IITAKA CONJECTURE $C_{n,m}$ IN DIMENSION SIX

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Abstract. We prove that the Iitaka conjecture $C_{n,m}$ for algebraic fibre spaces holds up to dimension 6, that is, when $n \leq 6$.

1. Introduction

We work over an algebraically closed field k of characteristic zero. Let X be a normal variety. The canonical divisor K_X is one of the most important objects associated with X especially in birational geometry. If another normal variety Z is in some way related to X, it is often crucial to find a relation between K_X and K_Z. A classical example is when Z is a smooth prime divisor on a smooth X in which case we have $(K_X + Z)|_Z = K_Z$.

An algebraic fibre space is a surjective morphism $f\colon X \to Z$ of normal projective varieties, with connected fibres. A central problem in birational geometry is the following conjecture which relates the Kodaira dimensions of X and Z. In fact, it is an attempt to relate K_X and K_Z.

Conjecture 1.1 (Iitaka)**.** *Let $f\colon X \to Z$ be an algebraic fibre space where X and Z are smooth projective varieties of dimension n and m, respectively, and let F be a general fibre of f. Then,*

$$\kappa(X) \geq \kappa(F) + \kappa(Z)$$

This conjecture is usually denoted by $C_{n,m}$. A strengthend version was proposed by Viehweg (cf. [19]) as follows which is denoted by $C^+_{n,m}$.

Conjecture 1.2 (Iitaka-Viehweg)**.** *Under the assumptions of 1.1,*

$$\kappa(X) \geq \kappa(F) + \max\{\kappa(Z), \mathrm{var}(f)\}$$

when $\kappa(Z) \geq 0$.

Kawamata [10] showed that these conjectures hold if the general fibre F has a good minimal model, in particular, if the minimal model and the abundance conjectures hold in dimension $n - m$ for varieties of

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nonnegative Kodaira dimension. However, at the moment the minimal model conjecture for such varieties is known only up to dimension 5 [3] and the abundance conjecture up to dimension 3 [15][12] and some cases in higher dimensions which will be discussed below. Viehweg [19] proved $C^+_{n,m}$ when Z is of general type. When Z is a curve $C_{n,m}$ was setteld by Kawamata [9]. Kollár [14] proved $C^+_{n,m}$ when F is of general type. The latter also follows from Kawamata [10] and the existence of good minimal models for varieties of general type by Birkar-Cascini-Hacon-M^{c}Kernan [4]. We refer the reader to Mori [16] for a detailed survey of the above conjectures and related problems. In this paper, we prove the following

Theorem 1.3. *Iitaka conjecture $C_{n,m}$ holds when $n \le 6$.*

Theorem 1.4. *Iitaka conjecture $C_{n,m}$ holds when $m = 2$ and $\kappa(F) = 0$.*

When $n \le 5$ or when $n = 6$ and $m \neq 2$, $C_{n,m}$ follows immediately from theorems of Kawamata and deep results of the minimal model program.

Iitaka conjecture is closely related to the following

Conjecture 1.5 (Ueno)**.** *Let X be a smooth projective variety with $\kappa(X) = 0$. Then, the Albanese map $\alpha\colon X \to A$ satisfies the following*

(1) $\kappa(F) = 0$ *for the general fibre F,*
(2) *there is an etale cover $A' \to A$ such that $X \times_A A'$ is birational to $F \times A'$ over A.*

Ueno conjecture is often referred to as Conjecture K. Kawamata [8] showed that α is an algebraic fibre space. See Mori [16, §10] for a discussion of this conjecture.

Corollary 1.6. *Part* (1) *of Ueno conjecture holds when* $\dim X \le 6$.

Proof. Immediate by Theorem 1.3. □

Concerning part (1) of Ueno conjecture, recently Chen and Hacon [6] showed that $\kappa(F) \le \dim A$.

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2. Preliminaries

Nef divisors. A Cartier divisor L on a projective variety X is called nef if $L \cdot C \geq 0$ for any curve $C \subseteq X$. If L is a $\mathbb{Q}$-divisor, we say that it is nef if lL is Cartier and nef for some $l \in \mathbb{N}$. We need a theorem about nef $\mathbb{Q}$-divisors due to Tsuji [18] and Bauer et al. [2].

Theorem 2.1. *Let L be a nef $\mathbb{Q}$-divisor on a normal projective variety X. Then, there is a dominant almost regular rational map $\pi\colon X \dashrightarrow Z$ with connected fibres to a normal projective variety, called the reduction map of L, such that*

(1) if a fibre F of π is projective and $\dim F = \dim X - \dim Z$, then $L|_F \equiv 0$,
(2) if C is a curve on X passing through a very general point $x \in X$ with $\dim \pi(C) > 0$, then $L \cdot C > 0$.

Here by almost regular we mean that some of the fibres of π are projective and away from the indeterminacy locus of π. Using the previous theorem, one can define the nef dimension $n(L)$ of the nef $\mathbb{Q}$-divisor L to be $n(L) := \dim Z$. In particular, if $n(L) = 0$, the theorem says that $L \equiv 0$.

