H^t invariant (since f is H^t-invariant). But by definition $Vol(H^t(F)) > \exp(t\epsilon)Vol(F)$. Therefore: $Vol(F) = 0$, and hence $f \leq 1$, since ϵ is arbitrary. Reversing the argument yields: $f = 1$, that is $[H, Z] = Z$. □

Fact 3.11 *H is a Killing field and is in particular smooth.*

Proof. Since we already know that H^t preserves the metric on $\mathcal{E}^\perp$, it is enough to show that it preserves the metric on $\mathcal{E}$. Observe firstly that H^t preserves the metric on $\{Y, Z\}$. Indeed, This is equivalent to that, H^t preserves the isotropic directions of $\{Y, Z\}$, which are in fact the directions of Y and Z, and that the product of the dilation coefficients is 1. This is the case for H^t, by the previous facts.

Observe now that H is orthogonal to $\{Y, Z\}$. Indeed this is the case for X, by 2.3, and H is nothing but the orthogonal projection of X on $\mathcal{E}$.

To finish to prove that H^t preserves the metric on $\mathcal{E}$, we have only to show that $< H, H >$ is constant along the H-orbits, that is $< H, H >$ is H-invariant. This is the case since $< H, H >$ is X-invariant, and hence by 2.10, $< H, H >$ is constant on the leaves of $\mathcal{E}$. □

Fact 3.12 *$\mathcal{E}$ is smooth. In fact if the length $< H, H >$ is constant, then $\mathcal{E}$ has geodesic leaves, and locally, M is isometric to the metric product of a leaf of $\mathcal{E}$ and a leaf of $\mathcal{E}^\perp$. It then follows that Z is smooth.*

Proof. From 2.8, Z has geodesic orbits. The same is true for any $Y^s Z$. Therefore if II is the second fundamental of the leaves of $\mathcal{E}$, then $II(Y^s Z, Y^s Z) = 0$, for $s \in \mathbf{R}$. So, for fixed x, II_x is a (vectorial) quadratic form on the 3-dimensional space $\mathcal{E}_x$ which vanishes on the cone determined by $\{Y\} \cup \{Y^s Z, s \in \mathbf{R}\}$. It is easy to see that $II_x = 0$, whence we verify that it furthermore vanishes for some element outside this last cone.

But by 2.9, if $< H, H >$ is constant, H will have geodesic orbits, and hence $II(H, H) = 0$. Thus the leaves of $\mathcal{E}$ are geodesic in this case.

In a standard way, one prove that the couple $(\mathcal{E}, \mathcal{E}^\perp)$ induces locally a metric splitting of M. For this just verify that the adapted (common) flow box given (locally), for fixed x by the canonical local diffeomorphism $\mathcal{E}_x \times \mathcal{E}_x^\perp \to M$, is isometric.

This implies that Z is smooth. Indeed, the orthogonal of H in $\mathcal{E}$ is a 2 dimensional lorentzian subbundle, which is smooth since $\mathcal{E}$ is. It has exactly 2 isotropic directions, which thus have the same regularity. But these directions are determined by Y and Z.

Observe now that one may change the metric in the smooth following way so that $< H, H >$ becomes constant. We keep the old metric on $\mathbf{R}H^{\perp}$, and decree that H is still orthogonal to (its old orthogonal) $\mathbf{R}H^{\perp}$ and that: $< H, H >= 1$. We easily verify that the same old vector fields play the same roles in the new situation. In particular H, Y, are Killing fields. Therefore $\mathcal{E}$ and Z are smooth in every case. □

3.3 A change of Z

We will replace Z by a collinear vector field U, such that $[Y, U] = H$. Indeed we know that $[Y, Z] = T = fH$, where f is a function, invariant by X^t, and hence f is invariant by all the present flows H^t, Y^t and Z^t by 2.10. We just take: $U = (1/f)Z$. By the fact above, U is a smooth vector field.

Taking into account the bracket identities satisfied by H, Y and U, we get:

Fact 3.13 *The three vector fields H, Y and U generate an algebra isomorphic to $sl(2, \mathbf{R})$.*

Fact 3.14 *U^t preserves $\mathcal{E}^{\perp}$ and the restriction of the metric on it.*

Proof. As in the case of H, we have just to prove that U^t preserves $\mathcal{E}^{\perp}$ (since this last foliation is geodesic).

Observe that $U^t(\mathcal{E}^{\perp})$ is contained in $\mathcal{E}^{\perp} \bigoplus \{H, U\}$. Indeed this last subbundle is just $(\mathbf{R}Z)^{\perp}$, which is integrable (2.6) and hence in particular, is invariant by Z, or equivalently U. Suppose that for some t, say $t = 1$, $U^1\mathcal{E}^{\perp}$ is not contained in $\mathcal{E}$, then its projection $\mathcal{E}'$ on $\mathcal{E}$ is contained in $\{H, U\}$.

Consider the image of $\mathcal{E}^{\perp}$ by $H^1 = U^{-1}Y^{-1}U^1Y^1$. We get: $Y^{-1}(U^1\mathcal{E}^{\perp}) = U^1\mathcal{E}^{\perp}$, since $\mathcal{E}^{\perp}$ is preserved by H^t and Y^t. Therefore, the projection $\mathcal{E}'$ is invariant by Y^{-1}, and hence by the integer powers Y^m. One easily see this is impossible (the only one non trivial subbundle in $\mathcal{E}$ invariant by Y^1 is $\mathbf{R}Y$). □

Fact 3.15 *U is a Killing field.*

Proof. We have to show that U^t preserves the metric along $\mathcal{E}$. Denote by $\mathcal{B}$ the basis (H, Y, U), and $ad_U A = [U, A]$, for $A \in \mathcal{B}$. Thus preserving the metric is equivalent to ad_U being antisymmetric. More precisely, we have to prove:

$$< ad_U A, B > + < A, ad_U B >= 0,$$

for A and B in $\mathcal{B}$.

We summarize what we know, about the scalar products of basis elements: H is orthogonal to $\{Y, U\}$ and that Y and U are isotropic. About brackets, we know: $ad_U H = -U$ and $ad_U Y = -H$.

We easily verify the antisymmetry condition if $A = U$, or $A = B$. So it remains to show: $< ad_U H, Y > + < H, ad_U Y >= 0$. That is $< U, Y > + < H, H >= 0$. To prove this, we use that Y is a Killing field, and so the antisymmetry condition is satisfied by ad_Y. In particular: $0 =< ad_Y U, H > + < U, ad_Y H >=< H, H > + < U, Y >$. But this is exactly the equality that we want to prove for ad_U. □

3.4 End of the proof

We have constructed a 3 dimensional algebra $\mathbf{R}H \bigoplus \mathbf{R}Y \bigoplus \mathbf{R}U$, isomorphic to $sl(2, \mathbf{R})$ of Killing fields on M. In fact if X does not belong to this algebra, that is $X \neq H$, then we have a Killing algebra $\mathbf{R}X \bigoplus \mathbf{R}H \bigoplus \mathbf{R}Y \bigoplus \mathbf{R}U$, isomorphic to $\mathbf{R} \bigoplus sl(2, \mathbf{R})$, containing our initial Killing algebra $\mathbf{R}X \bigoplus \mathbf{R}Y$ given by the isometric AG action.

It follows from [Zeg1], that the group determined by $\mathbf{R}H \bigoplus \mathbf{R}Y \bigoplus \mathbf{R}U$ is isomorphic to some $PSL_k(2, \mathbf{R})$ (the k cover of $PSL(2, \mathbf{R})$). The group generated by $\mathbf{R}X \bigoplus \mathbf{R}H \bigoplus \mathbf{R}Y \bigoplus \mathbf{R}U$ is contained in a product $PSL_k(2, \mathbf{R}) \times K$, where K is compact. Thus its closure is of the form $PSL_k(2, \mathbf{R}) \times \mathbf{T}$, for some torus $\mathbf{T} \subset K$ (possibly of dimension 0). This finishes the proof of Theorem 1.1.

4 Proof of Theorem 1.5

The direct part in Theorem 1.5 is a consequence of 1.1 and 1.4. It remains to check the constructive part.

First of all, let us notice that to add a given compact factor K to the isometry group of a Lorentz manifold M, one may take a product of M with a suitable compact Riemannian manifold N, with $K = Isom(N)$. So, in what follow, we will forget the compact factor.