Minimal models. Let X be a smooth projective variety. A projective variety Y with terminal singularities is called a minimal model of X if there is a birational map $\phi\colon X \dashrightarrow Y$, such that ϕ^{-1} does not contract divisors, K_Y is nef, and finally there is a common resolution of singularities $f\colon W \to X$ and $g\colon W \to Y$ such that $f^*K_X - g^*K_Y$ is effective and its support contains the birational transform of any prime divisor on X which is exceptional over Y. If in addition lK_Y is base point free for some $l \in \mathbb{N}$, we call Y a good minimal model.

The minimal model conjecture asserts that every smooth projective variety has a minimal model or a Mori fibre space, in particular, if the variety has nonnegative Kodaira dimension then it should have a minimal model. The abundance conjecture states that every minimal model is a good one.

Kodaira dimension. Campana and Peternell [5] made the following interesting conjecture.

Conjecture 2.2. *Let X be a smooth projective variety and suppose that $K_X \equiv A + M$ where A and M are effective and pseudo-effective $\mathbb{Q}$-divisors respectively. Then, $\kappa(X) \geq \kappa(A)$.*

They proved the conjecture in case $M \equiv 0$ [5, Theorem 3.1]. This result is an important ingredient of the proofs below.

3. Proofs

Proof. (of Theorem 1.4) We are given that the base variety Z has dimension 2 and that $\kappa(F) = 0$. We may assume that $\kappa(Z) \geq 0$ otherwise the theorem is trivial. Let $p \in \mathbb{N}$ be the smallest number such that $f_*\mathcal{O}_X(pK_X) \neq 0$. By Fujino-Mori [7, Theorem 4.5], there is a diagram

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\tau} & Z \end{array}$$

in which g is an algebraic fibre space of smooth projective varieties, σ and τ are birational, and there are $\mathbb{Q}$-divisors B and L on Z' and a $\mathbb{Q}$-divisor $R = R^+ - R^-$ on X' decomposed into its positive and negative parts satisfying the following:

(1) $B \geq 0$,

(2) L is nef,

(3) $pK_{X'} = pg^*(K_{Z'} + B + L) + R$,

(4) $g_*\mathcal{O}_{X'}(iR^+) = \mathcal{O}_{Z'}$ for any $i \in \mathbb{N}$,

(5) R^- is exceptional$/X$ and codimension of $g(\operatorname{Supp} R^-)$ in Z' is ≥ 2.

Thus for any sufficiently divisible $i \in \mathbb{N}$ we have

(6) $g_*\mathcal{O}_{X'}(ipK_{X'} + iR^-) = \mathcal{O}_{Z'}(ip(K_{Z'} + B + L))$

If the nef dimension $n(L) = 2$ or if $\kappa(Z) = \kappa(Z') = 2$, then $ip(K_{Z'} + L)$ is big for some i by Ambro [1, Theorem 0.3]. So, $ip(K_{Z'} + B + L)$ is also big and by (6) and by the fact that σ is birational and $R^- \geq 0$ is exceptional$/X$ we have

$$H^0(ipK_X) = H^0(ipK_{X'} + iR^-) = H^0(ip(K_{Z'} + B + L))$$

for sufficiently divisible $i \in \mathbb{N}$. Therefore, in this case $\kappa(X) = 2 \geq \kappa(Z)$.

If $n(L) = 1$, then the nef reduction map $\pi\colon Z' \to C$ is regular where C is a smooth projective curve, and there is a $\mathbb{Q}$-divisor D' on C such that $L \equiv \pi^* D'$ and $\deg D' > 0$ by [2, Proposition 2.11]. On the other hand, if $n(L) = 0$ then $L \equiv 0$. So, when $n(L) = 1$ or $n(L) = 0$, there is a $\mathbb{Q}$-divisor $D \geq 0$ such that $L \equiv D$. Now letting $M := \sigma_* g^*(D - L)$, for sufficiently divisible $i \in \mathbb{N}$, we have

$$\begin{aligned} H^0(ip(K_X + M)) &= H^0(ip(K_{X'} + g^*D - g^*L) + iR^-) \\ &= H^0(ip(K_{Z'} + B + D)) \end{aligned}$$

and by Campana-Peternell [5, Theorem 3.1]

$$\kappa(X) \geq \kappa(K_X + M) = \kappa(K_{Z'} + B + D) \geq \kappa(Z)$$

□

Proof. (of Theorem 1.3) We assume that $\kappa(Z) \geq 0$ and $\kappa(F) \geq 0$ otherwise the theorem is trivial.

If $m = 1$, then the theorem follows from Kawamata [9]. On the other hand, if $n - m \leq 3$, then the theorem follows from Kawamata [10] and the existence of good minimal models in dimension ≤ 3. So, from now on we assume that $n = 6$ and $m = 2$ hence $\dim F = 4$. By the flip theorem of Shokurov [17] and the termination theorem of Kawamata-Matsuda-Matsuki [13, 5-1-15] F has a minimal model (see also [3]). If $\kappa(F) > 0$, by Kawamata [11, Theorem 7.3] such a minimal model is good, so we can apply [10] again. Another possible argument would be to apply Kollár [14] when F is of general type and to use the relative Iitaka fibration otherwise.

Now assume that $\kappa(F) = 0$. In this case, though we know that F has a minimal model, abundance is not yet known. Instead, we use Theorem 1.4. □

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