Let S be a finite cover of $PSL(2, \mathbf{R})$ or a warped Heisenberg group, endowed with a bi-invariant metric. The isometry group of S is $S \times S/Center(S)$. Let $M = S/\Gamma$ be a compact quotient. Then $Isom^0(M)$ is identified with the centralizer of Γ in $S \times S$, which equals: $Centralizer_S \Gamma \times S$. A generic Γ, has exactly the center of S as a centralizer (in S). In this case, $Isom^0(M) = S$. Therefore, by taking, as above, products with Riemannian manifolds, we realize any product $K \times S$ as the identity component of the isometry group of a compact manifold.

4.1 Breaking symmetry

Now we consider the groups of the form $G = \mathbf{R}^k \times He_n$, where He_d is the quotient of the simply connected Heisenberg group $\tilde{He_d}$ of dimension $2n + 1$, by a lattice of its center.

In fact we will start with a manifold of the form $M = S/\Gamma$, where S is a warped Heisenberg group. So, if S is endowed with a bi-invariant metric, $Isom^0(M) = S$. Our purpose is to reduce the isometry group of M, by performing respectively local and global variation of the structure.

4.1.1 Local method

Let $M = S/\Gamma$. Instead of a bi-invariant metric, the warped Heisenberg group will be endowed with just a right invariant metric, which is furthermore left invariant under the action of some subgroup $G \subset S$. Such a structure is given by a Lorentz scalar product on $\mathcal{S}$, having exactly $Ad(G)$ as isotropy group in $Ad(S)$. Let us formulate an existence statement for these scalar products.

Consider the canonical warped Heisenberg algebra $\mathcal{S}$, generated by an Heisenberg algebra $\mathcal{HE}_n$ of dimension $2n+1$, together with an exterior element T, acting as follows. The Heisengerg algebra is identified with $\mathbf{C}^n \bigoplus \mathbf{R}Z$, with the bracket rule: $[X, Y] = \omega(X, Y)Z$, for X, Y in $\mathbf{C}^n$, where ω is the canonical symplectic form on $\mathbf{C}^n$ (the other brackets vanish). Now T commutes with Z (which will thus generate the center of $\mathcal{S}$), and acts as a complex multiplication by $i(=\sqrt{-1})$ on $\mathbf{C}^n$. Denote by h the hermitian metric on $\mathbf{C}^n$. Consider the metric g_0 extending h on $\mathcal{S}$, and such that: $\mathbf{C}^d$ is orthogonal to $\{Z, T\}$, Z and T are isotropic and finally $g_0(Z, T) = 1$. Then g_0 is bi-invariant, that is any ad_u for $u \in \mathcal{S}$ is antisymmetric in the sense of g_0. We will now consider metrics g that are invariant only by a subalgebra of the Heisenberg algebra $\mathcal{HE}_n$.

Fact 4.1 *Let g be a metric on $\mathcal{S}$, such that Z is orthogonal to $\mathcal{HE}_n$ and $g(Z,T)=1$. Suppose that T is not orthogonal to $\mathbf{C}^n$. Then the interior isometry algebra $\mathcal{G}$ of g (i.e. the isometry algebra in $ad(\mathcal{S})$) is $\mathbf{R}Z \bigoplus \mathcal{G}_0$, where: $\mathcal{G}_0 = \{X \in \mathbf{C}^d / g(iX,.) = h(iX,.)\}$. Equivalently: $\mathcal{G}_0 = \{iX/X \in Kernel(h-g)\}$. In particular any subalgebra of $\mathcal{HE}_n$ containing Z can occur as an interior isometry subalgebra for some metric g on $\mathcal{S}$.*

Proof.(briefly) Let $X \in \mathbf{C}^n$, and suppose that ad_X is g-antisymmetric. Then for $Y \in \mathbf{C}^n$: $0 = g(ad_X T, Y) + g(T, ad_X Y) = -g(iX,Y) + \omega(X,Y)g(T,Z) = -g(iX,Y) + h(iX,Y)g(T,Z)$. Therefore, $X \in \mathcal{G}_0$. On the other hand, the hypothesis that T is not orthogonal to $\mathbf{C}^n$ ensures that the interior isometry algebra is contained in $\mathcal{HE}_n$. □

Note however that to be sure that $\mathcal{G} = Isom(M,g)$ one must verify that there are no "exterior" isometries of (S,g), commuting with the right translation by the elements of Γ. That is, if $f \in Isom^0(S,g)$ satisfies: $f(x\gamma) = f(x)\gamma, \forall \gamma \in \Gamma$ and $\forall x \in S$, then f is a left translation.

We believe that this is true (at least for generic g), but the proof requires some analysis. So, we prefer the following synthetic approach.

4.1.2 Global method

Now we break the isometry group by deforming the geometric structure of $M = S/\Gamma$ endowed with the bi-invariant metric g_0. As in [Ghy2], small deformations of M are obtained by small deformations of the holonomy, which are given by homomorphisms of Γ.

The isometry group of (S, g_0) contains essentially $S \times S$, where the left (resp. right) S-factor act by left (resp. right) translation. There is also a compact semi-simple factor of $Isom^0(S, g_0)$, which is $SU(n)$ acting by exterior automorphisms. Therefore $Isom^0(S)$ will be a semi-direct product of $SU(n)$ with $S \times S$.

Consider a small deformation of $\Gamma \subset \{1\} \times S$ in $S \times S$, obtained by means of a small homomorphism $\rho : \Gamma \to S$. That is the new holonomy Γ_ρ is: $\gamma \in \Gamma \to (\rho(\gamma), \gamma)$.

The identity component of the isometry group of the geometric structure (i.e. a Lorentz metric locally isometric to (S, g_0)) obtained in this way is identified with the centralizer of Γ_ρ in $Isom^0(S, g_0)$. It is easy to see (since Γ is a lattice in S) that this is the same as the centralizer of $\rho(\Gamma)$ in S.

Finally one can easily construct examples of ρ, having as a centralizer, a given (connected) subgroup of He_n containing the center.

4.2 The isometry group

The following result describes in more detail the topology of the isometry group of a compact Lorentz manifold. It says essentially that this group is generally not simply connected.

Theorem 4.2 *Let G be the identity component of the isometry group of a compact Lorentz manifold.*

(i) If the Lie algebra $\mathcal{G}$ contains a factor $\mathcal{S}$ isomorphic to $sl(2,\mathbf{R})$, then the group that this factor determines has $\mathbf{Z}$ as a fundamental group (i.e. it is a finite cover of $PSL(2,\mathbf{R})$). The abelian factor of $\mathcal{G}$ determines a compact group.

(ii) If the Lie algebra $\mathcal{G}$ contains a warped Heisenberg algebra $\mathcal{S}$, then the group that it determines has $\mathbf{Z}^2$ as a fundamental group. (It is such groups and not the simply connected ones that we were called in [Zeg1], warped Heisenberg groups). The abelian factor of $\mathcal{G}$ determines a compact group.

Conversely, let G be a direct product $L \times S$, where L is a compact Lie group and S is a finite cover of $PSL(2, \mathbf{R})$ or a warped Heisenberg group. Then G is the identity component of some compact Lorentz manifold.

(iii) Let G be a direct product $L \times \mathbf{R}^k \times S$, where L is a compact Lie group, and S is isomorphic to a circle or isomorphic to the quotient of the simply connected Heisenberg group by $\mathbf{Z}$ (up to isomorphism there is a unique discrete normal subgroup of the simply connected Heisenberg group. It is isomorphic to $\mathbf{Z}$). Then G is the identity component of some compact Lorentz manifold.

We were not able to prove the converse of the last part of the theorem, which would give a complete characterization of the identity component of isometry groups. To do this, one has essentially to consider:

Question 4.3 *Let the simply connected Heisenberg group $\tilde{He}_d$ of dimension $2d+1$ act isometrically on a compact Lorentz manifold. Is the center Z of $\tilde{He}_d$ (isomorphic to $\mathbf{R}$) act equicontinuously, i.e. respecting a Riemannian metric ?*

Proof. The direct part of the Theorem follows from 1.1 and the actual statement of 1.4 in [Zeg1], which, in addition to the structure of the Killing Lie algebra, gives some information on the group that it generates.

The constructive part is already checked during the proof of 1.5 □

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